PH 707: Assignment #4

August 24, 2022

Rajat Kumar Mandal

Roll No - 226121014

1 Fourth Order Runge Kutta Error Term.

The Runge Kutta approximation for $\int_{t_n}^{t_{n+1}=t_n+h} \overrightarrow{f}\left(\overrightarrow{x}\left(t\right),t\right)dt$ to solve the equation $\overrightarrow{x}\left(t\right)=\overrightarrow{f}\left(\overrightarrow{x}\left(t\right),t\right)$ is,

$$\overrightarrow{x}_{n+1}=\overrightarrow{x}_n+\frac{h}{6}\left(\overrightarrow{k}_1+2\overrightarrow{k}_2+2\overrightarrow{k}_3+\overrightarrow{k}_4\right)$$

where,

$$\begin{aligned}
\overrightarrow{k}_1 &= \overrightarrow{f}\left(\overrightarrow{x}_n,t_n\right)\\
\overrightarrow{k}_2 &= \overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{k}_1,t_n+\frac{h}{2}\right)\\
\overrightarrow{k}_3 &= \overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{k}_2,t_n+\frac{h}{2}\right)\\
\overrightarrow{k}_4 &= \overrightarrow{f}\left(\overrightarrow{x}_n+\overrightarrow{k}_3,t_n+h\right).
\end{aligned}$$

Putting these together,

$$\begin{aligned}
\overrightarrow{x}_{n+1} = \overrightarrow{x}_n &+ \frac{h}{6}\overrightarrow{f}\left(\overrightarrow{x}_n,t_n\right)+\frac{h}{3}\overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{f}\left(\overrightarrow{x}_n,t_n\right),t_n+\frac{h}{2}\right)\\
&+\frac{h}{3}\overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{f}\left(\overrightarrow{x}_n,t_n\right),t_n+\frac{h}{2}\right),t_n+\frac{h}{2}\right)\\
&+\frac{h}{6}\overrightarrow{f}\left(\overrightarrow{x}_n+\overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{f}\left(\overrightarrow{x}_n+\frac{h}{2}\overrightarrow{f}\left(\overrightarrow{x}_n,t_n\right),t_n+\frac{h}{2}\right),t_n+\frac{h}{2}\right),t_n+h\right).
\end{aligned}$$

Now we can expand $E\left(h\right)=\int_{t_n}^{t_n+h}\overrightarrow{f}\left(\overrightarrow{x}\left(t\right),t\right)dt-\overrightarrow{x}_{n+1}\left(h\right)$ in a Taylor expansion of h up to 5th order in Mathematica. **The code and the results are given in the next page**. Note that the lowest order term is $o\left(h^5\right)$ so that the locally it is correct upto 4th order. Globally(in the whole problem range, not just in $\left(t_n,t_{n+1}\right)$), one upper bound of the error is $O\left(h^4\right)$(in general difficult to prove for arbitrary differential equations and arbitrary ranges).

Here is a cleaner Mathematica code for the error term

```
In[1]:= K1[h_] := h f[tn, x[tn]]
        K2[h_] := h f[tn + 1 / 2 h, x[tn] + 1 / 2 K1[h]]
        K3[h_] := h f[tn + 1 / 2 h,  x[tn] + 1 / 2 K2[h]]
        K4[h_] := h f[tn + h, x[tn] + K3[h]]
        RK4Approx[h_] := 1 / 6 (K1[h] + 2 K2[h] + 2 K3[h] + K4[h])
        Exact[h_] := Integrate[ f[t, x[t]], {t, tn, tn + h} ]

In[7]:= y[t_] := f[t, x[t]]
        x'[t_] := y[t]
        x''[t_] := y'[t]
        x'''[t_] := y''[t]
        x''''[t_] := y'''[t]
        FullSimplify[Series[Exact[h] - RK4Approx[h], {h, 0, 5}]]
```

Out[12]=

$$
\begin{aligned}
\frac{1}{2880} \Big(& -f[\text{tn}, x[\text{tn}]]^4 f^{(0,4)}[\text{tn}, x[\text{tn}]] + 24 f^{(0,1)}[\text{tn}, x[\text{tn}]]^3 f^{(1,0)}[\text{tn}, x[\text{tn}]] + f[\text{tn}, x[\text{tn}]]^3 \\
&\big(6 f^{(0,2)}[\text{tn}, x[\text{tn}]]^2 - 2 f^{(0,1)}[\text{tn}, x[\text{tn}]] f^{(0,3)}[\text{tn}, x[\text{tn}]] - 4 f^{(1,3)}[\text{tn}, x[\text{tn}]]\big) - \\
&6 f^{(0,1)}[\text{tn}, x[\text{tn}]]^2 f^{(2,0)}[\text{tn}, x[\text{tn}]] + 6 f^{(1,1)}[\text{tn}, x[\text{tn}]] f^{(2,0)}[\text{tn}, x[\text{tn}]] - \\
&6 f^{(1,0)}[\text{tn}, x[\text{tn}]] \big(3 f^{(0,2)}[\text{tn}, x[\text{tn}]] f^{(1,0)}[\text{tn}, x[\text{tn}]] + f^{(2,1)}[\text{tn}, x[\text{tn}]]\big) - \\
&6 f[\text{tn}, x[\text{tn}]]^2 \big(f^{(0,3)}[\text{tn}, x[\text{tn}]] f^{(1,0)}[\text{tn}, x[\text{tn}]] + \\
&\quad 3 f^{(0,2)}[\text{tn}, x[\text{tn}]] \big(2 f^{(0,1)}[\text{tn}, x[\text{tn}]]^2 - f^{(1,1)}[\text{tn}, x[\text{tn}]]\big) + f^{(2,2)}[\text{tn}, x[\text{tn}]]\big) + \\
&4 f^{(0,1)}[\text{tn}, x[\text{tn}]] \big(-3 f^{(1,0)}[\text{tn}, x[\text{tn}]] f^{(1,1)}[\text{tn}, x[\text{tn}]] + f^{(3,0)}[\text{tn}, x[\text{tn}]]\big) + \\
&2 f[\text{tn}, x[\text{tn}]] \\
&\big(3 \big(4 f^{(0,1)}[\text{tn}, x[\text{tn}]]^4 - 4 f^{(0,1)}[\text{tn}, x[\text{tn}]]^2 f^{(1,1)}[\text{tn}, x[\text{tn}]] + 2 f^{(1,1)}[\text{tn}, x[\text{tn}]]^2 - \\
&\quad 2 f^{(1,0)}[\text{tn}, x[\text{tn}]] f^{(1,2)}[\text{tn}, x[\text{tn}]] + f^{(0,2)}[\text{tn}, x[\text{tn}]] f^{(2,0)}[\text{tn}, x[\text{tn}]] + \\
&\quad f^{(0,1)}[\text{tn}, x[\text{tn}]] \big(-8 f^{(0,2)}[\text{tn}, x[\text{tn}]] f^{(1,0)}[\text{tn}, x[\text{tn}]] + f^{(2,1)}[\text{tn}, x[\text{tn}]]\big)\big) - \\
&2 f^{(3,1)}[\text{tn}, x[\text{tn}]]\big) - f^{(4,0)}[\text{tn}, x[\text{tn}]]\Big) h^5 + O[h]^6
\end{aligned}
$$

2 Physical Pendulum Using Runge Kutta 4th Order.

In the following pages, the codes and the results are presented in the following order:

1. Physical pendulum solution for stepsize $\frac{2\pi-0}{1000} = 0.00628319$ in the range $(0, 2\pi)$ and comparison with Taylor approximation.
2. The solution for various step-sizes and the optimal step size, by inspection.
3. A better, easier, less computationally expensive and very well-known method of controlling for the step-sizes in RK, the Adaptive Runge-Kutta method applied to the physical pendulum.

It helps to summarize the method using the Butcher tableaux as follows:

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	1/3	1/3	1/6

0						
1/4	1/4					
3/8	3/32	9/32				
12/13	1932/2197	−7200/2197	7296/2197			
1	439/216	−8	3680/513	-845/4104		
1/2	−8/27	2	−3544/2565	1859/4104	−11/40	
	16/135	0	6656/12825	28561/56430	−9/50	2/55
	25/216	0	1408/2565	2197/4104	−1/5	0

Figure 1: Butcher Tables for 4th order Runge Kutta and Adaptive Runge Kutta