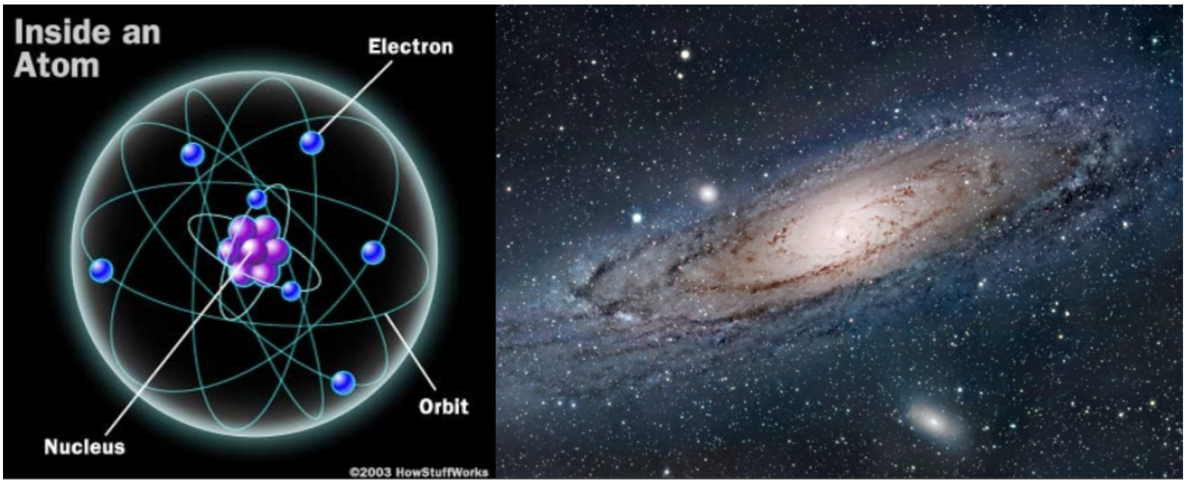
Kinematics of Rotational Motion

Background



What do the motion of a compact disc, a Ferris wheel, a circular saw blade, and a ceiling fan have in common? None of these can be represented adequately as a moving **point**; each involves a body that rotates about an axis that is stationary in some inertial frame of reference.

Rotation is occurs at all scales.

- motions of electrons in atoms
- motion of planets
- motion of galaxies

We need to develop some general methods for analyzing the motion of a rotating body.

How to analyze rotating bodies?

Real-world bodies are complicated - to simplify, we use **RIGID bodies**.

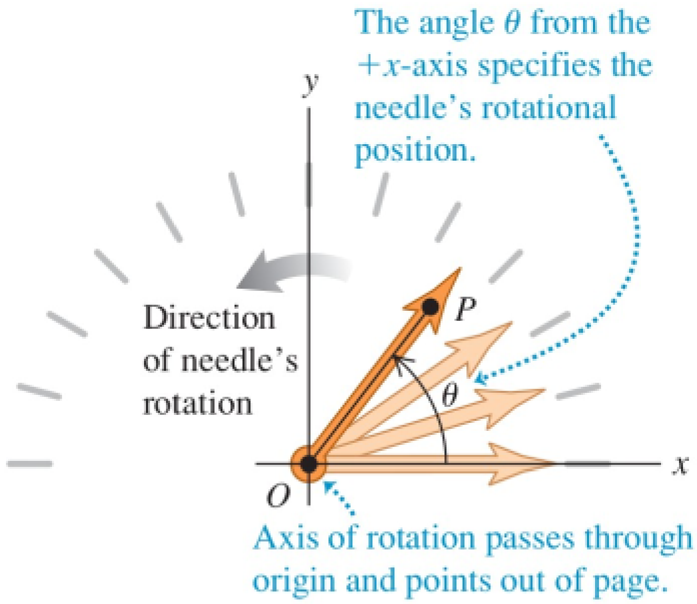
- ***Rigid bodies*** are idealized model that has definite and unchanging shape and size

We begin with kinematic language for describing rotational motion.

Angular Displacement

Guiding questions on how to model rotational motion:

1. How do we analyze a body in a rotational motion?
2. What quantities do we need to define?
    - How to track the rotation (i.e. point P)?



The simplest way is to provide x and y coordinate. But this is inconvenient. Note that x and y can be expressed using r and θ.

$$y = r\sin\theta \qquad (1)$$
$$x = r\cos\theta$$

Both have a dependence on the angle θ. Thus, using the angle of rotation, we have reduced the number of variables from two (x,y) to 1 (θ) in describing the rotational motion. This is a more convenient way to describe the rotational motion.

Angle of rotation (in radians):

We use the angle of rotation θ to quantify rotation. This angle is related to the radius and perimeter of the circular motion.

$$\theta = \frac{s}{r} \qquad (2)$$

Angular displacement (θ) is usually expressed in radians, in degrees, or in revolutions:

$$1 \text{ rev} = 360^\circ = 2\pi \text{ rad} \qquad (3)$$
$$1 \text{ rad} = 57.3^\circ$$

Angular Velocity

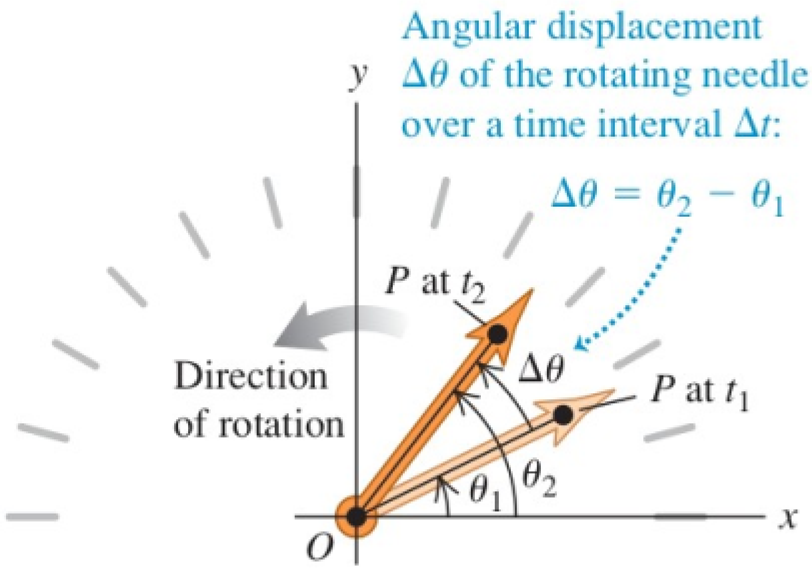
There are two ways to describe angular velocity: average angular velocity, instantaneous angular velocity.

Average angular velocity

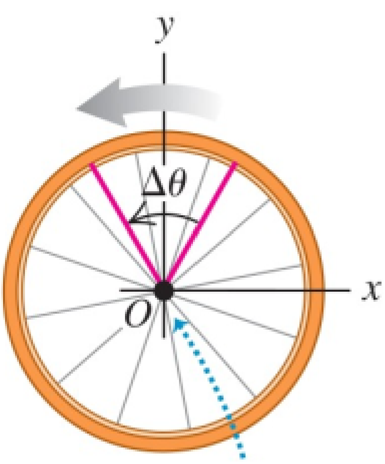
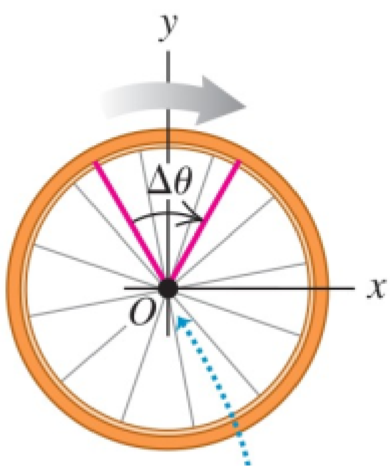
The average angular velocity is defined as,

$$\omega_{av-aor} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t} = \frac{[rad]}{[sec]} \tag{4}$$

where aor means the *axis of rotation*.



There are two types of rotation:

Counterclockwise rotation (positive)	Clockwise rotation (negative)
	
$\Delta\theta > 0$ $\omega_{av-z} = \frac{\Delta\theta}{\Delta t} > 0$	$\Delta\theta < 0$ $\omega_{av-z} = \frac{\Delta\theta}{\Delta t} < 0$

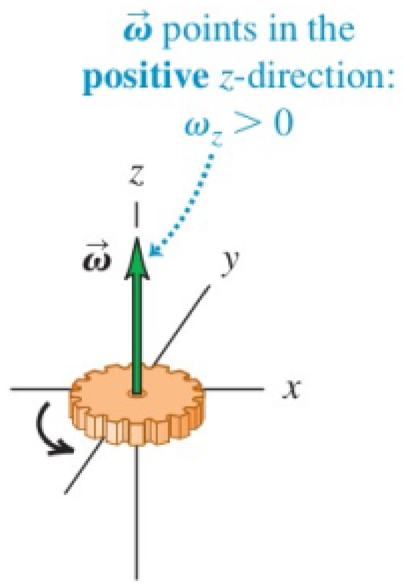
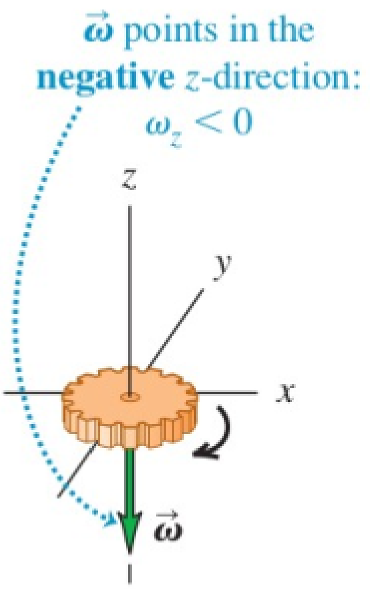
For the example above, note that subscript $av - z$ in the ω, the av means average while the z dictates the axis of rotation (aor). Thus the notation reads as,

$$\omega_{av-z} \equiv \text{average angular velocity along the z axis} \tag{5}$$

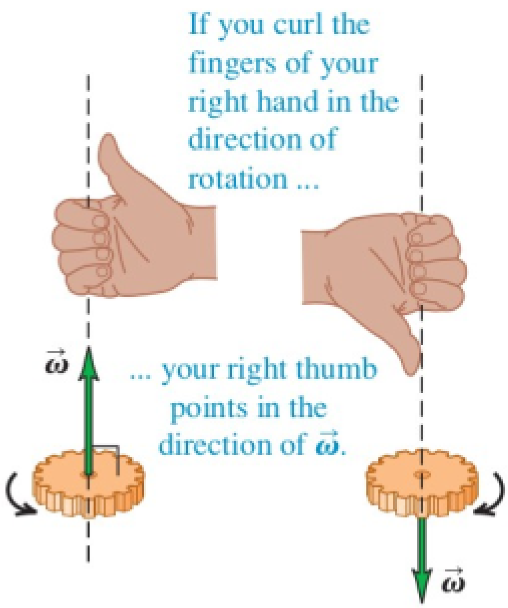
(Instantaneous) angular velocity

The instantaneous angular velocity is defined as,

$$\omega_{aor} = \frac{d\theta}{dt} = \frac{[rad]}{[sec]} \tag{6}$$

Positive z-direction (Counterclockwise)	Negative z-direction (Clockwise)
	

For both cases, the direction of the angular velocity (which is a vector) is dictated by the *axis of rotation*. To determine the direction of angular velocity use the **right-hand rule**:



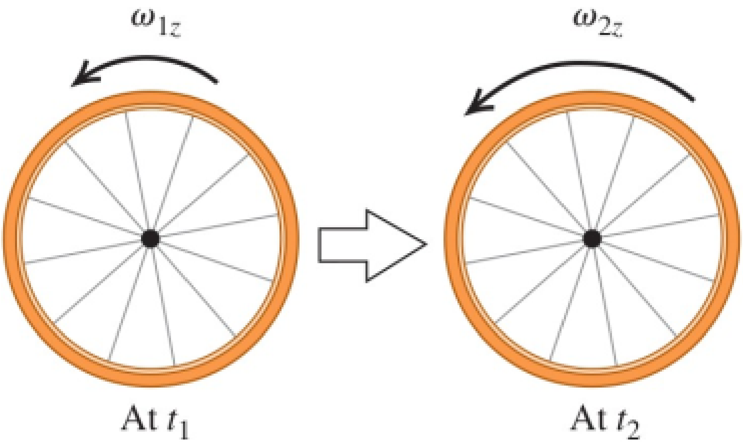
For instance, if the axis of rotation is z-axis, the angular velocity ($\hat{\omega}$) has only a z-component, e.g. $\vec{\omega} = \omega\hat{z}$.

Angular Acceleration

Average angular acceleration

The average angular acceleration is the change in angular velocity divided by the time interval:

$$\alpha_{av-z} = \frac{\omega_{2z} - \omega_{2z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t} \tag{7}$$

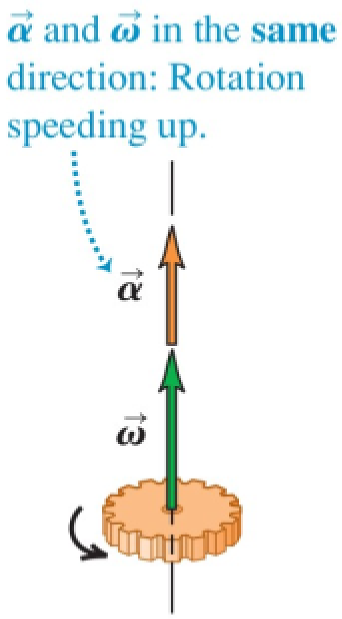
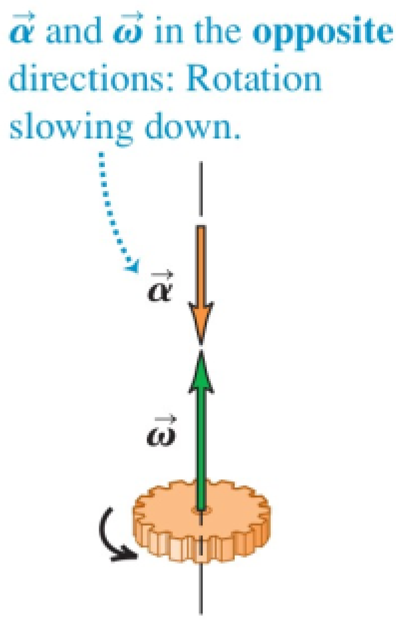


Instantaneous angular acceleration

The angular acceleration is defined as,

$$\alpha_{aor} = \frac{d\omega_{aor}}{dt} = \frac{d^2\theta}{dt^2} = \frac{[rad]}{[s^2]} \tag{8}$$

How fast is the rotation?

Angular acceleration	Angular Deceleration
	

Translational vs Rotational Motion

In previous discussions, we found that straight-line motion have a simple set of equations if acceleration is constant. This is also true in rotational motion about a fixed axis. We can derive equations for angular velocity and angular position using exactly the same procedure that we used for straight-line motion.

In fact, the set of equations are identical to the straight-line motion if we replace $x \to \theta$, $v_x \to \omega_z$, and $a_x \to \alpha_z$.

Straight-Line Motion with Constant Linear Acceleration	Fixed-Axis Rotation with Constant Angular Acceleration
$a_x = \text{constant}$	$\alpha_z = \text{constant}$
$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
$x = x_0 + v_{0x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$
$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

> Keep in mind that all of these results are valid only when the angular acceleration α_z is constant; be careful not to apply them to problems in which α_z is not constant.

Relating linear and Angular Kinematics

How do we find the linear speed and acceleration of a particular point in a rotating rigid body?

- we need to answer this question to proceed with our study of rotation

The key equation to derive the relationship between translational and rotational motion is the relationship between **arc length** (s) and the **angle of rotation** (θ) given by:

$$s = r\theta \tag{9}$$

Now, we take the time derivative and the absolute value,

$$\begin{aligned} \frac{d}{dt}\left\{s = r\theta\right\} \\ \underbrace{\left|\frac{ds}{dt}\right|}_{v} = r\underbrace{\left|\frac{d\theta}{dt}\right|}_{\omega} \end{aligned} \tag{10}$$

Thus, the relationship between linear and angular speeds is given by:

$$v = r\omega \tag{11}$$

From this, we can derive the tangential acceleration,

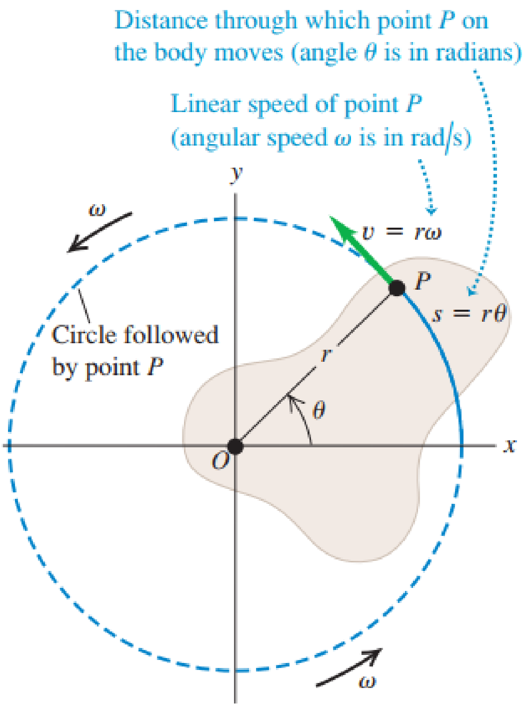
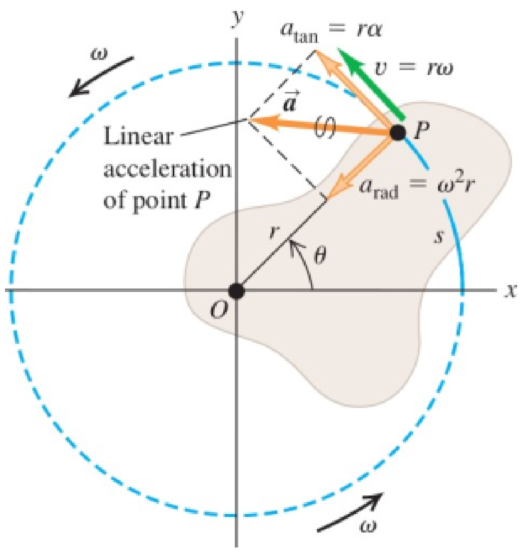
$$a_{tan} = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{12}$$

Remarks:

- the v and ω are **NEVER** negative, they are magnitudes of the vector $\vec{v}$ and $\vec{\omega}$
    - v_x and ω_z, on the other hand, can be either negative or positive; their signs tell you the direction of the motion
- the farther a point is from the axis, the greater its linear speed
- direction of linear velocity (v) vector is tangent to its circular path at each point.
- direction of the angular velocity ω is along the axis of rotation

- a_{tan} is parallel to the instantaneous velocity v, and acts to change the magnitude of particle's velocity

> Note that r is taken to be constant, this is an important assumption.

<table>
<tr><th>Linear and Angular Velocity</th><th>Linear and Angular Acceleration</th></tr>
<tr><td>
</td><td>
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In the discussion of uniform circular motion we have derived the equation for the radial acceleration:

$$a_{rad} = \frac{v^2}{r} \tag{13}$$

We can express this in terms of the angular velocity using the previous equation that we have derived, thus:

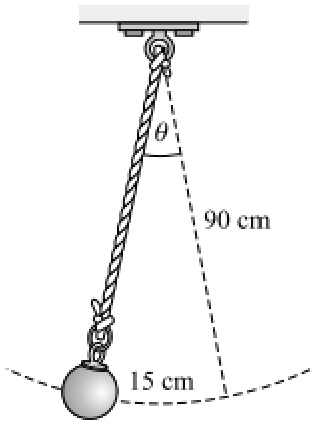
$$a_{rad} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{14}$$

The component of the particle's acceleration directed toward the rotation axis is also known as the centripetal component of the acceleration. The vector sum of the centripetal and tangential components of acceleration of a particle in a rotating body is the linear acceleration $\vec{a}$.

Implementation

Angular Displacement

> The bob of a pendulum 90 cm long swings through a 15-cm arc. Find the angle \theta, in radians and in degrees, through which it swings.



Recall that $s = r\theta$ applies only to angles in radian measure. Therefore, in radians,

$$\theta = \frac{s}{r} = \frac{0.15}{0.90} = 0.167 \text{ rad} \tag{15}$$
$$\theta = 0.167 \text{ rad} \times \frac{360^\circ}{2\pi \text{ rad}} = 9.6^\circ$$

Instantaneous Angular acceleration

> The angular velocity of a flywheel obeys the equation $\omega_z(t) = A + Bt^2$, where t is in seconds and A and B are constants having numerical values 2.75 (for A) and 1.50 (for B).
>
> (a) What are the units of A and B if ω_z is in rad/s?
>
> (b) What is the angular acceleration of the wheel at (i) t=0.00 and (ii) t=5.00 s?
>
> (c) Through what angle does the flywheel turn during the first 2.00 s?

`(a)` Since ω_z is in rad/s, the constant A should be in rad/s while B has to be in rad/s^3

`(b)` To find the angular acceleration, we take the time derivative of ω_z, thus we have,

$$\begin{aligned}
\frac{d}{dt}\left\{\omega_z(t) = A + Bt^2\right\} \\
\frac{d\omega_z(t)}{dt} &= 2Bt \\
\alpha_z(t) &= 2Bt
\end{aligned}$$

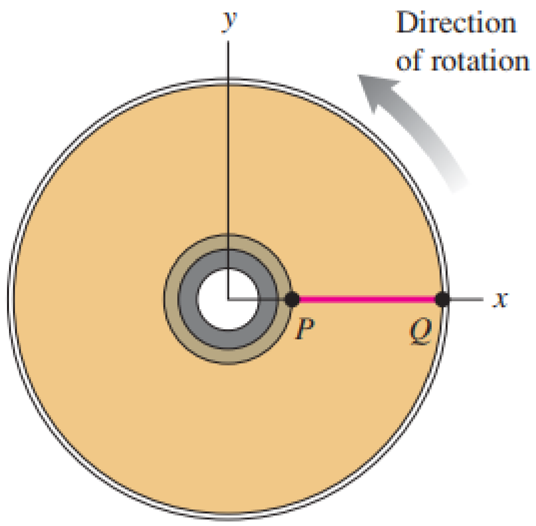
Thus, for (i) $t = 0.0$ the angular acceleration is zero while for (ii) $t = 2.00$ the angular acceleration is $\alpha_z = 2(1.50)(2.00) = 6\text{rad}/\text{sec}^2$.

`(c)` To find the angle during the first 2.0s from the angular velocity, we have to use integration:

$$\begin{aligned}
\omega_z(t) = \frac{d\theta}{dt} &= A + Bt^2 \\
d\theta &= (A + Bt^2)dt \\
\int d\theta &= \int (A + Bt^2)dt \\
\theta &= \int_0^{2.00} (A + Bt^2)dt \\
\theta &= At\Big|_0^{2.00} + \frac{1}{3}Bt^3\Big|_0^{2.00} \\
\theta &= 2.75\frac{[rad]}{[sec]} \times 2.00sec + \frac{1}{3} \cdot 1.50\frac{[rad]}{[sec]^3}(2.00)^3 sec^3 \\
\theta &= 9.5 \text{ rad}
\end{aligned}$$

Rotation with constant angular acceleration

> You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at $t=0$ is 27.5 rad/s and its angular acceleration is a constant -10.0 rad/s^2 . A line PQ on the disc's surface lies along the $+x$-axis at $t=0$. (a) What is the disc's angular velocity at t=0.300 sec?
>
> (b) What angle does the line PQ make with the $+x$-axis at this time?



`(a)` We can use the following equation to find for the disc's angular velocity:

$$\omega_z = \omega_{0z} + \alpha_z t \tag{16}$$

At t=0 the disc's angular velocity is said to be 27.5 rad/s this translates to $\omega_{0z} = 27.5 \ \text{rad/s}$. The angular acceleration is also given which is $\alpha_z = -10 \ \text{rad}/s^2$. Plugging this values into *equation (16)*, at $t = 0.3$ we have,

$$\begin{aligned}
\omega_z &= \omega_{0z} + \alpha_z t \\
\omega_z &= 27.5\text{rad/s} + (-10.0\text{rad/s}^2)(0.300s) \\
\omega_z &= 24.5 \ \text{rad/s}
\end{aligned}$$

`(b)` We can use the following equation to find for the disc's angular displacement at a particular time:

$$\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2 \tag{17}$$

The line PQ on the disc's surface lies along the $+x$-axis at $t=0$, which translates to $\theta_0 = 0$. Plugging this values into *equation (17)*, at $t = 0.3$ we have,

$$\begin{aligned}
&\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha_z t^2 \\
&\theta = 0 + (27.5 \ \text{rad/s})(0.300 \ \text{s}) + \frac{1}{2}(-10.0 \ \text{rad/s}^2)(0.300s)^2 \\
&\theta = 7.80 \ \text{rad}
\end{aligned}$$

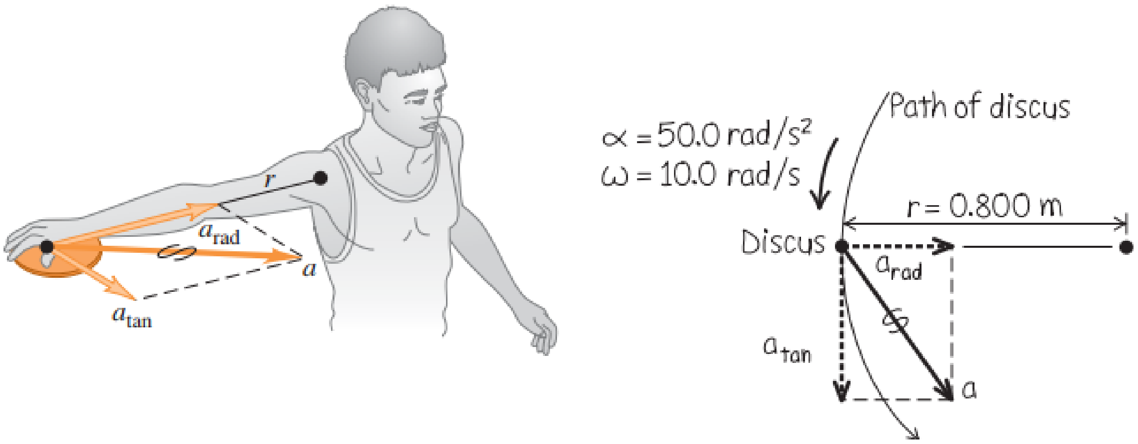
We can convert θ in terms of the number of revolution,

$$\theta = 7.80 \ \text{rad} \times \frac{1rev}{2\pi \ \text{rad}} = 1.24 \ \text{rev} \tag{18}$$

The disc has turned through one complete revolution plus an additional 0.24 revolution - that is, through $360°$ plus $0.24\,\mathrm{rev}\cdot(360°/\mathrm{rev}) = 87°$. Hence the line PQ makes an angle of $87°$ with the +x-axis.

Relating linear and Angular Kinematics

> An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10 rad/s and the angular speed is increasing at 50 rad/s^2. At this instant, (a) find the tangential and centripetal components of the acceleration of the discus and (b) the magnitude of the acceleration.



`(a)` From the equation of the tangential and centripetal components,

$$a_{tan} = r\alpha = (0.800\ \mathrm{m})(50.0\ \mathrm{rad}/s^2) = 40.0\ \mathrm{m/s}\hat{}2$$
$$a_{rad} = r\omega^2 = (0.800\ \mathrm{m})(10.0\ \mathrm{rad}/s)^2 = 80.0\ \mathrm{m/s}\hat{}2$$

`b` The magnitude of the acceleration then,

$$a = \sqrt{a_{tan}^2 + a_{rad}^2}$$
$$a = \sqrt{(40\ \mathrm{m/s}\hat{}2)^2 + (80\ \mathrm{m/s}\hat{}2)^2}$$
$$a = 89.4\ \mathrm{m/s}\hat{}2$$

Perspective

Things to remember:

- Note that all the angular motion we've considered are on a plane.
- Use angles in radians in all equations.