Dynamics of Rotational Motion

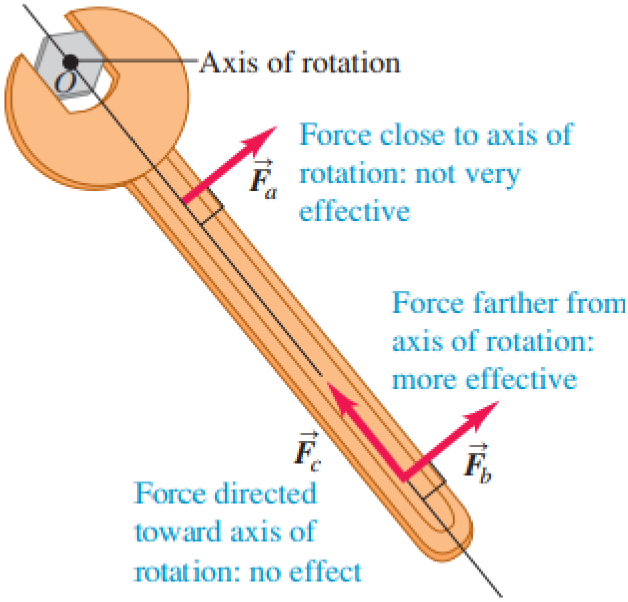
Background

We learned in previous discussions that a net force applied to a body gives that an acceleration. But what does it take to give a body an **angular acceleration**? That is, what does it take to start a stationary body rotating or to bring a spinning body to a halt?

A force is required, but it must be applied in a way that gives a twisting or turning action. In this section, we will define a new physical quantity that describes twisting or turning effort of a force - the **torque**.

Torque

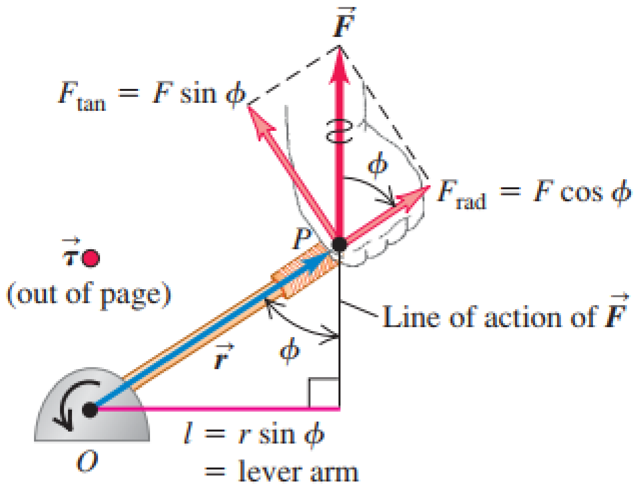
The magnitude and direction of the force are important, but so is the point on the body where the force is applied. In the figure below, a wrench is being used to loosen a tight bolt. Force $\vec{F}_b$ applied near the end of the handle is more effective than an equal force $\vec{F}_a$ applied near the bolt.



Force $\vec{F}_c$ doesn't do any good at all; its applied at the same point and has the same magnitude as $\vec{F}_b$, but it's directed along the length of the handle. The quantitative measure of the tendency of a force to cause or change a body's rotational motion is called torque; we say that $\vec{F}_a$ applies a torque about point O to the wrench, $\vec{F}_b$ applies a greater torque about O, and $\vec{F}_c$ applies zero torque about O.

How to calculate torque

There are three ways to calculate the torque of the force about the point O.



Solution 1: Find the lever arm l and use $\tau = Fl$

Solution 2: Determine the angle ϕ between the vectors $\vec{r}$ and $\vec{F}$; the lever arm is $r \sin\phi$, so $\tau = rF\sin\phi$.

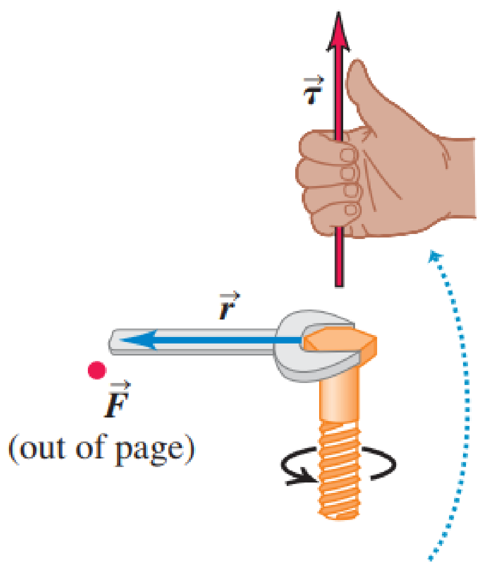
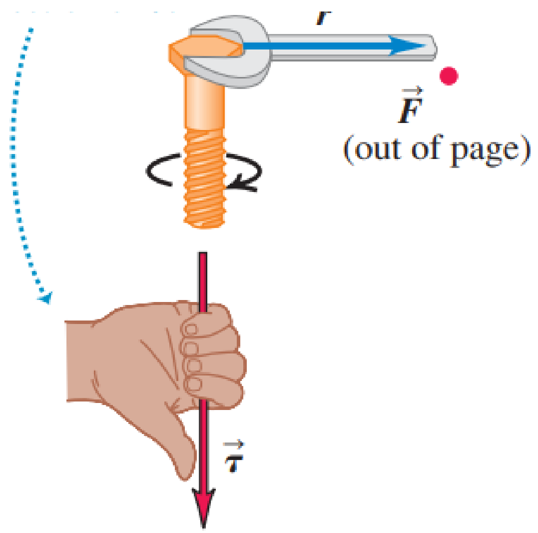
Solution 3: Decompose $\vec{F}$ in terms of a radial component F_{rad} (along the r) and a tangential component $\vec{F}_{tan}$ (perpendicular to r). Then $F_{tan} = F\sin\phi$ and the torque is $\tau = r(F\sin\phi) = F_{tan}r$.

- The component F_{rad} produces no torque with respect to O because its lever arm with respect to that point is zero.

Torque as a Vector

We now generalize the definition of torque as follows: when a force $\vec{F}$ acts at a point having a position vector $\vec{r}$ with respect to an origin O, the torque $\vec{\tau}$ of the force with respect to O is the vector quantity,

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{1}$$

<table>
<tr><th>Case 1</th><th>Case 2</th></tr>
<tr><td>
</td><td>
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The direction of torque must be perpendicular to the plane of vectors r and F.

Net Torque

For a rigid body as a whole, the rotational analog of Newton's second law is,

$$\Sigma\tau_z = I\alpha_z \tag{2}$$

Remarks:

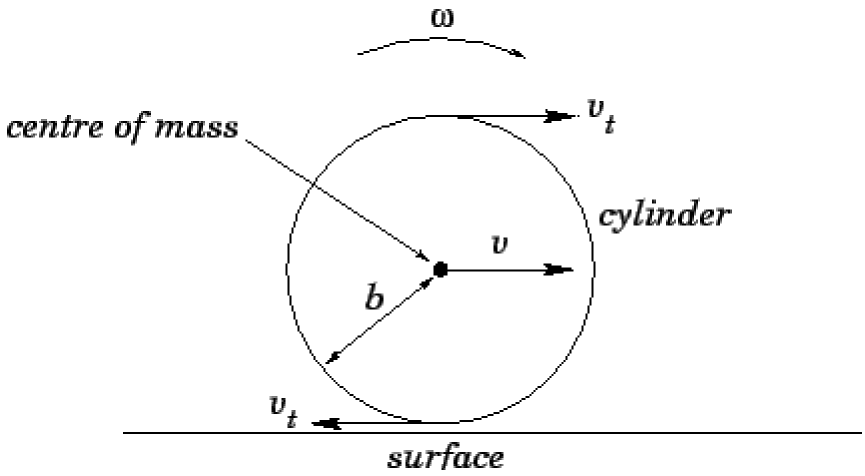
- α_z is the same for every particle in a rigid body
- I is the moment of inertia
- The left hand side $\Sigma\tau_z$ is the summation of all external torques

This is the relationship for the rotational dynamics of a rigid body. The angular acceleration of a rotating rigid body is directly proportional to the sum of the torque components along the axis of rotation. The proportionality factor is the moment of inertia.

Rotation about a moving axis

We can extend our analysis of the dynamics of rotational motion to some cases in which the axis of rotation moves. When that happens, the motion of the body is combined translation and rotation.

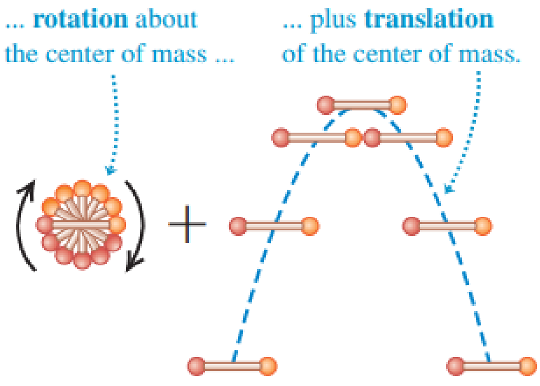
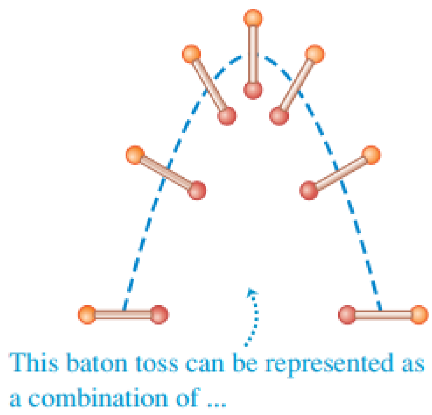
Combined Translation and Rotation: Dynamics



Every possible motion of a rigid body can be represented as a combination of

1. translational motion of the center of mass
2. rotation about an axis through the center of mass

The following figure illustrates the motion of translational and rotation:

Individual Motion	Combined Motion
	

For a body with a total mass M, the acceleration $\vec{a}_{cm}$ of the center of mass is the same as that of a point mass M acted on by all the external forces on the actual body:

$$\Sigma\vec{F}_{ext} = M\vec{a}_{cm} \tag{3}$$

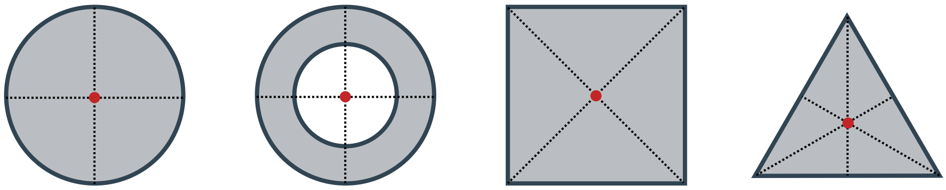
The rotational motion about the center of mass is described by the rotational analog of Newton's second law:

$$\Sigma\tau_z = I_{cm}\alpha_z \tag{4}$$

where I_{cm} is the moment of inertia with respect to an axis through the center of mass and the sum $\Sigma\tau_z$ includes all external torques with respect to this axis.

Center of Mass

The *center of mass (cm)* is a position defined relative to an object or system of objects. It is the average position of all the parts of the system, weighted according to their masses. For simple geometric shapes, the center of mass is indicated by the red dots in the figure below



For more complicated shapes, we need a more general mathematical definition of the center of mass. In general the center of mass can be found by the weighted position vectors which point to the center of mass of each object in a system.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i \vec{r}_i}{\Sigma_i m_i} \tag{5}$$

One quick technique which lets us avoid the use of vector arithmetic is finding the center of mass separately for components along each axis:

$$\begin{aligned} x_{cm} &= \frac{m_1x_1 + m_2x_2 + m_2x_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i x_i}{\Sigma_i m_i} \\ y_{cm} &= \frac{m_1y_1 + m_2y_2 + m_2y_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i y_i}{\Sigma_i m_i} \end{aligned} \tag{6}$$

When a rigid body moves, its translational motion is reduced to the motion of the center of mass. For the rotational motion, it rotates about an axis through the center of mass.

Combined Translation and Rotation: Energy Relationship

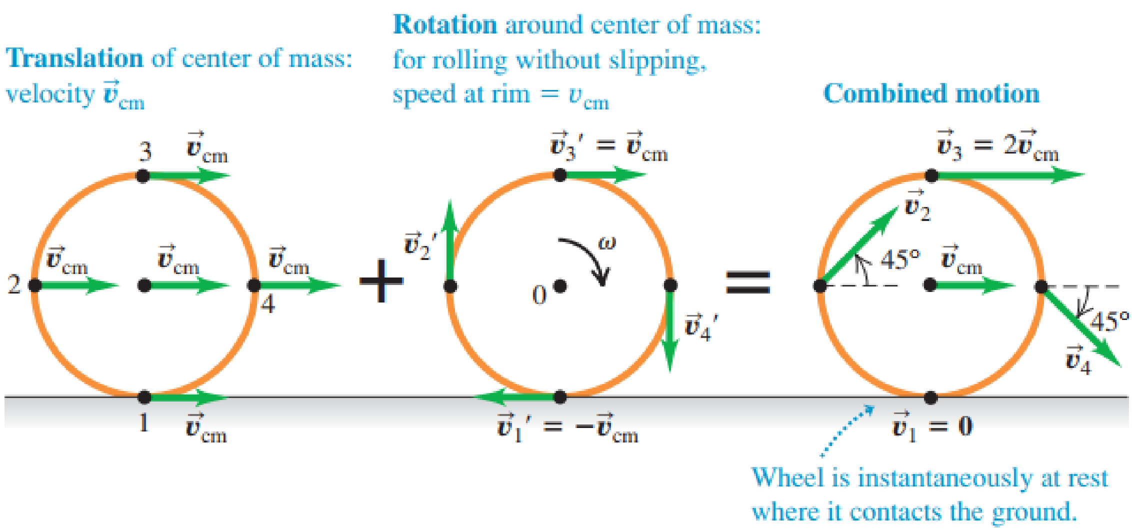
What happens to the kinetic energy of rigid body that has both translational and rotational motions?

In this case, the body's kinetic energy is the sum of the a part of K_{trans} associated with motion of the center of mass and a part K_{rot} associated with rotation about an axis through the center of mass:

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \tag{7}$$

Rolling without Slipping

An important case of combined translation and rotation is **rolling without slipping**, such as the motion of the wheel.



If the radius of the wheel is R and its angular speed about the center of mass is ω, then the magnitude of $\vec{v_1}'$ is $R\omega$; hence we must have

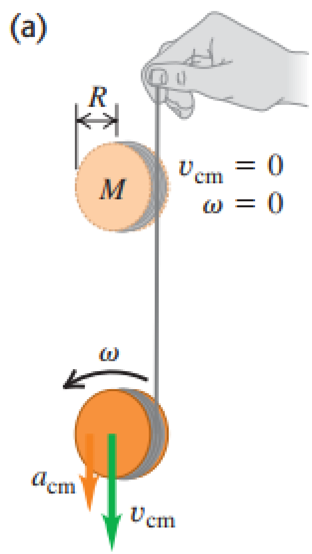
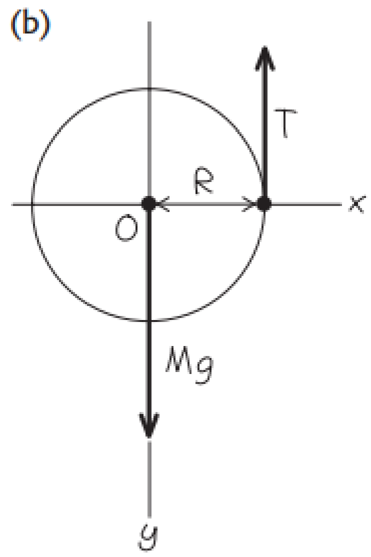
$$v_{cm} = R\omega \tag{8}$$

The figure above shows the velocity of a point on the wheel is the vector sum of the velocity of the center of mass and the velocity of the point relative to the center of mass.

Implementation

Acceleration of a primitive yo-yo

> For the primitive yo-yo in the figure below, find the (a) downward acceleration of the cylinder and the (b) tension in the string.

Dynamics of a yo-yo	Free-body diagram
	

Solution:

`(a)` We want to find a_{cm-y}. For the translation motion use equation (3),

$$\Sigma \vec{F}_{ext} = M\vec{a}_{cm} \tag{9}$$

We have to know the external forces acting on the body to fully solve the problem. One external force acting on the yo-yo is gravity, trying to pull down the mass M. Another external force is the tension due to the rope attached to the yo-yo. From this external forces, we have:

$$\Sigma \vec{F}_{ext} = M\vec{a}_{cm} \tag{10}$$
$$Mg + (-T) = Ma_{cm-y} \tag{11}$$

If you look at the free-body diagram in figure (b), one can see that the weight does not contribute to the torque on the rigid body because it passes through the axis of rotation (at O). On the other hand, the tension contributes, $\tau_T = Fl = TR$, thus we have:

$$\Sigma \tau_z = I_{cm}\alpha_z \tag{12}$$
$$\Sigma \tau_z = 0 + TR = I_{cm}\alpha_z = \frac{1}{2}MR^2\alpha_z \tag{13}$$

How do we eliminate α_z from equation (13)? We can use the relationship of the linear and rotational angular speed given in equation (8) and take its derivative.

$$\frac{d}{dt}\left\{v_{cm-y} = R\omega_z\right\}$$
$$a_{cm-y} = R\frac{d\omega_z}{dt} = R\alpha_z$$

We put equation (11) and equation (13) together to solve for a_{cm-y}:

$$\begin{aligned}
Mg + (-T) &= Ma_{cm-y} \\
MgR + (-T)R &= MRa_{cm-y} \\
MgR - I_{cm}\alpha_z &= MRa_{cm-y} \\
MgR - I_{cm}\frac{a_{cm-y}}{R} &= MRa_{cm-y} \\
MgR - \frac{1}{2}MR^2\frac{a_{cm-y}}{R} &= MRa_{cm-y} \\
MgR - \frac{1}{2}MRa_{cm-y} &= MRa_{cm-y}
\end{aligned}$$

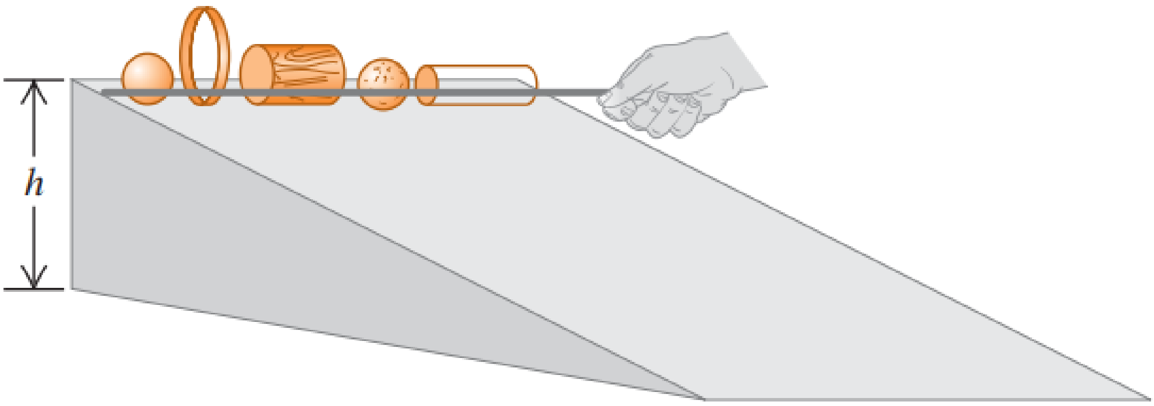
$$\begin{aligned}
MgR &= MRa_{cm-y} + \frac{1}{2}MRa_{cm-y} \\
MgR &= \left(MR + \frac{1}{2}MR\right)a_{cm-y} \\
MgR &= \frac{3}{2}MRa_{cm-y} \\
\frac{2}{3}g &= a_{cm-y}
\end{aligned}$$

b To solve for the tension, we start with equation (13):

$$\begin{aligned}
TR = I_{cm}\alpha_z &= \frac{1}{2}MR^2\alpha_z \\
T &= \frac{1}{2}MR\alpha_z \\
T &= \frac{1}{2}MR\frac{a_{cm-y}}{R} \\
T &= \frac{1}{2}Ma_{cm-y} \\
T &= \frac{1}{2}M\frac{2}{3}g \\
T &= \frac{1}{3}Mg
\end{aligned}$$

Conservation of Mechanical Energy

> In a physics demonstration, an instructor "races" various bodies that roll without slipping from rest down an inclined plane. What shape should a body have to reach the bottom of the incline first?



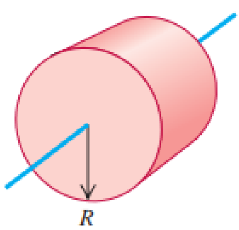
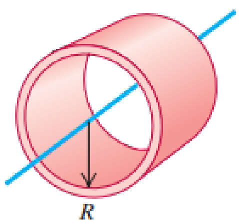
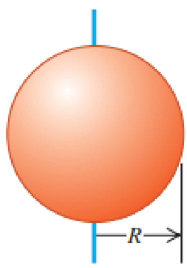
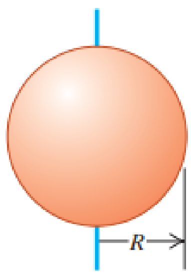
Solution:

From the conservation of energy,

$$\begin{aligned} K_1 + U_1 &= K_2 + U_2 \\ 0 + Mgh &= \left(\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2\right) + 0 \\ Mgh &= \left(\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\left(\frac{v_{cm}}{R}\right)^2\right) \\ Mgh &= \frac{1}{2}\left(Mv_{cm}^2 + \frac{I_{cm}}{R^2}v_{cm}^2\right) \\ Mgh &= \frac{1}{2}\left(M + \frac{I_{cm}}{R^2}\right)v_{cm}^2 \end{aligned}$$

To determine the moment of inertia of each body we can refer to the table in the previous lecture notes. We can express the four round bodies *cases (f)-(i)* as, $I_{cm} = cMR^2$, where c is a number less than or equal to one that depends on the shape of the body.

$$\begin{aligned} Mgh &= \frac{1}{2}\left(M + \frac{I_{cm}}{R^2}\right)v_{cm}^2 \\ Mgh &= \frac{1}{2}\left(M + \frac{cMR^2}{R^2}\right)v_{cm}^2 \\ Mgh &= \frac{1}{2}(M + cM)\,v_{cm}^2 \\ Mgh &= \frac{1}{2}M\,(1 + c)\,v_{cm}^2 \\ gh &= \frac{1}{2}(1 + c)\,v_{cm}^2 \\ \therefore v_{cm} &= \sqrt{\frac{2gh}{1 + c}} \end{aligned}$$

$c = \frac{1}{2}$	$c = 1$	$c = \frac{2}{5}$	$c = \frac{2}{3}$
			
$v_{cm} = \sqrt{\frac{2gh}{1+1/2}}$	$v_{cm} = \sqrt{\frac{2gh}{1+1}}$	$v_{cm} = \sqrt{\frac{2gh}{1+2/5}}$	$v_{cm} = \sqrt{\frac{2gh}{1+2/3}}$

For a given value of c, the speed v_{cm} after descending a distance h is independent of the body's mass M and radius R. The values of c tell us that the order of finish for uniform bodies will be as follows.

1. $(c = 2/5)$ any solid sphere
2. $(c = 1/2)$ any solid cylinder
3. $(c = 2/3)$ thin-walled, hollow sphere
4. $(c = 1)$ any thin-walled, hollow cylinder

Small-c bodies always beat large-c bodies because less f their kinetic energy is tied up in rotation and so more is available for translation.

Perspective

- For rotational motion, there is also an associated work done by the torque. The power associated with work done by torque on a rotating body can also be obtained.
- Note that equation (4) is valid even when the axis of rotation moves, provided the following conditions are met:
    1. The axis through the center of mass must be an axis of symmetry
    2. The axis must not change direction
- Variety of new physical phenomena, some quite unexpected, can occur when the axis of rotation can change direction. This surprising, nonintuitive motion of the axis is called **precession**. This physical phenomena is exploited in rotating machines such as gyroscopes.