Static Mechanical Equilibrium

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- Conditions for Equilibrium
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- Finding and using the Center of Gravity

Implementation
- Walking the plank
- Weight distribution on a beam
- Weight Distribution for a Car

Perspective

Objective

By studying this section, you will learn:

- the conditions that must be satisfied for a body or structure to be in equilibrium
- what is meant by the center of gravity of a body, and how it relates to the body's stability
- how to solve problems that involve rigid bodies in equilibrium

Background

A body that can be modeled as a particle is in equilibrium whenever the vector sum of the forces acting on it is zero. That condition isn't enough. If forces act at different points on an extended body, an additional requirement must be satisfied to ensure that the body has no tendency to rotate: the sum of the torques about any point must be zero.

This requirement is based on the principles of rotational dynamics developed in the previous lectures. To calculate the torque due to the weight of a body we can use the concept of center of gravity which we will introduce in this chapter.

Conditions for Equilibrium

A particle is in equilibrium if the vector sum of all the forces acting on the particle is zero $\Sigma\vec{F} = 0$. For an extended body, the equivalent statement is that the center of mass of the body has zero acceleration if the vector sum of all external forces acting on the body is zero. This is often called the first condition for equilibrium. In vector and component forms,

$$\Sigma\vec{F} = 0 \tag{1}$$

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{2}$$

A second condition for an extended body to be in equilibrium is that the body must have no tendency to rotate. This means that the sum of torques due to all the external forces acting on the body must be zero. A rigid body in equilibrium can't have any tendency to start rotating about any point, so the sum of external torques must be zero about any point. This is the second condition for equilibrium:

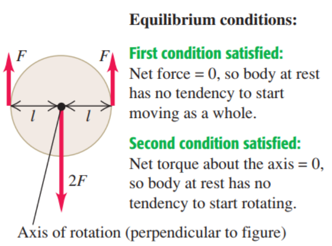
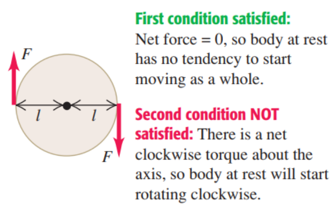
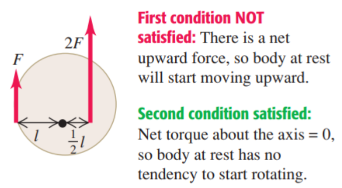
$$\Sigma\vec{\tau} = 0 \tag{3}$$

> The sum of the torques due to all external forces acting on the body, with respect to any specified point, must be zero.

A body which satisfy the first and second condition (no translation and rotation) is said to be in **static equilibrium**.

Note, however, that the same conditions apply to a rigid body in uniform translational motion (without rotation), such as airplane in flight with constant speed,direction and altitude. Such a body is in equilibrium but is not static.

To be in static equilibrium, a body at rest must satisfy both conditions for equilibrium.

In static equilibrium	Not in static equilibrium	Not in static equilibrium
		

Center of Gravity

In most equilibrium problems, one of the forces acting on the body is its weight. We need to be able to calculate the torque of this force. The weight doesn't act at a single point; it is distributed over the entire body. But we can always calculate the torque due to the body's weight by assuming that the entire force of gravity is concentrated at a point called the **center of gravity.**

In the lecture notes of Dynamics of Rotational Motion, we've covered the notion of center of mass. For a collection of particles with masses $m_1, m_2, ...$ and coordinate $(x_1, y_1, z_1), (x_1, y_1, z_1)...$ the coordinates of the center of mass are given by,

$$\begin{aligned} x_{cm} &= \frac{m_1x_1 + m_2x_2 + m_2x_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i x_i}{\Sigma_i m_i} \\ y_{cm} &= \frac{m_1y_1 + m_2y_2 + m_2y_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i y_i}{\Sigma_i m_i} \\ z_{cm} &= \frac{m_1z_1 + m_2z_2 + m_2z_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i z_i}{\Sigma_i m_i} \end{aligned} \tag{4}$$

These components x_{cm}, y_{cm}, z_{cm} are the components of the position vector $\vec{r}_{cm}$ of the center of mass,

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_2\vec{r}_2 + \cdots}{m_1 + m_2 + m_3 + \cdots} = \frac{\Sigma_i m_i \vec{r}_i}{\Sigma_i m_i} \tag{5}$$

Every particle in the body experiences a gravitational force, and the total weight of the body is the vector sum of a large number of parallel forces. A typical particle has mass m_i and weight $\vec{w}_i = m_i\vec{g}$. If $\vec{r}_i$ is the position vector of this weight $\vec{w}_i$ with respect to O is,

$$\tau_i = \vec{r}_i \times \vec{w}_i = \vec{r}_i \times m_i\vec{g} \tag{6}$$

The **total torque** due to the gravitational forces on all the particles is,

$$
\begin{aligned}
\vec{\tau} = \Sigma_i \vec{\tau_i} &= \vec{r}_1 \times m_1 \vec{g} + \vec{r}_2 \times m_2 \vec{g} + \cdots \\
&= (\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots) \times \vec{g} \\
&= \left(\Sigma m_i \vec{r}_i \right) \times \vec{g}
\end{aligned}
$$

When we multiply and divide both this by the total mass of the body we get,

$$\vec{\tau} = \vec{r}_{cm} \times M\vec{g} = \vec{r}_{cm} \times \vec{w} \tag{7}$$

where $M\vec{g}$ is the total weight of the body.

Remarks:

- the **total gravitational torque**, given by Eq. (7), is the same as though the total weight $\vec{w}$ were acting on the position $\vec{r}_{cm}$ of the center of mass, which we also call the *center of gravity*.
- If $\vec{g}$ has the same value at all points on a body, its center of gravity is identical to its center of mass.
    - Note, however, that the center of mass is defined independently of any gravitational effect

Finding and using the Center of Gravity

We can often use symmetry considerations to locate the center of gravity of a body, just as we did for the center of mass. The center of gravity of a homogeneous sphere, cube, circular sheet, or rectangular plate is at its geometric center. The center of gravity of a right circular cylinder or cone is on its axis of symmetry.

When a body acted on by gravity is supported or suspended at a single point, the center of gravity is always at or directly above or below the point of suspension. If it were anywhere else, the weight would have a torque with respect to the point of suspension, and the body could not be in rotational equilibrium. Table below shows how to use this fact to determine experimentally the location of the center of gravity of an irregular body.

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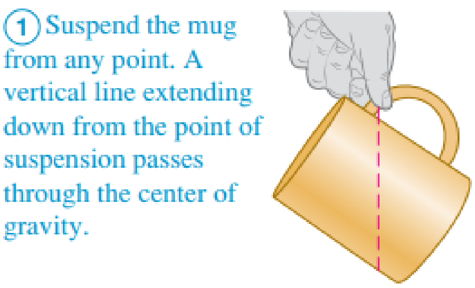
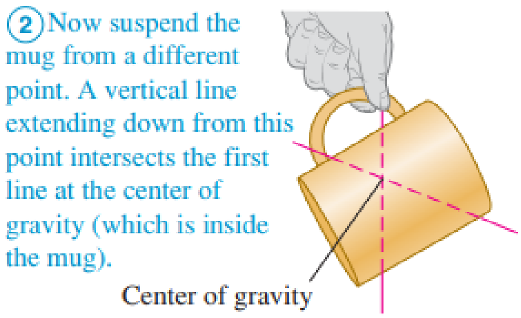
<tr><td>
</td><td>
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The lower the center of gravity and the larger the area of support, the more difficult it is to overturn a body. Four legged animals such as deer and horses have a large area of support bounded by their legs; hence they are naturally stable and need only small feet or hooves. Animals that walk erect on two legs, such as humans and birds, need relatively large feet to give them a reasonable area of support.

Implementation

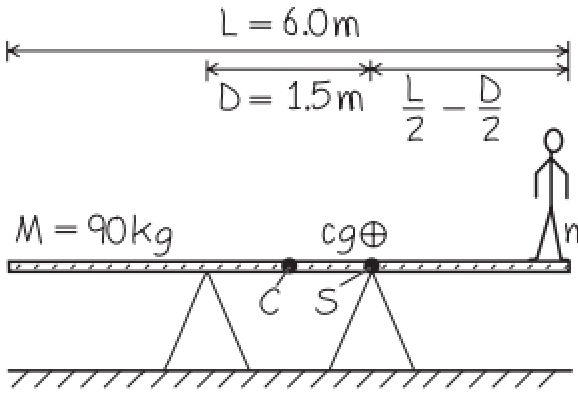
> To keep things simple, we'll restrict our attention to situations in which we can treat all forces as acting in a single plane, which we'll call the xy-plane.

The challenge is to apply these simple conditions to specific problems.

Walking the plank

> A uniform plank of length $L = 6.0m$ and mass $M = 90$kg rests on sawhorses separated by $D = 1.5$m and equidistant from the center of the plank. Cousin Joe wants to stand on the right-hand end of the plank. If the plank is to remain at rest, how massive can Throckmorton be?

The sketch of the problem is given below,



To just balance, Joe's mass m must be such that the center of gravity of the plank-Joe's system is directly over the right-hand sawhorse. Thus the center of gravity position in the x-axis is given by,

$$x_{cg} = \frac{M(0) + m(L/2)}{M+m} = \frac{m}{M+m}\frac{L}{2} \tag{8}$$

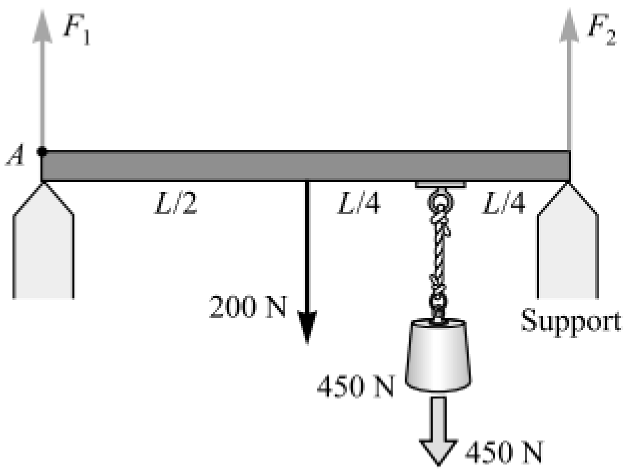
The center of gravity of the plank and Joe are at $x_p = 0$ and $x_J = L/2 = 3.0$. respectively, and the right-hand sawhorse is at $x_S = D/2$.

Since $x_s = x_{cg}$, we have:

$$\begin{aligned}
\frac{m}{M+m}\frac{L}{2} &= \frac{D}{2} \\
\frac{m}{M+m} &= \frac{D}{L} \\
m &= \frac{D}{L}(M+m) \\
m &= \frac{D}{L}M + \frac{D}{L}m \\
m(1-\frac{D}{L}) &= \frac{D}{L}M \\
m\left(\frac{L-D}{L}\right) &= \frac{D}{L}M \\
m &= M\frac{D}{L-D} = (90\text{kg})\frac{1.5\text{m}}{6.0\text{m} - 1.5\text{m}} \\
\therefore m &= 30\text{kg}
\end{aligned}$$

Weight distribution on a beam

> A uniform beam of length L weighs 200N and holds a 450-N object as shown in the figure below. Find the magnitudes of the forces exerted on the beam by the two supports at its ends. Assume the lengths are exact.



For the first condition, the summation of all forces along the y-axis is:

$$\begin{aligned} \Sigma F_y &= 0 \\ F_1 + F_2 - 200 - 450 &= 0 \\ F_1 + F_2 &= 650 \end{aligned} \tag{9}$$

Before the torque is written, an axis must be chosen. We choose it at A, so that the unknown force F_1 will pass through it and exert no torque. The torque equation is then,

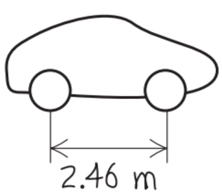
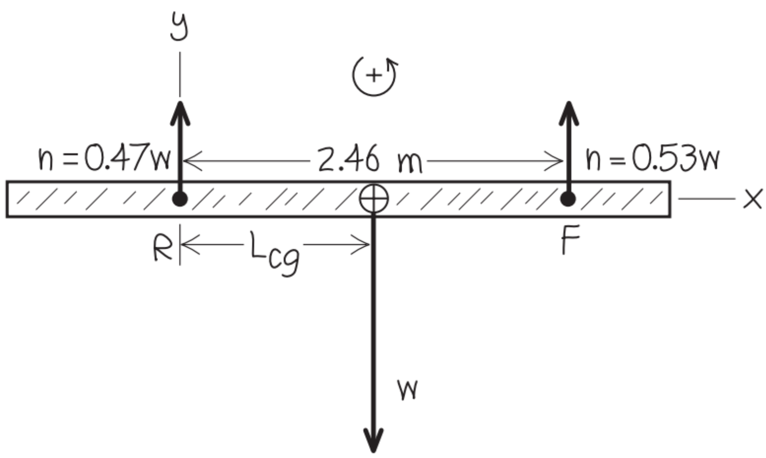
$$\Sigma\tau = F_1(0) - \frac{L}{2}(200)\sin(90) - (3L/4)(450)\sin(90) + LF_2 \sin 90 = 0 \tag{10}$$

$$F_2 = \frac{1}{2}(200) + (3/4)(450) = 437.5N$$

To find F_1 we substitute the value of F_2 in the force equation in Eq. (9),obtaining $F_1 = 212$N.

Weight Distribution for a Car

> An auto magazine reports that a certain sports car has 53% of its weight on the front wheels and 47% on its rear wheels. (That is, the total normal forces on the front and rear wheels are $0.53w$ and $0.47w$, respectively, where w is the car's weight.) The distance between the axles is 2.46m. How far in front of the rear axle is the car's center of gravity?

System Sketch	Free-body Diagram
	

The weight w acts at the center of gravity. Our target variable is the distance L_{cg}, the lever arm of the weight with respect to the rear axle R, so it is wise to take torques with respect to R. The torque due to the weight w is negative because it tends to cause a clockwise rotation about R.

We write the torque equation and solve for L_{cg}:

$$\begin{aligned} \Sigma\tau_R &= 0.47w(0) - wL_{cg} + 0.53w(2.46) \\ 0 &= -wL_{cg} + 0.53w(2.46) \\ wL_{cg} &= 0.53w(2.46) \\ L_{cg} &= 0.53(4.56) = 1.30\mathrm{m} \end{aligned}$$

Perspective

- Rigid bodies don't bend, stretch, or squash when forces act on them. But the rigid body is an idealization; all real materials are realistic and do deform to some extent.
- Elastic properties of materials are tremendously important. The concepts of stress, strain, and elastic modulus and a simple principle called Hooke's law helps predict what deformations will occur when forces are applied to a real (not perfectly rigid) body.