Energy in Rotational Motion

Background

A rotating rigid body consists of mass in motion, so it has kinetic energy. We can express this kinetic energy in terms of the body's angular speed and a new quantity, called the moment of inertia, that depends on the body's mass and how the mass is distributed.

Rotational Kinetic Energy

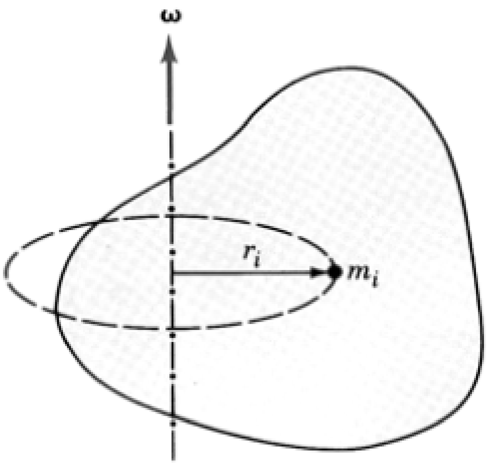


Figure 1

The ith particle in a rotating rigid body has a velocity of $v_i = r_i\omega$ and its kinetic energy is

$$K_i = \frac{1}{2}m_i v_i^2 \qquad (1)$$
$$K_i = \frac{1}{2}m_i r_i^2 \omega^2$$

Note that the angular speed ω doesn't have the ith subscript, this is due to the fact that all the ith particle in a rigid body has the same angular speed.

The kinetic energy equation (1) is only for a single particle in the rigid body. For the *kinetic energy of the whole rigid body,* we have to sum up all the ith kinetic energy.

$$K = \sum_i K_i \tag{2}$$

$$K = \sum_i \frac{1}{2} m_i r_i^2 \omega^2$$

$$K = \frac{1}{2}\left[\sum_i m_i r_i^2\right]\omega^2$$

$$K = \frac{1}{2} I \omega^2$$

Equation (2) gives the total **rotational kinetic energy** of the body, where we have to introduce the concept of ***moment of inertia*** (I).

Moment of Inertia

The *moment of inertia* I is defined as,

$$I = \sum_i m_i r_i^2 \tag{3}$$

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots$$

> The word "moment" means that I depends on how the body's mass is distributed in space; it has nothing to do with a "moment" of time.

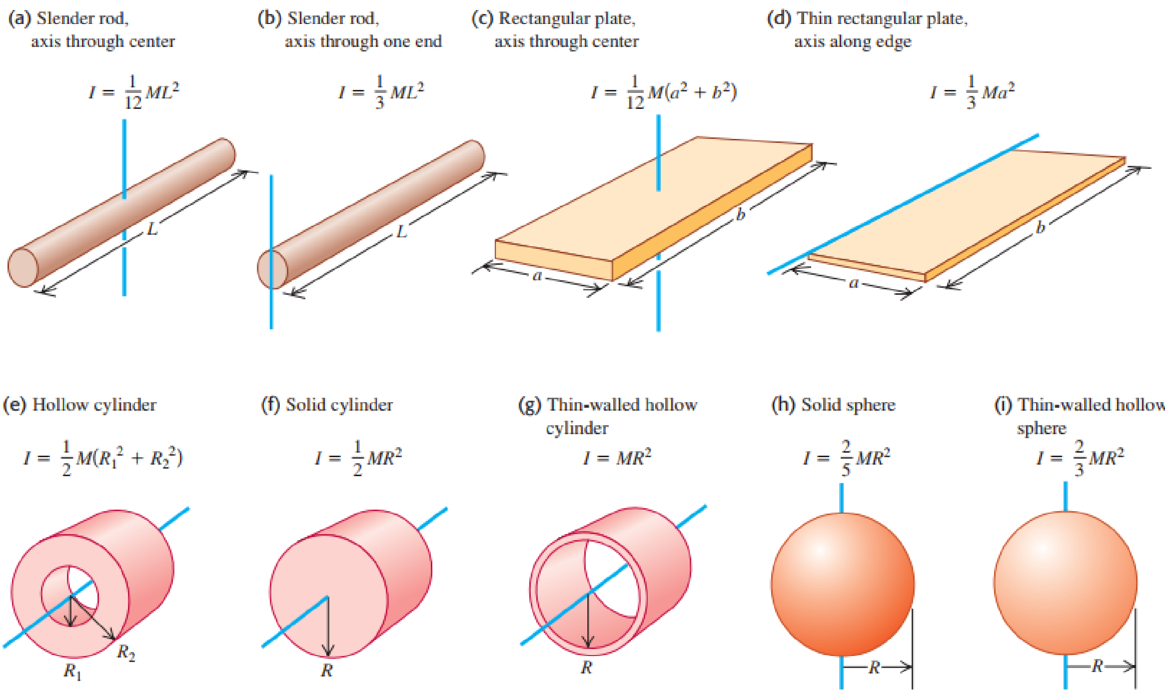
Remarks:

- the greater the moment of inertia (I)
  - the greater the kinetic energy of a rigid body rotating with a given angular speed ω
  - the harder it is to start the body rotating if it's at rest and the harder it is to stop its rotation if it's already rotating. For this reason, I is also called the ***rotational inertia***.

Rotational inertia is a property of any object which can be rotated. It is a scalar value which tells us how difficult it is to change the rotational velocity of the object around a given rotational axis.

The rotation inertia/moment of inertia has SI units of $\mathrm{kg \cdot m^2}$.

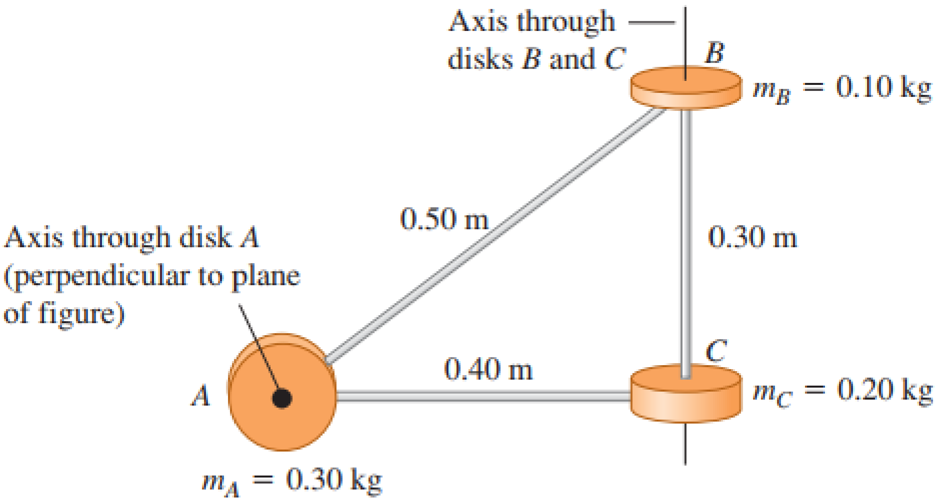
Moments of Inertia of Various Bodies



Implementation

Moment of inertia for different rotation axes

> A machine part consists of three disks linked by lightweight struts.
>
> (a) What is this body's moment of inertia about an axis through the center of disk A, perpendicular to the plane of the diagram?
>
> (b) What is its moment of inertia about an axis through the centers of disks B and C?
>
> (c) What is the body's kinetic energy if it rotates about the axis through A with angular speed of 4.0 rad/s



Solution:

a To calculate the moment of inertia about an axis use equation (3), for this case our axis runs through the center of disk A. Mass A does not contribute to the moment of inertia because its distance from the axis is zero. Only mass B and mass C contribute, thus,

$$I_A = \Sigma m_i r_i^2 = (0.10 \text{ kg})(0.50 \text{ m})^2 + (0.20 \text{ kg})(0.40 \text{ m})^2$$
$$I_A = 0.057 \text{ km} \cdot \text{m}^2$$

b Both mass B and C lie on axis BC, so neither particle contributes to the moment of inertia. Only A contributes, thus:

$$I_{BC} = \Sigma m_i r_i^2 = (0.30\text{kg})(0.40\text{m})^2 = 0.048 \text{ km} \cdot \text{m}^2 \tag{4}$$

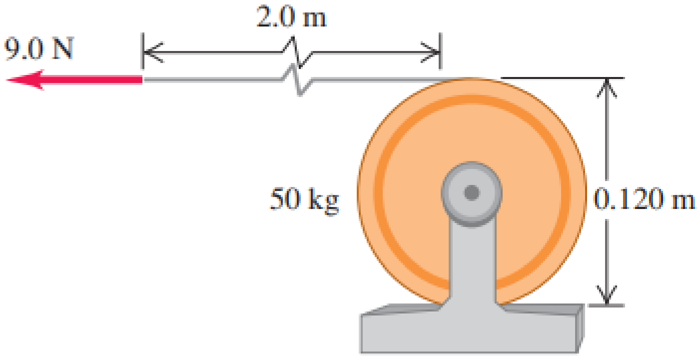
c To evaluate the body's kinetic energy we use equation 2. If it rotates about the axis A, the moment of inertia we should use is I_A and since we're given with the angular speed, the kinetic energy is,

$$K_A = \frac{1}{2} I_A \omega^2 = \frac{1}{2}(0.057 \text{ km} \cdot \text{m}^2)(4.0\text{rad/s})^2 \tag{5}$$

An unwinding cable

> We wrap a light, nonstretching cable around a solid cylinder of mass 50 kg and diameter 0.120 m, which rotates in frictionless bearings about a stationary horizontal axis. We pull the free end of the cable with a constant 9.0 N force for a distance of 2.0 m; it turns the cylinder as it unwinds without slipping. The cylinder is initially at rest. Find its (a) angular speed and the (b) final speed of the cable.

A cable unwinds from a cylinder (side view).



Solution:

a For this problem we have to use the previous definition of work which is $W = Fs$. The work done on the cylinder by an external force is just $W_{other} = (9.0\text{ N})(2.0\text{ m}) = 18\text{ J}$. Note that since the cylinder is initially at rest $K_1 = 0$ while at the final state the cylinder has a full rotational kinetic energy $K_2 = \frac{1}{2}I\omega^2$. From the conservation of energy,

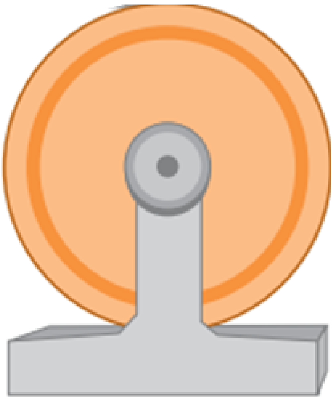
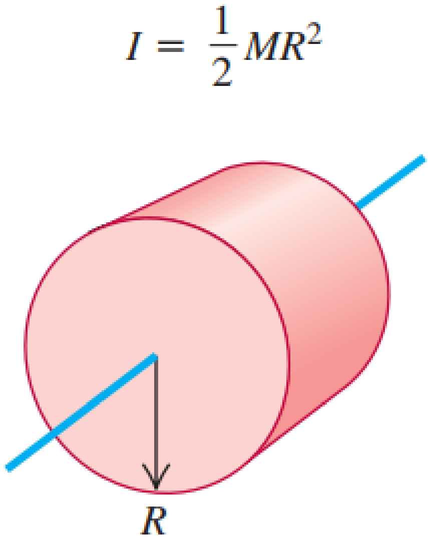
$$\begin{aligned} K_1 + U_1 + W_{others} &= K_2 + U_2 \\ 0 + 0 + 18\text{ J} &= \frac{1}{2}I\omega^2 + 0 \\ 18\text{ J} &= \frac{1}{2}I\omega^2 \\ \sqrt{\frac{36\text{ J}}{I}} &= \omega \end{aligned}$$

From the table of *Moments of Inertia of Various Bodies,* a cylinder shape body has a moment of inertia $I = \frac{1}{2}mR^2$. The radius R is half the diameter (0.120 m), thus the moment of inertia is,

$$I = \frac{1}{2}mR^2 = \frac{1}{2}(50\text{ kg})(0.060\text{ m})^2 = 0.090\text{ kg}\cdot\text{m}^2 \tag{6}$$

and we have,

$$\begin{aligned} \sqrt{\frac{36\text{ J}}{I}} &= \omega \\ \sqrt{\frac{36J}{0.090\text{ kg}\cdot\text{m}^2}} &= 20\text{rad/s} \\ \omega &= 20\text{rad/s} \end{aligned}$$

Cylinder (side view)	Solid Cylinder
	

b The final speed of the cable is the final tangential speed of the cylinder, and hence,

$$v = R\omega = (0.060\text{ m})(20\text{ rad/s}) = 1.2\text{ m/s} \tag{7}$$

Perspective

- Moment of inertia (rotational inertia) takes the place of mass in the rotational kinetic energy
$$K = \frac{1}{2}mv^2 \rightarrow K = \frac{1}{2}I\omega^2 \tag{8}$$
- A body doesn't have just one moment of inertia - it has infinitely many, because there infinitely many axes about which it might rotate. All this moment of inertia are related to the moment of inertia about the center of mass. *(see Parallel-Axis theorem)*

- If a rigid body is a continuous distribution mass - like a solid cylinder or a solid sphere - it cannot be represented by a few point masses. The sum of masses and distances that defines the moment of inertia in equation (3) has to be replaced by an *integral formulation*.
$$I = \sum_i m_i r_i^2 \rightarrow \int r^2\, dm \tag{9}$$
This is how you derive most of the moment of inertia of various bodies. This is where you skills in integration comes in handy.