PHYS332: Instructions for Project 3:

Low-dimensional Chaos

For this project, you can use any chaotic system to demonstrate the role of accurate ODE integration. Below, an example for the Lorenz system is given.

Part 1: Exploration of parameter space: The Lorenz equations were derived to describe a simplified model for atmospheric dynamics, reduced to the appearance of convective instability in the presence of an inverse temperature gradient. Despite their simplicity, they show a range of interesting behavior, and similar equations emerge also from other physical systems. The standard parameter values for the Lorenz system are $\sigma = 10, b = 8/3$. Initial conditions are $x(0) = 10,\ y(0) = 10,\ z(0) = 10$. The aim of this project is to explore and characterize the behavior of the Lorenz system for different values of the parameter r, which represents the steepness of the temperature gradient. These are the governing ODEs:

$$\frac{dx}{dt} = \sigma(y - x) \quad (1)$$

$$\frac{dy}{dt} = rx - y - xz \quad (2)$$

$$\frac{dz}{dt} = xy - bz. \quad (3)$$

Implement an algorithm to integrate the Lorenz system. You should develop your own set of integrators that you want to test. You will probably have to use an adaptive stepsize integrator.

Perform simulations for different values of the parameter r. "Interesting" parameter values are $r = 10, 22, 24.5, 100, 126.52, 150, 166.3, 212$. Study the dynamics of the system by plotting x, y, z against t, or the pairs $(x(t), y(t))$, $(x(t), z(t))$ and $(y(t), z(t))$. Finally, I suggest a three-dimensional plot $(x(t), y(t), z(t))$. Think "phase space". Discuss the following issues:

- What is the influence of the initial conditions on the model's long-term behavior? How long does it take to converge to that behavior (how long does the "transient" last)?
- For what parameters is the solution periodic?
- For what parameters does the system converge to a fixed point (this means that $(x(t), y(t), z(t))$ converges to a point (x_0, y_0, z_0) and stays there for the remainder of the simulation)? How many fixed points are there? When there are multiple fixed points, what determines which one is reached?

The solution $z(t)$ oscillates with time. The maximum value reached in z varies between oscillation cycles. The Lorenz map is obtained by determining the maximum z_n on cycle n and then plotting z_{n+1} against z_n. This map can be used to figure out whether periodic solutions would be stable. Calculate the Lorenz map for a few parameter values and determine whether there are periodic solutions for each of the parameter values.

Part 2: Communicating with chaos: Chaotic processes can be used to transmit signals. Take two Lorenz systems (see equations below): the sender (x, y, z) and the receiver (u, v, w). The sender can be brought in a chaotic state and one of its output variables is transmitted to the receiver. In the (receiver) ODEs for v and w, replace now the variable u by the output of the transmitted x. Then the receiver will synchronize with the sender. Therefore, (u, v, w) will be equal to (x, y, z) after a short transient (see Pecora & Caroll 1990). The key idea now is to perturb $x(t)$ with the signal $s(t)$ (e.g. an audio signal): $X = x + s$. Because of the chaotic nature of x, it is hard to extract (the small) s from $X = x + s$. Yet, since the variable u will synchronize with x, s can be extracted by $s = X - u$. The aim of this project is to determine which parameters of r work best and what type of signal can be transmitted and reconstructed with high fidelity. Would this work on binary data?

Here are the equations for the unperturbed system:

$$\frac{dx}{dt} = \sigma(y - x) \quad \text{sender} \tag{4}$$

$$\frac{dy}{dt} = rx - y - xz \tag{5}$$

$$\frac{dz}{dt} = xy - bz \tag{6}$$

$$\frac{du}{dt} = \sigma(v - u) \quad \text{unperturbed receiver} \tag{7}$$

$$\frac{dv}{dt} = rx(t) - v - x(t)w \tag{8}$$

$$\frac{dw}{dt} = x(t)v - bw. \tag{9}$$

The equations for the perturbed receiver then are

$$\frac{du}{dt} = \sigma(v - u) \tag{10}$$

$$\frac{dv}{dt} = rX(t) - v - X(t)w \tag{11}$$

$$\frac{dw}{dt} = X(t)v - bw. \tag{12}$$

- Implement an algorithm to integrate the two coupled Lorenz systems (6 degrees of freedom). Again, you should write your own integrators.
- Determine the step size necessary to achieve an accuracy of 10^{-6} when integrating up to $t = 100$. Your report should describe how you determined the step size.
- The standard parameter set is $\sigma = 10, b = 8/3, s(t) = 0$. Determine for which values $0 \leq r \leq 250$ the two Lorenz systems synchronize. The two systems are synchronized when $|u(t) - x(t)| \rightarrow 0$, $|v(t) - y(t)| \rightarrow 0$, $|w(t) - z(t)| \rightarrow 0$ for large values of t. How long do the systems take to synchronize?
- Generate a binary signal, with a given amplitude A and pulse duration T_p. Try a number of different values, and several binary sequences (e.g. generated by a random number generator). How well does the signal transmit?

- Record your own voice to obtain a time-varying signal (or use a song etc). Reconstruct the signal from $X(t)$ and $u(t)$ as described above. This only works if the timescale of the signal is matched to the timescale of the Lorenz system. Explicitly motivate your choice and provide enough details on the preliminary experiments you performed to arrive at that choice. In MatLab, you can record with `sound, wavrecord, wavplay`.