Physics 220 notes – Group Theory for Physicists (Mostly discrete groups)

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Contents

Introduction

This is a set of notes written in conjunction with a UCSD Group Theory for Physicists course. The actual course notes are available online here. The "theorems" in this document aren't necessarily theorems and the "proofs" just have to be convincing enough.

There's also an emphasis on applications and examples. I try to write most of the actual theory in "definition-theorem-proof" style, and the examples/applications in a more informal style.

1 Groups

1.1 Basic Theory of Groups

Definition 1.1. A *group* is a set G together with a binary operation $\cdot : G \times G \to G$ such that:

1. $\forall_{a,b,c \in G}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. $\exists_{e \in G} \forall_{g \in G}$, $e \cdot g = g$.
3. $\forall_{g \in G} \exists_{g^{-1} \in G} : g \cdot g^{-1} = e$.

Theorem 1.2. Basic theorems about groups:

- The identity element $e \in G$ is unique.
- Inverse elements are unique.
- $ge = g$
- $g^{-1}g = e$.

Definition 1.3. Given a group G, a *subgroup* $H \subseteq G$ is a subset of G which itself is a group. This is often written $H \leq G$.

Definition 1.4. Given a group G, subgroup $H \leq G$, and group element $g \in G$, a *coset* is the set $gH = \{gh_1, gh_2, \ldots\}$. Note that $|g_1 H| = |H|$. (This is also called a left-coset. You can also define right-cosets $Hg = \{h_1 g, \ldots\}$ but we don't need them.)

Theorem 1.5. $g_1 H$ and $g_2 H$ either share all their elements or none of their elements.

Proof. Suppose there is a common element $f \in g_1 H$ and $f \in g_2 H$. Then for some $h_1, h_2 \in H$, $f = g_1 h_1 = g_2 h_2$. Now let $g_2 h$ be any element of $g_2 H$. Using the formulas for f: $g_2 h = f h_2^{-1} h = g_1 \left(h_1 h_2^{-1} h\right) = g_1 h'$. So $g_2 H \subseteq g_1 H$. The proof is symmetric, so in fact $g_1 H \subseteq g_2 H$ and the sets are equal. $\square$

Definition 1.6. Let $H \subseteq G$. The *index* of the subgroup H with respect to G, denoted $[H : G]$, is the number of cosets gH.

Theorem 1.7. (Lagrange). For finite groups, if $H \leq G$, then $|G|/|H|$ is an integer and is equal to the number of cosets of H. ie $|G| = [G : H]|H|$

Proof. Every element of G belongs to some coset gH. These cosets cannot overlap (by the previous theorem), and the cosets are all of the same order. So we have partitioned the group G into n sets of size $|H|$, giving the relation $|G| = n|H|$. $\square$

Definition 1.8. Given group G, a *normal subgroup* $H \trianglelefteq G$ is a subgroup H such that for every $g \in G$, $gH = Hg$.

Definition-Theorem 1.9. Given a normal subgroup $H \trianglelefteq G$, the *quotient group* is the set of cosets $G/H = \{gH | g \in G\}$ with the group operation $(g_1 H) \cdot (g_2 H) = (g_1 \cdot g_2)H$. This is a well-defined group operation and is independent of which group elements you choose.

Definition 1.10. Given two groups A, B, a function $f : A \to B$ is called a *group homomorphism* if $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$. The *kernel* of a group homomorphism is the set of all elements of A that map to the identity in B, $\ker f = \{a : f(a) = e\}$.

Definition-Theorem 1.11. The kernel of a group homomorphism is a normal subgroup.

Note 1.12. The easiest way to prove that a subgroup $N \trianglelefteq G$ is normal is to find a homomorphism from G to some other group with N as its kernel.

Definition 1.13. A *simple group* is a group G with no normal subgroups except for the trivial cases $\{e\} \trianglelefteq G$ and $G \trianglelefteq G$.

Definition 1.14. Given a group element $g \in G$, the *conjugacy class* of g is the set of all xgx^{-1} where $x \in G$. That is: $C_g = \{xgx^{-1} : x \in G\}$.

1.2 Example: permutation groups

Definition 1.15. The symmetric group S_n is the set of all permutations on n elements. We have $|S_n| = n!$.

Definition 1.16. Every permutation is either even or odd, depending on whether you need an even or odd number of swaps to form the permutation. The group of all even permutations is known as the *alternating group* A_n. It's a normal subgroup of S_n of order $n!/2$.

Theorem 1.17. A_n is normal.

Proof. Construct the map $f : S_n \to S_2$, which sends a permutation to the identity if it's even, and to the other element if it's odd. Then $\ker f$ is precisely the alternating group A_n. Kernels of homomorphisms are normal subgroups, so A_n is normal in S_n. $\square$

Note 1.18. (Cycle notation). A permutation that sends $1 \mapsto 2$, $2 \mapsto 3$ and $3 \mapsto 1$ is a cycle of length 3, and can be denoted by $(1\ 2\ 3)$. You can also compose cycles: $(1\ 2\ 3)(1\ 2)$ is the cycle which first swaps 1 and 2, and then applies the three-cycle from before. We find $(1\ 2\ 3)(1\ 2) = (1\ 3)$. Every permutation can be written as the product of disjoint cycles, $(i_1\ i_2\ \cdots)(i_3\ i_4\ \cdots)\cdots \in S_n$.

Note 1.19. (Two-row notation). The above permutation $(1\ 2\ 3)$ can be written as $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$. The numbers in the first row get send to the corresponding numbers in the second row. This notation makes it easier to multiply permutations by hand.

Theorem 1.20. If $g = (i_1\ i_2\ i_3\ \cdots)$ is a permutation written in cycle notation, then given a permutation τ, $\tau g \tau^{-1} = (\tau(i_1)\ \tau(i_2)\ \tau(i_3)\ \cdots)$.

Proof. $\tau g \tau^{-1}$ is the permutation which sends $\tau(i_1)$ to $\tau(g(i_1)) = \tau(i_2)$. □

Corollary 1.21. Conjugacy classes in S_n are permutations with same cycle pattern.

Proof. The previous theorem shows that conjugation can't change the cycle pattern. Furthermore, for any two permutations with the same cycle structure, we can find a third permutation to relate the two by conjugation. For example if $a = (1\ 2\ 3)(4\ 5)$ and $b = (5\ 4\ 3)(2\ 1)$, then the permutation $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ has $\tau a \tau^{-1} = b$. □

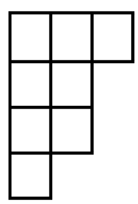
Corollary 1.22. There are $p(n)$ conjugacy classes of S_n, where $p(n)$ is the number of unordered integer partitions of n. For example, $p(3) = 3$ because 3 can be written as 3, $2+1$, or $1+1+1$.

Proof. Each partition corresponds one-to-one to a particular cycle structure, which corresponds one-to-one to a conjugacy class. □

Note 1.23. There isn't a general formula for $p(n)$, but there's an asymptotic formula given by Hardy and Ramanujan:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Definition 1.24. A Young diagram is a diagram in one-to-one correspondence with an unordered partition of N. It consists of N squares in rows such that there are n_1 squares in the first row, $n_2 \leq n_1$ squares in the second row, and so on. This is a trivial definition, but it will be indispensible for thinking about combinatorics later on.



A Young diagram corresponding to the partition $8 = 3 + 2 + 2 + 1$

Theorem 1.25. (Conjugacy classes using Young tableaux). Let c be a conjugacy class of S_N. Suppose the cycle structure of c will have n_1 cycles of length 1, n_2 cycles of length 2, and so on. Then

$$|c| = \frac{N!}{\prod_j (j)^{n_j}(n_j!)}$$

Proof. Think about filling in the boxes of the corresponding Young diagram. There are $N!$ ways to fill in the N boxes with the numbers one through N, but some correspond to the same permutations. If there are n_j cycles of length j, then we have counted each of these $n_j!$ times (we can swap any two rows of the same length without changing the permutation). Furthermore, for every cycle of length j like $(1\ 2\ \cdots\ j)$, we've included labelings like $(j\ 1\ 2\ \cdots)$, $(2\ \cdots\ j\ 1)$, and so on. There are j of these equivalent labelings, so for each cycle of length j we have overcounted by another factor of j. This means that overall, we have overcounted by a factor of $\prod_j j^{n_j}(n_j!)$. □

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&\multicolumn{1}{c}{}\\\cline{1-2} 6&7&\multicolumn{1}{c}{}\\\cline{1-2} 8&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 6&7&\multicolumn{1}{c}{}\\\cline{1-2} 4&5&\multicolumn{1}{c}{}\\\cline{1-2} 8&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 2&3&1\\\hline 5&4&\multicolumn{1}{c}{}\\\cline{1-2} 6&7&\multicolumn{1}{c}{}\\\cline{1-2} 8&\multicolumn{2}{c}{}\\\cline{1-1}\end{array} \quad = \quad (1\ 2\ 3)(4\ 5)(6\ 7)(8)$$

Several ways of labeling the Young diagram to get the same permutation

2 Representations

We'll only consider vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

2.1 Theory and the Schur theorems

Definition 2.1. Let V be a vector space. Then $\mathrm{GL}(V)$, the *general linear group* on V is the group of all invertible linear operators on V.

Definition 2.2. Given a group G and vector space V, a *representation* is some group homomorphism $M : G \to \mathrm{GL}(V)$. The representation is often denoted $M(G)$.

Definition 2.3. Let G be a group, and let $M : G \to \mathrm{GL}(V_1)$ and $N : G \to \mathrm{GL}(V_2)$ be two representations of G. A linear map $L : V_1 \to V_2$ is called an *intertwining map* if, for all $g \in G$, $L(M(g))v_1 = N(g)L(v_1)$.

In matrix notation, if $M(g)$ has dimensions $m \times m$ and $N(g)$ has dimensions $n \times n$, then L is an $n \times m$ matrix such that $LM(g) = N(g)L$.

Intertwining maps provide the way to relate representations on different vector spaces.

Definition 2.4. Let G be a group, and let $M : G \to \mathrm{GL}(V_1)$ and $N : G \to \mathrm{GL}(V_2)$ be two representations of G. The representations M and N are said to be *equivalent* if there exists an invertible intertwining map $L : V_1 \to V_2$. We also say that the two representations are *isomorphic*.

Theorem 2.5. If $V_1 = V_2 = V$, then two representations are equivalent if and only if for some matrix L and all $g \in G$, $N(g) = LM(g)L^{-1}$.

Proof. Using the same notation as in the last two definitions, L is an invertible linear map. We have $LM(g) = N(g)L$, or $LM(g)L^{-1} = N(g)$. So the whole business with intertwining maps is just a fancy way of saying the matrices are conjugate to each other. □

Definition 2.6. Given a representation $M : G \to \mathrm{GL}(V)$, a vector subspace $U \subseteq V$ is called an invariant subspace if $M(g)U = U$ for all $g \in G$. Representations always have the trivial invariant subspaces $\{\vec{0}\}$ and V.

Definition 2.7. A representation $M : G \to \mathrm{GL}(V)$ is called *reducible* if it has a nontrivial invariant subspace. That is, if it has an invariant which is neither $\{\vec{0}\}$ or V.

A representation that is not reducible is called *irreducible*. The word "irrep" is short for irreducible representation.

Definition 2.8. Take a group G, representation R_1 over vector space V_1, and representation R_2 over vector space V_2, the *direct sum representation* $R = R_1 \oplus R_2$ is the representation of G over vector space $V_1 \oplus V_2$, with action $R(g)(v_1, v_2) = (R_1(g)v_1, R_2(g)v_2)$.

Definition 2.9. Take a group G, representation R_1 over vector space V_1, and representation R_2 over vector space V_2, the *tensor product representation* $R = R_1 \otimes R_2$ is the representation of G over vector space $V_1 \otimes V_2$, with action $R(g)(v_1 \otimes v_2) = (R_1(g)v_1) \otimes (R_2(g)v_2)$. (To prove this is well defined, use the universal property of the tensor product).

Note 2.10. We can also define the direct sum of two representations of two different groups. In that case we wind up with a representation of $G_1 \times G_2$. This is not used too often in physics; usually we just care about different representations of the same group G.

Definition 2.11. A representation which is a direct sum of irreps is called *completely reducible.*

Example 2.12. (A reducible representation which is not completely reducible). The group of real numbers under addition has a representation by the two-by-two matrices $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$. The vector subspace spanned by $(1, 0)$ is fixed under the action of these matrices, so this representation is reducible. However, there is no other invariant subspace and therefore this representation cannot be a direct sum of invariant subspaces.

We don't need to worry much about cases like this though. We will prove that any representation of a finite group is equivalent to a unitary representation, and that any reducible unitary representation is also completely reducible.

Theorem 2.13. Schur's lemmas:

1. Let $M_1 : G \to V_1$ and $M_2 : G \to V_2$ be two irreps. Let ϕ be an intertwining map. Then either $\phi = 0$ or ϕ is an isomorphism.

2. Let $M : G \to V$ be an irrep, and let $\phi : V \to V$ be an intertwining map from V to itself. Then $\phi = \lambda I$ for some $\lambda \in \mathbb{C}$.

Proof. **Part 1**. Note that the kernel of ϕ is an invariant subspace of M_1, and the image of ϕ is an invariant subspace of M_2. So we either have $\ker \phi = V_1$ and $\operatorname{img} \phi = 0$ (so $\phi = 0$), or $\ker \phi = 0$ and $\operatorname{img} \phi = V_2$ (so ϕ is an isomorphism). □

Part 2. ϕ must have some eigenvalue λ for which $\det(\phi - \lambda I) = 0$. But then the map $\phi - \lambda I$ has a nonempty kernel, so that its kernel must be the entirety of V_1. Then $\phi - \lambda I = 0$. □

Corollary 2.14. Schur's lemmas in matrix notation:

1. Let $M_1(g)$ be an $m_1 \times m_1$ irrep, and let M_2 be an $m_2 \times m_2$ dimensional irrep. Let L be an $m_1 \times m_2$ matrix such that for all g, $M_1 L = L M_2$. Then either $L = 0$ or M_1 and M_2 are equivalent representations.

2. Let M be an irrep, and let L be a matrix that commutes with all $M(g)$. Then L is proportional to the identity matrix.

Definition 2.15. (Unitary representation) Let M be a representation on a vector space H with inner product $\langle cv_1, v_2 \rangle = c^* \langle v_1, v_2 \rangle$. The adjoint operator $M^\dagger$ is the operator for which $\langle M^\dagger v_1, v_2 \rangle = \langle v_1, M v_2 \rangle$ for all $v_1, v_2 \in H$. If each $M^\dagger = M^{-1}$, then M is called a *unitary representation.*

Theorem 2.16. Any linear representation of a finite group is equivalent to a unitary transformation.

Proof. Let H be a Hilbert space with inner product $\langle v_1, v_2\rangle_1$. Let $M(g)$ be a representation on a finite group. Define a new inner product $\langle v_1, v_2\rangle_2 = \sum_g \langle M(g)v_1, M(g)v_2\rangle_1$. The representation $M(g)$ is unitary with respect to this inner product, because:

$$\begin{aligned}
\langle v_1, M(h)v_2\rangle_2 &= \sum_g \langle M(g)v_1, M(gh)v_2\rangle_1 \\
&= \sum_{g'} \langle M(g'h^{-1})v_1, M(g')v_2\rangle_1 \\
&= \sum_g \langle M(g)M(h^{-1})v_1, M(g)v_2\rangle_1 \\
&= \langle M(h)^{-1}v_1, v_2\rangle_2
\end{aligned}$$

□

Note 2.17. It's tedious to link the abstract definition/theorem to actual matrices. An alternate proof of theorem 2.16 would involve finding a matrix X such that $XM(g)X^{-1}$ is a unitary matrix.

In this case the intertwining map was trivial, since all we did was reinterpret our Hilbert space as a space with a different inner product. If we to write this in terms of matrices then we have to implement a change of basis, and there's a small mountain of notation.

We can use theorems from linear algebra to find our orthonormal bases: $\langle e_i, e_j\rangle_1 = \delta_{ij}$, and $\langle \hat{e}_i, \hat{e}_j\rangle_2 = \delta_{ij}$. Then, using the repeated index summation convention:

1. $v = \langle e_i, v\rangle_1 e_i$ and $v = \langle \hat{e}_i, v\rangle_2 \hat{e}_i$ *(The orthonormal bases are complete)*
2. $M^g_{ij} = \langle e_i, M(g)e_j\rangle_1$ and $\hat{M}^g_{ij} = \langle \hat{e}_i, M(g)\hat{e}_j\rangle_2$ *(Matrix definitions)*
3. $M(g)e_j = e_i M^g_{ij}$ *(Consequence of completeness)*
4. $X_{ij} = \langle e_i, \hat{e}_j\rangle_1$ (Definition of the change of basis matrix)
5. $(X^{-1})_{ij} = \langle \hat{e}_i, e_j\rangle_1$ *(Change of basis matrix inverse. As a proof consider* $e_i = e_k\langle e_k, \hat{e}_j\rangle_1\langle \hat{e}_j, e_i\rangle_1$*)*
6. $\hat{e}_j = e_i X_{ij}$ and $e_j = \hat{e}_i (X^{-1})_{ij}$ *(consequence of completeness)*
7. $\langle e_i, e_j\rangle_2 = (X^{-1})^*_{ki}(X^{-1})_{kj} = ((X^{-1})^\dagger X^{-1})_{ij}$

Then, in totally exhaustive detail:

$$
\begin{aligned}
\hat{M}_{ai}(h) &= \langle \hat{e}_a, M(h)\hat{e}_i\rangle_2 \\
&= \sum_g \langle M(g)\hat{e}_a, M(gh)\hat{e}_i\rangle_1 \\
&= \sum_g \langle M(g)e_b X_{ba}, M(gh)e_j X_{ji}\rangle_1 \\
&= \sum_g \langle M(g)e_b X_{ba}, M(g)e_k M^h_{kj} X_{ji}\rangle_1 \\
&= X^*_{ba} M^h_{kj} X_{ji} \sum_g \langle M(g)e_b, M(g)e_k\rangle_1 \\
&= X^*_{ba} M^h_{kj} X_{ji} \langle e_b, e_k\rangle_2 \\
&= X^*_{ba} M^h_{kj} X_{ji} ((X^{-1})^\dagger X^{-1})_{bk} \\
&= X^*_{ba} ((X^{-1})^\dagger X^{-1})_{bk} M^h_{kj} X_{ji} \\
&= (X^\dagger)_{ab} ((X^{-1})^\dagger X^{-1})_{bk} M^h_{kj} X_{ji} \\
&= (X^{-1} M^h X)_{ai}
\end{aligned}
$$

Corollary 2.18. If M^g_{ij} is a matrix representation of a finite group, there exists a matrix X such that $X_{ij} M^g_{jk} X^{-1}_{k\ell}$ is a unitary matrix representation of G.

Theorem 2.19. *(Irrep orthogonality relation)* Let M and N be two irreducible unitary matrix representations of a group G. Then the following two relations hold:

1. if M and N are inequivalent representations, $\sum_{g\in G} M^*_{\mu\rho}(g) N_{\nu\lambda}(g) = 0$.

2. $\sum_{g\in G} M^*_{\mu\rho}(g) M_{\nu\lambda}(g) = \frac{|G|}{m}\delta_{\mu\nu}\delta_{\rho\lambda}$, where m is the dimension of the representation.

Proof. Define the matrices $Y(\mu\nu)$ to have components $Y(\mu\nu)_{\rho\lambda} = \delta_{\mu\rho}\delta_{\nu\lambda}$. Then define the matrices $X(\mu\nu) = \sum_{g\in G} M(g^{-1}) Y(\mu\nu) N(g)$. First, note that $X(\mu\nu)_{\rho\lambda} = \sum_{g\in G} M_{\mu\rho}(g)^* N_{\nu\lambda}(g)$ (expand into indices and apply the fact that M is a unitary matrix). Next, note:

$$
\begin{aligned}
M(h)X(\mu\nu) &= \sum_g M(hg^{-1}) Y(\mu\nu) N(g) \\
&= \sum_{g'} M(g'^{-1}) Y(\mu\nu) N(g'h) \\
&= X(\mu\nu) N(h)
\end{aligned}
$$

where the substitution $g'^{-1} = hg^{-1}$ was made.

In the case that M and N are inequivalent representations, Schur's lemma gives $X(\mu\nu) = 0$ so that $\sum_{g\in G} M_{\mu\rho}(g)^* N_{\nu\lambda}(g) = 0$ and the theorem is proved.

In the case that $M = N$, we have shown that $X(\mu\nu)$ commutes with all elements $M(h)$, and Schur's other lemma gives that each X must be proportional to the identity matrix: $X(\mu\nu) = c(\mu\nu) I$. We can take the trace of $X(\mu\nu)$ to get the constant:

$$\begin{aligned} mc(\mu\nu) &= \mathrm{tr}(X(\mu\nu)) \\ &= \sum_g \mathrm{tr}(M(g^{-1})Y(\mu\nu)M(g)) \\ &= \sum_g \mathrm{tr}(Y(\mu\nu)) \\ &= |G|\delta_{\mu\nu} \end{aligned}$$

So we have shown $X(\mu\nu) = \frac{|G|}{m}\delta_{\mu\nu}I$, and therefore

$$\sum_{g\in G} M_{\mu\rho}(g)^* N_{\nu\lambda}(g) = X(\mu\nu)_{\rho\lambda} = \frac{|G|}{m}\delta_{\mu\nu}\delta_{\rho\lambda}$$

□

Note 2.20. Conspicuously missing is the case $N(g) = QM(g)Q^{-1}$ for all g. If you plug this into the orthogonality relation, you get $\sum_{g\in G} M^*_{\mu\rho}(g)(QM(g)Q^{-1})_{\nu\lambda} = \frac{|G|}{m}Q_{\nu\mu}(Q^{-1})_{\rho\lambda}$. So you can't apply this relation to two equivalent irreps. However, if you add the caveat that M and N are either inequivalent or exactly equal, we can write The Great Orthogonality Relation in one line:

$$\sum_{g\in G} M^*_{\mu\rho}(g)N_{\nu\lambda}(g) = \frac{|G|}{m}\delta_{MN}\delta_{\mu\nu}\delta_{\rho\lambda} \tag{1}$$

3 Characters

3.1 Basic Theorems and Theory

We'll consider unitary matrix representations unless otherwise stated.

Definition 3.1. Let $f : G \to \mathbb{C}$ be some function. If $f(g) = f(h^{-1}gh)$ for all $h, g \in G$, then f is called a *class function*. The space of all class functions is an n_c-dimensional vector space, where n_c is the number of conjugacy classes of G.

Definition 3.2. For a matrix representation $M(g)$, define the *character* $\chi_M(g) = \mathrm{tr}\, M(g)$. χ_M is a class function, because $\mathrm{tr}\, M(hgh^{-1}) = \mathrm{tr}\, M(h)M(g)M(h^{-1}) = \mathrm{tr}\, M(g)$.

Theorem 3.3. (Orthogonality of characters).

$$\sum_{g\in G} \chi_M(g)^*\chi_N(g) = |G|\delta_{MN}$$

Proof.

$$\begin{aligned}\sum_{g\in G}\chi_M(g)^*\chi_N(g) &= \sum_g M^*_{\rho\rho}(g)N_{\mu\mu}(g)\\ &= \frac{|G|}{m}\delta_{MN}\delta_{\rho\mu}\delta_{\rho\mu}\\ &= \frac{|G|}{m}\delta_{MN}\delta_{\rho\rho}\\ &= |G|\delta_{MN}\end{aligned}$$

□

Corollary 3.4. Two irreps are equivalent if and only if they have the same characters.

Proof. If two irreps are equivalent, the cyclicity of the trace immediately means they have the same characters. ($\operatorname{tr} N^{-1}MN = \operatorname{tr} MNN^{-1}$)

If they have the same characters, then $\sum_g \chi_M(g)^*\chi_{M'}(g) > 0$, so that they can't be inequivalent. □

Corollary 3.5. Suppose $D(g)$ is a reducible representation, and let M label the irreducible representations of G. We may write write $X^{-1}D(g)X = \oplus a_M M(g)$, a_M being the number of times the irrep M appears in D. Then $\chi_D(g) = \sum_M a_M\chi_M(g)$, and orthogonality gives $a_M = \frac{1}{|G|}\sum_g \chi^*_M(g)\chi_D(g)$. That is, we can figure out what a reducible representation is built out of by taking inner products of the characters.

Corollary 3.6. A representation of a finite group is irreducible if and only if $\sum_g |\chi_M(g)|^2 = |G|$.

Proof. If we have an irrep, then the orthogonality theorem 3.3 shows the formula holds.

Now suppose $\sum_g |\chi_M(g)|^2 \neq |G|$. Using the previous corollary we can take inner products with all the irreps, and we must find that two or more irreps appear in the direct sum, so that M is reducible. □

Theorem 3.7. Let j run over all unitary irreps of a finite group G, where each nonequivalent irrep has dimension m_j. Then $\sum m_j^2 = |G|$.

Proof. Let's construct a $|G|$-dimensional representation of G called the regular representation M^R. Our $|G|$-dimensional vector space is called "group space". We let every element of G correspond to another orthonormal basis vector, so we can write vectors as $\sum_g c_g g$.

Let $f, g, h \in G$, and note that we can use these as indices as well as arguments. Define $M^R_{f,g}(h) = 1$ if $g = fh$ and zero otherwise. ie, $M^R_{f,g}(h) = \delta_{g,fh}$. This forms a representation, because...

$$\begin{aligned}M^R_{f,g}(h)M^R_{g,i}(k) &= \delta_{g,fh}\delta_{i,gk}\\ &= \delta_{i,fhk}\\ &= M^R_{f,i}(hk)\end{aligned}$$

The only way to get a nonzero trace for this is if $f = fh$ for some h, f, in which case h is the identity and the trace is $|G|$.

Now decompose M^R into irreps as above. We have $a_j = \frac{1}{|G|}\sum_g \chi^*_j(g)\chi_c(g) = \frac{1}{|G|}\chi^*_j(e)\chi_c(e) = m_j$, where m_j is the dimension of the jth irrep.

So, from corollary 3.5, $|G| = \chi_c(e) = \sum_j a_j\chi_j(e) = \sum_j m_j^2$. □

Theorem 3.8. Characters form a complete basis of class function space.

Proof. Let $F(g)$ be any class function, ans suppose $F(g)$ is a class function orthogonal to every character χ_M for irreps M. We will show $F(g) = 0$ identically.

Define $\phi_M = \sum_g F(g)^* M(g)$. Note:

$$M(h)\phi_M M(h)^{-1} = \sum_g F(g)^* M(hgh^{-1}) = \sum_g F(h^{-1}gh)^* M(g) = \phi_M$$

If M is an irrep, Schur's lemma gives $\phi_M = \lambda I$, and if we take traces we find $m\lambda = \sum_g F(g)^* \chi_M(g)$, which is zero by assumption. So for every irrep, $\phi_M = 0$.

Now consider the regular representation on group space again. Consider ϕ^R_M acting on the basis vector corresponding to the identity element e. $M^R(g)e = g$, so $\phi^R_M e = \sum_g F(g)^* g$. But M^R is conjugate to a direct sum of irreps, so we can write $X^{-1}\phi^R_M X = \sum_g F(g)^* \bigoplus_M M(g) = \bigoplus_M \phi_M$ where each irrep occurs multiple times in the direct sum. But each term in the sum is equal to zero! So $0 = \sum_g F(g)^* g$, and by linear independence $F(g) = 0$ for all g. □

Corollary 3.9. (Completeness relation):

$$\frac{|G|}{|C_\beta|}\delta_{\alpha,\beta} = \sum_j \chi_j(C_\alpha)\chi_j(C_\beta)^*$$

Where C_α are the conjugacy classes of G, the label α runs over conjugacy classes, and the label j runs over all irreps.

Proof. Since the characters form a complete basis of class function space, we can write $F(g) = \sum_j a_j \chi_j(g)$ for some coefficients a_j. Choose $F(C_\alpha) = \delta_{\alpha,\beta}$, that is $F(g)$ is one if $g \in C_\beta$ and zero otherwise.

Multiply both sides by $\chi_k(g)^*$ and sum over g to figure out a_j. On the righthand side we can use the orthogonality relation 3.3: $\sum_g \sum_j \chi^*_k(g) a_j \chi_j(g) = \sum_j |G| a_j \delta_{kj} = |G| a_k$. Now work on the left-hand side:

$$\begin{aligned}
\sum_g \chi_k(g)^* F(g) &= \sum_\alpha |C_\alpha| \chi_k(C_\alpha)^* F(C_\alpha) \\
&= \sum_\alpha |C_\alpha| \chi_k(C_\alpha)^* \delta_{\alpha\beta} \\
&= |C_\beta| \chi_k(C_\beta)^*
\end{aligned}$$

Setting the two sides equal, we find $a_j = \frac{|C_\beta|}{|G|}\chi_k(C_\beta)^*$, and so:

$$\frac{|G|}{|C_\beta|}\delta_{\alpha,\beta} = \sum_j \chi_j(C_\alpha)\chi_j(C_\beta)^*$$

□

Definition 3.10. (Self-conjugate representations). If $M(g)$ is an irrep, then $M(g)^*$ is also an irrep. If $M(g)^*$ is similar to $M(g)$, then M is called a *self-conjugate* representation. Otherwise M is called a *complex representation.*

If $M(g)$ is self-conjugate and there is a basis where all of its entries are real, then $M(g)$ is called a *real representation.* If no such basis exists, $M(g)$ is called a *pseudo-real representation.* (because $X^{-1}M(g)^*X = M(g)$ looks a bit like the equation $c^* = c$)

Theorem 3.11. If $M(g)$ is a self-conjugate irrep, and $X^{-1}M(g)^*X = M(g)$, then X is symmetric if M is a real representation and antisymmetric if M is a pseudoreal representation.

Theorem 3.12.

$$\frac{1}{|G|}\sum_g \chi(g^2) = \begin{cases} 1 & M \text{ is real} \\ -1 & M \text{ is pseudoreal} \\ 0 & M \text{ is complex} \end{cases}$$

Definition 3.13. A conjugacy class C_g is called self-conjugate if for all $h \in C_g$, $h^{-1} \in C_g$.

Theorem 3.14. The number of self-conjugate classes is equal to the number of self-conjugate irreps.

Definition 3.15. (Character table) A Character table is a square table. Each column represents a different conjugacy class, and each row is a different irrep. Its entries are $\chi_M(C_\alpha)$. For example:

	$\{e\}$	C_1	C_2
M_1	$\chi_1(e)$	$\chi_1(C_1)$	$\chi_1(C_2)$
M_2	$\chi_2(e)$	$\chi_2(C_1)$	$\chi_2(C_2)$
M_3	$\chi_3(e)$	$\chi_3(C_1)$	$\chi_3(C_2)$

There are lots of rules we've derived, summarized here. Let j be an index that goes over irreps, m_j be the dimension of the jth irrep, n_c be the number of conjugacy classes of G. Let α be an index that goes over conjugacy classes and let C_α be the αth conjugacy class.

1. $\chi_j(e) = m_j$ (The first column of the character table gives the dimension of each irrep)
2. $|G| = \sum_j m_j^2$ (the sum of squares of each dimension is $|G|$)
3. $|G| = \sum_\alpha |C_\alpha||\chi_j(C_\alpha)|^2$ (the sum of squares of each row, weighted by $|C_\alpha|$, is $|G|$)
4. $0 = \sum_\alpha |C_\alpha|\chi_j(C_\alpha)^*\chi_k(C_\alpha)$ for $j \neq k$ (the rows, weighted by $|C_\alpha|$, are orthogonal.)
5. $|G| = \sum_j |C_\alpha||\chi_j(C_\alpha)|^2$ (the sum of squares of each column, weighted by $|C_\alpha|$, is $|G|$)
6. $0 = \sum_j \chi_j(C_\alpha)\chi_j(C_\beta)^*$ for $\alpha \neq \beta$ (the columns are orthogonal)

Note that items 2, 5, and 6 are just the completeness relation 3.9, and that items 3 and 4 are just the orthogonality relation 3.3.

Note 3.16. If $M(g)$ is a representation of G, and $H \leq G$, then M is also a representation of H. In general $M(g)$ will be a reducible representation of H, even if it was an irreducible representation of G.

Note 3.17. If M_1 is an one-dimensional irrep of G and M_2 is some other irrep, then $M_1(g)M_2(g)$ is also an irrep with character $\chi_1(g)\chi_2(g)$. So if we have a one-dimensional irrep, we can take products with other rows of the character table to get other rows of the character table.

3.2 Characters and products

There is a lot of confusion on the use of the words direct product, tensor product, and direct sum, especially as used in physics. In considering representations of a group G, we have already spelled out the direct sum and tensor product representations where "direct sum" and "tensor product" refer to whether we consider representations on $V_1 \oplus V_2$ or representations on $V_1 \otimes V_2$. The characters of the direct sum representation are $\chi^1(g) + \chi^2(g)$, and the characters of the tensor product representation are $\chi^1(g) \cdot \chi^2(g)$.

There is another possibility. Given two groups G and H, a "direct product" representation can refer to a representation of a group $G \times H$. We can naturally form representations of $G \times H$ on both $V_G \oplus V_H$ and $V_G \otimes V_H$. These have characters $\chi^G(g) + \chi^H(h)$ and $\chi^G(g) \cdot \chi^H(h)$. The irreps of $G \times H$ are tensor products of irreps of G with irreps of H.

Group	Vector Space	Character
G	V	$\chi^V(g)$
H	U	$\chi^U(h)$
G	$V_1 \oplus V_2$	$\chi^{V_1}(g) + \chi^{V_2}(g)$
G	$V_1 \otimes V_2$	$\chi^{V_1}(g)\chi^{V_2}(g)$
$G \times H$	$V \oplus U$	$\chi^V(g) + \chi^V(h)$
$G \times H$	$V \otimes U$	$\chi^V(g)\chi^V(h)$

3.3 Characters of Abelian and Dihedral Groups

Example 3.18. (Characters of Abelian groups) If C_N is the cyclic group of order N, then we just have a bunch of one-dimensional representations.

$$\begin{array}{c||cccccc}
C_N & \{g^0\} & \{g\} & \{g^2\} & \dots & \{g^b\} & \dots \\
\hline
R_0 & 1 & 1 & 1 & & & \\
R_1 & 1 & e^{2\pi i/N} & e^{4\pi i/N} & & & \\
R_2 & 1 & e^{4\pi i/N} & e^{8\pi i/N} & & & \\
\vdots & & & & \ddots & & \\
R_a & & & & & e^{2\pi i \frac{ab}{N}} & \\
\vdots & & & & & & \ddots
\end{array}$$

For groups with more generators, we have to take product of these Fourier-like factors:

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\{e\}$	$\{a\}$	$\{b\}$	$\{ab\}$
$R_{0,0}$	1	1	1	1
$R_{1,0}$	1	-1	1	-1
$R_{0,1}$	1	1	-1	-1
$R_{1,1}$	1	-1	-1	1

You can go more in depth and do Fourier analysis with this.

TODO: Character table for the even dihedral groups
TODO: Character table for the odd dihedral groups

3.4 Character tables of T, O, and I

Example 3.19. (Character table for the proper tetrahedral group) The improper tetrahedral group is equivalent to the set of even permutations on 4 elements, A_4. There's the trivial permutation, 3 permutations of the form $(\cdot\ \cdot)(\cdot\ \cdot)$, and 8 permutations of the form $(\cdot\ \cdot\ \cdot)$.

The 3 permutations of the first form are in their own class, $C_{2,2}$ (Proof: Consider conjugating (1 2)(3 4) by (1 2 3)).

The 8 permutations split into two conjugacy classes, C_3 and C_3', because for example (1 2 3) and (3 2 1) are only conjugate if you allow the odd permutation (1 3). This also means that C_3 is not self-conjugate, so the character table doesn't have to be real.

$T \cong A_4$	$\{e\}$	$3C_{2,2}$	$4C_3$	$4C_3'$
A	1	1	1	1
E	1	1	ω	ω^2
E'	1	1	ω^2	ω
T	3	-1	0	0

where $\omega^3 = 1$. Proof: First, $12 = 1^2 + 1^2 + 1^2 + 3^2$ is the only way to add 1 and 3 squares to get 12. Next, since we know there are three one-dimensional representations, we should try roots of unity. We might try to send $C_{2,2}$ to -1, but this is inconsistent because $(1\ 3)(4\ 2) \cdot (1\ 2)(3\ 4) = (1\ 4)(2\ 3)$. The only option left is ω and ω^2

Finally, column orthogonality gives us the characters of the 3-dimensional representation.

Example 3.20. (Character table for the Octahedral group) The octahedral group is isomorphic to the set of permutations on 4 elements, S_4. The partitions of 4 are $1+1+1+1$, $1+1+2$, $2+2$, $3+1$, and 4. The class sizes are 1, 6, 3, 8, 6. Let's call these $\{e\}$, C_2, $C_{2,2}$, C_3, and C_4 respectively. $C_{2,2}$ and C_3 are the only conjugacy classes of even permutations, so we'll put them in the first two columns. They're all self-conjugate, so we know our character table will be real. I find:

$O \cong S_4$	$\{e\}$	$3C_{2,2}$	$8C_3$	$6C_2$	$6C_4$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	2	-1	0	0
T_1	3	-1	0	1	-1
T_2	3	-1	0	-1	1

A_1 and A_2 can be found by guessing and checking. The dimensions $2, 3, 3$ are the only solutions of the equation $a^2 + b^2 + c^2 = 22$. After that, it helps to look at the representation defined by permutations on 4 basis vectors. This representation has character $(4, 0, 1, 2, 0)$, so it has one copy of A_1 and zero copies of A_2. The only option left is that it contains one of the T representations. Subtract $(1, 1, 1, 1, 1)$ from this to get T_1. Row A_2 multiplied by row T_1 gives us T_2. Finally, column orthogonality gives us E.

Example 3.21. (Character table for the Icosahedral group) The icosahedral group is isomorphic to the set of even permutations on 5 elements, A_5.

Consider an icosahedron. There are 12 axes of fivefold rotation (the edges), 20 axes of threefold rotation (the faces), and 15 axes of twofold rotation (there are 30 edges and each axis passes through two edges). Rotation by $2\pi/5$ about a fivefold axis is conjugate to rotation by

$-2\pi/5$ about the same axis (conjugate by the twofold rotations!). But rotation by $2\pi/5$ isn't conjugate to rotation by $4\pi/5$. So our conjugacy classes are $\{e\}$, $12C_5$, $12C_5^2$, $20C_3$, and $15C_2$.

$I \cong A_5$	$\{e\}$	$12C_5$	$12C_5^2$	$20C_3$	$15C_2$
A	1	1	1	1	1
T_1	3	$1+2\cos(\frac{2\pi}{5})$	$1+2\cos(\frac{4\pi}{5})$	0	-1
T_2	3	$1+2\cos(\frac{4\pi}{5})$	$1+2\cos(\frac{2\pi}{5})$	0	-1
G	4	-1	-1	1	0
H	5	0	0	-1	1

Some notes on deriving this character table: $1+2\cos(2\pi/5) = -2\cos(4\pi/5)$, so you often see it written differently. The trace of any three-dimensional matrix with rotation by theta is always $1+2\cos(\theta)$, so T_1 and T_2 can be found by guessing and considering the actual point group of the icosahedron. Next, if you consider the representation of A_5 as it acts on the basis of 5 vectors, you find that it contains one copy of A and no copies of T_1 or T_2. The subspace of this representation such that $x_1+x_2+x_3+x_4+x_5 = 0$ is the four-dimensional irrep of $A_5 \cong I$ (I found this in Goodman and Wallach).

After all that, you can get H by orthogonality.

3.5 Decomposition of Spin(N) into various irreps

It is assumed the reader is familiar with the fact that Spin(ℓ) is a $2\ell+1$ dimensional irreducible representation of $SO(3)$. The group is composed of elements like $\exp(i\vec{\theta}\cdot\vec{L})$ where L_x, L_y, and L_z are matrices. All we really need is L_z, consisting of the diagonal matrix $(\ell, \ell-1, \ell-2, \cdots, 1-\ell, -\ell)$.

We can consider the action of subgroups of $SO(3)$ like T, O, and I on Spin(ℓ), and we find that in most cases Spin(ℓ) is not irreducible under these groups. The characters are easy to determine. For example, $\chi(\{e\})$ will always be $2\ell+1$. If we were considering rotation by $\theta = 2\pi/5$, we'd look at $\chi(12C_5) = \mathrm{tr}(e^{iL_z\theta}) = 1+2\cos(\theta)+2\cos(2\theta)+\cdots+2\cos(\ell\theta)$. Then we can take inner products with the previous three character tables to find out how the group decomposes! Using Mathematica, we can take inner products easily.

Spin rep	T decomposition	O decomposition	I decomposition
Spin(0)	A	A_1	A
Spin(1)	T	T_2	T_1
Spin(2)	$E\oplus E'\oplus T$	$E\oplus T_1$	H
Spin(3)	$A\oplus 2T$	$A_2\oplus T_1\oplus T_2$	$T_2\oplus G$
Spin(4)	$A\oplus E\oplus E'\oplus 2T$	$A_1\oplus E\oplus T_1\oplus T_2$	$G\oplus H$
Spin(5)	$E\oplus E'\oplus 3T$	$E\oplus T_1\oplus 2T_2$	$T_1\oplus T_2\oplus H$
Spin(6)	$2A\oplus E\oplus E'\oplus 3T$	$A_1\oplus A_2\oplus E\oplus 2T_1\oplus T_2$	$A\oplus T_1\oplus G\oplus H$
Spin(7)	$A\oplus E\oplus E'\oplus 4T$	$A_2\oplus E\oplus 2T_1\oplus 2T_2$	$T_1\oplus T_2\oplus G\oplus H$

You can visualize Spin(ℓ) using spherical harmonics. For example, the vector $(1,0,\cdots,0)$ corresponds to the spherical harmonic Y_ℓ^ℓ, the vector $(0,1,\cdots,0)$ represents $Y_\ell^{\ell-1}$, and so on. What this means is that we can visualize the decompositions by using spherical harmonics as well!

Take the icosahedral group for example. If $R(g)$ is a $2\ell+1$ dimensional matrix, we have to be able to find some unitary X for which $XR(g)X^\dagger = D$ is block-diagonal for all $g \in I$ – that's just the definition of a reducible representation. But we know from linear algebra that we can look at X as a change of basis matrix. If we take a column of $X^\dagger$ (call it ξ), we can consider $R(g)\xi$. That's the same as $X^\dagger D X \xi$, but $X\xi$ is just a column vector with a 1 in some index and zeros elsewhere.

This tells us that columns ξ of $X^\dagger$ transform under irreducible representations of I. So if we take $\sum_{-\ell}^{\ell} \xi_m Y_\ell^m$, this will give us a wavefunction on a sphere which transforms under some irrep! You can go on about clebsch gordan coefficients, but I'll just give some examples. In the following plots, color represents phase and radius represents amplitude.

Example 3.22. Considering tetrahedral irreps, $\text{Spin}(3) \cong A \oplus 2T$. The following 7 wavefunctions are an orthonormal basis that span the same function space as Y_3^m for $-3 \leq m \leq 3$. The decomposition between the two copies of T is not unique.

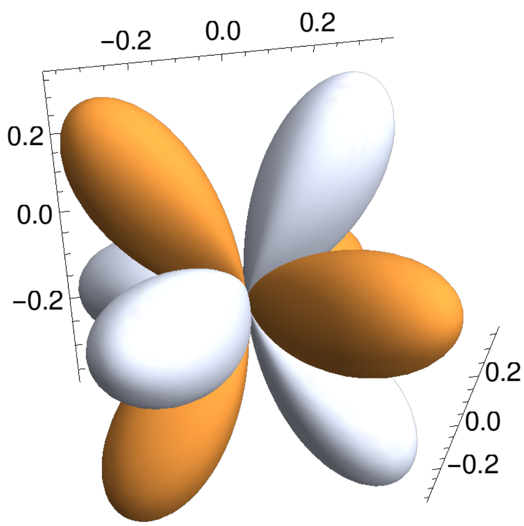


Figure 1: Transforms under A (the identity). Clear tetrahedral symmetry.

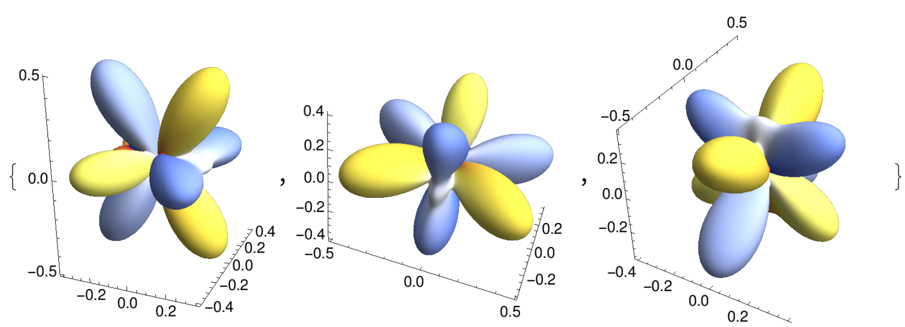


Figure 2: Transforms under one copy of T

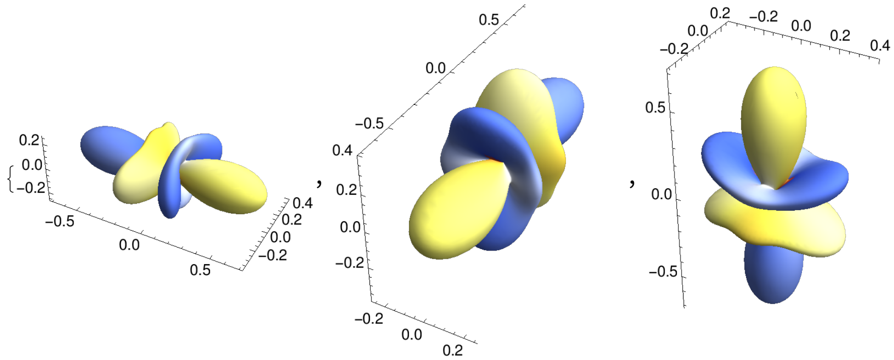


Figure 3: Transforms under a different copy of T

Example 3.23. Considering icosahedral irreps, Spin(6) $\cong A \oplus T_1 \oplus G \oplus H$. We've found 13 wavefunctions which are a good alternative orthonormal basis for the 13 spherical harmonics.

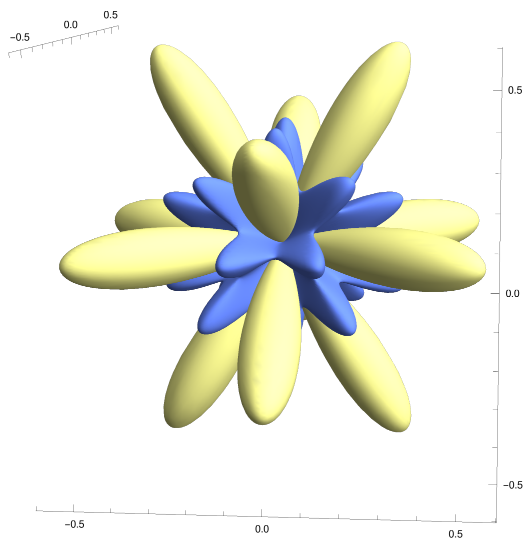


Figure 4: Transforms under A (the identity). This is the first one out of spin 0 through 5 whose wavefunction has such clear icosahedral symmetry.

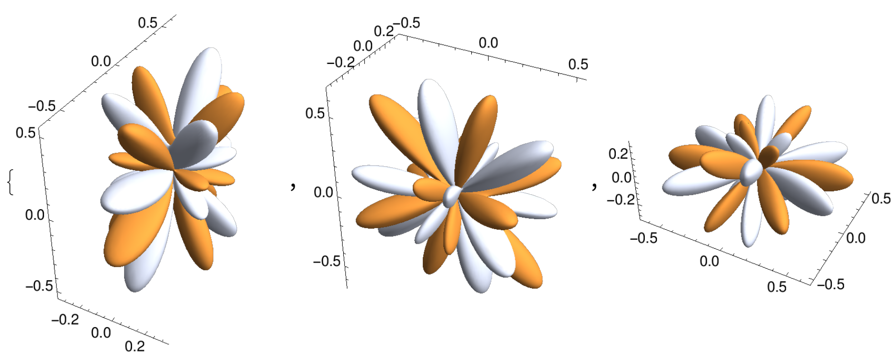


Figure 5: Transforms under the 3-dimensional irrep T_1

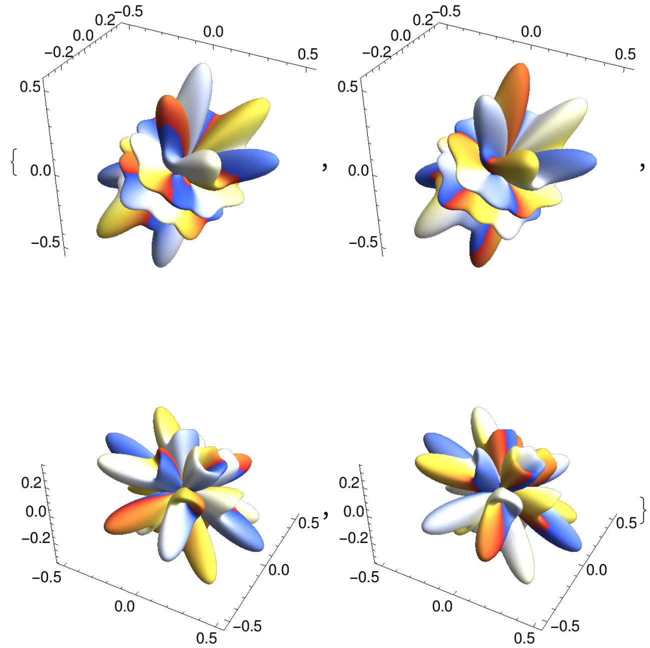


Figure 6: Transforms under the 4-dimensional irrep G

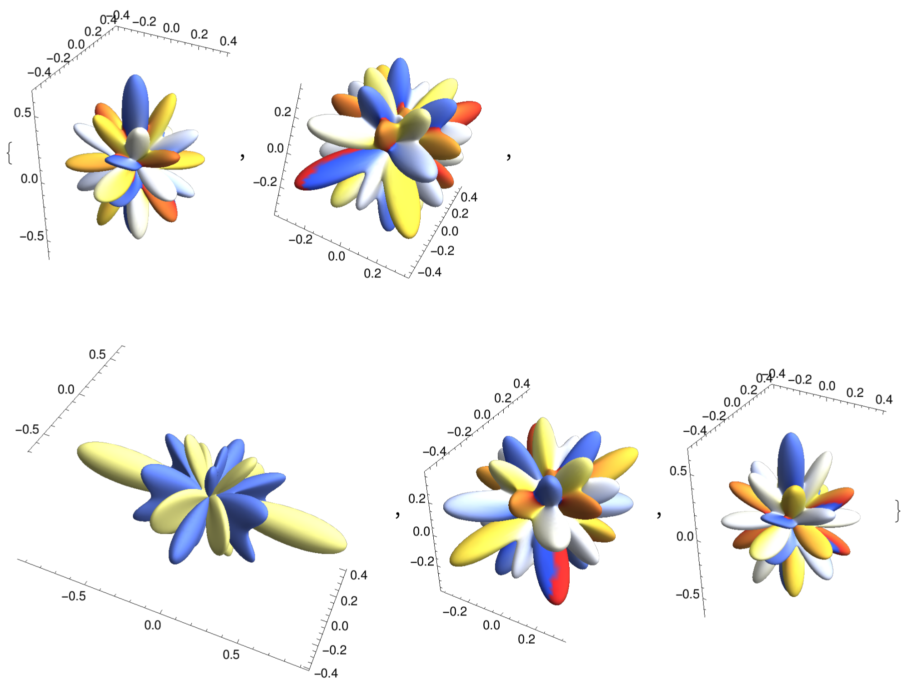


Figure 7: Transforms under the 5-dimensional irrep H

4 3 Dimensional Point Groups

4.1 Basic Definitions

Definition 4.1. $O(N)$ is the group of all orthonormal real matrices in N dimensions.

Definition 4.2. $SO(N) \leq O(N)$ is the group of all elements of $O(N)$ with determinant 1.

Definition 4.3. A *point group* or *finite point group* is a finite subgroup of $O(N)$. Two point groups are equivalent if they are related by conjugation.

A *proper point group* is a finite subgroup of $O(N)$.

Theorem 4.4. The only point groups in 2 dimensions are C_n (the cyclic group of order n) and D_n (the dihedral group on n points).

Theorem 4.5. The only finite subgroups of the rotation group in 3 dimensions are C_n, D_n, T, O, and I. T is the tetrahedral group, O the octahedral group, and I the icosahedral group.

Proof. The n-fold axes of a finite subgroup of $SO(3)$ form a regular polyhedron. There are five regular polyhedra, and the cube and octahedron correspond to the octahedral group while the icosa- and dodeca- hedron correspond to the icosahedral group. The tetrahedral group remains. □

Note 4.6. The subgroups of $O(3)$ are much more exhausting! This is because equivalent group structures can have different realizations in 3-space even if they have the same group structure. For example, we could rotate by 90° about the $\hat{z}$ axis to get C_4. We could also rotate by 90 degrees about the $\hat{z}$ axis, then reflect across the $\hat{x}\hat{y}$ plane to get a different cyclic group S_4. As groups they're equivalent (and so their character tables are the same), but their 3D representations are different.

Definition 4.7. Schönflies or Schoenflies notation is a convention very useful in writing about 3-dimensional point groups. It involves the following symbols, each representing a different point group or family of point groups:

$$C_n \quad C_{nh} \quad C_{nv} \quad S_{2n} \quad D_n \quad D_{nd} \quad D_{nh} \quad T \quad T_d \quad T_h \quad O \quad O_h \quad I \quad I_h$$

C_n, D_n, T, O, and I are the proper point groups mentioned earlier.

h is for "horizontal" plane reflection, "d" is for "diagonal" plane reflection, and "v" is for "vertical" plane reflection. The "C" is for "cyclic" (even though C_{nh} and C_{nv} aren't cyclic), and the "S" is for Spiegel, which is the German word for mirror. The notation is sort of easy to memorize and specific to three dimensions.

4.2 Cn, Cnh, Cnv, and S2n

Definition 4.8. For this section and the next, define the following four matrices:

$$c_n = \begin{bmatrix} \cos(2\pi/n) & -\sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \sigma_h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \sigma_v = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c_n is the cyclic generator of order n, σ_h is a reflection through the horizontal $\hat{x}\hat{y}$ plane, and σ_v is a reflection through the vertical $\hat{y}\hat{z}$ plane. These will be our generators.

Definition 4.9. If X is a shape in 3D space, the *orbit* of X under a group G is the set of all gX where $g \in G$.

Definition 4.10. C_n is the cyclic group of rotations about the $\hat{z}$ axis by an angle $\frac{2\pi}{n}$. It's generated by the single matrix c_n. C_n has the same group structure as $\mathbb{Z}_n$.

Definition 4.11. C_{nh} is the group C_n together with reflections about the horizontal plane. It's generated by the matrices c_n and σ_h. C_{nh} has the same group structure as $\mathbb{Z}_n \times \mathbb{Z}_2$ (note σ_h commutes with c_n).

Definition 4.12. C_{nv} is the group C_n together with reflections about the horizontal plane. It's generated by the matrices c_n and σ_v. C_{nv} has the same group structure as the dihedral group D_n.

Definition 4.13. S_{2n} is cyclic group generated by the matrix $\sigma_h c_n$. These are called "rotoreflections". S_{2n} has the same group structure as $\mathbb{Z}_{2n}$.

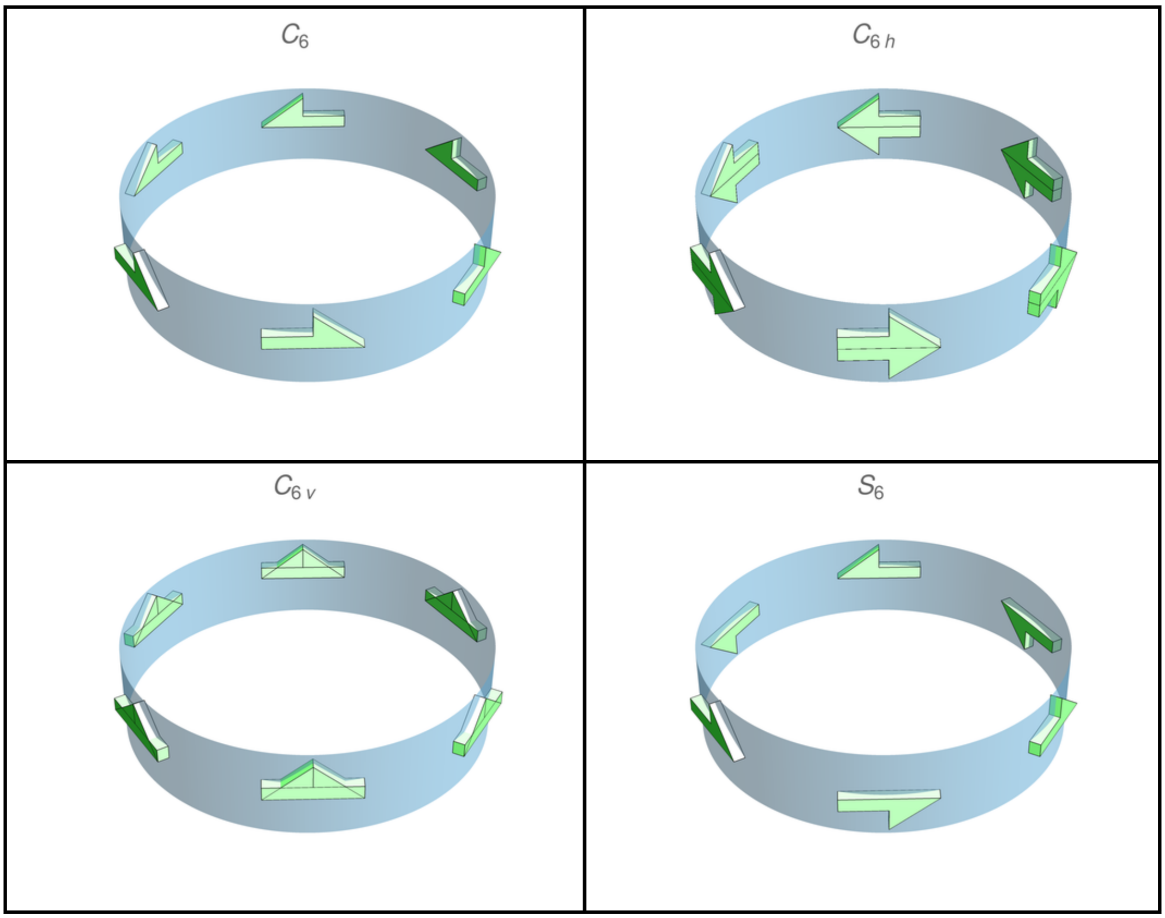


Figure 8: Orbits of an arrow under the given C groups

4.3 Dn, Dnd, and Dnh

Definition 4.14. D_n is dihedral group. It's generated by the matrix c_n, together with rotation by 180 degrees about the $\hat{y}$ direction:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that this matrix is equal to $\sigma_h \sigma_v$

Definition 4.15. D_{nd} is the point group generated by $c_{2n}\sigma_h$ and σ_v. It contains C_n, C_{nh}, C_{nv}, S_n, and D_n. D_{nd} actually has the same group structure as the dihedral group on $2n$ points, D_{2nd}.

Definition 4.16. D_{nh} is the point group generated by c_n, σ_v, and σ_h. It's a subgroup of D_{2n}. As a group, D_{nh} is isomorphic to $D_n \times \mathbb{Z}_2$.

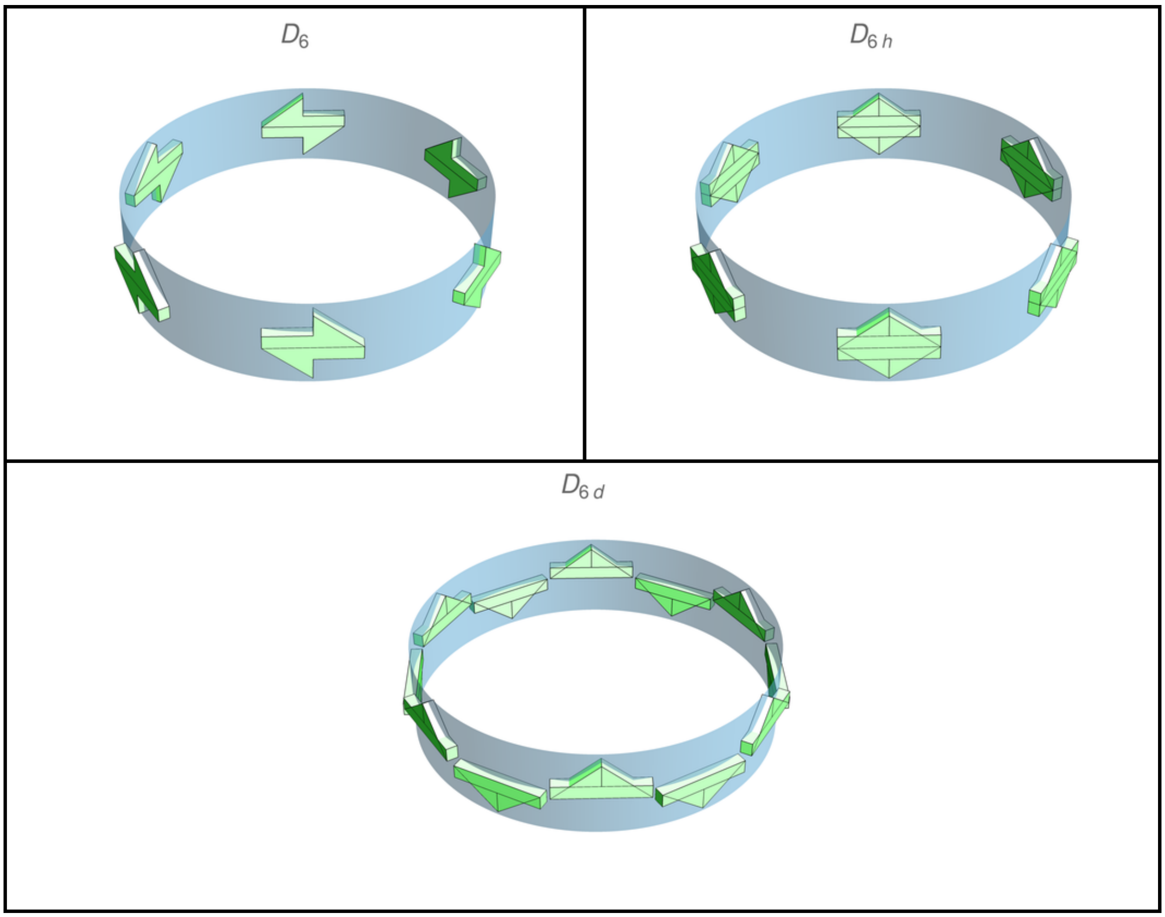


Figure 9: Orbits of an arrow under the given D groups

4.4 T, Td, and Th

Definition 4.17. For this section, we care about different rotations and translations. Consider the tetrahedron with vertices at $(1,1,1)$, $(-1,-1,1)$, $(-1,1,-1)$, and $(1,-1,-1)$. The symmetries of this set of points are: rotation by $180°$ about the $\hat{z}$ axis; rotation by $120°$ about the $(1,1,1)$ axis; reflection through the plane spanned by $(1,1,0)$ and $(0,0,1)$; and finally inversion. These four matrices are, respectively:

$$r_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad r_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \iota = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Definition 4.18. T is the point group generated by r_2 and r_3. As a group it is isomorphic to A_4. It has twelve elements.

Definition 4.19. T_d is the point group generated by r_2, r_3, and σ. As a group it is isomorphic to S_4. It has 24 elements.

Definition 4.20. T_h is the point group generated by r_2, r_3, and ι. As a group it is isomorphic to $A_4 \times \mathbb{Z}_2$ (Note that ι commutes with r_2 and r_3). It has 24 elements.

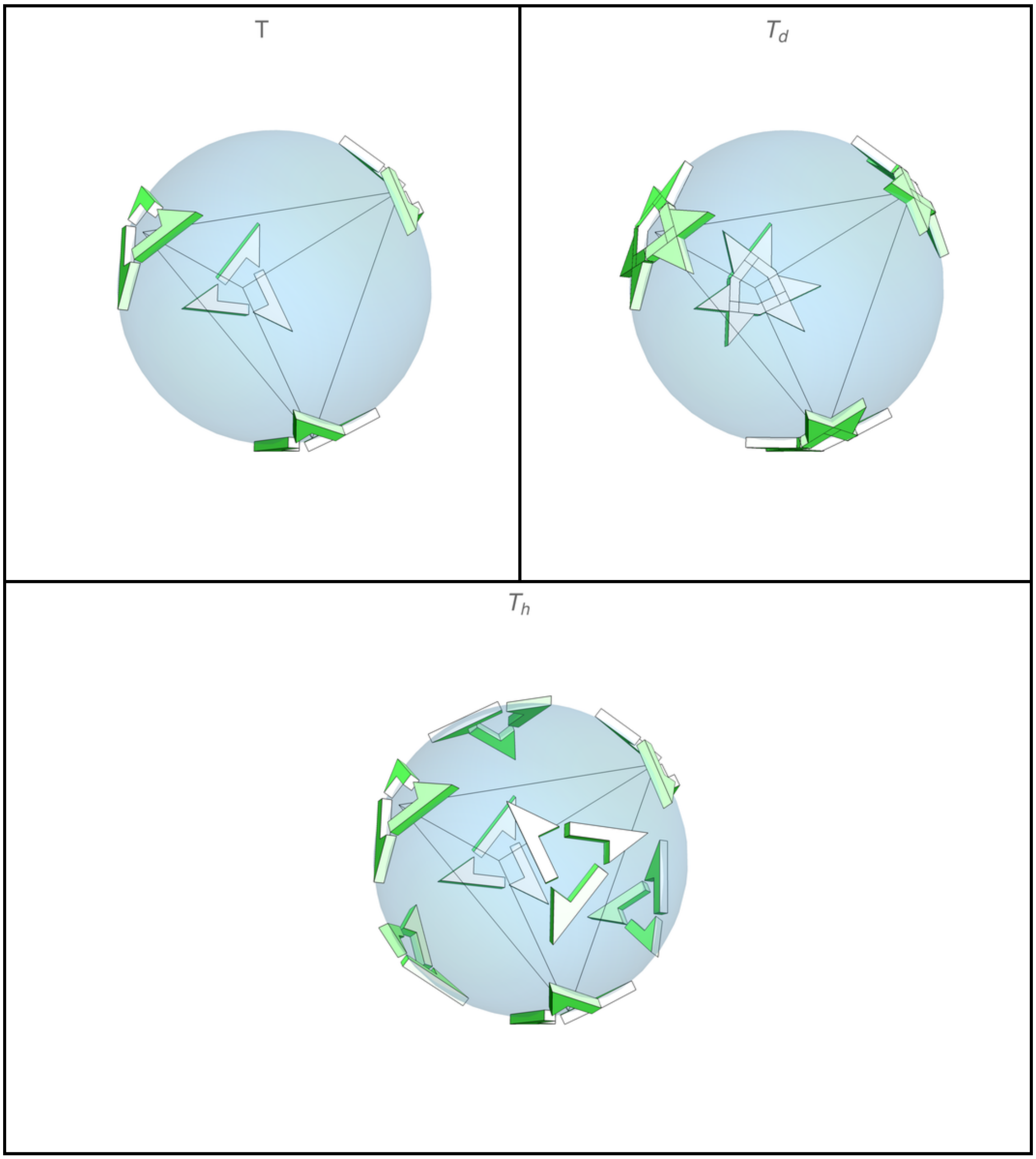


Figure 10: Orbits of an arrow under the given T groups, together with a plot of the underlying tetrahedron.

4.5 O and Oh

Definition 4.21. The proper octahedral point group O can be generated by rotation by $\pi/2$ about any two principal axes. For example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

As a group it is isomorphic to S_4.

Definition 4.22. The improper octahedral point group O_h can be generated by the generators of O as well as inversion symmetry ι. As a group it is isomorphic to $S_4 \times \mathbb{Z}_2$.

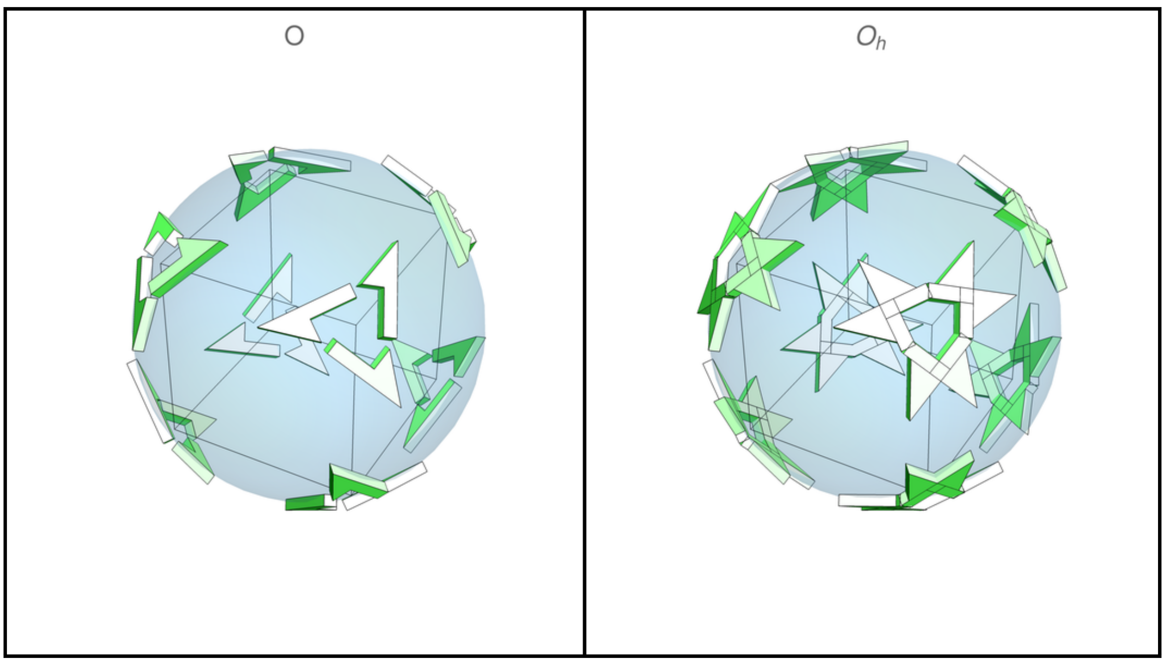


Figure 11: Orbits of an arrow under the given O groups. A cube is also plotted for reference. Note how the diagram for O differs from that for T_h.

4.6 I and Ih

Definition 4.23. The proper Icosahedral point group I can be generated by rotation by π in the $\hat{x}\hat{y}$ plane, as well as a rotation by 72°. If we denote $f = (1+\sqrt{5})/2$, then we have the following two generators:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 & -f & 1/f \\ f & 1/f & -1 \\ 1/f & 1 & f \end{bmatrix}$$

As a group it is isomorphic to A_5.

Definition 4.24. The improper icosahedral point group O_h is generated by the generators of I as well as inversion symmetry ι. As a group it is isomorphic to $A_5 \times \mathbb{Z}_2$.

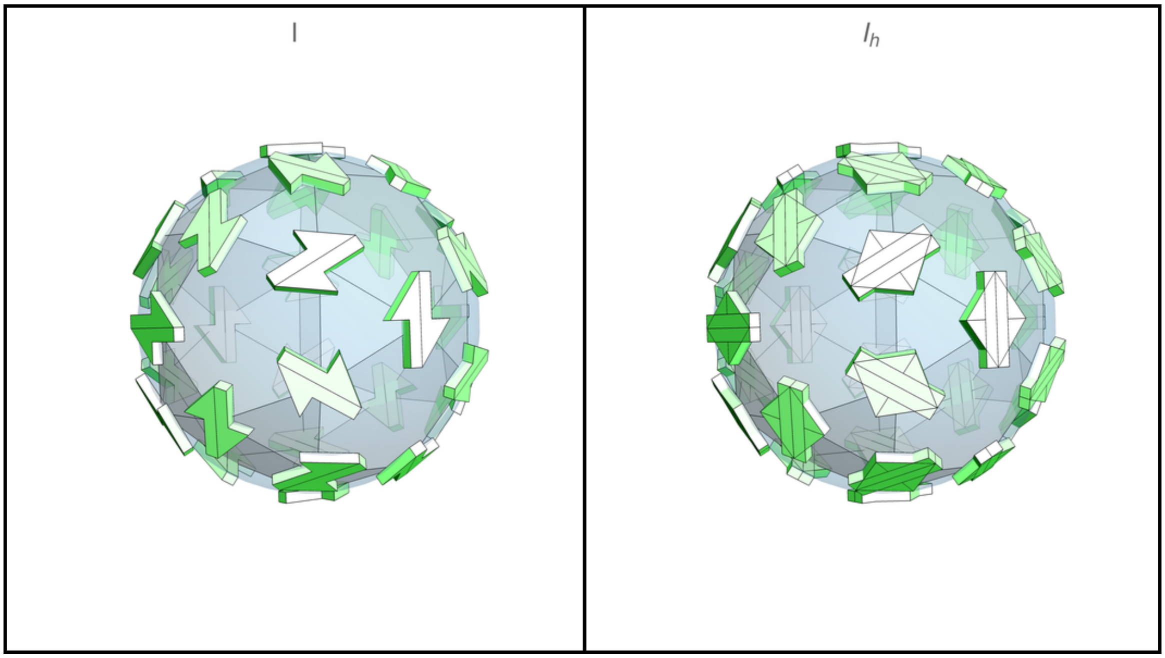


Figure 12: Orbits of an arrow under the given I groups. A partially opaque icosidodecahedron is also plotted, in an attempt to make the diagram more clear.

5 An Elementary look at Carbon C60

5.1 A Sphere with a Weak Icosahedral Potential

5.2 A Tight Binding Model

5.3 A Spring-and-Ball Model

6 Wigner-Eckart theorem

7 Lattices

7.1 Basic Theory

7.2 Application to condensed matter systems

test