EQ2341 Pattern Recognition
Assignment 1: HMM Random Source

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A.1.1

Please see the attached m-files.

A.1.2.1

$$
\begin{aligned}
P(S_{t+1} = j) &= \sum_{i=1}^{N} P(S_{t+1} = j, S_t = i) \\
&= \sum_{i=1}^{N} \underbrace{P(S_{t+1} = j | S_t = i)}_{a_{ij}} P(S_t = i)
\end{aligned} \tag{1}
$$

In the specific case, the relation between states and probabilities can be rewrite like below:

$$
\begin{bmatrix} P(S_{t+1} = 1) \\ P(S_{t+1} = 2) \end{bmatrix} = A^T \begin{bmatrix} P(S_t = 1) \\ P(S_t = 2) \end{bmatrix} \Longleftrightarrow \boldsymbol{p}_{t+1} = A^T \boldsymbol{p}_t \tag{2}
$$

Now we can calculate $\boldsymbol{p}_1$ given $\boldsymbol{q}$ and A.

$$
\boldsymbol{p}_1 = A^T \boldsymbol{q} = \begin{pmatrix} 0.99 & 0.03 \\ 0.01 & 0.97 \end{pmatrix} \cdot \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} \Longrightarrow \boldsymbol{p}_1 = A^T \boldsymbol{q} \tag{3}
$$

Since A is time-invariant, we can iterating the calculating above to conclude that $\boldsymbol{p}_t = \begin{bmatrix} 0.75 & 0.25 \end{bmatrix}^T$ for all t.

To verify that $P(S_t = j)$ is constant is equivalent to prove that the distribution is stationary:

$$
\boldsymbol{p} = A^T \boldsymbol{p} \tag{4}
$$

If equation 4 holds, then we can find a eigenvector of the transposed matrix with eigenvalue 1, as the stationary probability vector.

```
>> [V,D]=eig(A')

V =

    0.9487   -0.7071
    0.3162    0.7071


D =

    1.0000         0
         0    0.9600
```

After calculation, we get the eigenvector with eigenvalue 1: $\begin{bmatrix} 0.9487 & 0.3162 \end{bmatrix}^T$, which is a scale of the initial state probability vector. Then we can prove that our case is a stationary probability distribution and $\boldsymbol{p}_t$ is constant for all t.

A.1.2.2

After we generate a sequence of Markov chain, we observed that the frequencies of $S_t = 1$ and $S_t = 2$ are 0.7531 and 0.2469, which is approximately equal to $P(S_t)$.

A.1.2.3

Then we investigate the HMM model.

$$\begin{aligned} E[X_t] &= E_{S_t}[E_{X_t}[X_t|S_t]] \\ &= P(S_1)\mu_1 + P(S_2)\mu_2 \\ &= 0.75 \cdot 0 + 0.25 \cdot 3 = 0.75 \end{aligned} \tag{5}$$

$$\begin{aligned} var[X_t] &= E_{S_t}[var_{X_t}[X_t|S_t]] + var_{S_t}[E_{X_t}[X_t|S_t]] \\ &= \sum_{j=1}^{2} P(S_t = j) \cdot \sigma_j^2 + \sum_{j=1}^{2} P(S_t = j) \cdot (\mu_j - E[X_t])^2 \\ &= 0.75 \cdot 1 + 0.25 \cdot 4 + 0.75 \cdot (0.75 - 0)^2 + 0.25 \cdot (0.75 - 3)^2 \\ &= 3.4375 \end{aligned} \tag{6}$$

```
MU =

    0.7434


VA =

    3.4338
```

The result agrees approximately with the theoretical values.

A.1.2.4

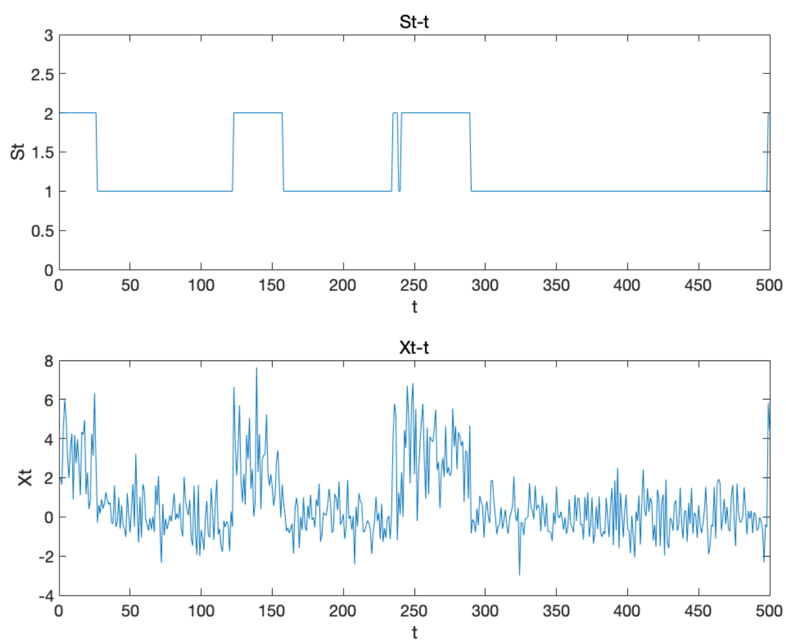

Figure 1: 500 continuous samples from the HMM source

From Fig.1, we can observe that X_t changes its value as S_t transits to a different state and get trapped in a state for a long time since the diagonal value of A

is close to 1. The signal mainly fluctuates between 0 and 3. The transition of state and output probability distribution jointly characterizes the output of the HMM.

A.1.2.5

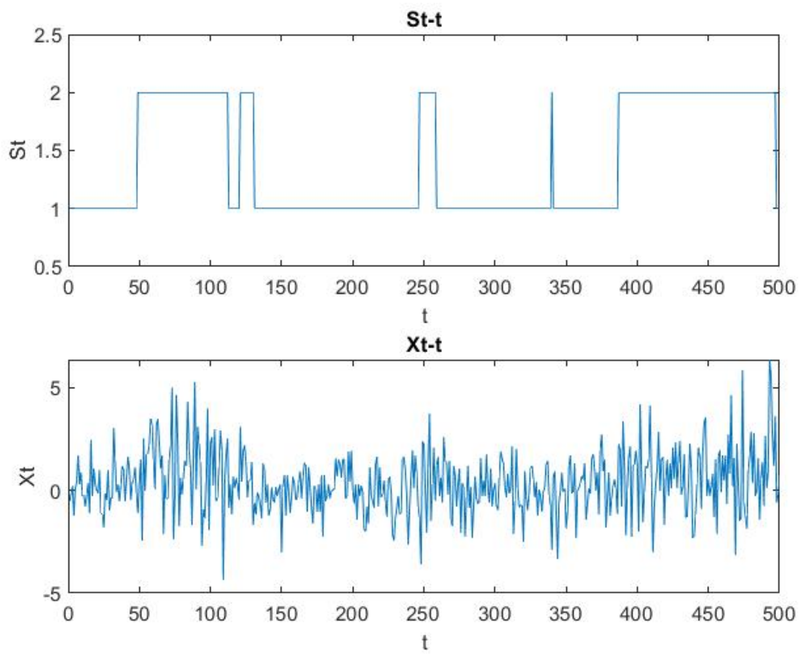

Figure 2: HMM with $\mu_2 = \mu_1 = 0$.

As we can see in Fig.2, the new HMM has the same variance but different mean. It's possible to estimate the state sequence if we measure the difference over time.

A.1.2.6

Now we consider

$$A = \begin{pmatrix} 0.99 & 0.0299 & 0.0001 \\ 0.01 & 0.97 & 0 \end{pmatrix} \tag{7}$$

,which indicates it's a finite-duration HMM. The lengths of some test sequences are 18985, 19495, 11662. They are reasonable because the HMM exits with possibility of 0.0001 in state 1. So the total possibility

$$P(exit) = P(exit|S_1)P(S_1) \tag{8}$$

, which is approximately 0.000075 though $P(S_1)$ increases a little in the stable distribution.

```
q = [0.75 0.25];
%infinite duration
%A = [0.99 0.01;0.03 0.97];
%finite duration
A = [0.99 0.0099 0.0001;0.03 0.97 0];
%X is random variable
b1 = GaussD;
b2 = GaussD('Mean',1,'StDev',2);
B = [b1;b2];
%X is random vector
% b1 = GaussD('Mean',[0;0],'Covariance',[2,1;1,4]);
% b2 = GaussD('Mean',[0;0],'Covariance',[2,1;1,4]);
B = [b1;b2];
mc = MarkovChain(q,A);
hm = HMM(mc,B);
sample = rand(mc,100000);
mu_mc = mean(sample);
%calculate stable distribution
[V, D] = eig(A(1:2,1:2)');
p_cov = V*[1,0;0 0]*inv(V)*q';
sample2 = [];
pd = hm.OutputDistr(sample);
for i = 1:length(sample)
    sample2 = [sample2, rand(pd(i),1)];
end
```

A.1.2.7

Then we generate random vectors and estimate the covariance using the following code.

```
cov = [0,0;0,0];
for i = 1:2
    for j= 1:2
        cov(i,j) = (sample2(i,:)-mu_hm(i))*(sample2(j,:)-mu_hm(j))'/length(sample);
    end
end
```

The estimation $\hat{\Sigma} = \begin{pmatrix} 1.9669 & 1.0162 \\ 1.0162 & 3.9895 \end{pmatrix}$ when $\Sigma = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$