Exact Solutions in the 3+1 Split

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February 3, 2020

1 Introduction

This began as some documentation Erik Schnetter wrote for the Penn State Maya code. I wanted to enter some more simple sample (3+1 splits of) exact space-times. It *may* save someone, somewhere, some work. On the flip side, if you see an obvious error, or have something to suggest, let me know.

There is, of course, a definite bias towards black hole space-times. I may add cosmological ones when/if I get the chance.

A few words about notation: In what follows, Greek indices such as α, β, μ, ν are four-vector indices and run from 0 to 3. Latin indices such as i, j, k, l, m, n are three-vector indices and run from 1 to 3.

When using spherical polar coordinates, in general R will denote the standard "areal" radial coordinate; when dealing with a conformally flat solution, I'll use r to denote the radial coordinate, as then it will *not* be areal. Additionally, I often use the letter q to denote the cylindrical polar quantity $\sqrt{x^2+y^2} = r\sin\theta$; most references I know use the Greek letter ρ for this purpose, but I've found ρ to be used for too many other purposes.

2 3+1 Decomposition and Conventions

The standard reference for the 3+1 decomposition is [1]. I'll repeat only the most important resulting equations here.

The line element ds is given by:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

This leads directly to the relationship between the four-metric $g_{\mu\nu}$ on the one hand and the three-metric γ_{ij}, the lapse α, and the shift β^i on the other hand (see [2], section 21.4):

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_m\beta^m & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \Rightarrow g^{\mu\nu} = \frac{1}{\alpha^2}\begin{pmatrix} -1 & \beta^j \\ \beta^i & \gamma^{ij}\alpha^2 - \beta^i\beta^j \end{pmatrix}$$

The metric connection ${}^4\Gamma^a_{bc}$ is defined in terms of the metric inverse, and the metric derivatives:

$${}^4\Gamma^\mu_{\nu\rho} \equiv \frac{1}{2} g^{\mu\sigma}\left[g_{\nu\sigma,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma}\right].$$

We can write this in terms of 3+1 quantities by looking separately at the time and space components. Starting with the $\mu = 0$:

$$\begin{aligned}
{}^4\Gamma^0_{00} &= -\frac{1}{2\alpha^2}\left[\left(-\alpha^2\right)_{,0} - \gamma_{mn,0}\beta^m\beta^n\right] + \frac{1}{2\alpha^2}\beta^m\left[-\left(-\alpha^2 + \beta_n\beta^n\right)_{,m}\right], \\
{}^4\Gamma^0_{0i} &= -\frac{1}{2\alpha^2}\left[\left(-\alpha^2\right)_{,i} + \beta_m\beta^m_{,i}\right] + \frac{1}{2\alpha^2}\beta^m\left[\gamma_{mi,0} - \beta_{i,m}\right], \\
{}^4\Gamma^0_{ij} &= -\frac{1}{2\alpha^2}\left[\beta_{i,j} + \beta_{j,i} - \gamma_{ij,0}\right] + \frac{1}{2\alpha^2}\beta^m\left[\gamma_{im,j} + \gamma_{mj,i} - \gamma_{ij,m}\right].
\end{aligned}$$

Now for the spatial $\mu = i$ components:

$$
\begin{aligned}
{}^4\Gamma^i_{00} &= \frac{1}{2\alpha^2}\beta^i\left[\left(-\alpha^2+\beta_m\beta^m\right)_{,0}\right] + \frac{1}{2\alpha^2}\left(\alpha^2\gamma^{im}-\beta^i\beta^m\right)\left[2\beta_{m,0}-\left(-\alpha^2+\beta_n\beta^n\right)_{,m}\right],\\
{}^4\Gamma^i_{0j} &= \frac{1}{2\alpha^2}\beta^i\left[\left(-\alpha^2+\beta_m\beta^m\right)_{,j}\right] + \frac{1}{2\alpha^2}\left(\alpha^2\gamma^{im}-\beta^i\beta^m\right)\left[\beta_{m,j}+\gamma_{mj,0}-\beta_{j,m}\right],\\
{}^4\Gamma^i_{jk} &= \frac{1}{2\alpha^2}\beta^i\left[\beta_{j,k}+\beta_{k,j}-\gamma_{jk,0}\right] + \frac{1}{2\alpha^2}\left(\alpha^2\gamma^{im}-\beta^i\beta^m\right)\left[\gamma_{jm,k}+\gamma_{mk,j}-\gamma_{jk,m}\right]\\
&= \frac{1}{2\alpha^2}\beta^i\left[\beta_{j,k}+\beta_{k,j}-\gamma_{jk,0}\right] + \Gamma^i_{jk} + \frac{1}{2\alpha^2}\left(-\beta^i\beta^m\right)\left[\gamma_{jm,k}+\gamma_{mk,j}-\gamma_{jk,m}\right].
\end{aligned}
$$

To get the extrinsic curvature from the three-metric, we must take the future-pointing unit time-like normal to the slice, $\hat{n}^\mu$:

$$
\begin{aligned}
\hat{n}_\mu &= (-\alpha, 0, 0, 0)\\
\Rightarrow \hat{n}^\mu &= g^{\mu\alpha}\hat{n}_\alpha = \left[\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right].
\end{aligned}
$$

The "projected" four-metric is $h_{\mu\nu} \equiv g_{\mu\nu} + \hat{n}_\mu\hat{n}_\nu$. Then we define the extrinsic curvature ([1] eq. (19),(35))[1]:

$$
\begin{aligned}
K_{ij} &\equiv -\frac{1}{2}\mathcal{L}_{\hat{n}}h_{ij}\\
&= \frac{1}{2\alpha}\left(\beta_{i|j}+\beta_{j|i}-\partial_t\gamma_{ij}\right) = \frac{1}{2\alpha}\left(\mathcal{L}_\beta h_{ij}-\mathcal{L}_t h_{ij}\right)\\
&= \frac{1}{2\alpha}\left(\beta_{i,j}+\beta_{j,i}-\partial_t\gamma_{ij}-2\Gamma^p_{ij}\beta_p\right)\\
&= -\alpha\,{}^4\Gamma^0_{ij}.
\end{aligned}
$$

This can be inverted to give an expression for the three-metric's time-derivative in terms of the extrinsic curvature:

$$\gamma_{ij,0} = \beta_{i,j} + \beta_{j,i} - 2\alpha K_{ij} - 2\Gamma^p_{ij}\beta_p.$$

In the special case of trivial gauge ($\alpha = 1$, $\beta^i = 0$), the expressions for the four-connection simplify enormously:

$$
\begin{aligned}
{}^4\Gamma^0_{00} &= 0, & {}^4\Gamma^i_{00} &= 0,\\
{}^4\Gamma^0_{0j} &= 0, & {}^4\Gamma^i_{0j} &= \frac{1}{2}\gamma^{im}\left[\gamma_{mj,0}\right] = -\gamma^{im}K_{im},\\
{}^4\Gamma^0_{ij} &= \frac{1}{2}\gamma_{ij,0} = -K_{ij}, & {}^4\Gamma^i_{jk} &= \frac{1}{2}\gamma^{im}\left[\gamma_{jm,k}+\gamma_{mk,j}-\gamma_{jk,m}\right] = \Gamma^i_{jk},
\end{aligned}
$$

Smarr [4] defines the *electric* and *magnetic* parts of the Weyl curvature in the 3+1 split. These are spatial tensors (that is, they are orthogonal to the unit normal to the hypersurface), with components given by

$$
\begin{aligned}
E_{ij} &= -R_{ij} - KK_{ij} + K_{mi}K_j{}^m,\\
B_{ij} &= D_m K_{n(i}\varepsilon^{mn}{}_{j)}.
\end{aligned}
\tag{1}
$$

It can be seen from this definition that the tracelessness of the electric and magnetic tensors is equivalent to the satisfying of the Hamiltonian and momentum constraints.

The Riemann and Ricci tensors (in any dimension) are given by:

$$
\begin{aligned}
\mathrm{R}^a{}_{bcd} &\equiv \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^a_{mc}\Gamma^m_{bd} - \Gamma^a_{md}\Gamma^m_{bc},\\
\mathrm{R}_{ab} &\equiv \mathrm{R}^c{}_{acb},
\end{aligned}
$$

the former following the Landau-Lifshitz Spacelike Convention (LLSC), as with [2].

Note that Maple's Tensor package follows the conventions of [2] for Riemann, but *not* for Ricci.

[1] This agrees with [2] and Cook. Wald [3] uses the opposite sign (eq. (10.2.13),(E.2.30)) but is self-consistent.

2.1 Kerr-Schild Space-Times

An important sub-class of black-hole space-times can be written in *Kerr-Schild* form. The four-metric is written

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H\ell_\mu\ell_\nu \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - 2H\ell^\mu\ell^\nu,$$

where ℓ_μ is a flat-space null vector:

$$\ell_\mu\ell^\mu \equiv \eta^{\mu\nu}\ell_\mu\ell_\nu = 0.$$

From this generic form, we can deduce something of the 3+1 decomposition:

$$\begin{aligned}
\alpha &= \frac{1}{\sqrt{1+2H\ell_0^2}}, \\
\beta_i &= 2H\ell_0\ell_i, \\
\beta^i &= \frac{2H\ell_0\ell^i}{1+2H\ell_0^2}, \\
\gamma_{ij} &= \eta_{ij} + 2H\ell_i\ell_j, \\
\gamma^{ij} &= \eta^{ij} - \frac{2H}{1+2H\ell_0^2}\ell^i\ell^j, \\
\Rightarrow \gamma_{ij,k} &= 2\left[H_{,k}\ell_i\ell_j + H\ell_{i,k}\ell_j + H\ell_i\ell_{j,k}\right], \\
K_{ij} &= \alpha\left[\ell_i H_{,j} + \ell_j H_{,i} + H\ell_{i,j} + H\ell_{j,i} + 2H^2\left(\ell_i\ell_m\ell_{j,m} + \ell_j\ell_m\ell_{i,m}\right) + 2H\ell_i\ell_j\ell_m H_{,m}\right].
\end{aligned}$$

This agrees with the formula presented in eqn (35) of [5].

2.2 Bowen-York Solution

Bowen & York [6] introduced a set of partial solutions to the constraint equations, where the momentum constraint is solved automatically by extrinsic curvature that can incorporate bulk ADM linear and/or angular momentum. The data is not completely determined – there is a conformal factor ψ that must be found by solving the Hamiltonian constraint elliptic equation. The physical metric quantities are:

$$\gamma_{ij} = \psi^4\delta_{ij} \ \ , \ \ K_{ij} = \psi^{-2}\hat{K}_{ij}, \tag{2}$$

where the conformal extrinsic curvature contains angular- and/or linear-momentum terms:

$$\begin{aligned}
\hat{K}_{ij} =& \frac{3}{r^3}\left[\epsilon_{kim}\, S^m\, n^k\, n_j + \epsilon_{kjm}\, S^m\, n^k\, n_i\right] \\
&+ \frac{3}{2r^2}\left[P_i\, n_j + P_j\, n_i - (\delta_{ij} - n_i\, n_j)P^k\, n_k\right] \\
&\mp \frac{3a^2}{2r^4}\left[P_i\, n_j + P_j\, n_i + (\delta_{ij} - 5n_i\, n_j)P^k\, n_k\right],
\end{aligned} \tag{3}$$

where r is a conformal radial coordinate, centred on the singularity. The factor ψ then must be found by solving the Poisson-like vacuum Hamiltonian constraint:

$$\Delta\psi + \frac{1}{8}\psi^{-7}\hat{K}_{ij}\hat{K}^{ij} = 0 \tag{4}$$

Bowen-York data relates very neatly to the ADM quantities, but has two main disadvantages: (a) the Hamiltonian constraint (4) must be solved explicitly to obtain a solution of the Einstein equations, and (b) even once solved, the data does *not* describe a “clean” black hole – there's always some radiation on top of the hole, which will radiate away to infinity, or into the hole, over time. In contrast, Kerr-Schild data (subsection 2.1) represents both spinning and boosted black holes very cleanly.

The real advantage of conformal data like Bowen-York is its convenient extension to multiple black holes. Since the momentum constraint is linear in the extrinsic curvature, the exact B-Y solutions (3) can be repeated – with different S and/or P, and centred at a different point in coordinate space – to represent multiple holes. The number of holes present doesn't affect the only numerical problem, that of solving the Hamiltonian constraint (4).

3 Flat Space

This is the default, nothing happening here – move on, boring as heck zilch space-time. It can, of course, still be represented in different ways.

3.1 Minkowski Coordinates

In spherical polar coordinates, the four-metric is:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

This yields non-vanishing connection components

$$\begin{aligned} {}^4\Gamma^r_{\theta\theta} &= -r, \quad {}^4\Gamma^r_{\phi\phi} = -r\sin^2\theta \\ {}^4\Gamma^\theta_{r\theta} = {}^4\Gamma^\theta_{\theta r} &= \frac{1}{r}, \quad {}^4\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \\ {}^4\Gamma^\phi_{r\phi} = {}^4\Gamma^\phi_{\phi r} &= \frac{1}{r}, \quad {}^4\Gamma^\phi_{\theta\phi} = {}^4\Gamma^\phi_{\phi\theta} = \cot\theta \end{aligned}$$

In cylindrical polars, the metric is:

$$ds^2 = -dt^2 + dq^2 + q^2 d\phi^2 + dz^2$$

This yields non-vanishing connection components

$${}^4\Gamma^q_{\phi\phi} = -q, \quad {}^4\Gamma^\phi_{q\phi} = {}^4\Gamma^\phi_{\phi q} = \frac{1}{q}$$

In Cartesian coordinates, the metric is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$

yielding the ADM fields:

$$\begin{aligned} \alpha &= 1 \ \ , \ \ \beta^i = 0 \\ \gamma_{ij} &= \delta_{ij} \\ K_{ij} &= 0 \end{aligned}$$

3.2 Rindler Coordinates

Minkowski spacetime can be recast ...

4 Static, Spherically Symmetric Black Holes – The Schwarzschild Solution

First and simplest, apart from Minkowski, of course. If we may assume spherical symmetry, the line element can be written as:

$$ds^2 = \left(-\alpha^2 + a^2\beta^2\right) dt^2 + 2a^2\beta dR\, dt + a^2 dR^2 + b^2 d\Omega^2$$

so that the extrinsic curvature is

$$K_{ij} = \mathrm{diag}\left(\frac{a}{\alpha}[a\,\beta' + \beta\,a' - \dot{a}], \frac{b}{\alpha}[\beta\,b' - \dot{b}], \frac{b}{\alpha}[\beta\,b' - \dot{b}]\sin^2\theta\right)$$

4.1 Schwarzschild Coordinates

The simplest set of coordinates used to describe spherically-symmetric space-time is that due to Schwarzschild himself. Using spherical polar coordinates, the metric is:

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2\, d\Omega^2,$$

where M is the mass of the black hole. In Cartesian coordinates, this becomes:

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dt^2 + \left(\delta_{ij} + \frac{2M}{R - 2M}\frac{x^i}{R}\frac{x^j}{R}\right) dx^i dx^j,$$

yielding the 3+1 fields

$$\begin{aligned}
\alpha &= \left(1 - \frac{2M}{R}\right)^{\frac{1}{2}} \quad , \quad \beta^i = 0, \\
\gamma_{ij} &= \delta_{ij} + \frac{2M}{R - 2M}\frac{x_i}{R}\frac{x_j}{R} \Rightarrow \gamma^{ij} = \delta^{ij} - \frac{2M}{R}\frac{x^i}{R}\frac{x^j}{R}, \\
K_{ij} &= 0,
\end{aligned}$$

where $x_i \equiv \delta_{ij}x^j$ and $R = \sqrt{\delta_{ij}x^i x^j}$.

The 4-Christoffel symbol has non-zero components:

$$\begin{aligned}
&{}^4\Gamma^t_{tR} = {}^4\Gamma^t_{Rt} = \frac{M}{R(R - 2M)}, \\
&{}^4\Gamma^R_{tt} = \frac{M}{R^2}\left(1 - \frac{2M}{R}\right), \quad {}^4\Gamma^R_{RR} = -\frac{M}{R(R - 2M)}, \quad {}^4\Gamma^R_{\theta\theta} = -(R - 2M), \quad {}^4\Gamma^R_{\phi\phi} = -(R - 2M)\sin^2\theta, \\
&{}^4\Gamma^\theta_{R\theta} = {}^4\Gamma^\theta_{\theta R} = \frac{1}{R}, \quad {}^4\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
&{}^4\Gamma^\phi_{R\phi} = {}^4\Gamma^\phi_{\phi R} = \frac{1}{R}, \quad {}^4\Gamma^\phi_{\theta\phi} = {}^4\Gamma^\phi_{\phi\theta} = \cot\theta.
\end{aligned}$$

The four-Riemann tensor in spherical polars has non-zero components:

$${}^4\mathrm{R}_{tRtR} = \frac{2M}{R^3}.$$

Moving to the spatial quantities, in spherical polars, the 3-Christoffel symbol has non-zero components (trivially the same as the spatial parts of the 4-Christoffel above, because of zero shift terms):

$$\begin{aligned}
&\Gamma^R_{RR} = -\frac{M}{R(R - 2M)}, \quad \Gamma^R_{\theta\theta} = -\left(R - 2M\right), \quad \Gamma^R_{\phi\phi} = -\left(R - 2M\right)\sin^2\theta, \\
&\Gamma^\theta_{R\theta} = \Gamma^\theta_{\theta R} = \frac{1}{R}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
&\Gamma^\phi_{R\phi} = \Gamma^\phi_{\phi R} = \frac{1}{R}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.
\end{aligned}$$

In Cartesian coordinates, this becomes

$$\Gamma^i_{jk} = \frac{2M}{R^3} x^i \delta_{jk} - \frac{M(3R-4M)}{R^5(R-2M)} x^i x_j x_k$$

The three-Ricci tensor is, in spherical polars,

$$\mathrm{R}_{ij} = \mathrm{diag}\left(\frac{2M}{R^2(R-2M)}, -\frac{M}{R}, -\frac{M\sin^2\theta}{R}\right).$$

The electric and magnetic tensors are, then,

$$\begin{aligned} E_{ij} &= \left(-\frac{2M}{R^2(R-2M)}, \frac{M}{R}, \frac{M\sin^2\theta}{R}\right) \\ B_{ij} &= 0. \end{aligned}$$

In Cartesian coordinates, these become

$$\begin{aligned} \mathrm{R}_{ij} &= \frac{M}{R^3}\delta_{ij} - \frac{M(3R-2M)x_i x_j}{R^5(R-2M)}, \\ E_{ij} &= -\frac{M}{R^3}\delta_{ij} + \frac{M(3R-2M)x_i x_j}{R^5(R-2M)}, \\ B_{ij} &= 0. \end{aligned}$$

4.2 Ingoing Eddington-Finkelstein Coordinates

More useful are the horizon-penetrating Ingoing Eddington-Finkelstein coordinates [7, 8][2] This is the simplest non-trivial instance of a Kerr-Schild-type metric; here $H = M/R$, and in spherical polars, $\ell_\mu = (1,1,0,0)$, leading to the four-metric:

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \frac{4M}{R} dt\, dR + \left(1 + \frac{2M}{R}\right) dR^2 + R^2 d\Omega^2,$$

implying covariant components

$$g_{tt} = -\frac{R-2M}{R}, \;\; g_{tR} = \frac{2M}{R}, \;\; g_{RR} = \frac{R+2M}{R}, \;\; g_{\theta\theta} = R^2, \;\; g_{\phi\phi} = R^2\sin^2\theta,$$

and inverse-metric components

$$g^{tt} = -\frac{R+2M}{R}, \;\; g^{tR} = \frac{2M}{R}, \;\; g^{RR} = \frac{R-2M}{R}, \;\; g^{\theta\theta} = \frac{1}{R^2}, \;\; g^{\phi\phi} = \frac{1}{R^2\sin^2\theta}.$$

The 4-Christoffel symbol has components:

$$\begin{aligned} &{}^4\Gamma^t_{tt} = \frac{2M^2}{R^3}, \;\; {}^4\Gamma^t_{tR} = \frac{M(R+2M)}{R^3}, \;\; {}^4\Gamma^t_{RR} = \frac{2M(R+M)}{R^3}, \;\; {}^4\Gamma^t_{\theta\theta} = -2M, \;\; {}^4\Gamma^t_{\phi\phi} = -2M\sin^2\theta, \\ &{}^4\Gamma^R_{tt} = \frac{M(R-2M)}{R^3}, \;\; {}^4\Gamma^R_{tR} = -\frac{2M^2}{R^3}, \;\; {}^4\Gamma^R_{RR} = -\frac{M(R+2M)}{R^3}, \;\; {}^4\Gamma^R_{\theta\theta} = -(R-2M), \;\; {}^4\Gamma^R_{\phi\phi} = -(R-2M)\sin^2\theta, \\ &{}^4\Gamma^\theta_{R\theta} = \frac{1}{R}, \;\; {}^4\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\ &{}^4\Gamma^\phi_{R\phi} = \frac{1}{R}, \end{aligned}$$

all other non-zero components being determined by symmetry in the lower indices.

[2] This is simply related to the Schwarzschild version by a change of coordinate time: $t_S \equiv t_{IEF} - 2M\ln((R-2M)/D)$, where D is an arbitrary constant with dimensions of length/time/mass.

The 4-Riemann tensor has components

$$\begin{aligned}
{}^4\mathrm{R}_{tRtR} &= -\frac{2M}{R^3}, \\
{}^4\mathrm{R}_{t\theta t\theta} &= \frac{(R-2M)M}{R^2}, \\
{}^4\mathrm{R}_{t\theta R\theta} &= -\frac{2M}{R^3}, \\
{}^4\mathrm{R}_{t\phi t\phi} &= \frac{(R-2M)M}{R^2}\sin^2\theta, \\
{}^4\mathrm{R}_{t\phi R\phi} &= -\frac{2M}{R^3}\sin^2\theta, \\
{}^4\mathrm{R}_{R\theta R\theta} &= -\frac{(R+2M)M}{R^2}, \\
{}^4\mathrm{R}_{R\phi R\phi} &= -\frac{(R+2M)M}{R^2}\sin^2\theta, \\
{}^4\mathrm{R}_{\theta\phi\theta\phi} &= 2MR\sin^2\theta,
\end{aligned}$$

all other non-zero components being obtained by index swapping etc.

The four-metric decomposes into:

$$\begin{aligned}
\alpha &= \sqrt{\frac{R}{R+2M}} \;\; , \;\; \beta^i = \left(\frac{2M}{R+2M}, 0, 0\right) \\
\gamma_{ij} &= \mathrm{diag}\left[\frac{R+2M}{R}, R^2, R^2\sin^2\theta\right] \\
K_{ij} &= \mathrm{diag}\left[-\frac{2M(R+M)}{R^2\sqrt{R(R+2M)}}, 2M\sqrt{\frac{R}{R+2M}}, 2M\sqrt{\frac{R}{R+2M}}\sin^2\theta\right]
\end{aligned}$$

Moving to the spatial quantities, in spherical polars, the 3-Christoffel symbol has non-zero components:

$$\begin{aligned}
\Gamma^R_{RR} &= -\frac{M}{R(R+2M)}, \;\; \Gamma^R_{\theta\theta} = -\frac{R^2}{R+2M}, \;\; \Gamma^R_{\phi\phi} = -\frac{R^2\sin^2\theta}{R+2M}, \\
\Gamma^\theta_{R\theta} &= \Gamma^\theta_{\theta R} = \frac{1}{R}, \;\; \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
\Gamma^\phi_{R\phi} &= \Gamma^\phi_{\phi R} = \frac{1}{R}, \;\; \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.
\end{aligned}$$

This yields a three-Ricci tensor:

$$\mathrm{R}_{ij} = \mathrm{diag}\left[-\frac{2M}{R^2(R+2M)}, \frac{M(R+4M)}{(R+2M)^2}, \frac{M(R+4M)}{(R+2M)^2}\sin^2\theta\right]$$

In cylindrical coordinates $[q, \phi, z]$, the four-metric becomes:

$$ds^2 = -dt^2 + dq^2 + q^2 d\phi^2 + dz^2 + \frac{2M}{R}\left(dt + \frac{q}{R}dq + \frac{z}{R}dz\right)^2,$$

and the field quantities become:

$$\begin{aligned}
\left(\beta^q, \beta^\phi, \beta^z\right) &= \frac{2M}{R+2M}\left(\frac{q}{R}, 0, \frac{z}{R}\right) \\
\left(\beta_q, \beta_\phi, \beta_z\right) &= \frac{2M}{R^2}\left(q, 0, z\right) \\
\gamma_{qq} &= 1 + \frac{2Mq^2}{R^3} \;\; , \;\; \gamma_{qz} = \frac{2Mqz}{R^3} \\
\gamma_{zz} &= 1 + \frac{2Mz^2}{R^3} \;\; , \;\; \gamma_{\phi\phi} = q^2
\end{aligned}$$

$$K_{qq} = -\frac{2M\left[Mq^2 + R(q^2 - z^2)\right]}{R^4\sqrt{R(R+2M)}} \quad , \; K_{qz} = -\frac{2Mqz(2R+M)}{R^4\sqrt{R(R+2M)}}$$
$$K_{zz} = -\frac{2M\left[Mz^2 + R(z^2 - q^2)\right]}{R^4\sqrt{R(R+2M)}} \; , \; K_{\phi\phi} = \frac{2Mq^2}{R\sqrt{R(R+2M)}}$$

In Cartesian coordinates the metric becomes:

$$ds^2 = -\left(1 - \frac{2M}{R}\right)dt^2 + \frac{4Mx_i}{R^2}dt\,dx^i + \left(\delta_{ij} + \frac{2M}{R}\frac{x_i}{R}\frac{x_j}{R}\right)dx^i dx^j,$$

which leads to the standard ADM fields:

$$\alpha = \sqrt{\frac{R}{R+2M}} \quad , \; \beta^i = \frac{2M}{R+2M}\frac{x^i}{R} \Rightarrow \beta_i = \frac{2Mx_i}{R^2}$$
$$\gamma_{ij} = \delta_{ij} + \frac{2M}{R}\frac{x_i x_j}{R^2} \Rightarrow \gamma^{ij} = \delta^{ij} - \frac{2M}{R+2M}\frac{x^i x^j}{R^2}$$
$$K_{ij} = \frac{2M}{\sqrt{R^3\,(R+2M)}}\left(\delta_{ij} - \frac{2R+M}{R}\frac{x_i x_j}{R^2}\right)$$

We can write down some additional useful quantities:

- the three-metric determinant: $|\gamma| = 1 + \frac{2M}{R}$,
- the three-Christoffel symbol: $\Gamma^i_{jk} = \frac{2Mx^i}{R^2(R+2M)}\left(\delta_{jk} - \frac{3x_j x_k}{2R^2}\right)$,
- the mixed-index extrinsic curvature: $K^i{}_j = \frac{2M}{\sqrt{R^3\,(R+2M)}}\left(\delta^i{}_j - \frac{2R+3M}{R+2M}\frac{x^i x_j}{R^2}\right)$,
- the three-Ricci tensor: $\mathrm{R}_{ij} = \frac{M(R+4M)}{R^2(R+2M)^2}\delta_{ij} - \frac{M(3R+8M)x_i x_j}{R^4(R+2M)^2}$,
- the three-Ricci scalar: $\mathrm{R} \equiv \gamma^{mn}R_{mn} = \frac{8M^2}{R^2(R+2M)^2}$,
- the trace-free extrinsic curvature: $A_{ij} \equiv K_{ij} - \frac{K}{3} = \frac{2M(2R+3M)}{3\sqrt{R^3(R+2M)^3}}\left[\delta_{ij} - \frac{3R+4M}{R}\frac{x_i x_j}{R^2}\right]$,
- the electric tensor (1): $E_{ij} = -\frac{M}{R^3}\delta_{ij} + \frac{M(3R+4M)}{R^6}x_i x_j$.

We can deduce from these what the BSSN-decomposed fields are:

$$\phi = \frac{1}{12}\log\left(\frac{R+2M}{R}\right) \Rightarrow e^{6\phi} = \sqrt{\frac{R+2M}{R}}$$
$$\Rightarrow Q = \alpha e^{-6n\phi} = \left(\sqrt{\frac{R}{R+2M}}\right)^{n+1}$$
$$\tilde{\gamma}_{ij} \equiv e^{-4\phi}\gamma_{ij} = \left(1 + \frac{2M}{R}\right)^{-1/3}\left[\delta_{ij} + \frac{2M}{R}\frac{x_i x_j}{R^2}\right]$$
$$K \equiv \gamma^{mn}K_{mn} = \frac{2M(R+3M)}{\sqrt{R^3(R+2M)^3}}$$
$$\tilde{A}_{ij} \equiv e^{-4\phi}A_{ij}$$
$$= \left(1 + \frac{2M}{R}\right)^{-1/3}\frac{2M(2R+3M)}{3\sqrt{R^3(R+2M)^3}}\left[\delta_{ij} - \frac{3R+4M}{R}\frac{x_i x_j}{R^2}\right]$$
$$\tilde{\Gamma}^i \equiv -\tilde{\gamma}^{im}{}_{,m} = \left(1 + \frac{2M}{R}\right)^{1/3}\frac{8M(R+3M)x^i}{3R^2(R+2M)^2}$$

Q here is the *densitised lapse*, currently popular in BSSN-type evolutions for its stabilising properties.

4.3 Isotropic Schwarzschild Coordinates

The Schwarzschild solution in Isotropic Schwarzschild coordinates is (see [2], exercise 31.7):

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \left(dr^2 + r^2\, d\Omega^2\right)$$

In Cartesian coordinates, this is:

$$ds^2 = -\left(\frac{2r-M}{2r+M}\right)^2 dt^2 + \left(1+\frac{M}{2r}\right)^4 \delta_{ij}\, dx^i\, dx^j,$$

yielding 3+1 fields

$$\begin{aligned}
\alpha &= \frac{2r-M}{2r+M} \quad , \quad \beta^i = 0, \\
\gamma_{ij} &= \left(1+\frac{M}{2r}\right)^4 \delta_{ij}, \\
K_{ij} &= 0.
\end{aligned}$$

We can write down some additional useful quantities:

- the (spherical coords) three-Ricci tensor: $\mathrm{R}_{ij} = \frac{M}{r(2r+M)^2}\mathrm{diag}\left(-8M, 4Mr^2, 4Mr^2\sin^2(\theta)\right)$
- the three-Ricci scalar: $\mathrm{R} \equiv \gamma^{mn}R_{mn} = 0$.

These coordinates are designed to emphasise the conformally-flat nature of the Schwarzschild solution - apart from an overall conformal factor, the three-metric is that of flat space three-space.

The BSSN-decomposed fields are then:

$$\begin{aligned}
\phi &= \log\left(1+\frac{M}{2r}\right) \\
\tilde{\gamma}_{ij} &= \delta_{ij} \\
K &= 0 \\
\tilde{A}_{ij} &= 0 \\
\tilde{\Gamma}^i &= 0
\end{aligned}$$

4.4 Painlevé-Gullstrand Coordinates

In Painlevé-Gullstrand [9, 10] coordinates, the Schwarzschild metric takes the form:

$$ds^2 = -\left(1-\frac{2M}{R}\right) dt^2 + 2\sqrt{\frac{2M}{R}} dt\, dR + dR^2 + R^2 d\Omega^2$$

This decomposes into:

$$\begin{aligned}
\alpha &= 1 \quad , \quad \beta^i = \left(\sqrt{\frac{2M}{R}}, 0, 0\right) \\
\gamma_{ij} &= \mathrm{diag}\left(1, R^2, R^2\sin^2\theta\right) \\
K_{ij} &= \mathrm{diag}\left(-\frac{1}{R}\sqrt{\frac{M}{2R}}, \sqrt{2MR}, \sqrt{2MR}\sin^2\theta\right)
\end{aligned}$$

In Cartesian coordinates, this yields 3+1 fields:

$$\begin{aligned}
\alpha &= 1 \quad , \quad \beta^i = \sqrt{\frac{2M}{R}}\frac{x^i}{R}, \\
\gamma_{ij} &= \delta_{ij}, \\
K_{ij} &= \sqrt{\frac{2M}{R^3}}\left(\delta_{ij} - \frac{3x_i x_j}{2R^2}\right),
\end{aligned}$$

In cylindrical coordinates $[q,z,\phi]$, these quantities become:

$$\begin{aligned}
g_{ij} &= \mathrm{diag}\left(1,1,q^2\right), \\
K_{qq} &= -\frac{q^2-2z^2}{R^3}\sqrt{\frac{M}{2R}}, \quad K_{qz} = -\frac{3qz}{R^3}\sqrt{\frac{M}{2R}} \\
K_{zz} &= \frac{2q^2-z^2}{R^3}\sqrt{\frac{M}{2R}}, \quad K_{\phi\phi} = \frac{2q^2}{R}\sqrt{\frac{M}{2R}}
\end{aligned}$$

We can calculate the expansion Θ of outgoing (l^a) and ingoing (n^a) null normals to spherical surfaces for these coordinates. Here, the spatial unit normal is $s^i = \frac{x^i}{R}$, and

$$\begin{aligned}
\Theta_{(l)} &\equiv \nabla_i s^i + K_{ij}\left(s^i s^j - \gamma^{ij}\right) \\
&= \partial_i \frac{x^i}{R} + \sqrt{\frac{2M}{R^3}}\left(\delta_{ij} - \frac{3x_i x_j}{2R^2}\right)\left(\frac{x^i}{R}\frac{x^j}{R} - \delta^{ij}\right) \\
&= \frac{2}{R} - 2\sqrt{\frac{2M}{R^3}},
\end{aligned}$$

which vanishes at $R = 2M$, as we would expect. Similarly,

$$\Theta_{(n)} = -\frac{2}{R} - 2\sqrt{\frac{2M}{R^3}}.$$

4.5 Trumpet Coordinates

Due to [11] [see also recent work by [12, 13, 14]]. The idea is that a numerically useful evolution of initially Schwarzschild data using something like maximal slicing will result in a late-time slicing quite different to the original one. The general Estabrook form of the final four-metric is

$$ds^2 = \left(-\alpha^2 + \beta^2/\gamma\right) d\tau^2 + 2\,\beta\, d\tau\, dR + \gamma\, dR^2 + R^2 d\Omega^2.$$

Solving the maximal slicing condition and the constraints at any time τ yields the following general solutions:

$$\begin{aligned}
\alpha \equiv f(\tau,R) &= \left(1 - \frac{2M}{R} + \frac{C(\tau)^2}{R^4}\right)^{1/2}, \\
\gamma_{ij} &= \mathrm{diag}\left(\gamma, R^2, R^2\sin^2\theta\right) = \mathrm{diag}\left(\frac{1}{f(\tau,R)^2}, R^2, R^2\sin^2\theta\right) \\
\Rightarrow \gamma^{ij} &= \mathrm{diag}\left(f(\tau,R)^2, \frac{1}{R^2}, \frac{1}{R^2\sin^2\theta}\right), \\
\beta_i &= \left(\frac{\alpha\gamma_{RR}C(\tau)}{R^2}, 0, 0\right) = \left(\frac{C(\tau)}{R^2 f(\tau,R)}, 0, 0\right), \\
\beta^i &= \left(\frac{C(\tau)f(\tau,R)}{R^2}, 0, 0\right).
\end{aligned}$$

At late times, the "slicing parameter" $C(\tau)$ asymptotes to the value $3\sqrt{3}\,M^2/4$, and $f(\tau,R) \to f(R) = \left(1 - 2\,M/R + 27M^4/16R^4\right)^{1/2}$.

The nonvanishing three-Christoffel symbols are:

$$\begin{aligned}
&\Gamma^R_{RR} = -\frac{f'}{f}, \quad \Gamma^R_{\theta\theta} = -Rf^2, \quad \Gamma^R_{\phi\phi} = -Rf^2\sin^2\theta, \\
&\Gamma^\theta_{R\theta} = \Gamma^\theta_{\theta R} = \frac{1}{R}, \quad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \\
&\Gamma^\phi_{R\phi} = \Gamma^\phi_{\phi R} = \frac{1}{R}, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta,
\end{aligned}$$

where f' is just $\partial_R f$. Then at late times, we can use the τ-independent relation

$$K_{ij} = \frac{1}{2\alpha}\left(\beta_{i,j} + \beta_{j,i} - 2\Gamma^p_{ij}\beta_p\right)$$

to get the extrinsic curvature:

$$K_{ij} \to \mathrm{diag}\left(-\frac{2C}{f^2R^3}, \frac{C}{R}, \frac{C\sin^2\theta}{R}\right).$$

It's straightforward to verify that the mixed-index extrinsic curvature is trace-free: $\gamma^{ij}K_{ij} = 0$.

4.6 Conformal Trumpet Coordinates

At this late time, the Estabrook solution can be recast into a conformally flat form by introducing a "conformal radial coordinate" r, related to the original Schwarzschild coordinate via $R = \psi^2(R)r$, where the conformal factor $\psi(R)$ is [13]:

$$\psi(R)^2 = \left[\frac{4R}{2R + M + \sqrt{4R^2 + 4MR + 3M^2}}\right]\left[\frac{8R + 6M + 3\sqrt{8R^2 + 8MR + 6M^2}}{(4 + 3\sqrt{2})(2R - 3M)}\right]^{1/\sqrt{2}}$$

Numerically, we'll be working with the isotropic r as our primary coordinate, so it's preferable to have $\psi[r]$ instead. However, this is not available analytically. But by plotting and fitting in `gnuplot`, we can come up with a pretty good approximation:

$$\psi[r]^2 = 1 + A_1\frac{M}{r} + A_2\frac{M^2}{r^2} + A_3\frac{M^3}{r^3} + A_4\frac{M^4}{r^4},$$

where $A_1 = 1.04225$, $A_2 = 0.15676$, $A_3 = -0.0234512$, and $A_4 = 0.00116509$.

The radial shift in isotropic coordinates will be

$$\beta^r = \frac{3\sqrt{3}M^2}{4}\frac{r}{R^3} = \frac{3\sqrt{3}M^2}{4\psi^6[r]r^2} \Rightarrow \beta^x = \frac{3\sqrt{3}}{4\psi^6[r]}\frac{M^2}{r^2}n^x$$

Since the slicing is maximal, the extrinsic curvature is still trace-free. In mixed form, and isotropic coordinates $\{r, \theta, \phi\}$, it is given by:

$$K^i{}_j = -\frac{3\sqrt{3}}{4}\frac{M^2}{R^3}\mathrm{diag}\,[2, -1, -1] = -\frac{3\sqrt{3}}{4}\frac{M^2}{\psi^6 r^3}\mathrm{diag}\,[2, -1, -1]$$

$$\Rightarrow K_{ij} = -\frac{3\sqrt{3}}{4}\frac{M^2}{\psi^2 r^3}\mathrm{diag}\left[2, -r^2, -r^2\sin^2\theta\right]$$

Or, in Cartesian coordinates,

$$K_{ij} = \frac{3\sqrt{3}}{4}\frac{M^2}{\psi^2 r^3}(\delta_{ij} - 3n_i n_j),$$

$$K^i{}_j = \frac{3\sqrt{3}}{4}\frac{M^2}{\psi^6 r^3}(\delta^i{}_j - 3n^i n_j).$$

For initial data purposes, we often wish to work with a conformally transformed extrinsic curvature:

$$\hat{K}_{ij} = \psi^2 K_{ij} = \frac{3\sqrt{3}}{4}\frac{M^2}{r^3}(\delta_{ij} - 3n_i n_j),$$

$$\Rightarrow \hat{K}^i{}_j = \psi^6 K^i{}_j = \frac{3\sqrt{3}}{4}\frac{M^2}{r^3}(\delta^i{}_j - 3n^i n_j)$$

4.6.1 Moving to Binary Data

The expressions above apply to single black holes with Estabrook slicing. One may try to extend these with an ansatz for combining terms for two black holes at different locations.

The basic idea is to take expressions originally appearing as M/r, and make the replacement $(M1/r1 + M2/r2)$. Working in isotropic coordinates, this needs to be done for the conformal factor ψ and the extrinsic curvature – either physical or conformal. Additionally, we might want the lapse and shift, since they are a large part of the attraction of these coordinates.

For the shift and extrinsic curvature, we will also need to do a spherical-to-Cartesian transformation at some stage. This will happen at the end.

Combine the mixed-form conformal extrinsic curvatures, before retransforming with the overall conformal factor.

5 Stationary, Axisymmetry - The Kerr Solution

After Schwarzschild, the most common black-hole space-time encountered.

5.1 Kerr-Schild Coordinates

5.1.1 Arbitrary Spin Direction

In Kerr-Schild coordinates [15] (see end of section 2), a black-hole with arbitrarily-directed spin has a four-metric given by (see also [16]):

$$H = \frac{Mr^3}{r^4 + (\vec{a}\cdot\vec{x})^2} \ , \ \ell_\mu = \left(1, \frac{r^2\vec{x} - r\vec{a}\times\vec{x} + (\vec{a}\cdot\vec{x})\,\vec{a}}{r\,(r^2+a^2)}\right) = (1, \vec{\ell}),$$

and r is the solution of $r^4 - \left(R^2 - a^2\right) r^2 - (\vec{a}\cdot\vec{x})^2 = 0$ (where $R \equiv |\vec{x}|$):

$$r = \sqrt{\frac{R^2 - a^2 + \sqrt{(R^2-a^2)^2 + 4(\vec{a}\cdot\vec{x})^2}}{2}} = \sqrt{\frac{R^2 - a^2 + \omega}{2}}.$$

For zero spin, this reduces to the *Ingoing Eddington-Finkelstein* solution (Section 4.2) – the coordinates are horizon-penetrating.

Apart from the scalar ("dot") and vector ("cross") products, we shall use the antisymmetric tensor formed by contracting a vector with the alternator on its middle index:

$$\hat{a}_{ij} \equiv \varepsilon_{ipj} a_p$$

The most difficult part is the calculation of the extrinsic curvature. We do this in stages. First, the derivative of the "ring" distance w.r.t. the coordinate positions:

$$r_{,i} = \frac{1}{2r}\left(1 + \frac{R^2 - a^2}{\omega}\right) x_i + \frac{\vec{a}\cdot\vec{x}}{\omega r} a_i.$$

Then the derivatives of the metric functions are

$$H_{,i} = \frac{H}{r}\frac{(3(\vec{a}\cdot\vec{x})^2 - r^4) r_{,i} - 2r(\vec{a}\cdot\vec{x}) a_i}{r^4 + (\vec{a}\cdot\vec{x})^2},$$

$$\ell_{i,j} = \frac{(r^2\delta_{ij} - r\hat{a}_{ij} + a_i a_j)}{(r^3 + a^2 r)} + \frac{\left[(a^2r^2 - r^4)x_i + 2r^3(\vec{a}\times\vec{x})_i - (3r^2 + a^2)(\vec{a}\cdot\vec{x})a_i\right] r_{,j}}{(r^3 + a^2 r)^2}.$$

The (outer) horizon of the hole is the surface $r(R,\theta) = M + \sqrt{M^2 - a^2}$. This corresponds to positions

$$R_+ = \frac{\sqrt{-6a^2M^2\cos^2\theta + 2a^2M^2 + 2a^2M\sqrt{M^2-a^2}\sin^2\theta + 8M^3\sqrt{M^2-a^2}\cos^2\theta + 8M^4\cos^2\theta}}{\sqrt{a^2\cos^4\theta - 2a^2\cos^2\theta + a^2 + 4M^2\cos^2\theta}}$$

For an extremal hole ($a \to M$), this simplifies to $R_+ = M\sqrt{3 + \cos 2\theta}/(1 + \cos^2\theta)$.

5.1.2 z-directed Spin

Usually, we assume a z-directed spin $\vec{a} = a\hat{k}$, in which case, the above metric functions simplify to:

$$H = \frac{Mr^3}{r^4 + a^2z^2} \ , \ \ell_\mu = \left(1, \frac{rx + ay}{(r^2+a^2)}, \frac{ry - ax}{(r^2+a^2)}, \frac{z}{r}\right).$$

and r is the solution of $r^4 - \left(R^2 - a^2\right) r^2 - a^2z^2 = 0$ (where $R \equiv |\vec{x}|$):

$$r = \sqrt{\frac{R^2 - a^2 + \sqrt{(R^2-a^2)^2 + 4a^2z^2}}{2}} = \sqrt{\frac{R^2 - a^2 + \omega}{2}}.$$

This yields a 3+1 decomposition of:

$$\alpha = \sqrt{\frac{r^4 + a^2z^2}{r^4 + 2Mr^3 + a^2z^2}} \tag{5}$$

$$\beta^i = \frac{2Mr^3}{r^4 + 2Mr^3 + a^2z^2}\left(\frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right) \tag{6}$$

$$\gamma_{ij} = \begin{pmatrix} 1 + \frac{2Mr^3}{r^4+a^2z^2}\frac{(rx+ay)^2}{(r^2+a^2)^2} & \frac{2Mr^3}{r^4+a^2z^2}\frac{(rx+ay)(ry-ax)}{(r^2+a^2)^2} & \frac{2Mr^3}{r^4+a^2z^2}\frac{(rx+ay)z}{(r^2+a^2)r} \\ --- & 1 + \frac{2Mr^3}{r^4+a^2z^2}\frac{(ry-ax)^2}{(r^2+a^2)^2} & \frac{2Mr^3}{r^4+a^2z^2}\frac{(ry-ax)z}{(r^2+a^2)r} \\ --- & --- & 1 + \frac{2Mr^3}{r^4+a^2z^2}\frac{z^2}{r^2} \end{pmatrix} \tag{7}$$

In cylindrical coordinates $\{t, q, \phi, z\}$, the null vector becomes:

$$\ell^\mu = \left(-1, \frac{rq}{r^2+a^2}, -\frac{a}{r^2+a^2}, \frac{z}{r}\right) \text{ (contravariant)} \tag{8}$$

$$\ell_\mu = \left(1, \frac{rq}{r^2+a^2}, -\frac{aq^2}{r^2+a^2}, \frac{z}{r}\right) \text{ (covariant)} \tag{9}$$

This yields a 3+1 decomposition of:

$$\alpha = \sqrt{\frac{r^4 + a^2z^2}{r^4 + 2Mr^3 + a^2z^2}} \tag{10}$$

$$\beta^i = \frac{2Mr^3}{r^4 + 2Mr^3 + a^2z^2}\left(\frac{qr}{r^2+a^2}, -\frac{a}{r^2+a^2}, \frac{z}{r}\right) \tag{11}$$

$$\gamma_{ij} = \begin{pmatrix} 1 + \frac{2Mq^2r^5}{(a^2+r^2)^2(r^4+a^2z^2)} & -\frac{2aMr^3q^4}{(a^2+r^2)^2(r^4+a^2z^2)} & \frac{2Mqr^3z}{(a^2+r^2)(r^4+a^2z^2)} \\ --- & q^2 + \frac{2a^2Mq^4r^3}{(a^2+r^2)^2(r^4+a^2z^2)} & -\frac{2aMq^2r^2z}{(a^2+r^2)(r^4+a^2z^2)} \\ --- & --- & 1 + \frac{2Mrz^2}{r^4+a^2z^2} \end{pmatrix} \tag{12}$$

In spherical coordinates, the null vector becomes:

$$\ell^\mu = \left(-1, \frac{R(r^2+a^2\cos^2\theta)}{r(r^2+a^2)}, -\frac{a^2\cos\theta\sin\theta}{r(r^2+a^2)}, -\frac{a}{(r^2+a^2)}\right) \text{ (contravariant)}$$

$$\ell_\mu = \left(1, \frac{R(r^2+a^2\cos^2\theta)}{r(r^2+a^2)}, -\frac{a^2R^2\cos\theta\sin\theta}{r(r^2+a^2)}, -\frac{aR^2\sin^2\theta}{(r^2+a^2)}\right) \text{ (covariant)}$$

5.2 Boyer-Lindquist Coordinates

In Boyer-Lindquist Coordinates [17], for a hole of mass M and spin a along the z axis, the Kerr line element looks like this:

$$ds^2 = -\left(1 - \frac{2MR}{\varrho^2}\right)dt^2 - \frac{4MaR\sin^2\theta}{\varrho^2}dt\,d\phi + \frac{\varrho^2}{\Delta}dR^2 + \varrho^2 d\theta^2 + \frac{\Sigma}{\varrho^2}\sin^2\theta\,d\phi^2, \tag{13}$$

where $\varrho^2 \equiv R^2 + a^2\cos^2\theta$, $\Delta \equiv R^2 - 2MR + a^2$, and $\Sigma \equiv (R^2+a^2)\,\varrho^2 + 2Ma^2R\sin^2\theta$ [3]. This reduces to the Schwarzschild solution, *in Schwarzschild coordinates* (section 4.1), when $a = 0$.

[3]Notation here is far from standard; for instance, Σ here corresponds to Σ^2 in [18].

This yields Christoffel symbols:

$$
\begin{aligned}
\Gamma^t_{tt} &= 0 \\
\Gamma^t_{tR} = \Gamma^t_{Rt} &= -\frac{2M(R^2+a^2)(-2R^2+a^2+a^2\cos 2\theta)}{(R^2-2MR+a^2)(2R^2+a^2+a^2\cos 2\theta)^2} \\
\Gamma^t_{t\theta} = \Gamma^t_{\theta t} &= -\frac{4Ma^2R\sin 2\theta}{(2R^2+a^2+a^2\cos 2\theta)^2} \\
\Gamma^t_{t\phi} = \Gamma^t_{\phi t} &= 0 \\
\Gamma^t_{RR} &= 0 \\
\Gamma^t_{R\theta} = \Gamma^t_{\theta R} &= 0 \\
\Gamma^t_{R\phi} = \Gamma^t_{\phi R} &= -\frac{2Ma(6R^4+3a^2R^2+(R^2-a^2)a^2\cos 2\theta - a^4)\sin^2\theta}{(R^2-2MR+a^2)(2R^2+a^2+a^2\cos 2\theta)^2} \\
\Gamma^t_{\theta\theta} &= 0 \\
\Gamma^t_{\theta\phi} = \Gamma^t_{\phi\theta} &= \frac{8Ma^3R\cos\theta\sin^3\theta}{(2R^2+a^2+a^2\cos 2\theta)^2} \\
\Gamma^t_{\phi\phi} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^R_{tt} &= \frac{M(R^2-2MR+a^2)((2R^2-a^2-a^2\cos 2\theta))}{2(R^2+a^2\cos^2\theta)^3} \\
\Gamma^R_{tR} = \Gamma^R_{Rt} &= 0 \\
\Gamma^R_{t\theta} = \Gamma^R_{\theta t} &= 0 \\
\Gamma^R_{t\phi} = \Gamma^R_{\phi t} &= -\frac{Ma(R^2-a^2\cos^2\theta)(R^2-2MR+a^2)\sin^2\theta}{(R^2+a^2\cos^2\theta)^3} \\
\Gamma^R_{RR} &= -\frac{(R-M)a^2\cos^2\theta - R(a^2-MR)}{(R^2+a^2\cos^2\theta)(R^2-2MR+a^2)} \\
\Gamma^R_{R\theta} = \Gamma^R_{\theta R} &= -\frac{a^2\cos\theta\sin\theta}{R^2+a^2\cos^2\theta} \\
\Gamma^R_{R\phi} = \Gamma^R_{\phi R} &= 0 \\
\Gamma^R_{\theta\theta} &= -\frac{R(R^2-2MR+a^2)}{R^2+a^2\cos^2\theta} \\
\Gamma^R_{\theta\phi} = \Gamma^R_{\phi\theta} &= 0 \\
\Gamma^R_{\phi\phi} &= -\frac{(R^2-2MR+a^2)\sin^2\theta\left(R(R^4-a^4-a^2(MR-a^2)\sin^2\theta)+(2R(R^2+a^2)-a^2(R-M)\sin^2\theta)a^2\cos^2\theta\right)}{(R^2+a^2\cos^2\theta)^3}
\end{aligned}
$$

$$\Gamma^{\theta}_{tt} = -\frac{Ma^2 R \sin 2\theta}{(R^2 + a^2\cos^2\theta)^3}$$
$$\Gamma^{\theta}_{tR} = \Gamma^{\theta}_{Rt} = 0$$
$$\Gamma^{\theta}_{t\theta} = \Gamma^{\theta}_{\theta t} = 0$$
$$\Gamma^{\theta}_{t\phi} = \Gamma^{\theta}_{\phi t} = \frac{MaR(R^2 + a^2)\sin 2\theta}{(R^2 + a^2\cos^2\theta)^3}$$
$$\Gamma^{\theta}_{RR} = \frac{a^2\cos\theta\sin\theta}{(R^2 + a^2\cos^2\theta)(R^2 - 2MR + a^2)}$$
$$\Gamma^{\theta}_{R\theta} = \Gamma^{\theta}_{\theta R} = \frac{R}{(R^2 + a^2\cos^2\theta)}$$
$$\Gamma^{\theta}_{R\phi} = \Gamma^{\theta}_{\phi R} = 0$$
$$\Gamma^{\theta}_{\theta\theta} = -\frac{a^2\cos\theta\sin\theta}{(R^2 + a^2\cos^2\theta)}$$
$$\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = 0$$
$$\begin{aligned}\Gamma^{\theta}_{\phi\phi} = &-\frac{\sin 2\theta}{16(R^2 + a^2\cos^2\theta)^3}\big(8R^6 + 16a^2R^4 + 16Ma^2R^3 + 11a^4R^2 + 10Ma^4R + 3a^6 \\ &+ a^4(R^2 - 2MR + a^2)\cos 4\theta + 4a^2(2R^2 + a^2)(R^2 - 2MR + a^2)\cos 2\theta\big)\end{aligned}$$

$$\Gamma^{\phi}_{tt} = 0$$
$$\Gamma^{\phi}_{tR} = \Gamma^{\phi}_{Rt} = \frac{2M^2aR(2R^2 - a^2 - a^2\cos 2\theta) + Ma(R^2 - a^2\cos^2\theta)(2R^2 - 4MR + a^2 + a^2\cos 2\theta)}{2(R^2 + a^2\cos^2\theta)^3(R^2 - 2MR + a^2)}$$
$$\Gamma^{\phi}_{t\theta} = \Gamma^{\phi}_{\theta t} = -\frac{4M^2a^3R^2\sin 2\theta + MaR\csc^2\theta(R^2 + a^2)(2R^2 - 4MR + a^2 + a^2\cos 2\theta)\sin 2\theta}{2(R^2 + a^2\cos^2\theta)^3(R^2 - 2MR + a^2)}$$
$$\Gamma^{\phi}_{t\phi} = \Gamma^{\phi}_{\phi t} = 0$$
$$\Gamma^{\phi}_{RR} = 0$$
$$\Gamma^{\phi}_{R\theta} = \Gamma^{\phi}_{\theta R} = 0$$
$$\Gamma^{\phi}_{R\phi} = \Gamma^{\phi}_{\phi R} = \frac{-a^4\cos 4\theta(M - R) + a^4M + 3a^4R - 12a^2MR^2 + 4a^2R\cos 2\theta(R^2 - 2MR + a^2) + 8a^2R^3 - 16MR^4 + 8R^5}{8(R^2 - 2MR + a^2)(R^2 + a^2\cos^2\theta)^2}$$
$$\Gamma^{\phi}_{\theta\theta} = 0$$
$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{\cot\theta\left(a^4\cos 4\theta + 3a^4 + 4a^2\cos 2\theta(R^2 - 2MR + a^2) + 8a^2MR + 8a^2R^2 + 8R^4\right)}{8(R^2 + a^2\cos^2\theta)^2}$$
$$\Gamma^{\phi}_{\phi\phi} = 0$$

In spherical polars, the hypersurface quantities are[4]:

$$\begin{aligned}\alpha &= \sqrt{\frac{\Delta\varrho^2}{\Sigma}} \ , \ \beta^i = \left[0, 0, -\frac{2aMR}{\Sigma}\right] \\ \gamma_{ij} &= \mathrm{diag}\left(\frac{\varrho^2}{\Delta}, \varrho^2, \frac{\Sigma}{\varrho^2}\sin^2\theta\right) \\ K_{R\phi} &= \frac{(R\partial_R\Sigma - \Sigma)Ma\sin^2\theta}{\sqrt{\varrho^6\Sigma\Delta}} \ , \ K_{\theta\phi} = \frac{(\partial_\theta\Sigma)MaR\sin^2\theta}{\sqrt{\varrho^6\Sigma\Delta}}\end{aligned} \tag{14}$$

We note that $\partial_\theta\Sigma \equiv -2a^2\sin\theta\cos\theta\Delta$, so the latter quantity simplifies to

$$K_{\theta\phi} = -\frac{2Ma^3R\sqrt{\Delta}\sin^3\theta\cos\theta}{\sqrt{\varrho^6\Sigma}}. \tag{15}$$

[4] I have used the identity $\Sigma(\varrho^2 - 2MR) + 4M^2a^2R^2\sin^2\theta \equiv \varrho^4\Delta$.

Working instead in Cartesian coordinates (and using expressions in [19]), the metric quantities for spin vector $\vec{a}$ in *any* direction are:

$$\begin{aligned}
\beta^i &= \frac{2MR}{\Sigma}(\vec{a}\times\vec{X})^i \\
\gamma_{ij} &= \frac{\varrho^2}{R^2}\delta_{ij} + \frac{\varrho^2}{\Delta R^2}(2MR - a^2)n_i n_j + \frac{\varrho^2(\Sigma - 4M^2R^2)}{R^4\left[\Delta\Sigma + a^2(4M^2R^2 - \Sigma)\right]}(\vec{a}\times\vec{X})_i(\vec{a}\times\vec{X})_j \\
\beta_i &= \frac{2M\Delta\varrho^2}{R\left[\Delta\Sigma + a^2(4M^2R^2 - \Sigma)\right]}(\vec{a}\times\vec{X})_i
\end{aligned}$$

5.3 Accretion Kerr-Schild

This version of Kerr is used by accretion codes that employ the "Cowling approximation" (that is, treating the background as stationary). See, e.g. [20].

$$\begin{aligned}
ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 + \frac{4Mr}{\rho^2}dt\,dr - \frac{4Mar\sin^2\theta}{\rho^2}dt\,d\phi \\
&+ \left(1 + \frac{2Mr}{\rho^2}\right)dr^2 - 2a\left(1 + \frac{2Mr}{\rho^2}\right)\sin^2\theta\,dr\,d\phi \\
&+ \rho^2 d\theta^2 + \left[\rho^2 + a^2\left(1 + \frac{2Mr}{\rho^2}\right)\sin^2\theta\right]\sin^2\theta\,d\phi^2.
\end{aligned} \tag{16}$$

...where $\Delta \equiv r^2 - 2Mr + a^2$, $\rho^2 \equiv r^2 + a^2\cos\theta^2$, and $A = ((r^2 + a^2)^2) - a^2\Delta\sin\theta^2$. As the radial coordinate here is unchanged from that of the original Boyer-Lindquist form, the horizon is still a coordinate sphere defined by $\Delta(r) = 0$: $r_+ = M + \sqrt{M^2 - a^2}$.

The inverse four-metric is:

$$\begin{aligned}
g^{00} &= -1 - \frac{2Mr}{\rho^2} \\
g^{01} &= \frac{2Mr}{\rho^2} \\
g^{11} &= \frac{\Delta}{\rho^2} \\
g^{13} &= \frac{a}{\rho^2} \\
g^{22} &= \frac{1}{\rho^2} \\
g^{33} &= \frac{1}{\rho^2\sin^2\theta}
\end{aligned}$$

This yields a "3+1" decomposition into lapse function, shift vector, and three-metric:

$$\alpha = \frac{1}{\sqrt{-g^{00}}} = \sqrt{\frac{\rho^2}{\rho^2 + 2Mr}} \tag{17}$$

$$\beta^a = -\frac{g^{01}}{g^{00}} = \left(\frac{2Mr}{\rho^2 + 2Mr}, 0, 0\right) \tag{18}$$

$$\gamma_{ij} = g_{ij} = \begin{pmatrix} 1 + 2\frac{Mr}{\rho^2} & 0 & -a\sin^2\theta\left(1 + \frac{2Mr}{\rho^2}\right) \\ 0 & \rho^2 & 0 \\ -a\sin^2\theta\left(1 + \frac{2Mr}{\rho^2}\right) & 0 & A\frac{\sin^2\theta}{\rho^2} \end{pmatrix}. \tag{19}$$

Converting to cylindrical polar coordinates $q \equiv r\sin\theta, \phi \equiv \phi, z \equiv r\cos\theta$, we obtain

$$\beta^a = \left(\frac{2Mq}{\rho^2 + 2Mr}, 0, \frac{2Mz}{\rho^2 + 2Mr}\right) \tag{20}$$

$$\gamma_{ij} = \begin{pmatrix} \frac{\rho^2}{r^2}\cos^2\theta + \left(1 + \frac{2Mr}{\rho^2}\right)\sin^2\theta & -a\left(1 + \frac{2Mr}{\rho^2}\right)\sin^3\theta & \left(1 + \frac{2Mr}{\rho^2} - \frac{\rho^2}{r^2}\right)\sin\theta\cos\theta \\ -a\left(1 + \frac{2Mr}{\rho^2}\right)\sin^3\theta & \frac{A}{\rho^2}\sin^2\theta & -a\left(1 + \frac{2Mr}{\rho^2}\right)\sin^2\theta\cos\theta \\ \left(1 + \frac{2Mr}{\rho^2} - \frac{\rho^2}{r^2}\right)\sin\theta\cos\theta & -a\left(1 + \frac{2Mr}{\rho^2}\right)\sin^2\theta\cos\theta & \frac{\rho^2}{r^2}\sin^2\theta + \left(1 + \frac{2Mr}{\rho^2}\right)\cos^2\theta \end{pmatrix}. \tag{21}$$

Converting to Cartesian coordinates $x \equiv r\sin\theta\cos\phi$, $y \equiv r\sin\theta\sin\phi$, $z \equiv r\cos\theta$, we obtain

$$\beta^a = \left(\frac{2Mx}{\rho^2 + 2Mr}, \frac{2My}{\rho^2 + 2Mr}, \frac{2Mz}{\rho^2 + 2Mr}\right) \tag{22}$$

5.4 Quasi-Isotropic Kerr

In the mid-1990s, Steve Brandt and collaborators introduced an analogue to the Isotropic Schwarzschild (Section 4.3) solution. This is covered, for instance, in [21]. One starts with Kerr written in Boyer-Lindquist coordinates (Section 5.2), and introduce a new radial coordinate η given in terms of the original Boyer-Lindquist $R \equiv R_{BL}$:

$$R \equiv R_+ \cosh^2\left(\frac{\eta}{2}\right) - R_- \sinh^2\left(\frac{\eta}{2}\right) = M + \sqrt{M^2 - a^2}\cosh\eta, \tag{23}$$

where $R_\pm \equiv M \pm \sqrt{M^2 - a^2}$ are the outer and inner horizons, respectively. With this new coordinate, the spatial metric takes the form:

$$\gamma_{ij} = \mathrm{diag}\left(\gamma_{\eta\eta}, \gamma_{\theta\theta}, \gamma_{\phi\phi}\right) = \Psi_0^4\, \mathrm{diag}\left(e^{-2q_0}, e^{-2q_0}, \sin^2\theta\right), \tag{24}$$

where we define the "pseudo-conformal factor" Ψ_0 and q_0 by

$$\Psi_0^4 \equiv \frac{\gamma_{\phi\phi}}{\sin^2\theta} = \frac{\Sigma}{\varrho^2}\ ,\ e^{-2q_0} \equiv \gamma_{\theta\theta}\Psi_0^{-4} = \frac{\varrho^4}{\Sigma}. \tag{25}$$

We can also define a more normal isotropic radial coordinate r as:

$$r \equiv \frac{\sqrt{M^2 - a^2}}{2} e^{\eta}. \tag{26}$$

This is related to the original Boyer-Lindquist radial coordinate R through:

$$\begin{aligned} R &= r\left(1 + \frac{M + a}{2r}\right)\left(1 + \frac{M - a}{2r}\right) = r + M + \frac{M^2 - a^2}{4r} \\ \Rightarrow \frac{\partial R}{\partial r} &= \frac{\sqrt{\Delta}}{r}\ , r = \frac{1}{2}\left[R - M \pm \sqrt{\Delta}\right], \end{aligned} \tag{27}$$

where the + sign is appropriate for $R > 2M$. Then the three-metric can be written:

$$\gamma_{ij} = \mathrm{diag}\left(\gamma_{rr}, \gamma_{\theta\theta}, \gamma_{\phi\phi}\right) = \Phi_0^4\, \mathrm{diag}\left(1, r^2, r^2\,\xi^2\sin^2\theta\right), \tag{28}$$

where the new conformal factor is $\Phi_0^4 \equiv \varrho^2/r^2$, and $\xi^2 \equiv \Sigma/\varrho^4$. Now the horizon is at

$$r_+ = \frac{1}{2}\left(R_+ - M\right) = \frac{1}{2}\sqrt{M^2 - a^2}. \tag{29}$$

Note that in the limit $a \to 0$, $\Psi_0^4 \to (1 + M/2r)^4$, the correct factor for isotropic Schwarzschild. The extrinsic curvature components are now:

$$K_{r\phi} = \frac{(R\partial_R\Sigma - \Sigma)Ma\sin^2\theta}{r\sqrt{\varrho^6\Sigma}}\ \ ,\ \ K_{\theta\phi} = -\frac{2Ma^3R\sqrt{\Delta}\sin^3\theta\cos\theta}{\sqrt{\varrho^6\Sigma}}.$$

Expressed in Cartesian coordinates, the three-metric is

$$\gamma_{ij} = \frac{\Phi_0^4}{q^2} \begin{pmatrix} x^2 + y^2\xi^2 & (1-\xi^2)xy & 0 \\ (1-\xi^2)xy & y^2 + x^2\xi^2 & 0 \\ 0 & 0 & q^2 \end{pmatrix}$$

The determinant of this is surprisingly simple:

$$\det(\gamma_{ij}) = \Phi_0^{12}\xi^2 \Rightarrow \phi = \frac{1}{12}\log\det(\gamma_{ij}) = \log\Phi_0 + \frac{1}{12}\log\xi^2,$$

which leads to the following BSSN conformal splitting

$$\tilde{\gamma}_{ij} = \frac{\xi^{-2/3}}{q^2} \begin{pmatrix} x^2 + y^2\xi^2 & (1-\xi^2)xy & 0 \\ (1-\xi^2)xy & y^2 + x^2\xi^2 & 0 \\ 0 & 0 & q^2 \end{pmatrix}$$

For extending to general spin direction, it might be of use to re-express the conformal factor in terms of the vector $\vec{a}$:

$$\begin{aligned} \Psi_0^4 = \frac{\Sigma}{\varrho^2} &= \frac{(R^2+a^2)\,\varrho^2 + 2Ma^2R\sin^2\theta}{\varrho^2} \\ &= (R^2+a^2) + \frac{2Ma^2R\sin^2\theta}{R^2+a^2\cos^2\theta} = (R^2+a^2) + \frac{2MR|\hat{n}\times\vec{a}|^2}{R^2+(\hat{n}\cdot\vec{a})^2} \\ &= (R^2+a^2) + \frac{2MR|\vec{x}\times\vec{a}|^2}{R^2r^2+(\vec{x}\cdot\vec{a})^2} = (R^2+a^2) + \frac{2MR[r^2a^2-(\vec{x}\cdot\vec{a})^2]}{R^2r^2+(\vec{x}\cdot\vec{a})^2} \end{aligned}$$

Now we can take radial derivatives more easily:

$$\begin{aligned} \partial_r\Psi_0^4 =& 2R\partial_rR + \partial_r\left[\frac{2MR[r^2a^2-(\vec{x}\cdot\vec{a})^2]}{R^2r^2+(\vec{x}\cdot\vec{a})^2}\right] \\ =& 2R\partial_rR + 2M\frac{\partial_rR[r^2a^2-(\vec{x}\cdot\vec{a})^2] + R[2a^2r - 2(\vec{x}\cdot\vec{a})(\hat{n}\cdot\vec{a})}{R^2r^2+(\vec{x}\cdot\vec{a})^2} \\ &- \frac{2MR[r^2a^2-(\vec{x}\cdot\vec{a})^2]}{[R^2r^2+(\vec{x}\cdot\vec{a})^2]^2}\left(2Rr^2\partial_rR + 2R^2r + 2(\vec{x}\cdot\vec{a})(\hat{n}\cdot\vec{a})\right) \end{aligned}$$

5.5 Spinning Bowen-York

One special case of the Bowen-York solution (2) is that of a black hole with spin only (the "S" term of (3)). This is parametrised by the ADM angular momentum S

$$\begin{aligned} \gamma_{ij} &= \psi^4\delta_{ij} \;\;,\;\; K_{ij} = \psi^{-2}\hat{K}_{ij} \\ \hat{K}_{ij} &= \frac{3}{r^3}\left[\epsilon_{kim}S^m\,n^k\,n_j + \epsilon_{kjm}S^m\,n^k\,n_i\right] \end{aligned}$$

where the conformal factor ψ must satisfy the Hamiltonian constraint. Gleiser et al [22] have found an approximate solution for ψ, accurate to $O(\hat{a}^3)$, where $\hat{a}\equiv S/M^2$, assuming the angular momentum vector points along the polar (z) axis:

$$\psi(r,\theta) = 1 + \frac{a}{r} + S^2\left[\frac{(a^2+3ar+r^2)}{40a^3(a+r)^3} + \frac{3r^2}{40a(a+r)^5}\sin^2\theta\right]$$

where $a = m_p/2$ is the apparent horizon location. Additionally, note that in spherical polars, the only non-vanishing component of the (conformal) extrinsic curvature is

$$\hat{K}_{r\phi} = \frac{3\,M\,a\,\sin^2\theta}{r^2}.$$

Note that M here is the ADM mass of the system, which is non-local in nature. However, their perturbative evolutions of this data indicates that energy emission occurs only at $O(S^4)$, so to $O(S^2)$, the horizon and ADM masses are identical.

Gleiser *et al.*'s solution was first derived not interms of the ADM mass, but in terms of the "throat radius" a, equal to one-half the "bare mass" m_p. This is not a physically meaningful quantity, especially if the topology chosen doesn't even possess a throat, but for a single hole, the Sommerfeld-like inner boundary condition at the throat is in fact consistent with the demand of finiteness at the puncture location when the puncture method is used. Thus the Gleiser et al solution is consistent with punctures.

Moreover, the throat $r = a$ is a *minimal surface* – that is, an apparent horizon, and hence a coordinate sphere in these coordinates.

We can push the solution to the next perturbative order: Postulating a conformal factor of the form:

$$\psi(r,\theta) = \psi^{(0)}(r,\theta) + S^2\,\psi^{(2)}(r,\theta) + S^4\,\psi^{(4)}(r,\theta),$$

where

$$\psi^{(4)}(r,\theta) = \psi_0^{(4)}(r)\,P(0,\cos\theta) + \psi_2^{(4)}(r)\,P(2,\cos\theta) + \psi_4^{(4)}(r)\,P(4,\cos\theta),$$

and

$$\begin{aligned}
\psi_0^{(4)}(r) &= -\frac{7}{105600\,a^7\,(r+a)^{11}}\left[19\,r^{10} + 209\,a\,r^9 + 1045\,a^2\,r^8 + 3135\,a^3\,r^7 + 6072\,a^4\,r^6\right.\\
&\quad \left.+7656\,a^5\,r^5 + 6072\,a^6\,r^4 + 3135\,a^7\,r^3 + 1045\,a^8\,r^2 + 209\,a^9\,r + 19\,a^{10}\right]\\
\psi_2^{(4)}(r) &= \frac{r^2}{26400\,a^5\,(r+a)^{11}}\left[31\,r^6 + 341\,a\,r^5 + 1210\,a^2\,r^4 + 1980\,a^3\,r^3 + 1210\,a^4\,r^2\right.\\
&\quad \left.+341\,a^5\,r + 31\,a^6\right]\\
\psi_4^{(4)}(r) &= -\frac{r^4}{4400\,a^3\,(r+a)^{11}}\left[r^2 + 11\,a\,r + a^2\right]
\end{aligned}$$

Using this solution, solving the Apparent Horizon equation still yields a null result: $r_{AH} = a$ (no angular variation). Calculating the surface area of the horizon now yields

$$m_{irr} = m_p + \frac{11}{40\,m_p^3}\,S^2 - \frac{115163}{528000\,m_p^7}\,S^4,$$

leading to a horizon (or black hole) mass

$$m = m_p + \frac{2}{5\,m_p^3}\,S^2 - \frac{86869}{264000\,m_p^7}\,S^4$$

This is different to the ADM mass at $O(S^4)$, yielding an initial radiation energy content of

$$M_{ADM} - m = \frac{53\,S^4}{8000\,m_p^7}$$

This is very much an upper bound on the energy that will actually be radiated to infinity, as the bulk of it will actually be reabsorbed by the horizon during evolution.

Addendum: from a tabulated list of $\hat{a} = S/m^2$ against the $\hat{a}_p \equiv S/m_p^2$ that generated them, we can construct a semi-empirical rule:

$$\hat{a} = 0.928\,\left(1 - e^{-1.288\,\hat{a}_p + 0.1135\,\hat{a}_p^2}\right),$$

where the overall asymptotic value $\hat{a}_\infty = 0.928$ was drawn from a plot in [23].

6 Boosted Single Black Holes

It should be straightforward to add a boost to a spherically-symmetric solution, but Maple folds like a cheap hooker who's been punched in the stomach by a fat man with spots. Or something.

The idea of boosting black holes is a mixture of Special and General Relativity ideas, and is not necessarily very well defined. Nevertheless it's used a lot. We take a set of lab and rest-frame coordinates connected by the general transformation (labelling lab-frame coordinates with primes, and rest-frame ones as unprimed):

$$x^\mu = f^\mu(x^{\alpha'}) \Rightarrow dx^\mu = \Lambda^\mu{}_{\alpha'} dx^{\alpha'},$$

where the Lorentz transformation matrix is given by eqn (2.44) of [2]:

$$\begin{aligned}
\Lambda^0{}_{0'} &= \gamma \\
\Lambda^0{}_{j'} &= -\gamma v_j \\
\Lambda^i{}_{0'} &= -\gamma v^i \\
\Lambda^i{}_{j'} &= \delta^i_j + (\gamma - 1)\frac{v^i v_j}{v^2}
\end{aligned}$$

Then the four-metric transforms according to:

$$g_{\alpha'\beta'} = \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'} g_{\mu\nu}$$

In particular, the three-metric, lapse and shift at a point $x^{\alpha'}$ in the lab frame are:

$$\begin{aligned}
\gamma_{i'j'} =& g_{i'j'} = \Lambda^\mu{}_{i'}\Lambda^\nu{}_{j'} g_{\mu\nu} \\
=& \Lambda^0{}_{i'}\Lambda^0{}_{j'} g_{00} + \Lambda^m_{i'}\Lambda^0{}_{j'} g_{m0} + \Lambda^0{}_{i'}\Lambda^n{}_{j'} g_{0n} + \Lambda^m_{i'}\Lambda^n{}_{j'} g_{mn} \\
=& \gamma^2 v_i v_j(-\alpha^2 + \beta_p\beta^p) - \gamma v_j\left[\delta^m_i + (\gamma-1)\frac{v^m v_i}{v^2}\right]\beta_m \\
& - \gamma v_i\left[\delta^n_j + (\gamma-1)\frac{v^n v_j}{v^2}\right]\beta_n + \left[\delta^m_i + (\gamma-1)\frac{v^m v_i}{v^2}\right]\left[\delta^n_j + (\gamma-1)\frac{v^n v_j}{v^2}\right]\gamma_{mn} \\
=& \gamma^2 v_i v_j(-\alpha^2 + \beta_p\beta^p) - \gamma v_j\left[\beta_i + (\gamma-1)\frac{v^m v_i}{v^2}\beta_m\right] - \gamma v_i\left[\beta_j + (\gamma-1)\frac{v^n v_j}{v^2}\beta_n\right] \\
& + \gamma_{ij} + (\gamma-1)\frac{v^n v_j}{v^2}\gamma_{in} + (\gamma-1)\frac{v^m v_i}{v^2}\gamma_{mj} + (\gamma-1)^2\frac{v^m v_i}{v^2}\frac{v^n v_j}{v^2}\gamma_{mn} \\
=& \gamma_{ij} + (\gamma-1)\frac{v^n v_j}{v^2}\gamma_{in} + (\gamma-1)\frac{v^m v_i}{v^2}\gamma_{mj} - \gamma v_j\beta_i - \gamma v_i\beta_j \\
& + \left[\gamma^2(-\alpha^2 + \beta_p\beta^p) - 2\gamma(\gamma-1)\frac{v^m}{v^2}\beta_m + (\gamma-1)^2\frac{v^m}{v^2}\frac{v^n}{v^2}\gamma_{mn}\right] v_i v_j
\end{aligned}$$

$$\begin{aligned}
\beta_{i'} =& g_{i'0'} = \Lambda^\mu{}_{i'}\Lambda^\nu{}_{0'} g_{\mu\nu} \\
=& \Lambda^0{}_{i'}\Lambda^0{}_{0'} g_{00} + \Lambda^m_{i'}\Lambda^0{}_{0'} g_{m0} + \Lambda^0{}_{i'}\Lambda^n{}_{0'} g_{0n} + \Lambda^m_{i'}\Lambda^n{}_{0'} g_{mn} \\
=& -\gamma^2 v_i(-\alpha^2 + \beta_p\beta^p) + \gamma\left[\delta^m_i + (\gamma-1)\frac{v^m v_i}{v^2}\right]\beta_m \\
& + \gamma^2 v_i v^n\beta_n - \gamma v^n\left[\delta^m_i + (\gamma-1)\frac{v^m v_i}{v^2}\right]\gamma_{mn} \\
=& \gamma\beta_i - \gamma v^n\gamma_{in} \\
& + \left[-\gamma^2(-\alpha^2 + \beta_p\beta^p) + \gamma(\gamma-1)\frac{v^m}{v^2}\beta_m + \gamma^2 v^n\beta_n - \gamma(\gamma-1)\frac{v^m v^n}{v^2}\gamma_{mn}\right] v_i
\end{aligned}$$

$$\begin{aligned}
-\alpha'^2 + \beta_{b'}\beta^{b'} &= g_{0'0'} = \Lambda^\mu{}_{0'}\Lambda^\nu{}_{0'} g_{\mu\nu} \\
&= \Lambda^0{}_{0'}\Lambda^0{}_{0'} g_{00} + \Lambda^m_{0'}\Lambda^0{}_{0'} g_{m0} + \Lambda^0{}_{0'}\Lambda^n{}_{0'} g_{0n} + \Lambda^m_{0'}\Lambda^n{}_{0'} g_{mn} \\
&= \gamma^2(-\alpha^2 + \beta_p\beta^p) - 2\gamma^2 v^m\beta_m + \gamma^2 v^m v^n\gamma_{mn}
\end{aligned}$$

For the special case of Kerr-Schild-type slicings, the rest-frame four-metric is given by a combination of scalars and (flat-space) vectors:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H\ell_\mu\ell_\nu.$$

Then in the lab frame,

$$\begin{aligned}
g_{\alpha'\beta'} &= \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'} g_{\mu\nu} \\
&= \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'}\eta_{\mu\nu} + 2H\Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'}\ell_\mu\ell_\nu \\
&= \eta_{\alpha\beta} + 2H\ell_{\alpha'}\ell_{\beta'}
\end{aligned}$$

$$\begin{aligned}
\ell_{0'} &= \Lambda^\mu{}_{0'}\ell_\mu = \gamma\left(1 - v^m\ell_m\right) \\
\ell_{i'} &= \Lambda^\mu{}_{i'}\ell_\mu \\
&= -\gamma v_i + \left[\delta_i^m + (\gamma-1)\frac{v^m v_i}{v^2}\right]\ell_m \\
&= \ell_i + \left[-\gamma + (\gamma-1)\frac{v^m\ell_m}{v^2}\right]v_i
\end{aligned}$$

The derivatives of the four-metric can be determined from the derivatives of the scalar and vector:

$$\begin{aligned}
H_{,\alpha'} &= \Lambda^\mu{}_{\alpha'}H_{,\mu} = \Lambda^m{}_{\alpha'}H_{,m} \\
\Rightarrow H_{,0'} &= \Lambda^m_{0'}H_{,m} \\
&= -\gamma v^m H_{,m} \\
H_{,i'} &= \Lambda^m_{i'}H_{,m} \\
&= \left[\delta_i^m + (\gamma-1)\frac{v^m v_i}{v^2}\right]H_{,m} \\
&= H_{,i} + (\gamma-1)\frac{v^m v_i}{v^2}H_{,m}
\end{aligned}$$

$$\begin{aligned}
\ell_{\alpha',\beta'} &= \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'}\ell_{\mu,\nu} = \Lambda^m{}_{\alpha'}\Lambda^n{}_{\beta'}\ell_{m,n} \\
\Rightarrow \ell_{0',0'} &= \Lambda^m_{0'}\Lambda^n{}_{0'}\ell_{m,n} \\
&= \gamma^2 v^m v^n \ell_{m,n} \\
\ell_{i',0'} &= \Lambda^m_{i'}\Lambda^n{}_{0'}\ell_{m,n} \\
&= -\left[\delta_i^m + (\gamma-1)\frac{v^m v_i}{v^2}\right]\gamma v^n\ell_{m,n} \\
&= -\gamma v^n\ell_{i,n} - \gamma(\gamma-1)\frac{v^m v^n v_i}{v^2}\ell_{m,n} \\
\ell_{0',j'} &= \Lambda^m_{0'}\Lambda^n{}_{j'}\ell_{m,n} \\
&= -\left[\delta_j^n + (\gamma-1)\frac{v^n v_j}{v^2}\right]\gamma v^m\ell_{m,n} \\
&= -\gamma v^m\ell_{m,j} - \gamma(\gamma-1)\frac{v^m v^n v_j}{v^2}\ell_{m,n} \\
\ell_{i',j'} &= \Lambda^m_{i'}\Lambda^n{}_{j'}\ell_{m,n} \\
&= \left[\delta_i^m + (\gamma-1)\frac{v^m v_i}{v^2}\right]\left[\delta_j^n + (\gamma-1)\frac{v^n v_j}{v^2}\right]\ell_{m,n} \\
&= \left[\delta_i^m + (\gamma-1)\frac{v^m v_i}{v^2}\right]\left[\ell_{m,j} + (\gamma-1)\frac{v^n v_j}{v^2}\ell_{m,n}\right] \\
&= \ell_{i,j} + (\gamma-1)\frac{v^n v_j}{v^2}\ell_{i,n} + (\gamma-1)\frac{v^m v_i}{v^2}\ell_{m,j} + (\gamma-1)^2\frac{v^m v_i}{v^2}\frac{v^n v_j}{v^2}\ell_{m,n}
\end{aligned}$$

6.1 z-Boosted Isotropic Schwarzschild

We introduce a boost to the Isotropic Schwarzschild system, v in the positive z direction. In the rest frame of the black hole, the center is at $(0,0,0)$, while in the lab frame, it's at $(0,0,vT)$. Take $X^\mu \equiv (t,x,y,z)$ as the lab-frame coordinates, with $r \equiv \sqrt{x^2+y^2+z^2}$. Under a boost in the z-direction, the rest-frame coordinates of the hole are related to the lab frame ones by:

$$T = \gamma(t - vz)\ ,\ X = x\ ,\ Y = y\ ,\ Z = \gamma(z - vt).$$

If we write $\Psi = 1 + \frac{M}{2r}$, $\Omega = 1 - \frac{M}{2r}$, the non-boosted Isotropic Schwarzschild interval can be written as:

$$ds^2 = -\Omega^2\Psi^{-2}dt^2 + \Psi^4\left(dx^2 + dy^2 + dz^2\right)$$

Then the boosted four-metric is (in the lab frame)

$$g^4_{\mu\nu} = \begin{pmatrix} -\gamma^2\Psi^{-2}\left(\Omega^2 - v^2\Psi^6\right) & 0 & 0 & \gamma^2 v\Psi^{-2}\left(\Omega^2 - \Psi^6\right) \\ 0 & \Psi^4 & 0 & 0 \\ 0 & 0 & \Psi^4 & 0 \\ \gamma^2 v\Psi^{-2}\left(\Omega^2 - \Psi^6\right) & 0 & 0 & \gamma^2\Psi^{-2}\left(\Psi^6 - v^2\Omega^2\right) \end{pmatrix}$$

This yields a lapse function

$$\alpha = \frac{\Omega\Psi^2}{\gamma\sqrt{\Psi^6 - v^2\Omega^2}},$$

a shift vector

$$\beta^i = \left(0, 0, -v\frac{\Psi^6 - \Omega^2}{\Psi^6 - v^2\Omega^2}\right),$$

and a three-metric

$$\gamma_{ij} = \mathrm{diag}\left(\Psi^4, \Psi^4, \gamma^2\Psi^{-2}\left(\Psi^6 - v^2\Omega^2\right)\right).$$

The non-zero extrinsic curvature components are

$$\begin{aligned} K_{xx} = K_{yy} =& v\frac{2\Psi\Omega\partial_z\Psi}{\gamma\sqrt{\Psi^6 - v^2\Omega^2}} \\ K_{xz} =& -v\frac{\gamma\Psi\left(3\Omega + \Psi\right)\partial_x\Psi}{\sqrt{\Psi^6 - v^2\Omega^2}} \\ K_{yz} =& -v\frac{\gamma\Psi\left(3\Omega + \Psi\right)\partial_y\Psi}{\sqrt{\Psi^6 - v^2\Omega^2}} \\ K_{zz} =& -v\frac{\gamma\left(2\Psi^6\left(\Psi + 2\Omega\right) - v^2\Omega^2\left(\Psi + \Omega\right)\right)\partial_z\Psi}{\Psi^5\sqrt{\Psi^6 - v^2\Omega^2}} \end{aligned}$$

6.2 z-Boosted Ingoing Eddington-Finkelstein

We introduce a boost to the iEF system, v in the positive z direction. For convenience, we'll work in cylindrical polar coordinates. In the rest frame of the black hole, the center is at $(0,0,0)$, while in the lab frame, it's at $(0,0,vT)$. Take $x^\mu \equiv (t,q,\phi,z)$ as the lab-frame coordinates, with $r \equiv \sqrt{q^2+z^2}$. Under a boost in the z-direction, the rest-frame coordinates of the hole are related to the lab frame ones by:

$$\begin{aligned} T =& \gamma(t - vz)\ ,\ Q = q\ ,\ \Phi = \phi\ ,\ Z = \gamma(z - vt) \\ \Rightarrow dT =& \gamma dt - v\gamma dz\ ,\ dQ = dq\ ,\ d\Phi = d\phi\ ,\ dZ = \gamma dz - v\gamma dt \end{aligned}$$

In the boosted frame, the metric is

$$
\begin{aligned}
ds^2 =& - dT^2 + dQ^2 + Q^2 d\Phi^2 + dZ^2 + \frac{2M}{R}\left(dT + \frac{Q}{R}dQ + \frac{Z}{R}dZ\right)^2 \\
=& - \gamma^2(dt - vdz)^2 + dq^2 + q^2 d\phi^2 + \gamma^2(dz - vdt)^2 \\
&+ \frac{2M}{R}\left[\gamma(dt - vdz) + \frac{q}{R}dq + \gamma\frac{Z}{R}(dz - vdt)\right]^2 \\
=& - dt^2 + dq^2 + q^2 d\phi^2 + dz^2 \\
&+ \frac{2M}{R}\left[\gamma\left(1 - \frac{vZ}{R}\right)dt + \frac{q}{R}dq + \gamma\left(\frac{Z}{R} - v\right)dz\right]^2 \\
=& - dt^2 + dq^2 + q^2 d\phi^2 + dz^2 + \frac{2M}{R}\left[\frac{\rho}{R}dt + \frac{q}{R}dq + \frac{\zeta}{R}dz\right]^2
\end{aligned}
$$

This implies a four-metric given by $g_{\mu\nu} = \eta_{\mu\nu} + 2H\ell_\mu\ell_\nu$, where

$$
H = \frac{M}{R}, \quad \ell_\mu = (\ell_t, \ell_q, \ell_\phi, \ell_z) = \left(\frac{\rho}{R}, \frac{q}{R}, 0, \frac{\zeta}{R}\right),
$$

and $\rho \equiv \gamma(R - vZ)$, $\zeta \equiv \gamma(Z - vR)$. Note that

$$
\begin{aligned}
q^2 + \zeta^2 =& R^2 - Z^2 + \gamma^2(Z^2 - 2vZR + v^2R^2) \\
=& \gamma^2\left((1 - v^2)(R^2 - Z^2) + Z^2 - 2vZR + v^2R^2\right) \\
=& \gamma^2(R^2 - 2vZR + v^2Z^2) \\
=& \rho^2.
\end{aligned}
$$

As ℓ_μ is still null, the inverse four-metric is very simply $g^{\mu\nu} = \eta^{\mu\nu} - 2H\ell^\mu\ell^\nu$. Now the three-metric is

$$
(\gamma_{ij}) = \begin{pmatrix} 1 + \frac{2Mq^2}{R^3} & 0 & \frac{2Mq\zeta}{R^3} \\ 0 & q^2 & 0 \\ \frac{2Mq\zeta}{R^3} & 0 & 1 + \frac{2M\zeta^2}{R^3} \end{pmatrix},
$$

the lapse function is

$$
\alpha = \frac{1}{\sqrt{1 + \frac{2M\rho^2}{R^3}}},
$$

the shift vector is

$$
(\beta_q, \beta_\phi, \beta_z) = \frac{2M\rho}{R^3}(q, 0, \zeta) \Rightarrow \left(\beta^q, \beta^\phi, \beta^z\right) = \frac{2M\rho}{R^3 + 2M\rho^2}(q, 0, \zeta),
$$

and the extrinsic curvature is:

$$
\begin{aligned}
K_{qq} =& -\frac{M\left[2Mq^2\rho + \gamma R^2 q^2 + \rho R(q^2 - 2Z^2)\right]}{R^4\sqrt{R^4 + 2MR\rho^2}} \\
K_{qz} =& -\frac{Mq\left[2M\rho\zeta + 3\gamma^2 Zq^2 + \gamma(3Z^2 + r^2)\zeta\right]}{R^4\sqrt{R^4 + 2MR\rho^2}} \\
K_{\phi\phi} =& \frac{2Mq^2\rho}{R\sqrt{R^4 + 2MR\rho^2}} \\
K_{zz} =& \frac{M\left[(R - 2M)\rho\zeta^2 - R\left(R\gamma\zeta^2 + 2\gamma R\rho^2 - 4q^2\gamma^2\rho\right)\right]}{R^4\sqrt{R^4 + 2MR\rho^2}}
\end{aligned}
$$

Then the three-metric determinant is

$$
|\gamma| = 1 + \frac{2M\rho^2}{R^3},
$$

and the trace of the extrinsic curvature is

$$K = \frac{M\rho^2\,[6M\rho + \gamma R(2R + 3vZ)]}{R(R^3 + 2M\rho^2)\sqrt{R(R^3 + 2M\rho^2)}}$$

To linear order in v, this can be approximated as:

$$K \approx \frac{2M(r + 3M)}{\sqrt{r^3(r + 2M)^3}} - v\frac{Mz(r + 8M)}{\sqrt{r^3(r + 2M)^5}}.$$

Having calculated these, we can transform back to Cartesian coordinates. The Cartesian shift vector is

$$(\beta^x, \beta^y, \beta^z) = \frac{2M\rho}{R^3 + 2M\rho^2}\,(x, y, \zeta)\,,$$

The Cartesian three-metric is

$$(\gamma_{ij}) = \begin{pmatrix} 1 + \frac{2Mx^2}{R^3} & \frac{2Mxy}{R^3} & \frac{2Mx\zeta}{R^3} \\ \frac{2Mxy}{R^3} & 1 + \frac{2My^2}{R^3} & \frac{2My\zeta}{R^3} \\ \frac{2Mx\zeta}{R^3} & \frac{2My\zeta}{R^3} & 1 + \frac{2M\zeta^2}{R^3} \end{pmatrix},$$

In practice, the boost transformation will be calculated numerically using the boost matrix. However, the above derivation has proven useful in certain semi-analytic calculations.

6.3 Linearly Moving Bowen-York

The Bowen-York solution (2) with only linear momentum is given by taking only the "P" term of (3). This is parametrised by the ADM linear momentum P:

$$\begin{aligned} \gamma_{ij} =& \psi^4\delta_{ij} \;\;,\;\; K_{ij} = \psi^{-2}\,\hat{K}_{ij} \\ \hat{K}_{ij} +& \frac{3}{2R^2}\left[P_i\,n_j + P_j\,n_i - (\delta_{ij} - n_i\,n_j)P^k\,n_k\right] \\ \mp& \frac{3a^2}{2R^4}\left[P_i\,n_j + P_j\,n_i + (\delta_{ij} - 5n_i\,n_j)P^k\,n_k\right] \end{aligned}$$

where the conformal factor ψ must satisfy the Hamiltonian constraint. Gleiser et al [24] have produced a slow-moving approximate solution for the conformal factor ψ, with two types of inner boundary conditions: inversion symmetric throat conditions, and puncture conditions. I quote here only the result of the latter choice, assuming the hole has momentum P in the positive-z direction:

$$\psi = \psi^{(0)}(R) + P^2\,\left[\psi_0^{(2)}(R)\,P_0(\cos\theta) + \psi_2^{(2)}(R)\,P_2(\cos\theta)\right] + O(P^4),$$

where the radial functions are:

$$\begin{aligned} \psi^{(0)} =& 1 + \frac{M}{2\,R}, \\ \psi_0^{(2)} =& \frac{M^4 + 10\,M^3\,R + 40\,M^2\,R^2 + 80\,M\,R^3 + 80\,R^4}{8\,M(2\,R + M)^5}, \\ \psi_2^{(2)} =& \frac{120\,R^5 + 768\,R^4\,M + 1078\,R^3\,M^2 + 658\,R^2\,M^3 + 189\,R\,M^4 + 21\,M^5}{20\,R^2\,(2\,R + M)^5} \\ &- \frac{21\,M}{40\,R^3}\ln\left(\frac{2\,R + M}{M}\right). \end{aligned}$$

Note that the conformal extrinsic curvature has non-vanishing components:

$$\begin{aligned} \hat{K}_{RR} =& \frac{3\,P}{R^2}\,\cos\theta, \\ \hat{K}_{R\theta} = \hat{K}_{\theta R} =& -\frac{3\,P}{2\,R}\,\sin\theta, \\ \hat{K}_{\theta\theta} =& -\frac{3\,P}{2}\,\cos\theta, \\ \hat{K}_{\phi\phi} =& -\frac{3\,P}{2}\,\cos\theta\,\sin^2\theta. \end{aligned}$$

7 Binary Black-Hole Solutions

7.1 Brill-Lindquist

The simplest binary data solution is a conformal one due to Brill and Lindquist [25]. In Cartesian coordinates, the four-metric is given by[5]

$$ds^2 = -dt^2 + \left(1 + \frac{m_1}{2R_1} + \frac{m_2}{2R_2}\right)^4 \delta_{ij}\, dx^i\, dx^j$$

yielding 3+1 fields

$$\begin{aligned} \alpha =& 1 \;\;, \;\; \beta^i = 0, \\ \gamma_{ij} =& \left(1 + \frac{m_1}{2R_1} + \frac{m_2}{2R_2}\right)^4 \delta_{ij}, \\ K_{ij} =& 0 \end{aligned}$$

Things to note about Brill-Lindquist data:

- as the extrinsic curvature and the shift vector both vanish, this data is *time-symmetric*;
- this data is only instantaneously valid – it applies for one time slice only in a foliation of space-time;
- the data is conformally flat;
- the underlying topology is 3-sheeted – the two hole interiors correspond to separate sheets

7.2 Binary Bowen-York

Bowen-York data (Section 2.2) for two holes with initial spins and/or boosts can be taken to represent a binary black hole system.

This data reduces to Brill-Lindquist (section 7.1) when there's no spin or boost. In that limit, the Hamiltonian constraint is identically satisfied. In all other cases, an elliptic solver must be used.

[5]In fact, Brill & Lindquist's solution included charge also; I'm ignoring this here, so their χ and ψ are just ψ here.

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