# Consequences of the Lorentz Transformation


Arnav Wadalkar

*Physics Undergraduate Society*


3rd August 2025


## Abstract

In our previous session, we discussed how the **Lorentz transformation** is derived from basic principles and constraints, as well as through Maxwell's Equations. We then formulated the Lorentz Transformation matrix.

In this article, we will discuss the **consequences of the Lorentz transformation** and how it has altered previous theories and assumptions, specifically **Simultaneity, Causality, Length Contraction, and Time Dilation.**


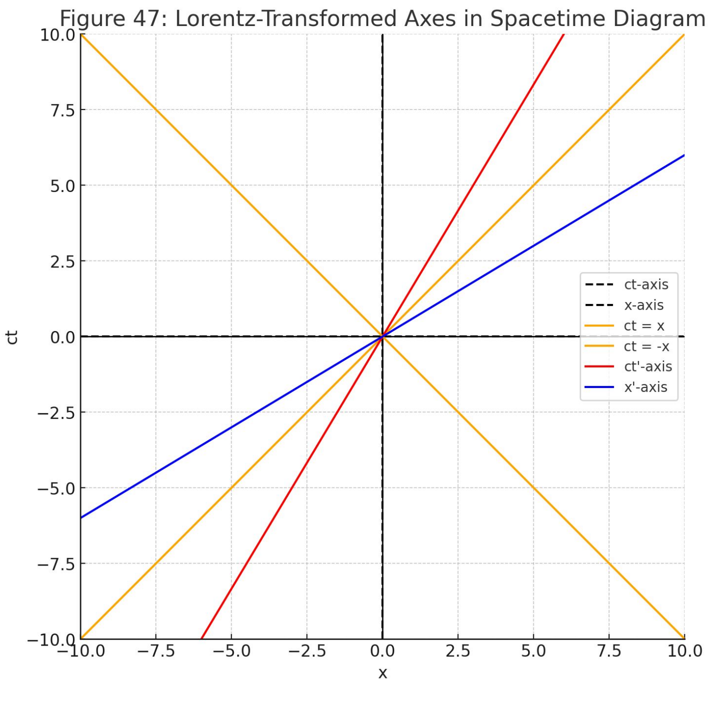


Figure 1: Spacetime diagram of Lorentz transformation

## 1. Simultaneity

Before the introduction of Lorentz Transformation, the Galilean Transformation stated that events occurring simultaneously in one inertial frame of reference occurred simultaneously in all other inertial frames of reference, which is a consequence of time being absolute. Before we go ahead and talk about how our new set of transformations breaks this continuity, let's understand what Simultaneous events imply.

**An Event states the position and time for a particle in spacetime geometry.** It is analogous to specifying the position of a particle in Cartesian coordinates, except that here we are working in a 4-dimensional space, i.e. spacetime typically$(ct, x, y, z)$, and hence each particle's position and time are defined by a unique event in spacetime. Using this concept we define simultaneity, *'Two events are said to be simultaneous if they occur at the same time in same inertial frames of reference, regardless of their position in space.'* Hence, every event contains four independent co-ordinates where three contain information regarding its position in space and the fourth one being time. The old Galilean definition of simultaneity was 'Two events are said to be simultaneous if they occur at the same time in two inertial frames of reference, regardless of their position in space.' We will work towards disproving this now.

Let's now understand how simultaneity of two frames break under Lorentz transformation via spacetime diagrams. Take an inertial frame of reference $S$. Now, we draw horizontal lines in the spacetime diagram of $S$, depicting a specific time. Further, we define two events $P$ and $Q$ which occur simultaneously in $S$ frame. Hence, they will lie on the same horizontal line by the definition of simultaneity. Now, suppose $S$ frame goes under the Lorentz transformation, we call this new frame $S^{'}$. Following the same procedure, i.e. drawing horizontal lines depicting constant time for $S^{'}$ frame, we observe that the events $P$ and $Q$ aren't lying on the same horizontal line in $S^{'}$ which were on the same horizontal time (same time) in the $S$ frame. From this observation, we can conclude that *the events occurring at the same time in the $S$ frame are not occurring at the same time in the $S^{'}$ frame.* **This breaking of simultaneity is a consequence of the constancy of light**, what I mean by that is that the speed of light is the same in every frame of reference. Let's understand this better with an example.

Suppose there are two observers, A and B, where A is stationary on the ground while B is an observer inside a train moving at a speed comparable to the speed of light. The event in consideration is the striking of two lightning bolts. This event is simultaneously occurring for observer A, and and under Galilean transformations, it was assumed this would also be the case for B. Though the physics of relativity says otherwise, intuitively you can think of the situation with respect to B's frame as that *B is moving 'towards' one lightning bolt and 'away' from the other.* Earlier, since there was no bound for light's speed, it mathematically adjusted itself such that the event retains its simultaneity in both frames of reference A and B. Applying the proven postulate which states that the speed of light is a constant in every frame of reference, there's no mathematical adjustment and the simultaneity breaks for the two observers. Hence, in $S'$ frame the lightning bolts are no longer simultaneous because the light-arrival times differ and simultaneity surfaces tilt under a boost.

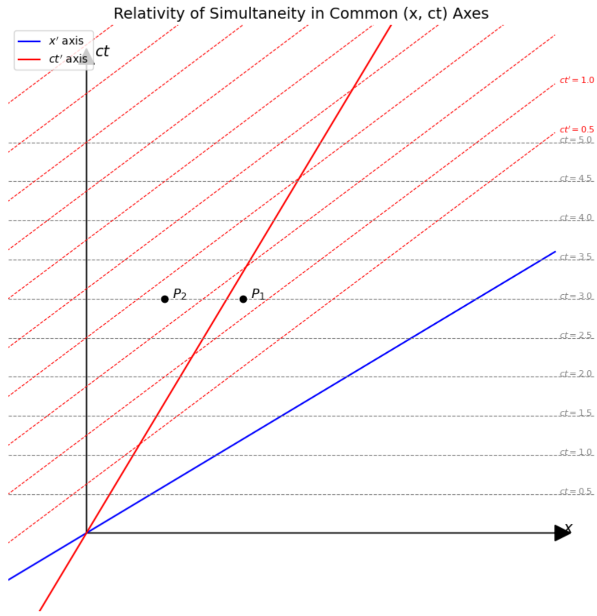


Figure 2: Depiction of breaking of simultaneity via spacetime diagram

## 2. Causality

Causality is a fundamental concept and a very intuitive one. It states that for any event, *the cause of the event should be before the effect of the event with respect to time.* To understand this, lets understand the concept of '*light-cones*', light-cones are structures in the spacetime which depict causality and the constraint of information light has.

From the two dimensional spacetime diagram and also the Lorentz Boost equation, we know that the **speed of light has an angle of $\frac{\pi}{4}$.** *All events having world-line steeper than $\frac{\pi}{4}$ are called 'time-like' (speed less than the speed of light) and all events having world-line lower than $\frac{\pi}{4}$ are called 'space-like' (speed more than the speed of light).* To depict this constraint in three dimensional depiction of spacetime (xy plane being the spatial coordinates and the z axis being time) we construct a cone with the apex angle of $\frac{\pi}{2}$ ,i.e. $\frac{\pi}{4}$ inclined by both sides of the time axis. This cone is called a 'light-cone'. To further investigate the light-cone, we define the past light-cone and the future light-cone. From the name it is understandable that the past light-cone will be below the event in consideration $P$ containing all the events that can possibly influence $P$ and the future light-cone will be above the event considered $P$ containing all the events $P$ can possibly affect(using the words 'above' and 'below' is with respect to the time axis). More specifically, we construct the light-cone considering event $P$ such that *the past light-cone is a set of all the events that can potentially affect $P$ whereas the future light-cone is a set of all the events that can potentially be affected by the event $P$.* Any event outside the past light-cone has no possibility to influence the event $P$, similarly any event outside the future light-cone of $P$ cannot be affected by $P$.

Lets take this constructed light-cone and consider three events $Q, R,\ and\ S$The event $Q$ which is inside the past light-cone, i.e. an element of the past light-cone set can by definition influence $P$. On the other hand $R$ being an element of the future light-cone can be influenced by the event $P$. When we consider the event $S$, since it is neither inside the future light-cone nor the past light-cone, by definition it can neither influence $P$ nor be influenced by it. This concept is a clear depiction of causality stating clear boundaries

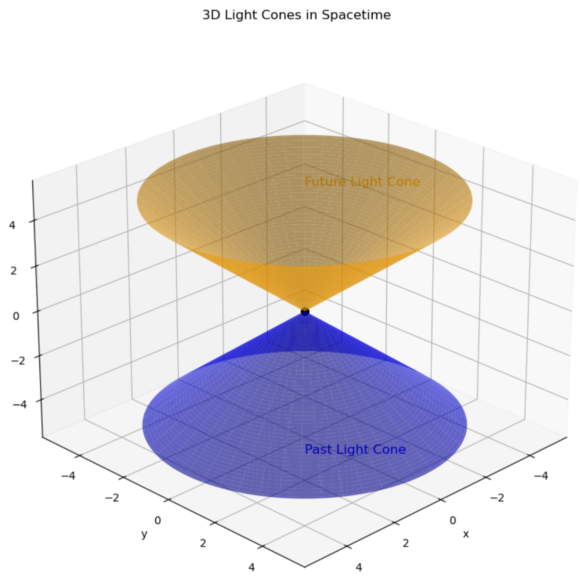


Figure 3: 3D light cones in spacetime showing the past and future light cones.

and a directional flow in time stating that the cause(past light-cone) is preceding the effect(future light-cone).

Light-cones are crucial for Information theory as it states a constraint on the information $P$ can transmit. From the law of Causality, we can state that '*Events outside the light-cone are non-communicable*' , in other words, this statement states that no signal can travel faster than the speed of light. In information theory, the future light-cone is a region where information can be sent to. On the other hand, the past light-cone is the region from which information can arrive. Another statement which can be thought of as a corollary for the law of Causality is '*A message can only be transmitted from one event to another if they are time-like or light-like separated.*'

## 3. Time Dilation

A mind-bending consequence of the constancy of speed of light is that time is no longer absolute! It is different for different frame of references. We will prove this phenomenon by two methods, first by the already available Lorentz boost equations and the its implications in the spacetime diagram

We start with considering two frames of reference $S$ and $S'$ such that their initial conditions are same, i.e. $t = t' = 0$ and $x = x' = 0$. A particle exists in the moving frame $S'$. Assume the particle is at rest in $S'$, its position in $S'$ is constant, this gives and freedom to choose any constant position. Here we choose $x' = 0$ for simplicity.

Let the particle emit two ticks, once at time $t'_1$ and then at time $t'_2$. The time interval measured in the moving frame is:

$$\Delta t' = t'_2 - t'_1$$

We want to find the corresponding time interval $\Delta t = t_2 - t_1$ as measured in the stationary frame $S$.

From the Lorentz Boost:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Since $x' = 0$

$$t' = \gamma\left(t - \frac{\beta(vt)}{c}\right) = \gamma\left(t - \frac{v^2 t}{c^2}\right) = \gamma t\left(1 - \frac{v^2}{c^2}\right)$$

Thus,

$$t' = t \cdot \frac{1}{\gamma} \quad \Rightarrow \quad t = \gamma t'$$

Hence, the time interval in $S$ is:

$$\Delta t = \gamma \Delta t'$$

Therefore:

$$\boxed{\Delta t = \gamma \Delta t'}$$

We see *there is factor of* $\gamma$ *differentiating the time interval between the two frame of references* and as $v \to c \Rightarrow \gamma \to \infty$. Therefore faster the frame of reference, more is the time being 'dilation'. Hence the faster an object is moving, the slower it will perceive time. Therefore, we can state that '**moving clocks run slower from the lab's perspective**'

We will now derive time dilation through an example. Let frame $A$ be the rest observer and frame $B$ be the frame of a car moving at constant speed $v$. The separation between the initial position of A and B is $L$, and the time between ticks measured in $A$ (the proper time) is

$$\Delta t_A = \frac{2L}{c}.$$

relative to $A$ (to the right). Observers in $B$ see the clock moving, so the light takes a slanted path. Let the time between ticks measured in $B$ be $\Delta t_B$.

In $B$:

Horizontal displacement of the clock: $v\frac{\Delta t_B}{2}$,

Length of light's path: $c\frac{\Delta t_B}{2}$,

Vertical separation: $L$

By the Pythagoras theorem:

$$\left(c\frac{\Delta t_B}{2}\right)^2 = L^2 + \left(v\frac{\Delta t_B}{2}\right)^2.$$

$$(c^2 - v^2)\Delta t_B^2 = 4L^2.$$

Thus:

$$\Delta t_B^2 = \frac{4L^2}{c^2 - v^2} = \frac{4L^2}{c^2\left(1 - \frac{v^2}{c^2}\right)} \Rightarrow \Delta t_B = \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}}.$$

Since $\frac{2L}{c} = \Delta t_A$, we find:

$$\boxed{\Delta t_B = \frac{\Delta t_A}{\sqrt{1 - v^2/c^2}} = \gamma(v)\, \Delta t_A}.$$

Here $\gamma(v) = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the Lorentz factor. This shows that the moving clock (as seen by $B$) runs slower: $\Delta t_B > \Delta t_A$.

Finally, let's try to approach time dilation via spacetime diagrams. For a constant $v$, $\gamma$ is constant since $\gamma$ is is a function of $v$

$$\gamma = f(v)$$

Therefore, the Lorentz factor converts to a **scaling factor** where the multiplication of $\gamma$ only changes the scaling of the transformed co-ordinates. Hence, using the previous example, for the moving frame $S'$, time is moving slower meaning that less time is passed in the $S$ frame and hence according to $S$ frame, the $t'$ axis is closer together compared to $t$. In an intuitive sense, the $t'$ axis has 'shrunk' in size. This is also depicted by the formula ( $\gamma > 1$)

$$\Delta t = \gamma \Delta t'$$

An experimental evidence of this event is **Muon scattering**. The mean lifetime of a Muon is $2.197\mu s$ and the speed Muon has to achieve (non-relativistically) to reach from the lower stratosphere (where they are formed) to the see level before it decays is $\approx 4.55 * 10^9 m/s$ which is greater than the speed of light and hence not possible. The shocking observation was that *it was observed there was a non zero flux of muon particles found at sea level.* This phenomenon then could have only been explained by physics of relativity. Let's understand the mechanism of this phenomenon by time dilation, From the Muon's perspective, i.e. a moving frame of reference, time is dilated and hence the mean lifetime of muon is increased when observed from the Laboratory's frame of reference. The amount of time muon required to reach from its atmospheric layer of origin to the sea level is $\approx 33.4\mu s$ and the mean lifetime from Lab's perspective for the particle is dilated to $\approx 34.8\mu s$. Hence, the presence of Muons in sea level is explained via time dilation which is a consequence of constancy of light.

## 4. Length Contraction

Another equally mind bending consequence due to the universal speed limit of light is that the faster an object moves, the shorter in length it gets! We will now derive it via Lorentz boost equations.

Take a **rigid rod** in $S$ frame(Lab frame) and the observer in $S'$ frame(Moving frame) moving with speed $v$ comparable to the speed of light with respect to $S$ frame. Here $S'$ frame is achieved when $S$ goes under Lorentz transformation. Rigidity is assumed such that length of the rod will be difference between the position of the start and the end point.

The Lorentz transformations are:

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

We know,

$$x' = \gamma(x - vt) \Rightarrow x = \gamma^{-1}x' + vt$$

Now, we define the position of the *end point* and the *start point* of rod in $S$ as $x_1$ and $x_2$ respectively. Also the lengths of the rod will be $L$ and $L_0$ for $S'$ and $S$ frame respectively. Under $S'$ frame (the Lorentz transformed frame), $x_1$ and $x_2$ positions will transform to:

$$x_1 = \gamma^{-1}x'_1 + vt, \quad x_2 = \gamma^{-1}x'_2 + vt$$

where $x'_1$ and $x'_2$ represent the position of both the rods in $S'$ frame of reference. Now, for a rigid rod:

$$L = x_2 - x_1 = \gamma^{-1}(x'_2 - x'_1) = \gamma^{-1}L_0$$

$$\boxed{L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore, we arrive at our equation stating that the length observed by the two observer, one in $S$ and another in $S'$ is not the same.

Coming back to Muon Scattering, it was understood from the previous section that from the Lab's perspective(stationary frame) the mean life time of Muon had increased, a consequence of time dilation. What about Muons perspective? Well the answer connects the two concepts of time dilation and length contraction in a beautiful manner. From the Muons perspective, there is no change in the mean lifetime of the particle. Rather the distance from the stratosphere to the sea level is 'contracted'. According to Muon, the thickness of the Earth's atmosphere is contracted, this contraction obeys the formula

$$L = \frac{L_0}{\gamma}$$

Assuming Muon to be moving at the speed $v = 0.998c$, $\gamma$ is calculated to be 15.82 and hence the distance from Muons perspective is drastically reduced by a factor of $\approx 16$. Hence, from Lab's frame of reference the distance is $10km$, from Muons perspective, the particle has to travel only $632m$ which it can in $2.11\mu s$(mean lifetime of muon is $2.2\mu s$)

## 5. Conclusion

In this article, we delved into the consequences of relativity and constancy of speed of light in every frame of reference. We first depicted how simultaneity is not maintained under Lorentz transformation, then we went ahead towards Causality and its direct implications in spacetime diagrams - 'light-cones'. Further we investigated two mathematical structures derived from Lorentz boost namely time dilation and length contraction, and discussed the consequences and examples for both mathematical structures. We will conclude this article by explaining how relativity solves the Car-garage paradox.

**The Car-Garage Paradox** Suppose you have a garage of length $L$, take the garage's perspective as Lab's frame since its at rest. Suppose a Car (moving frame of reference) of length $1.2L$ is moving with a speed comparable to the speed of light. Suppose the car moves fast enough that its Lorentz-contracted length in the garage frame is exactly equal to the garage's length. Mathematically, we want the car to move at a speed where $\gamma = 1.2$. Another assumption is made that from the Lab frame, as soon as the car is perfectly inside the garage, the car instantly stops so that it does not clash with the end wall of the garage and the door instantly closes as the car is inside. Now, the paradox is that one would think 'How can a car which is bigger than the garage fit into it and the door closes even though the car is bigger? The calculations suggest that this is possible but my head cannot wrap around the idea of this occurring' You will be right, a car which is bigger than the garage cannot in-fact fit inside the garage, why so? The answer is contained in simultaneity. For the Lab's frame the door does close but for the moving frame, i.e. the car's frame of reference the door does not close at the same time since it is a different frame of reference. Thus, the car never sees itself fully inside the garage.

# References


[1] David Tong, *Special Relativity*, University of Cambridge Part II Notes, http://www.damtp.cam.ac.uk/user/tong/special.html

[2] A. Einstein, *On the Electrodynamics of Moving Bodies*, Annalen der Physik, 17 (1905), pp. 891–921. English translation available at: fourmilab.ch

[3] E. F. Taylor and J. A. Wheeler, *Spacetime Physics: Introduction to Special Relativity*, 2nd ed., W. H. Freeman, 1992.

[4] David J. Griffiths, *Introduction to Electrodynamics*, 4th ed., Pearson, 2013.

[5] W. Rindler, *Introduction to Special Relativity*, 2nd ed., Oxford University Press, 1991.