Lorentz transformation

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Abstract

This work presents a detailed discussion of the Lorentz transformations, which form the foundation of Einstein's Special Theory of Relativity. Beginning with the limitations of Galilean transformations, it highlights the inconsistency between classical mechanics and the constancy of the speed of light demonstrated by Maxwell's equations and the Michelson–Morley experiment. Two derivations of the Lorentz transformations are provided: one from Einstein's postulates of relativity and another from Lorentz's attempt to adjust Maxwell's equations with the aether theory. The analysis provides the crucial difference between Lorentz's attachment to the aether concept and Einstein's revolutionary rejection of it, which led to a new understanding of space and time as relative and interconnected, which is known as "Spacetime". Finally, the reduction of Lorentz transformations to Galilean transformations in the low-velocity limit is shown, addressing the compatibility of classical mechanics within its domain. This study underscores the pivotal role of Lorentz transformations in bridging classical and relativistic physics.

1 Why Galilean Transformations aren't enough

Maxwell's equations implicitly say that light travels at a constant speed in vacuum-regardless of the motion of the source. But he believed it was constant only in an aether frame, not necessarily in all frames. But there is no experimental proof for aether's existence. So, Michelson- Morley tried to detect the aether. In order to detect aether, they disproved aether's existence and concluded that speed of light is constant for all inertial observers. There is no such thing as an "aether" frame. This is a big surprise for scientific community even for Albert Michelson and Edward Morley (who performed this experiment). Because, according to classical (Newtonian) mechanics, velocities transform between inertial frames using Galilean addition:

$$v' = v - u$$

where:

- v is the velocity of an object in the stationary frame,
- u is the relative velocity between the two frames,
- v' is the velocity of the object in the moving frame.

Let apply this logic to light. Suppose a light beam moves at speed c in one frame. Then, using Galilean addition, an observer moving at velocity v relative to the source would measure:

$$c' = c - v$$

The Michelson–Morley experiment confirmed that the speed of light is invariant:

$$c' = c$$

So,the Galilean velocity transformation fails for light. It predicts $c' \neq c$, which contradicts both Maxwell's theory and experimental results. This inconsistency necessitated a new transformations. Here, Einstein came into story.He took Maxwell's equations seriously and rejected aether theory completely. With this he proposed his "Special theory of Relativity".The special theory of relativity is based on two postulates proposed by Albert Einstein in 1905:

1.The principle of relativity: The laws of physics are the same in all inertial reference frames.

2.The principle of the constancy of the speed of light: The speed of light in free space has the same value c in all inertial reference frames.

By using this postulates, Einstein independently formulated a new transformations known as Lorentz transformations which replace Galilean transformations.

2 Derivation of Lorentz transformations using postulates of Special Theory of Relativity

Let S and S' be two inertial frames moving with a relative velocity v. For simplicity, we will ignore the directions y and z, which are perpendicular to the direction of motion. Both inertial frames are equipped with Cartesian coordinates: (x, t) for frame S, and (x', t') for frame S'.

The Lorentz transformations are assumed to be of the form:

$$x' = f(x, t), \quad t' = g(x, t)$$

But using some physical principles, we can constrain these functions.

1. Law of inertia (uniform motion is preserved in absence of any external force): To preserve uniform motion, f and g must be **linear** functions of x and t. So we write:

$$x' = \alpha_1 x + \alpha_2 t \tag{1}$$

$$t' = \alpha_3 x + \alpha_4 t \tag{2}$$

2. Frame S' moves at velocity v relative to S: In frame S, the origin of S' moves along the line $x = vt$. But for S', its origin is always at rest: $x' = 0$. So, $x = vt$ must map to $x' = 0$. From (1),

$$0 = \alpha_1 vt + \alpha_2 t \quad \Rightarrow \quad \alpha_2 = -\alpha_1 v$$

Substitute this into equation (1):

$$x' = \alpha_1 x + \alpha_2 t = \alpha_1 x - \alpha_1 vt = \alpha_1 (x - vt) \tag{3}$$

So,

$$x' = \alpha_1 (x - vt) \quad \text{(Transformation from } S \text{ to } S')$$

Similarly, from S' to S, we write:

$$x = \alpha_1 (x' + vt') \tag{4}$$

3. Principle of relativity: The laws of physics are the same in all inertial reference frames.The transformations must be symmetric; so the constant α_1 is the same in both directions.

4. Constancy of speed of light: A light signal satisfies:

$$\text{In } S: \quad x = ct \quad \text{or} \quad x = -ct$$

$$\text{In } S': \quad x' = ct' \quad \text{or} \quad x' = -ct'$$

Using equation (4), substitute $x = ct$:

$$ct = \alpha_1 (ct' + vt') = \alpha_1 t'(c + v) \quad \Rightarrow \quad t' = \frac{ct}{\alpha_1 (c + v)} \tag{5}$$

Using equation (3), substitute $x' = ct'$, and $x = ct$:

$$ct' = \alpha_1 (ct - vt) = \alpha_1 t(c - v) \quad \Rightarrow \quad t' = \alpha_1 t(1 - \frac{v}{c}) \tag{6}$$

From equations (5) and (6), we get:

$$t' = \alpha_1 t \left(1 - \frac{v}{c}\right), \quad t = \alpha_1 t' \left(1 + \frac{v}{c}\right)$$

$$\frac{t'}{t} = \alpha_1 \left(1 - \frac{v}{c}\right), \quad \frac{t}{t'} = \alpha_1 \left(1 + \frac{v}{c}\right)$$

Now multiply both expressions:

$$\left(\frac{t'}{t}\right)\left(\frac{t}{t'}\right) = 1 \quad \Rightarrow \quad \alpha_1^2 \left(1 - \frac{v^2}{c^2}\right) = 1 \quad \Rightarrow \quad \boxed{\alpha_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Now from (5),

$$t' = \alpha_1 \left(t - \frac{v}{c} t\right) \Rightarrow t' = \alpha_1 \left(t - \frac{v(ct)}{c^2}\right)$$

This is for S frame. As we mentioned earlier, in S frame, $x = ct$.
Now put x value in above t' equation:

$$\Rightarrow t' = \alpha_1 \left(t - \frac{vx}{c^2}\right)$$

We reframe α_1 as γ and $\frac{v}{c} = \beta$
Now we've got our Lorentz transformations

$$\boxed{x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{\beta x}{c}\right)}$$

$$\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \beta = \frac{v}{c}$$

Here comes a small doubt that if Einstein did this , Why this transformations known as Lorentz transformations not like "Einstein transformations" ?... Because, Lorentz did this before Einstein without using SR postulates.Let dive into his idealogy.

3 Derivation of Lorentz transformation using Maxwell's equations

Lorentz wanted to preserve the aether hypothesis while explaining the null result of Michelson-Morley experiment. He tried to find the transformations which relate the coordinates of an object moving through the aether to those in the aether frame. This whole mess created due to implications of Maxwell's equations. So he starts with Maxwell's equations and define them for $\Phi(x, t)$ (for S frame) and $\Phi(x', t')$ (for S' frame).

For $\Phi(x, t)$;

$$\frac{\partial^2 \vec{\Phi}}{\partial x^2} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\Phi}}{\partial t^2} = 0 \text{ [For S frame]} \text{———————————} (1)$$

For $\Phi(x', t')$;

$$\frac{\partial^2 \vec{\Phi}}{\partial x'^2} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{\Phi}}{\partial t'^2} = 0 \text{ [For S' frame]} \text{———————————} (2)$$

Let us take Galilean transformations

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

These are consistent with Newtonian mechanics, but not with electromagnetism. Because Galilean transformations imply that the speed of light should vary for moving observers:

$$c' = c \pm v \text{ , where } v \text{ is the velocity of the observer.}$$

But, Maxwell's equations implicitly imply that the speed of light is constant in all inertial frames or independent of the observer's motion. So, Lorentz tried to find new transformations that leave the wave equation of light. He modified the Galilean transformation as follows:

$$x' = \gamma(x - Vt)$$

$$t' = \beta_1 t + \beta_2 x$$

You may have two doubts here: 1) Why did Lorentz assume Linear Transformations? 2) Why does he include the "x" term in "t'" which denies Newton's absolute time concept. The answer for the first one is "for conserving uniformity in motion (our preliminary assumption while dealing with S and S' frames) and the fundamental symmetries of space and time in inertial frames. And for the second one, actually Lorentz doesn't deny absoluteness of time but rather introduces "local time" as a mathematical adjustment to preserve the form of Maxwell's equations in moving frames, while still believing that time is universal and absolute. He includes the "x" term for light to appear to move at speed c in all directions in all inertial frames. So, the transformation of time must involve the position term. Remember this is just a mathematical "local time" as per Lorentz.

We have $x' = x'(x, t)$, $t' = t'(x, t)$

So we can connect $\Phi(x, t)$ and $\Phi(x', t')$ through the above transformation:

$$\frac{\partial}{\partial x}\Phi = \left[\frac{\partial x'}{\partial x}\right]\frac{\partial}{\partial x'}\Phi + \left[\frac{\partial t'}{\partial x}\right]\frac{\partial}{\partial t'}\Phi$$

$$\frac{\partial}{\partial t}\Phi = \left[\frac{\partial x'}{\partial t}\right]\frac{\partial}{\partial x'}\Phi + \left[\frac{\partial t'}{\partial t}\right]\frac{\partial}{\partial t'}\Phi$$

Given the transformations:

$$x' = \gamma x - \gamma V t \quad \Rightarrow \quad \frac{\partial x'}{\partial x} = \gamma, \quad \frac{\partial x'}{\partial t} = -\gamma V$$

$$t' = \beta_1 t + \beta_2 x \quad \Rightarrow \quad \frac{\partial t'}{\partial x} = \beta_2, \quad \frac{\partial t'}{\partial t} = \beta_1$$

$$\frac{\partial}{\partial x}(\phi) = \gamma\frac{\partial}{\partial x'}(\phi) + \beta_2\frac{\partial}{\partial t'}(\phi)$$
$$\frac{\partial}{\partial t}(\phi) = -\gamma V\frac{\partial}{\partial x'}(\phi) + \beta_1\frac{\partial}{\partial t'}(\phi)$$

Think $\frac{\partial}{\partial x} = D_1$, $\frac{\partial}{\partial t} = D_2$ as operators

$$\frac{\partial^2}{\partial x^2} = D_1^2, \quad \frac{\partial^2}{\partial t^2} = D_2^2$$

*** Remember the flow of finding complementary equation in solving of second order differential equation** $\left[(D^2 + D + 1)y = 0, \mathrm{etc}\right]$

$$\frac{\partial^2}{\partial x^2} = D_1^2 = \left(\gamma\frac{\partial}{\partial x'} + \beta_2\frac{\partial}{\partial t'}\right)^2$$
$$= \gamma^2\frac{\partial^2}{\partial x'^2} + \beta_2^2\frac{\partial^2}{\partial t'^2} + 2\gamma\beta_2\frac{\partial^2}{\partial x'\partial t'}$$

*
$$\frac{\partial^2}{\partial x^2}(\phi) = \gamma^2\frac{\partial^2}{\partial x'^2}(\phi) + \beta_2^2\frac{\partial^2}{\partial t'^2}(\phi) + 2\gamma\beta_2\frac{\partial^2}{\partial x'\partial t'}(\phi) ---------- (3)$$

$$\frac{\partial^2}{\partial t^2} = D_2^2 = \left(-\gamma V\frac{\partial}{\partial x'} + \beta_1\frac{\partial}{\partial t'}\right)^2$$
$$= \gamma^2 V^2\frac{\partial^2}{\partial x'^2} + \beta_1^2\frac{\partial^2}{\partial t'^2} - 2\gamma V\beta_1\frac{\partial^2}{\partial x'\partial t'}$$

*
$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\phi) = \frac{\gamma^2 V^2}{c^2}\frac{\partial^2}{\partial x'^2}(\phi) + \frac{\beta_1^2}{c^2}\frac{\partial^2}{\partial t'^2}(\phi) - \frac{2\gamma V\beta_1}{c^2}\frac{\partial^2}{\partial x'\partial t'}(\phi) ---------- (4)$$

From (1) and (2),

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2}$$

Substitute (3) and (4) in the above obtained relation:

$$
\begin{aligned}
&\left(\gamma^2 \frac{\partial^2}{\partial x'^2}(\phi) + \beta_2^2 \frac{\partial^2}{\partial t'^2}(\phi) + 2\gamma\beta_2 \frac{\partial^2}{\partial x' \partial t'}(\phi)\right) \\
&\quad - \frac{1}{c^2}\left(\gamma^2 V^2 \frac{\partial^2}{\partial x'^2}(\phi) + \beta_1^2 \frac{\partial^2}{\partial t'^2}(\phi) - 2\gamma V \beta_1 \frac{\partial^2}{\partial x' \partial t'}(\phi)\right) \\
&= \frac{\partial^2}{\partial x'^2}(\phi) - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}(\phi)
\end{aligned}
$$

Group the terms:

$$
\begin{aligned}
&\left(\gamma^2 - \frac{\gamma^2 V^2}{c^2}\right)\frac{\partial^2 \phi}{\partial x'^2} + \left(\beta_2^2 - \frac{\beta_1^2}{c^2}\right)\frac{\partial^2 \phi}{\partial t'^2} \quad + \left(2\gamma\beta_2 + \frac{2\gamma V \beta_1}{c^2}\right)\frac{\partial^2 \phi}{\partial x' \partial t'} \\
&= \frac{\partial^2}{\partial x'^2}(\phi) - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}(\phi)
\end{aligned}
$$

By comparing terms:

$$
\gamma^2\left(1 - \frac{V^2}{c^2}\right) = 1 \quad \Rightarrow \quad \gamma^2 = \frac{1}{1 - \frac{V^2}{c^2}} \quad \Rightarrow \quad \boxed{\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}}
$$

Also,

$$
\beta_2^2 - \frac{\beta_1^2}{c^2} = -\frac{1}{c^2} \tag{5}
$$

$$
2\gamma\beta_2 + 2\gamma\frac{V\beta_1}{c^2} = 0 \tag{6}
$$

From (6),

$$
\beta_2 + \frac{V\beta_1}{c^2} = 0 \Rightarrow \beta_2 = -\frac{V\beta_1}{c^2}
$$

Substitute β_2 in equation (5):

$$
\begin{aligned}
\frac{V^2\beta_1^2}{c^4} - \frac{\beta_1^2}{c^2} &= -\frac{1}{c^2} \\
\beta_1^2\left(\frac{V^2}{c^2} - 1\right) &= -1 \\
\beta_1^2\left[\left(1 - \frac{V^2}{c^2}\right)\right] &= 1 \\
\Rightarrow \beta_1^2 = \frac{1}{1 - \frac{V^2}{c^2}} &= \gamma^2 \quad \Rightarrow \quad \boxed{\beta_1 = \gamma}
\end{aligned}
$$

We can rewrite β_2 as:

$$
\Rightarrow \quad \boxed{\beta_2 = -\frac{V\gamma}{c^2}}
$$

Rewriting Galilean Transformations as per Lorentz

$$
\begin{aligned}
x' &= \gamma(x - Vt) \\
t' &= \gamma\left(t - \frac{V}{c^2}x\right) \\
\Rightarrow t' &= \gamma\left[t - \frac{v}{c^2}x\right] \\
\text{Consider } \frac{v}{c} = \beta &\Rightarrow \frac{v}{c^2} = \frac{\beta}{c} \\
\Rightarrow t' &= \gamma\left[t - \beta\frac{x}{c}\right]
\end{aligned}
$$

Now we've got our Lorentz transformations

$$
\boxed{x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{\beta x}{c}\right)}
$$

$$
\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \beta = \frac{v}{c}
$$

But here comes the big question: Why didn't Lorentz reach the Special Relativity (SR) postulates even after having his own transformations, which can redefine thoughts on time and space? **Why???** Because

Lorentz believed in aether's existence. So, he tried to explain Michelson-Morley's *"null result"* without rejecting the aether concept. So, he came up with these results and said:

> "The interferometer's arm aligned along Earth's motion contracts along the direction of motion relative to the ether just enough to cancel out the expected time difference between the two arms."

But Einstein rejected the whole ether theory, derived the Lorentz transformations with his SR postulates independently and realised consquences of Lorentz transformations were not apparent effects caused by motion relative to the aether, but by the relative motion between arbitrary observers.

Simply, Lorentz messed up with " his blind belief in aether's existence."

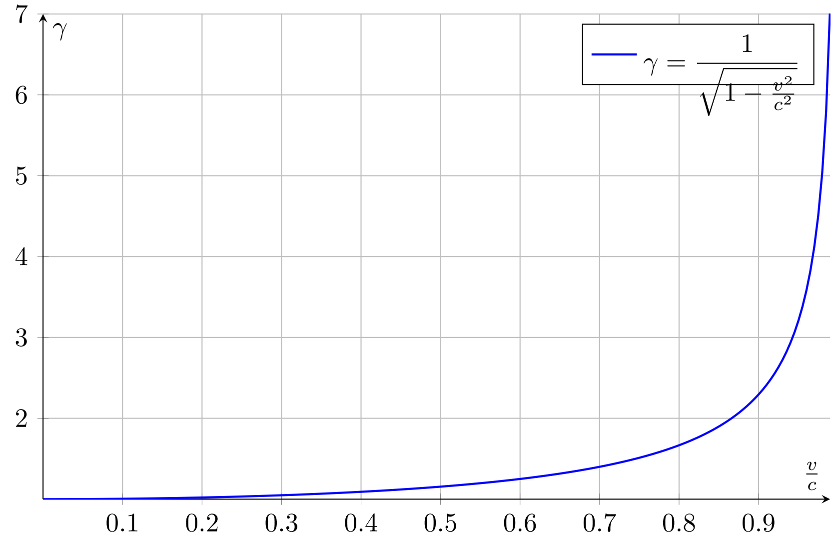
4 Are Galilean transformations completely wrong?

We know Galilean transformations failed to define how space and time coordinates transform between inertial frames but still in high school we were taught these Galilean transformations. Why??? Because Galilean transformations may not be the full picture but also not a fully wrong picture. They are valid up to a certain limit of speed. For the limit of low speeds (i.e., $v \lll c$)—the classical (Newtonian) limit—the ratios $\frac{v^2}{c^2}$ and $\frac{v}{c^2}$ become very small (almost negligible). So, they tend to zero. Then, γ equals one:

$$
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 1
$$

Lorentz vs Galilean Transformations

1. Lorentz Factor Graph



As $\frac{v}{c} \to 0$, the Lorentz factor $\gamma \to 1$, which means relativistic effects vanish. Hence, Lorentz transformations reduce to Galilean transformations in the classical limit.

So, the Lorentz transformations reduce to the Galilean transformations.

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \quad \text{with} \quad \begin{cases} \gamma \approx 1 \\ \dfrac{v}{c^2} \approx 0 \end{cases} \quad \Rightarrow \quad \begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}$$

5 Conclusion

The development of Lorentz transformations marks a fundamental shift in our understanding of space and time. While Lorentz actually formulated them to preserve Maxwell's equations in the presence of a hypothetical aether, his belief in the aether concept prevented him from fully grasping their deeper implications(i.e relativistic effects). Here, Einstein makes a smart move and rejected the aether entirely and derived the same transformations from two simple but powerful postulates: the principle of relativity and the constancy of the speed of light. At first, we thought time and space are absolute but this re-interpretation revolutionized physics by revealing that space and time are not absolute but interconnected aspects . Lorentz transformations naturally reduce to Galilean transformations at low velocities $v \lll c$), preserving classical mechanics in its classical domain, while accurately predicting relativistic effects at high velocities.

Thus, Lorentz transformations open the door to the universe where motion, time, and space become relative.

References

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