Optical effects analyzed by the slip-step Fourier method for femtosecond pulses in single-mode fibers

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Abstract: Optical fibers have become important in terms of material processing and data transmission. In the scope of optical communication systems propagation of ultra-short laser pulses considering various nonlinear and linear effects is already studied. To link to the current research, the goal of this paper is to implement the widely-used split-step Fourier method (SSFM) with which pulse propagation within optical fibers can be simulated. Besides dispersion, nonlinear effects like the self-phase modulation (SPM) are considered. I also discuss nonlinear pulse compression. In this context the fundamentals of dispersion and nonlinearity in optical fibers are introduced. Finally, I benchmarking the computations of ultra-short pulses in single-mode fibers.

1. Introduction

The propagation of ultra-short pulses in single-mode fibers is typically modelled with the generalized nonlinear Schrödinger equation (NLSE) [1], [2] by

$$\frac{\partial A}{\partial z} = -\frac{\alpha}{2}A + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} - i\gamma\left(|A|^2A - \frac{i}{\omega_0}\cdot\frac{\partial}{\partial T}(|A|^2A) - T_R A\frac{\partial}{\partial T}|A|^2\right) \qquad 1.$$

The nonlinear Schrödinger equation can be derived from the wave equation. However, I can choose between two (Fourier transform) notations. The derivation with the negative one can be found in Ursula Keller's book [1]. I used this, because the scipy library (a python library) uses the negative convention for the discrete Fourier transformation (DFT). In equation (1), A(z; T) denotes the slowly varying envelope of the field and z is the propagation distance. Equation (1) is valid in the frame of reference of the pulse traveling with field group velocity v_g. Using the relation $T = t - z/v_g$ between the present time t and the retarded time T, equation (1) can be transformed into the laboratory frame of reference. The parameter α models linear loss, β_3 and $\beta_2$are the coefficients of third- and second-order dispersion respectively and γ is the nonlinear parameter. The parameters i/ω_0 and T_R govern, respectively, the effects of SS and stimulated Raman scattering (SRS). Since analytic solutions of equation (1) are generally not possible, numerical techniques are required for modeling the pulse propagation. The dominant approach consists of split-step schemes that perform spatial sub-steps considering only the linearities on the left-hand side of equation (1) by DFT and sub-steps approximating only the influence of the nonlinear right-hand side terms (in the brackets) in an alternating fashion. When pulse widths are well in the pico-second regime, i/ω_0= 0 and $T_R = 0$ can be used and all nonlinear derivatives vanish [8]. For this specific regime, the construction of such SSFM is very well established [5], [3]. In this paper I used a program (developed in python) in order to solve the nonlinear Schrödinger equations (NLSE) in the femtosecond regime [4]. This software is originally coming from Ole Krarup [6], [7].

2. Fiber characterization

2.1 Fiber losses

An important fiber parameter provides a measure of power loss during transmission of optical signals. The attenuation constant α is a measure of total fiber losses from all sources. It is customary to express α in units of dB/km using the relation [5].

$$\alpha_{dB} = 4.343\alpha \qquad 2.$$

2.2 Chromatic dispersion

Fiber dispersion plays a critical role in propagation of optical pulses because different spectral components associated with the pulse travel at different speeds given by $c/n(\omega)$. Mathematically speaking, the effects of fiber dispersion are accounted for by expanding the mode-propagation constant β in a Taylor series around the frequency ω_0. The parameters β_2 and β_1 are related to the refractive index $n(\omega)$ and its derivatives. This is where the group index n_g and the group velocity v_g comes in. Physically speaking, the envelope of an optical pulse moves at the group velocity v_g, while the parameter β_2 represents dispersion of the group velocity and is responsible for pulse broadening. This phenomenon is known as the group-velocity dispersion (GVD), and β_2 is the GVD parameter. Nonlinear effects in fibers can manifest different behaviors depending on the sign of the parameter GVD. $\beta_2 > 0$ is normal dispersion with red light pulling ahead, causing a negative leading chirp. $\beta_2 < 0$ is anomalous dispersion with blue light pulling ahead, causing a positive leading chirp. A chirp is a variation of the instantaneous frequency inside the pulse. The coefficient β_3 appearing in that term is called the third-order dispersion (TOD) parameter. For ultra-short pulses (with $\tau < 1$ ps), it is necessary to include the β_3 term even when β_2 is not equal to zero because the expansion parameter $\Delta\,\omega/\omega_0$ is no longer small enough to justify the truncation of the Taylor expansion after the $\beta_2$term parameter.

2.3 Nonlinearity

In addition to dispersion there are various other effects acting on a pulse while propagating through the fiber. Particularly important are the nonlinear optical effects. According to R. W. Boyd nonlinear optics can be described as the study of phenomena that are caused by the modification of optical properties of the underlying material by the presence of sufficiently intense light [8]. The $\mathrm{n(I)} = \mathrm{n}_0 + \mathrm{n}_2\mathrm{I}$ dependence of the refractive index on the intensity is called the nonlinear Kerr effect. The Kerr effect modifies the phase of a pulse, where n_0 is the refractive index, n_2 is the nonlinear index coefficient. The frequency shift in a pulse due to the Kerr effect depends on the slope of its intensity. This effect, in which a pulse alters its own frequency and phase by altering the n refractive index locally through the Kerr effect, is called self-phase modulation (SPM). When the pulse travels through the medium with a positive n_2, it will introduce a redshift in the front of the pulse and a blueshift in the back of the pulse. Basically, it creates a chirp in the pulse. Besides SPM, cross-phase modulation (CPM) and four-wave-mixing (FWM) are also a direct consequence of the optical Kerr-effect, which causes a nonlinear refractive index. However, in this paper I am not going to consider these terms. The right hand side of equation (1) (in the brackets) describes the phase evolution due to nonlinearity (SPM + SS + SRS). When ultra-short pulses of light propagate in a nonlinear medium, they may experience SS, meaning a distortion of the temporal pulse shape such that the trailing slope gets increasingly steep. The phenomenon can be interpreted as a reduction in group velocity $\mathrm{v_g}$ proportional to the optical intensity [9]. SRS for ultra-short optical pulse, effectively shifting the spectral envelope of the pulse towards longer wavelengths. In optical fibers for intense pulses, SRS can be detrimental: it can transfer much of the pulse energy into a wavelength range where laser amplification does not occur. That effect can limit the peak power achievable with such devices [10]. One can observe that the nonlinear parameter is then defined by

$$\gamma = \frac{\omega_0 n_0}{c A_{eff}} \qquad 3.$$

where c is the speed of light in vacuum and A_{eff} is the effective mode area of the fiber. For Gaussian shaped modes the effective area can alternatively be calculated with $A_{eff} = (\pi/4\,)w_0^2$ whereby $\mathbf{w_0}$ is the beam waist of the Gaussian shaped pulse in the transverse direction.

2.4 Some useful parameters

Depending on the initial pulse width τ_0 and its peak power P_0 (or P_{peak}), either the linear or the nonlinear effects may dominate along the fiber [5]. In order to get a better feeling, which particular effect is more dominant, two corresponding scales are introduced. The first is called dispersion length L_D and is defined

$$L_D = \frac{\tau_0^2}{|\beta_2|} \qquad 4.$$

whereby τ_0 is the FWHM of the initial pulse, and β_2 is the GVD. On the other side the nonlinear length L_{NL} is defined as

$$L_{NL} = \frac{1}{\gamma P_0} \qquad 5.$$

Depending on the magnitude of these scales, overall four different regimes can be distinguished:

- $L \ll L_D$ and $L \ll L_{NL}$ → neither linear nor nonlinear effects can be observed
- $L \approx L_D$ and $L \ll L_{NL}$ → dispersion dominates
- $L \ll L_{NL}$ and $L \ll L_{LD}$ → nonlinear effects dominate
- $L \geq L_D$ and $L \geq L_{NL}$ → linear and nonlinear effects act together along pulse propagation

2.5 Fiber config

Table 1. Fiber config

Description	Value
Fiber length [m]	$L = 5 \cdot 10^{-2}$
Number of steps we divide the fiber into	$N_{fiber} = 2^{10}$
Effective mode diameter [m] [11]	$d_{eff} = 5 \cdot 10^{-6}$
Effective mode area $[m^2]$	$A_{eff} = \pi d_{eff}^2/4$
Nonlinear refractive coefficient $[m^2/W]$ [2]	$n_2 = 2.7 \cdot 10^{-20}$
Nonlinear parameter [1/Wm]	$\gamma = 2\pi n_2/\lambda_0\, A_{eff}$
GVD $[fs^2/mm]$ [13]	$\beta_2 = 36.16$
TOD $[fs^3/mm]$ [13]	$\beta_3 = 27.47$
Power attenuation coefficient [dB/km]	$\alpha_{dB} = 0.2$

In table (1) the GVD, TOD and n_2 parameters are @ 800 nm, the α_{dB} is @ 1550 nm, and the d_{eff} is @ 850 nm.

3. Simulation parameters

The implementation contains a calculation of pulse energy. This is performed alongside each propagation step. Therefore one can check whether the pulse energy is conserved, which can be used for validation purposes of the implementation. If the energy is no longer conserved, the step size is too big, such that the approximation about the constant integrand no longer holds. As a consequence, the step size dt needs to be lowered. As long as not explicitly stated differently, the initial pulse is Gaussian shaped. The central frequency is set to 0.375 PHz, and the number of discretization points N in time equals 2^{15}. The temporal window has to be large enough to resolve the pulse with its leading and trailing flank accordingly. Otherwise, important information is lost. Multiplication of the envelope with the term $e^{i2\pi f c t}$ generates an oscillating electric field and shifts the power spectrum to the central frequency, respectively. In advance it should be noted that this factor is neglected, which means only the envelope of the pulse is considered. As an advantage the temporal discretization can be coarsened, and thus N can be decreased. This speeds up the simulation, which is very important. On the negative side one has to be aware that the power spectrum needs to be shifted manually. Therefore relative and absolute frequencies have to be defined. The original software [7] (in the picosecond regime) defines a lot of parameters, for example the

- the time step
- the symmetric time array
- the (relative) frequency array
- the central frequency
- the absolute frequency

and we continue to count on that without any modifications. The parameters were coordinated in such a way that the energy calculated during each step did not disappear

Table 2. Simulation config

Description	Value
Pulse central wavelength [m] [14]	$\lambda_0 = 800 \cdot 10^{-9}$
Pulse central frequency [Hz]	$f_c = c/\lambda_0$
Pulse duration in FWHM [s]	$\tau_0 = 20 \cdot 10^{-15}$
Pulse repetition frequency [Hz] [14]	$f_{rep} = 85 \cdot 10^6$
Pulse average power [W] [14]	$P_{avg} = 600 \cdot 10^{-3}$
Pulse energy [J]	$E = P_{avg}/f_{rep}$
Pulse peak power [W]	$P_{peak} = E/\tau_0$
Amplitude [$\sqrt{W}$]	$A = \sqrt{P_{peak}}$
Number of points	$N = 2^{15}$
Time resolution [s]	$dt = 10^{-16}$

4. Simulation results

In this chapter we present the simulation results. According to the initial parameters, the dispersion length will be $L_D \approx 0.0110$ m, the nonlinear length will be $L_{NL} \approx 0.000262$ m, and the fiber length is $L = 0.05$ m. Since $L \geq L_D$ and $L \geq L_{NL} \rightarrow$ linear and nonlinear effects act together along pulse propagation. In figure (1) one can see the initial pulse's power and amplitude envelope. The amplitude's FWHM is broader than the power's FWHM. In figure (2) one can see the initial pulse's Fourier transformed spectrum. We can see that the (analytical) theory and the numerical calculations are the same. In figure (3) one can see the initial pulse's power envelope. In this simulation the SS and SRS effects are not considered. According to the theory about dispersion, a pulse broadening can be observed. Since the energy of the pulse needs to be conserved, the pulse peak power decreases simultaneously. In figure (4) and (5) one can see the initial and final pulse's spectrum. According to the theory about nonlinearity, a spectrum broadening can be observed due to SPM. Since the energy of the spectrum needs to be conserved, the peak of the spectrum decreases simultaneously. It can also be seen that the maximum of the spectrum elongates towards higher frequencies and lower wavelengths respectively. the simulation gives you the option to turn each effect on and off. This is due to the TOD. This type of elongation cannot be observed without the effect of TOD. In figure (6) and (7) one can see the distance-time and distance-spectrum evolution respectively. We see that the pulse broadens in the time domain (due to dispersion) and correspondingly broadens in the frequency domain (due to SPM). However the broadening in the frequency domain will not continue forever. It will eventually stops around 0.01 m. After that it will not change. The broadening in the time domain continues even after 0.01 m. In figure (8) and (9) one can see the initial and final pulse's spectrogram. The goal of a spectrogram is to represent the variation of the intensity of a signal in time and frequency, simultaneously [15], [16]. This is done by calculating the signal spectrum at consequent time gates. Spectrograms were also used to simulate nonlinear propagation in optical fibers [17], which exhibits complicated reshaping of initial electric field, as well. Mathematically, the spectrogram of a function, S(t) is [16]:

$$P(\tau,\omega) = \frac{1}{2\pi}\left|\int_{-\infty}^{\infty} e^{-i\omega t} S(t)h(t-\tau)dt\right|^2 \qquad 6.$$

where $P(\tau,\omega)$ is the spectrogram, $h(t-\tau)$ is the gate function at time delay τ and ω is the angular frequency. The gate function is a Gaussian with 20 fs FWHM. Figure (9) represents a negative chirped pulse with second order spectral phase coefficient of $\mathrm{GVD} \cdot \mathrm{L} = 1808\, fs^2$ and with third order spectral phase coefficient of $\mathrm{TOD} \cdot \mathrm{L} = 1373.5\, fs^3$ and duration of τ_{final}. In figure (10) one can see the initial and dispersion-compensated Fourier-limited final pulse. This pulse's FWHM is around 4.83 fs. This figure was created as follows:

1. Taking the absolute squared value of the final spectrum.
2. Taking the inverse Fourier transform of the given spectrum.
3. Taking the absolute squared value of the given pulse.
4. Taking the absolute squared value of the initial pulse and comparing it.

Since the spectrum is broadened by SPM, the final pulse will be shorter after dispersion compensation. In this simulation the SS and SRS are not considered, because they are not implemented yet. The latest version of Ole Krarup's code supports modelling the impact of the Raman effect. Since our pulse is below 1ps and has a high peak power, this should definitely be included in our simulation. Actually, Ole Krarup just double checked the impact of Raman (assuming a silica waveguide) given the simulation parameters I specified and it seems to have a small impact. The reason is that the pulse duration is initially quite small compared to the duration of the Raman response function. Furthermore, β_2 is positive, so the evolution is dominated by optical wave breaking, which causes the pulse to quickly flatten out. This reduces the peak power and makes Raman less noticeable. In figure (11) one can see the convergence of the solution. We can see that there are two different regions: the first region is a plateau, while the second region is monotonically decreasing. This figure was created as follows:

1. I declared an array with values for the N_{fiber} parameter.
2. Within a for loop:
    a. I went through the elements of the previously created array.
    b. I instantiated the Fiber class using the already known parameters and the i-th element of the array.

c. I ran the SSFM on the previously declared fiber config.
d. I defined the i-th last pulse of the solution.
e. I calculated the relative error of the pulses based on it's energy values, where the reference pulse was with $N_{fiber} = 1024$.
f. And finally, I stored the relative error values into an array.

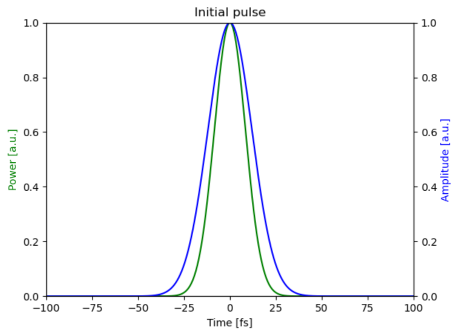

Fig. 1. Initial pulse

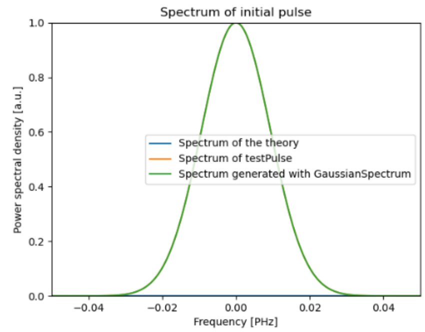

Fig. 2. Spectrum of the initial pulse

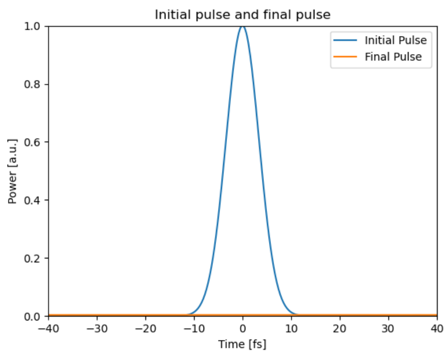

Fig. 3. Initial and final pulse

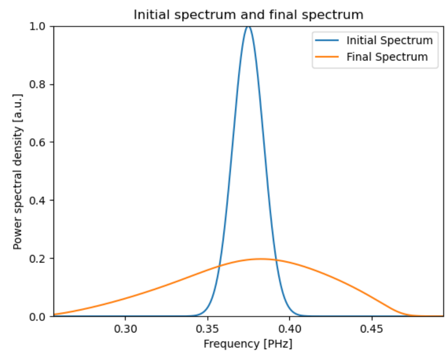

Fig. 4. Initial and final spectrum

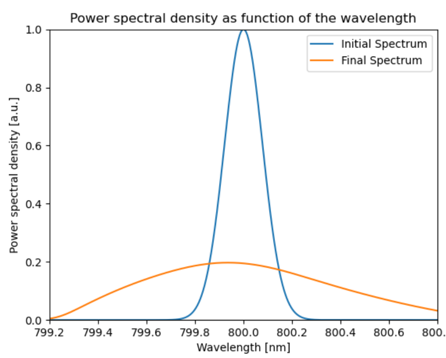

Fig. 5. Power spectral density in function of wavelength

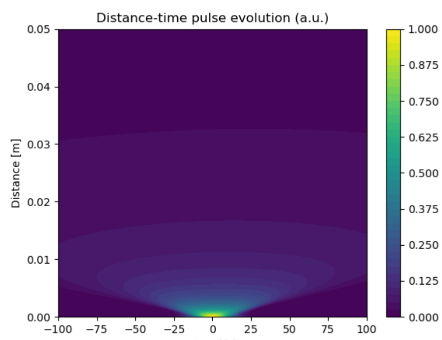

Fig. 6. Distance-time pulse evolution

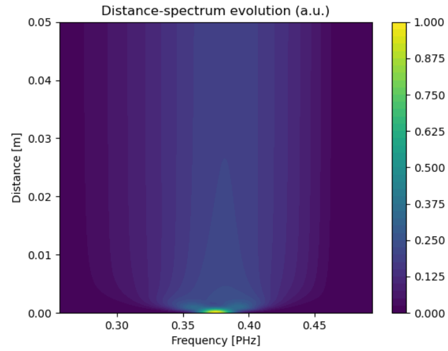

Fig. 7. Distance-spectrum evolution

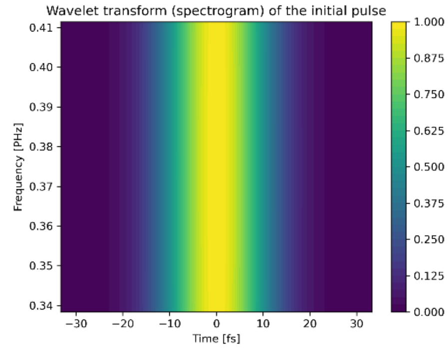

Fig. 8. Spectrogram of the initial pulse

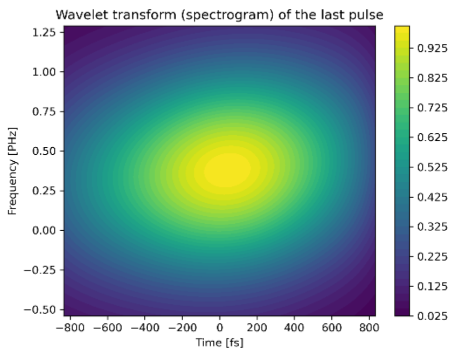

Fig. 9. Spectrogram of the last pulse

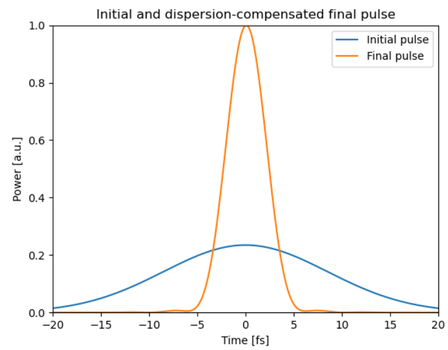

Fig. 10. Initial and dispersion-compensated final pulse

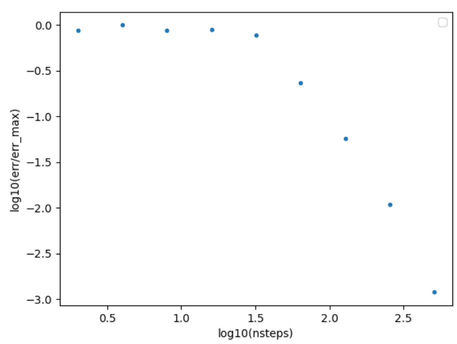

Fig. 11. Convergence of the solution

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