The magneto scalar field and its role

Balázs Tari

Szeged

2023.

Table of contents

1. Abstract

In the early 2000s, there was a minor flare-up in the history of physics due to the article by C. Monstein and J.P. Wesley (Monstein & Wesley, 2002), when they claimed to have experimentally supported the existence of longitudinal electromagnetic waves. Around this time, the first alternative electrodynamics was born with the help of quaternions (van Vlaenderen & Waser, 2001). Later, following K. Rębilas (Rębilas, 2008), it turned out that the experimental results can be explained within the framework of classical electrodynamics. The principle of conservation of charge was supported by many experiments. The experiments were characterized by a long observation time, which means that the processes could have been averaged. The question may arise, if we repeat the measurements on the same system at sufficiently short intervals, will we experience fluctuations in the total amount of charge in a closed system? For some condensed matter systems, admitting discontinuities in the probability current is the cause of the violation of local conservation. This leads us to ask a logical theoretical question: how should one compute the electromagnetic field of a microscopic current that is not locally conserved? In such cases, the Aharonov-Bohm theory offers a logically consistent approach.

The laws of the theory of electrodynamics were formulated in quaternion form. Formalism shows that Maxwell's equations can be derived from only a single quaternion equation. I distinguish between modified and unmodified Maxwell equations according to whether or not the Lorenz gauge is applied. I present the shape of the quaternion as the continuity equation, generalizing the "traditional" continuity equation with three equations. A magneto scalar field set in equations that propagates at the speed of light. I lay the ground pillars of the static magneto scalar field. I also deal with magneto scalar waves, the energy conservation, the constitutive relations in vacuum, the invariance to the Lorentz transformation, the Lagrangian density and some other topics.

Keywords: quaternion, electrodynamics, magneto scalar potential

2. Introduction

Classical electrodynamics is the cornerstone of modern physics. Classical electrodynamics provides the basis for many physical models. Michael Faraday wrote and published a series of thirty articles in the Philosophical Transactions of the Royal Society between 1832 and 1856 under the following titles: Experimental Researches in Electricity (Al-Khalili, 2015). James Clerk Maxwell (Maxwell, 1865) Based on Faraday's empirical results, he formulated classical electrodynamics in the form of 20 partial differential equations. Oliver Heaviside (Heaviside, 2007) using vector calculus, and also the introduction of the $\vec{A}$ vector potential and the φ scalar potential rewrote Maxwell's equations. Ludvig Lorenz (Lorenz, 1867) realized that the wave equations of $\vec{A}$ and φ they can be produced with an additional constraint, namely through the Lorenz gauge named after him The gauge transformation is: $\varphi \to \varphi - \frac{\partial \chi}{\partial t}$ és $\vec{A} \to \vec{A} + \nabla\chi$, where χ is a gauge function. By definition $\vec{E}$ electric field strength vector and the $\vec{B}$ magnetic induction are invariant under the gauge transformation. Nowadays, classical electrodynamics is considered a complete and closed theory, no need for reinterpretation. However, I have found some experiments that are difficult or impossible to explain with classical electrodynamics. In (here), I list these experiments and conclusions. Falsifiability states that an empirical theory cannot be verified but can be disproved by contrary test results (Popper, 1972). I distinguish between modified and unmodified Maxwell's equations according to whether I use the Lorenz gauge or not. The modified Maxwell equations predict the existence of the longitudinal electric field strength vector and the magneto scalar potential, as well as the local violation of charge conservation. In this study, I will delve into the modified Maxwell equations.

3. The initial point

Any physical theory that does not describe the desired physical effect needs to be modified. However, there are supposed errors in a theory that turn out to be correct after all. However, there are effects that are difficult or not at all to explain with classical electrodynamics. I deal with these topics in this chapter.

3.1. The supposed overdetermination of Maxwell's equation

Maxwell's equations appear to be overdetermined, containing six unknowns and eight equations. Maxwell's divergence equations are generally thought to be redundant, and both equations are taken as initial conditions for the rotational equations. For this reason, the two divergence equations are not usually solved in electromagnetic simulations. There is a circular fallacy in this explanation, and the two non-redundant but fundamental divergence equations cannot be ignored in electromagnetic simulations (Changli, 2015).

3.2. C. Monstein's and J. P. Wesley's experiment

In the early 2000s, there was a minor flare-up in the history of physics, under the influence of C. Monstein and J.P. Wesley (Monstein & Wesley, 2002) claimed to have experimentally supported the existence of longitudinal electromagnetic waves. Later then K. Rębilas (Rębilas, 2008) it turned out that the experimental results can be explained within the framework of classical electrodynamics. In the article by C. Monstein and J.P. Wesley, a curl-free vector potential is a requirement. However, in the experiments of C. Monstein and J.P. Wesley, they did not experimentally confirm the local violation of the charge conservation. The question may arise in us, whether we can change the nature of the electromagnetic interaction just by properly designed experimental conditions?

3.3. Lee M. Hively's and Andrew S. Loebl's experiment

Lee M. Hively and Andrew S. Loebl claimed experimental support for the existence of longitudinal electromagnetic waves in the macroscopic range (Hively & Loebl, Classical and extended electrodynamics, 2019). According to their published article, the curl-free vector potential is a requirement and their experimental results are explained with the modified Maxwell equations. However, in their experiment, they did not experimentally confirm the local violation of the charge conservation. The question may arise in us, whether we can change the nature of the electromagnetic interaction just by properly designed experimental conditions?

3.4. Quantum mechanical aspect

The principle of conservation of charge was supported by many experiments. The experiments were characterized by a long observation time, which means that the processes could have been averaged. The question may arise, if we repeat the measurements on the same system at sufficiently short intervals, will we experience fluctuations in the total amount of charge in a closed system? One of the consequences of the modified Maxwell equations is the local violation of charge conservation. It says that charges can transform into electromagnetic-scalar waves and vice versa.

Lev Borisovich Okun in his article (Okun, 1989) processed a total of 30 articles and came to the conclusion that this would violate the Pauli exclusion principle.

Later I deal with this topic

3.5. The initial point

In the literature of longitudinal electromagnetic interaction, Stueckelberg's work is often cited as a starting point (Stueckelberg, 1938). The following Lagrangian density is associated to him:

$$L = -\frac{\varepsilon_0 \cdot c^2}{4} \cdot F_{\mu\upsilon} \cdot F^{\mu\upsilon} + J^\mu \cdot A_\mu$$
$$-\frac{\gamma \cdot \varepsilon_0 \cdot c^2}{2} \cdot \left(\partial_\mu A^\mu\right)^2 - \frac{\gamma \cdot \varepsilon_0 \cdot c^2 \cdot k^2}{2} \cdot \left(A_\mu \cdot A^\mu\right) \tag{1}$$

For the full Lagrangian density, the parameter γ should be chosen as 1 and the parameter k as 0, where k is the Compton wavenumber of photons of mass m (Reed & Hively, 2020). The reality is that Stueckelberg's name is not associated with this Lagrangian density.
In field theory, the Stueckelberg action describes a massive spin-1 field as an R (the real numbers are the Lie algebra of U(1)). Yang–Mills theory coupled to a real scalar fieldφ. This scalar field takes on values in a real 1D affine representation of R with m as the coupling strength.

$$L = -\frac{1}{4} \cdot F_{\mu\upsilon} \cdot F^{\mu\upsilon} + \frac{1}{2} \cdot (\partial^\mu \varphi + m \cdot A^\mu) \cdot \left(\partial_\mu \varphi + m \cdot A_\mu\right) \tag{2}$$

This is a special case of the Higgs mechanism, where practically λ and thus the mass of the Higgs scalar excitation were taken to be infinite, so the Higgs is decoupled and can be ignored, which results in a non-linear, affine representation of the field, instead of a linear representation - in today's terminology, this a U(1) nonlinear σ-model. By choosing the measure φ=0, we get the Proca effect (Stueckelberg action, 2022).

Originally this Larangian is came from the Aharonov-Bohm electrodynamics (Aharonov & Bohm, 1963).

4. Quaternion mathematics – major relationships

Many literatures deal with the application of quaternions. I will highlight one of them and will use their marking system in the future (Hong & Kim, 2019). Let's denote quaternions as follows:

$$\tilde{A} = \left(A_0, A_1 \cdot \underline{i} + A_2 \cdot \underline{j} + A_3 \cdot \underline{k}\right) \tag{3}$$

, where A_0, A_1, A_2, A_3 are real numbers and $\underline{i}$, $\underline{j}$, $\underline{k}$ are the quaternion unit vectors for which the following relations hold:

$$\underline{i}^2 = \underline{j}^2 = \underline{k}^2 = -1 \tag{4}$$

$$\underline{i} \cdot \underline{j} = \underline{k} \;\&\; \underline{k} \cdot \underline{i} = \underline{j} \;\&\; \underline{j} \cdot \underline{k} = \underline{i} \;\&\; \underline{i} \cdot \underline{j} \cdot \underline{k} = -1 \tag{5}$$

$$\underline{i} \cdot \underline{j} = -\underline{j} \cdot \underline{i} \;\&\; \underline{j} \cdot \underline{k} = -\underline{k} \cdot \underline{j} \;\&\; \underline{k} \cdot \underline{i} = -\underline{i} \cdot \underline{k} \tag{6}$$

Equation (3) can be divided into two parts: scalar (s) and quaternion vector ($\vec{v}$):

$$\tilde{A} = (s, \vec{v}) \tag{7}$$

If $\tilde{A} = \left(A_0, \vec{A}\right)$ and $\tilde{B} = \left(B_0, \vec{B}\right)$ two quaternions, then their product is formed as follows:

$$\tilde{A} \cdot \tilde{B} = \left(A_0, \vec{A}\right) \cdot \left(B_0, \vec{B}\right) = \left(A_0 \cdot B_0 - \vec{A} \cdot \vec{B}, A_0 \cdot \vec{B} + \vec{A} \cdot B_0 + \vec{A} \times \vec{B}\right) \tag{8}$$

, where $\vec{A} \cdot \vec{B}$ is the scalar product of the two vectors and $\vec{A} \times \vec{B}$ is the vector product of the two vectors. Quaternions can be extended to a set of complex numbers, that is, complex quaternions can be interpreted:

$$\tilde{A} = \left(a + i \cdot b, \vec{c} + i \cdot \vec{d}\right) \tag{9}$$

, where a, b are real numbers, $\vec{c}$, $\vec{d}$ are vectors, and i is the complex unit. A complex quaternion is also called a biquaternion. Being complex quaternions, the complex conjugation operation can be:

$$\tilde{A}^* = \left(a - i \cdot b, \vec{c} - i \cdot \vec{d}\right) \tag{10}$$

In addition, the magnitude of a quaternion can be interpreted as follows:

$$|\tilde{A}| = \sqrt{\tilde{A} \cdot \tilde{A}^*} = \sqrt{A_0^2 + A_1^2 + A_2^2 + A_3^2} \tag{11}$$

The Nabla quaternion operator and it's complex conjugate can be defined as follows:

$$\tilde{\nabla} = \left(\frac{i}{c} \cdot \frac{\partial}{\partial t}, \vec{\nabla}\right) \;\&\; \tilde{\nabla}^* = \left(\frac{i}{c} \cdot \frac{\partial}{\partial t}, -\vec{\nabla}\right) \tag{12}$$

, where i is the imaginary unit and c is the speed of light in vacuum. Using equation (12), we can define the D'Alambert quaternion operator in the following way:

$$-\tilde{\nabla}\cdot\tilde{\nabla}^{*}=\left(\frac{i}{c}\cdot\frac{\partial}{\partial t},\vec{\nabla}\right)\cdot\left(\frac{i}{c}\cdot\frac{\partial}{\partial t},-\vec{\nabla}\right)=-\left|\tilde{\nabla}\right|^{2}=\tilde{\square} \tag{13}$$

5. Quaternion electrodynamics

5.1. The modified Maxwell equations

The maxwell equations can be derived from a single quaternion (wave) equation that connects two quaternion quantities: The $\tilde{A}$ quaternion vector potential and the $\tilde{J}$ quaternion current density.

$$\tilde{A} = \left(i \cdot \frac{\varphi}{c}, \vec{A}\right) \tag{14}$$

$$\tilde{J} = \left(i \cdot c \cdot \rho, \vec{J}\right) \tag{15}$$

, where φ the scalar potential, $\vec{A}$ is the vector potential, ρ is the charge density. Start by forming the product of the complex conjugated quaternion vector potential and the quaternion Nabla operator according to equation (12):

$$\tilde{\nabla} \cdot \tilde{A} = \left(\frac{i}{c} \cdot \frac{\partial}{\partial t}, \vec{\nabla}\right) \cdot \left(i \cdot \frac{\varphi}{c}, \vec{A}\right) = \left(-\left(\frac{1}{c^2} \cdot \frac{\partial \varphi}{\partial t} + \vec{\nabla} \cdot \vec{A}\right), \frac{i}{c} \cdot \left(-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi\right) + \left(\vec{\nabla} \times \vec{A}\right)\right) \tag{16}$$

At this point, three familiar relationships are immediately apparent: the Lorenz gauge (if the first term in parentheses is chosen to be zero), the electric field strength and the magnetic induction (Jackson, 1999).

$$\Lambda = \left(\frac{1}{c^2} \cdot \frac{\partial \varphi}{\partial t} + \vec{\nabla} \cdot \vec{A}\right) \tag{17}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi \tag{18}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{19}$$

The Lorenz gauge is not assumed in this case and we examine what it leads to. Using equation (17), (18), (19) simplifies to the following:

$$\tilde{G} = \tilde{\nabla} \cdot \tilde{A} = \left(-\Lambda, -\frac{i}{c} \cdot \vec{E} + \vec{B}\right) \tag{20}$$

Use the D’Alambert quaternion operator on the quaternion vector potential. Then the equation will be proportional to the quaternion current density $\tilde{J}$. The proportionality factor is μ_0 the vacuum permeability:

$$-\tilde{\nabla}^* \cdot \tilde{\nabla} \cdot \tilde{A} = -\left(\frac{i}{c} \cdot \frac{\partial}{\partial t}, -\vec{\nabla}\right) \cdot \left(-\Lambda - \frac{i}{c} \cdot \vec{E} + \vec{B}\right) = \tilde{\Box}\tilde{A} = \mu_0 \cdot \tilde{J} \tag{21}$$

$$\left(\frac{i}{c}\cdot\left(\vec{\nabla}\cdot\vec{E}+\frac{\partial\Lambda}{\partial t}\right)-\left(\vec{\nabla}\cdot\vec{B}\right),\left(-\vec{\nabla}\Lambda-\frac{1}{c^2}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\times\vec{B}\right)-\frac{i}{c}\cdot\left(\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}\right)\right) \tag{22}$$

Disassemble equation (22) into its scalar and vectorial components and form newer equations. Assume that magnetic monopoles do not exist.

$$\text{scalar part}\left(-\tilde{\nabla}^*\cdot\tilde{\nabla}\cdot\tilde{A}\right)=\text{scalar part}\left(\mu_0\cdot\tilde{J}\right) \tag{23}$$

Let's use the relationship between the speed of light in vacuum, the vacuum permeability and vacuum permittivity:

$$\frac{1}{c^2}=\mu_0\cdot\varepsilon_0 \tag{24}$$

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}-\frac{\partial\Lambda}{\partial t} \tag{25}$$

$$\vec{\nabla}\cdot\vec{B}=0 \tag{26}$$

Equation (25) shows that the electric field strength has a source. Its source can be not only the charge density, but also the time-varying magneto scalar potential (Λ).

$$\text{vectorial part}\left(-\tilde{\nabla}^*\cdot\tilde{\nabla}\cdot\tilde{A}\right)=\text{vectorial part}\left(\mu_0\cdot\tilde{J}\right) \tag{27}$$

$$\frac{\partial\vec{B}}{\partial t}+\vec{\nabla}\times\vec{E}=0 \tag{28}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\cdot\vec{J}+\frac{1}{c^2}\cdot\frac{\partial\vec{E}}{\partial t}+\vec{\nabla}\Lambda \tag{29}$$

Equation (29) shows that the curl of the magnetic induction generates not only the time-varying electric field strength but also the gradient of the magneto scalar potential.

So far I have dealt with the differential form of the modified Maxwell equations. In the next step, I define their integral form. Let's start from equations (25), (26), (28), (29). According to the Gauss theorem:

$$\int_V \vec{\nabla}\cdot\vec{F}dV=\oint_S \vec{F}\cdot\vec{n}dS \tag{30}$$

According to the Stokes theorem:

$$\int_S \left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{n}d\vec{S}=\oint_C \vec{F}d\vec{r} \tag{31}$$

According to the gradient theorem:

$$\int_C \vec{\nabla}\varphi d\vec{r}=\oint_t \left(\vec{\nabla}\varphi\right)\cdot\vec{v}dt \tag{32}$$

, where

- $\vec{F}$ is a vector potential
- φ is a scalar potential
- $\vec{v}$ is a vector potential

Take the volume integral of both sides of equation (25) We apply the Gauss theorem. Then we arrive at the following equation.

$$\oint_S \vec{E} \cdot \vec{n} dS = \frac{1}{\varepsilon_0} \int_V \rho dV - \frac{\partial}{\partial t} \int_V \Lambda dV \tag{33}$$

Take the volume integral of both sides of equation 26). We apply the Gauss theorem. Then we arrive at the following equation.

$$\oint_S \vec{B} \cdot \vec{n} dS = 0 \tag{34}$$

Take the surface integral of both sides of equation 28). We apply Stokes' theorem. Then we arrive at the following equation.

$$\oint_C \vec{E} d\vec{r} = -\frac{\partial}{\partial t} \int_S \vec{B} \cdot \vec{n} dS \tag{35}$$

Take the surface integral of both sides of equation (29) We apply Stokes' theorem. Then we arrive at the following equation.

$$\mu_0 \int_S \vec{J} \cdot \vec{n} \mathrm{d}S + \frac{1}{c^2} \cdot \frac{\partial}{\partial t} \int_S \vec{E} \cdot \vec{n} dS + \int_S \vec{\nabla} \Lambda \vec{n} dS = \oint_C \vec{B} d\vec{r} \tag{36}$$

With this, I derived the integral form of the modified Maxwell equations.

Usually, Maxwell equations are invariant with respect to a gauge transformation of the potentials and one can choose freely a gauge condition. For instance, the Lorenz gauge condition yields the potential Lorenz inhomogeneous wave equation. It is possible to introduce a scalar field in the Maxwell equations such that the generalised Maxwell theory, expressed in terms of the potentials, automatically satisfy the Lorenz inhomogeneous wave equation., without any gauge condition. This theory of electrodynamics is no longer gauge invariant with respect to the transformation of the potentials: it is electrodynamics with broken gauge symmetry. The appearance of the extra scalar field terms can be described as a conditional current regauge that does not violate the conservation of charge and it has several consequences (van Vlaenderen & Waser, 2001).

5.2. The Lorentz force density

The quaternion Lorentz force can be written from the product of two quaternions: the $\tilde{v}$ velocity quaternion and the complex conjugate of the $\tilde{G}$ quaternion.

$$\tilde{F} = \left(i \cdot \frac{p}{c}, \vec{f}\right) = \tilde{J} \cdot \tilde{\nabla} \cdot \tilde{A} = \left(i \cdot c \cdot \rho, \vec{J}\right)\left(-\Lambda, -\frac{i}{c} \cdot \vec{E} + \vec{B}\right) \tag{37}$$

, p the power density, $\vec{f}$ the Lorentz force density. The velocity vector and the magnetic induction are perpendicular to each other, therefore $\vec{J} \cdot \vec{B} = 0$. Let's disassemble equation (37) into its scalar and vectorial components and form newer equations.

$$\text{scalar part }(\tilde{F}) = \text{scalar part }\left(i \cdot \frac{p}{c}, \mathrm{f}\right) \tag{38}$$

$$p = \left(\vec{J} \cdot \vec{E} - \rho \cdot \Lambda \cdot c^2\right) \tag{39}$$

$$\text{vectorial part }(\tilde{F}) = \text{vectorial part }\left(i \cdot \frac{p}{c}, \vec{f}\right) \tag{40}$$

At equation (40) we assume only real force components; we equate the imaginary ones with zero. This can be used to express $\vec{B}$ as the function of $\vec{E}$. Then we get the following equation:

$$\vec{B} = \frac{\vec{v}}{c^2} \times \vec{E} \tag{41}$$

$$\vec{f} = \rho \cdot \vec{E} + \vec{J} \times \vec{B} - \Lambda \cdot \vec{J} \tag{42}$$

The following authors reached a similar result:

- (Arbab, Extended electrodynamics and its consequences, 2017)
- (Arbab & Satti, On the Generalized Maxwell Equations and Their Prediction of ElectroscalarWave, 2009)
- (van Vlaenderen & Waser, 2001)
- (Waser, 2015)

A force component $-q \cdot \Lambda \cdot \vec{v}$ appears, which is velocity dependent and has a negative sign. This force component is characteristics of viscous fluids, bit it exclusively the work of Λ and can also occur in a vacuum The liquid (ether) had once been introduced into which electromagnetic radiation could have spread, and was later discarded (Arbab, Do we need to modify Maxwell's equations?, 2020). According to the current experimental results, if a point charge moves with speed $\vec{v}$ in an external magnetic field $\vec{B}$ and the velocity vector and the magnetic induction are perpendicular to each other, then its trajectory will be circular. Let's start from equation (42). Let's neglect the electric field strength $\vec{E}$ and let the velocity and the magnetic induction vector be perpendicular to each other. The force on the point charge should be equal to the centripetal force. We write their magnitudes in place of vectorial quantities. Then we arrive the following equation:

$$m \cdot \frac{v^2}{r} = q \cdot v \cdot B - q \cdot \Lambda \cdot v \tag{43}$$

Simplifying the above equation to the speed and rearranging it to the radius (r) of the circular path, we arrive at the following relationship:

$$r = \frac{m}{q \cdot (B - \Lambda)} \cdot v \tag{44}$$

We conclude that the radius of the circular orbit will be larger in the presence of the magneto scalar field Λ than without it.

5.3. The continuity equation

One consequence of the Maxwell equations is the charge conservation law (continuity equation). The quaternion variant of this law can be written with the product of two quaternion quantities: the $\tilde{J}$ current density quaternion and the $\tilde{\nabla}$ Nabla quaternion. We will see that two additional equations are added to the continuity equation.

$$\tilde{\nabla} \cdot \tilde{J} = \left(\frac{i}{c} \cdot \frac{\partial}{\partial t}, \vec{\nabla}\right) \cdot \left(i \cdot c \cdot \rho, \vec{J}\right) =$$
$$\left(-\left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}\right), \frac{i}{c} \cdot \left(\frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\rho \cdot c^2\right) + i \cdot c \cdot \rho \cdot \left(\vec{\nabla} \times \vec{J}\right)\right) \tag{45}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \tag{46}$$

$$\frac{\partial \vec{J}}{\partial t} + \vec{\nabla}\rho \cdot c^2 = 0 \tag{47}$$

$$\vec{\nabla} \times \vec{J} = 0 \tag{48}$$

According to equation (46), if the charge leaves an infinitesimal volume through a given surface, then the amount of charge in the volume decreases. Applying Stokes’s theorem to equation (48) we get that the line integral of the current density along a closed curve is considered to be zero. The result obtained is analogous to Kirchhoff’s law. The following authors reached a similar result:

- (Arbab & Satti, On the Generalized Maxwell Equations and Their Prediction of ElectroscalarWave, 2009)
- (Waser, 2015)

5.4. Generalised energy and momentum theorems

In this chapter I introduce the generalised energy and momentum theorems.Within the generalised Lorenz force equation $\tilde{F} = \tilde{J} \cdot \tilde{\nabla} \cdot \tilde{A}$ we can substitute for $\tilde{J}$ its definition in terms of potentials, $(1/\mu_0) \cdot \tilde{\Box}\tilde{A}$. Then we get:

$$\tilde{\Box}\tilde{A} \cdot \tilde{\nabla} \cdot \tilde{A} = \mu_0 \cdot \tilde{F} \tag{49}$$

When we evaluate the imaginary scalar part of this equation in terms of fields and sources, we get the following energy equation:

$$\mu_0 \cdot \left(\vec{E} \cdot \vec{J} - \rho \cdot c^2 \cdot \Lambda\right) = -\frac{\partial}{2\partial t}\left[\Lambda^2 + \frac{\vec{E}^2}{c^2} + \vec{B}^2\right] - \vec{\nabla} \cdot \left(\vec{E} \times \vec{B} + \Lambda \cdot \vec{E}\right) \tag{50}$$

The real vector part of the biquaternion equation is:

$$\begin{gathered} \mu_0 \cdot \left(\rho \cdot \vec{E} + \vec{J} \times \vec{B} - \vec{J} \cdot \Lambda\right) = \\ \left[\frac{1}{c^2} \cdot \left(\left(\vec{\nabla} \cdot \vec{E}\right) \cdot \vec{E} + \left(\vec{\nabla} \times \vec{E}\right) \times \vec{E} + \left(\vec{\nabla} \times \vec{B}\right) \times \vec{B}\right)\right] + \\ \left[\Lambda \cdot \vec{\nabla}\Lambda - \vec{\nabla} \times \left(\Lambda \cdot \vec{B}\right)\right] + \\ \frac{1}{c^2} \cdot \frac{\partial\left(\vec{E} \cdot \Lambda - \vec{E} \times \vec{B}\right)}{\partial t} \end{gathered} \tag{51}$$

This equation is the extended momentum theorem in the generalised Maxwell theory (van Vlaenderen & Waser, 2001).

6. The linear elasticity theory and the relation to the Maxwell equations

In this chapter I introduce the article (Podgainy & Zaimidoroga, 2010) by Podgainy et al. Building of theoretical scheme will be started with consideration of an analogy between the linear elasticity theory equations and classic Maxwell equations. Let us note that it is the elastic continuum that supports propagation of both longitudinal and transverse waves, therefore, this analogy will give us a constructive indication of how the equations of generalized electrodynamics should look like. The basic equation of the linear elasticity theory is the Lame equation, which takes the following form in the absence of outside forces (Love, 1927):

$$-\ddot{\vec{u}} + c_l^2 \cdot \vec{\nabla}\vec{\nabla} \cdot \vec{u} - c_t^2 \cdot \vec{\nabla} \times \vec{\nabla} \times \vec{u} = 0 \tag{52}$$

where the vector $\vec{u}$ represents the vector of displacements in an elastic medium, c_l and c_t are the velocities of propagation of the longitudinal and transverse waves, correspondingly. The displacement vector is the main variable in the linear elasticity theory, although it is not a directly observable quantity. Physically observable quantities in this theory are first-order derivatives of $\vec{u}$, i.e. $\dot{\vec{u}}$ is the velocity of elastic displacements and $\sigma_{ij} = \lambda \cdot \vec{\nabla} \cdot \vec{u} + 2 \cdot \mu \cdot u_{ij}$ is the stress tensor, where λ and μ are the elastic constants of the medium and u_{ij} is the elastic deformation tensor (Landau, Pitaevskii, Kosevich, & Lifshitz, 1986). With these variables the Lame equation looks as follows:

$$-\frac{\partial^2 u_i}{\partial t^2} + \frac{\partial \sigma_{ik}}{\partial x_k} = 0 \tag{53}$$

However, the Lame equation for the observables presented in this form disguises the structure of wave processes that might arise in an elastic medium, namely, from this form one cannot see arrangement of the wave processes into transverse and longitudinal elastic waves. In order to separate the wave processes explicitly, the following designations are introduced (Dubrovskii, 1985):

$$\vec{E} = -\dot{\vec{u}} \quad \vec{H} = -c_t \cdot \vec{\nabla} \times \vec{u} = 0 \tag{54}$$

Now, let us employ the rotor and divergence operation to the vector $\vec{E}$:

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \dot{\vec{u}} = -\frac{1}{c_t} \cdot \dot{\vec{H}} \tag{55}$$

$$\vec{\nabla} \cdot \vec{E} = -\vec{\nabla} \cdot \dot{\vec{u}} = -\frac{1}{c_l} \cdot \dot{W} \tag{56}$$

Thus, for the introduced field denotations one can obtain from the LAME equation the following system of equations:

$$\dot{\vec{E}} + c_l \cdot \vec{\nabla} W - c_t \cdot \vec{\nabla} \times \vec{H} = 0 \tag{57}$$

$$\frac{1}{c_t} \cdot \dot{\vec{H}} + \vec{\nabla} \times \vec{E} = 0 \tag{58}$$

$$\frac{1}{c_l} \cdot \dot{W} + \vec{\nabla} \cdot \vec{E} = 0 \tag{59}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \tag{60}$$

which, in the event of $W = c_l \cdot \vec{\nabla} \cdot \vec{u} = 0$, corresponding to the incompressible elastic contimuum, coincides with the system of Maxwell equations in vacuum. If the continous medium does not support rotating motion, foe which $\vec{\nabla} \times \vec{u} = 0$ (e.g. liquid or gas), then the system of equations takes this form:

$$\dot{\vec{E}} + c_l \cdot \vec{\nabla} W = 0 \quad (61)$$

$$\vec{\nabla} \times \vec{E} = 0 \quad (62)$$

$$\frac{1}{c_l} \cdot \dot{W} + \vec{\nabla} \cdot \vec{E} = 0 \quad (63)$$

Such a system describes propagation of longitudinal (acoustic) waves in a continuous medium. The total vector field $\vec{E}$ will be defined as a sum of $\vec{E}_t$ and $\vec{E}_l$. Because the transverse and longitudinal waves propagate in the elastic continuum with different velocities, c_t for the transverse and c_l for the longitudinal ones, the fields $\vec{E}_t$ and $\vec{E}_l$ can be considered as independent.

7. The symmetric-assymmetric decomposition of the electromagnetic tensor

In this chapter I introduce the article (Spirichev, 2017) by Spirichev et al., where he presents the solution to the problem of scalar longitudinal waves within the framework of electromagnetic potential A_ν without introducing any additional members into the canonical field Lagrangian or the Maxwell equations that ensure the existence of a longitudinal wave component. The existence of longitudinal expansion/contraction waves implies the existence of an elastic continuum (electromagnetic vacuum). In the theory of continuous media, an elastic medium is described by a symmetric tensor. However, in Maxwell's theory the electromagnetic vacuum is incompressible. This property is reflected in the EMF description in the form of a canonical antisymmetric tensor the trace of which equals zero.

$$F_{[\mu\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (64)$$

The definition of a canonical antisymmetric tensor includes an asymmetric tensor of the second rank:

$$F_{\mu\nu} = \partial_\mu A_\nu \quad (65)$$

which is a four-dimensional derivative of the electromagnetic potential. This asymmetric tensor can be uniquely decomposed into symmetric and antisymmetric tensors:

$$F_{\mu\nu} = F_{[\mu\nu]}/2 + F_{(\mu\nu)} \quad (66)$$

where thy symmetric tensor of the second rank:

$$F_{(\mu\nu)} = \partial_\mu A_\nu + \partial_\nu A_\mu \quad (67)$$

More precisely, this tensor is implicitly contained in the canonical description of EMF:

$$F_{[\mu\nu]} = 2 \cdot F_{\mu\nu} - F_{(\mu\nu)} \quad (68)$$

In the theory of continuous media, the antisymmetric displacement tensor of a medium is associated with its rotation as a whole, and the symmetric tensor is connected by longitudinal and shear deformations of the medium. The diagonal components of this tensor describe a four-dimensional volumetric deformation of the EMF and represent a four-dimensional divergence of the electromagnetic potential $\partial^\nu A_\nu$. The Lorentz gauge condition $\partial^\nu A_\nu = 0$ is widely used in electrodynamics. Thus, the physical essence of the Lorentz gauge condition is the elimination of four-dimensional volume deformation from the EMF equations. Naturally, when the condition $\partial^\nu A_\nu = 0$, is imposed, longitudinal waves and interactions of longitudinal currents are excluded from electrodynamics. According to Spirichev et al. the propagation velocity of longitudinal EMF waves is $\sqrt{2} \cdot c$.

8. The static magneto scalar field

In this chapter, I examine the solution of the modified Maxwell equations, where the charge density does not change over time, and the presence of constant stationary currents is also allowed. Then the modified Maxwell equations will obviously have a solution where the spatial quantities are constant in time. Take equations (17), (18), (19), then substitute them into equation (29) and use equation (24). Then we get one of the familiar differential equations of magnetostatics.

$$-\nabla^2\vec{A} = \mu_0 \cdot \vec{J} \tag{69}$$

Note that the Coulomb gauge was not used. In other words, the neglect of the Lorenz gauge has a much deeper meaning than previously thought. Following the derivation of the Biot-Savart law, we already know the shape of the $\vec{A}$ vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\cdot\pi}\int_V \frac{\vec{J}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}dV' = \frac{\mu_0\cdot I}{4\cdot\pi}\oint_C \frac{d\vec{l}\,'}{|\vec{r}-\vec{r}\,'|} \tag{70}$$

Substitute equation (19) into equation (29). Since the physical quantities do not change in time, we obtain the following equation:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}\vec{\nabla}\cdot\vec{A} - \nabla^2\vec{A} = \mu_0\cdot\vec{J} + \vec{\nabla}\Lambda \tag{71}$$

Using equation (69) the term $\mu_0\cdot\vec{J}$ can be dropped. Integrating both sides of the above equation and choosing the integration constant as 0, we arrive at the following relationship:

$$\Lambda = \vec{\nabla}\cdot\vec{A} \tag{72}$$

According to equation (72), the divergence of the vector potential $\vec{A}$ must now be taken to obtain the scalar potential Λ. We can live with the freedom to put the differential operator behind the integration. Notice that we are trying to derive a scalar and a vector function. We can use the following identity for this:

$$\vec{\nabla}\cdot(\varphi\cdot\vec{a}) = \vec{a}\cdot\vec{\nabla}\varphi + \varphi\vec{\nabla}\cdot\vec{a} \tag{73}$$

$$\Lambda(\vec{r}) = \frac{\mu_0}{4\cdot\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}\,')}{|\vec{R}|}dV' + \frac{\mu_0}{4\cdot\pi}\int_V \vec{J}(\vec{r}\,')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{R}|}\right)dV' \tag{74}$$

$$\vec{R} = \vec{r}-\vec{r}\,' \quad R = |\vec{r}-\vec{r}\,'| \tag{75}$$

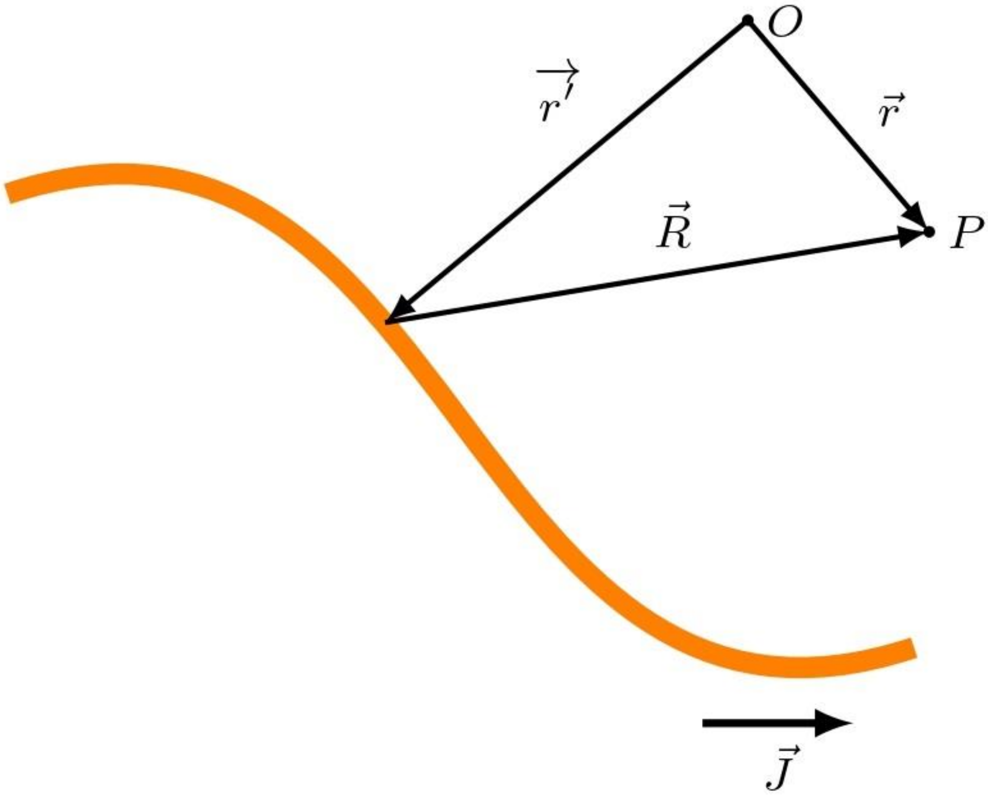

Figure 1

In addition to the magnetic and electric fields, a third field (called the magneto scalar field) appears, which together form the electro-magnetic-scalar field. Now let's turn for a moment to the topic of the next chapter: magneto scalar waves. Then, in the modified Maxwell equations, we also take into account the temporal change of the spatial quantities. After some derivation, we get the following equation:

$$\frac{1}{c^2} \cdot \frac{\partial^2 \Lambda}{\partial t^2} - \nabla^2 \Lambda = \mu_0 \cdot \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \right) \tag{76}$$

The above equation has a double meaning. On the one hand, charges can be sources and sinks of magneto scalar waves, and on the other hand, magneto scalar waves can be sources and sinks of charges. According to the equation, the charge is a locally non-conserving quantity. We again apply the conditions for the static magneto scalar field to the above equation. Then we get the following equation:

$$-\nabla^2 \Lambda = \mu_0 \cdot \vec{\nabla} \cdot \vec{J} \tag{77}$$

The solution of the above differential equation is as follows:

$$\Lambda(\vec{r}) = \frac{\mu_0}{4 \cdot \pi} \int_V \frac{\vec{\nabla}' \cdot \vec{J}(\vec{r}')}{|\vec{R}|} dV' \tag{78}$$

The starting points of the equation above and equation (74) are correct, but a different result was obtained. Let's write down the findings so far and compare them.

1. The inhomogeneous wave equation for Λ expresses that the charges can transform into a magneto scalar field and vice versa. That is, the charges are withdrawn from the conductor.
2. Examining equation (74) alone, we can draw the conclusion that the magneto scalar field consists not only of the removal of charges, but also of their flow.
3. If the left side of the inhomogeneous wave equation for Λ is set equal to 0, then charge conservation will be fulfilled.
4. If the right-hand side of the inhomogeneous wave equation for Λ is set equal to 0, we will obtain the freely propagating magneto scalar waves without a source.
5. In classical electrodynamics, the flow of charges creates the field and at the same time the conservation of charge is achieved.

Here the findings conflict. How would we resolve this paradox if all the statements were correct together? We can say that we do not know enough about the second term by examining equation (74) alone. All statements are correct individually and collectively if and only if the following relation holds:

$$\frac{\mu_0}{4\cdot\pi}\int_V \vec{J}(\vec{r}')\cdot\vec{\nabla}'\left(\frac{1}{|\vec{R}|}\right)dV' = 0 \tag{79}$$

Since the volume of integration contains all sources, no current transverses its boundary, so that the surface integral is zero. Also, $\vec{\nabla}'|\vec{R}| = -\vec{\nabla}|\vec{R}|$ (Minotti & Modanese, Gauge waves generation and detection in Aharonov-Bohm, 2023). So, the final spatial dependence of the static magneto scalar field can be described by the following equation:

$$\Lambda(\vec{r}) = \frac{\mu_0}{4\cdot\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}')}{|\vec{R}|}dV' \tag{80}$$

Equation (70) is suitable for determining the potential, but in the case of a complex $\vec{J}$ function, due to both practical and illustrative reasons, it is customary to use an approximation procedure (multipolar series decomposition). The nature of electromagnetic radiation depends on the system of charges or currents that create it. In case of a complex system, its direct definition is difficult and sometimes impossible. Since the Maxwell equations are linear, the electric and magnetic fields depend linearly on the distribution of their source. This linearity offers us the possibility to decompose the radiation (more precisely the structure of the source) into the sum of moments of increasing complexity (using the principle of superposition). Because the electromagnetic field is more strongly dependent on lower-order moments than on higher-order ones, the electromagnetic field can be approximated without a detailed knowledge of it's source. Suppose that the corresponding $\vec{J}$ currents are localized, that is, they disappear outside a finite volume. Take the origin within this range. To determine each member of the series, Let's take the Taylor series of the following function around the of location of $\vec{r}' = 0$ (i.e. r=$|\vec{r}| \gg$ r'=$|\vec{r}'|$):

$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\left(r^2 - 2\vec{r}\cdot\vec{r}' + r'^2\right)^{\frac{1}{2}}} = \frac{1}{r\left(1 + \frac{r'^2}{r^2} - 2\cdot\frac{r'}{r}\cdot cos\vartheta'\right)^{\frac{1}{2}}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(cos\vartheta') \tag{81}$$

$$cos\vartheta' = \frac{\vec{r}}{|\vec{r}|}\cdot\frac{\vec{r}'}{|\vec{r}'|} = \hat{r}\cdot\hat{r}' \tag{82}$$

, where $P_n(cos\vartheta')$ are the Legendre polynomials. Far enough from the localized currents $\vec{J}$ we can get a good approximation to $\vec{A}$ by taking the first (or the first two) non-disappearing term. To do this, use the equation (70). Then we obtain the magnetic monopole, dipole (and quadrupole) moments for the vector potential $\vec{A}$.

$$\begin{aligned}\vec{A} &= \frac{\mu_0}{4\pi}\cdot\frac{1}{r}\int_V \vec{J}(\vec{r}')dV' \\ &+ \frac{\mu_0}{4\pi}\cdot\frac{1}{r^2}\int_V \vec{J}(\vec{r}')\cdot r'\cdot cos\vartheta' dV' \\ &+ \frac{1}{r^3}\int_V \vec{J}(\vec{r}')\cdot r'^2\left(\frac{3}{2}\cdot cos^2\vartheta' - \frac{1}{2}\right)dV' + \cdots \\ &= \frac{\mu_0}{4\pi}\cdot\frac{1}{r}\int_V \vec{J}(\vec{r}')dV' + \\ &\frac{\mu_0}{4\pi}\cdot\frac{\hat{r}}{r^2}\int_V \vec{J}(\vec{r}')\cdot\vec{r}' dV + \\ &\frac{\mu_0}{4\pi}\cdot\frac{1}{r^3}\int_V \vec{J}(\vec{r}')\cdot\left(\frac{3}{2}(\hat{r}\cdot\vec{r}')^2 - \frac{1}{2}\cdot r'^2\right)dV' + \cdots\end{aligned} \tag{83}$$

Since there are no magnetic monopoles, we can expect this term to fall out. The dipole term will then dominant, which we will use in our calculations in the future. After some derivation we get the following expression:

$$\vec{A}_D(\vec{r}) = -\frac{\mu_0}{8\cdot\pi}\cdot\frac{\hat{r}}{r^2}\times\int_V \vec{r}'\times\vec{J}(\vec{r}')dV' = \frac{\mu_0}{4\cdot\pi}\cdot\frac{\vec{m}\times\hat{r}}{r^2} \tag{84}$$

, where $\vec{m}$ is the magnetic dipole momentum vector. To obtain the magneto scalar potential of the dipole, all we have to do is take the divergence of the vector potential $\vec{A}_D$ according to equation (72).

$$\Lambda_D(\vec{r}) = \vec{\nabla}\cdot\vec{A}_D = \frac{\mu_0}{4\cdot\pi}\cdot\frac{(\vec{\nabla}\times\vec{m})\cdot\hat{r}}{r^2} = \frac{\mu_0}{4\cdot\pi}\int_V \vec{J}(\vec{r})_m\cdot\frac{\vec{r}}{r^3}dV \tag{85}$$

By definition, the magnetic moment vector $\vec{m}$ is the volume integral of the magnetization vector $\vec{M}$. The curl of the magnetization vector gives the magnetization current density $\vec{J}_M$. Using these two definitions, we arrive at relation (85). Let's compute $\vec{\nabla} \times \left(\vec{r}' \times \vec{J}(\vec{r}')\right)$:

$$\vec{\nabla} \times \left(\vec{r}' \times \vec{J}(\vec{r}')\right) = \left(\vec{\nabla} \cdot \vec{J}(\vec{r}')\right) \cdot \vec{r}' - \left(\vec{\nabla} \cdot \vec{r}'\right) \cdot \vec{J}(\vec{r}') + \left[\vec{J}(\vec{r}') \cdot \vec{\nabla}\right] \cdot \vec{r}' - \left[\vec{r}' \cdot \vec{\nabla}\right] \cdot \vec{J}(\vec{r}') \qquad (86)$$

- The stationary case in terms of static magneto scalar field means $\vec{\nabla} \cdot \vec{J} = 0$ and $\partial\rho/\partial t = 0$.
- When we have a charge source/sink in case of static magneto scalar field means $\vec{\nabla} \cdot \vec{J} \neq 0$, but $\partial\rho/\partial t = 0$.

As you can see, the above equation can be extremely complex even though it is a dipole approximation. Not to mention that there are undesirable elements in it that you should get rid of. For this reason, I deal with the series expansion based on the solution of the wave equation.

9. The magneto scalar waves

In this chapter, in the modified Maxwell equations, I take into account the change of field quantities over time, and I show that it follows from the modified Maxwell theory that the electromagnetic and magneto scalar fields propagate in the form of waves. The task is thus given: to derive the inhomogeneous wave equations for $\vec{E}$, $\vec{B}$ and Λ. Let's start from connections (24), (25) and (29). Derive equation (25) with respect to time, form the divergence of both sides of equation (26) and use equation (24). We add relations (25) and (29) and obtain the inhomogeneous wave equation for Λ.

$$\frac{1}{c^2} \cdot \frac{\partial^2 \Lambda}{\partial t^2} - \vec{\nabla} \cdot \vec{\nabla} \Lambda = \mu_0 \cdot \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} \right) \tag{87}$$

The above equation has a double meaning. On the one hand, charges can be sources and sinks of magneto scalar waves, and on the other hand, magneto scalar waves can be sources and sinks of charges. According to the equation, the charge is a locally non-conserving quantity.

Let's continue the derivation with the inhomogeneous wave equation for $\vec{E}$. Let's start from equations (24), (25), (28) and (29). We form the time derivative of equation (29) Let's form the rotation of equation (28) Substitute equation 28) into equation (29). We form the gradient of both sides of equation (25). Equation (25) is added to equation (29). Let's use relation (24).

$$\frac{1}{c^2} \cdot \frac{\partial^2 \vec{E}}{\partial t^2} - \nabla^2 \vec{E} = -\frac{1}{\varepsilon_0} \cdot \left(\frac{1}{c^2} \cdot \frac{\partial \vec{j}}{\partial t} + \vec{\nabla} \rho \right) \tag{88}$$

Finally, the derivation of the inhomogeneous wave equation for $\vec{B}$ should follow. Let's start from equations (24), (26), (28) and (29). We form the rotation of equation (29). We form the time derivative of equation (28) Form the gradient of equation (26) Equation (28) is multiplied by equation (24). Subtract equation (26) from equation (29). Equations (28) and (29) are added together.

$$\frac{1}{c^2} \cdot \frac{\partial^2 \vec{B}}{\partial t^2} - \nabla^2 \vec{B} = \mu_0 \cdot \left(\vec{\nabla} \times \vec{j} \right) \tag{89}$$

If we take the right side of the three equations as 0 (assuming no charges and currents), we get back the generalized continuity equations, along with the homogeneous wave equations. Many solutions to homogeneous wave equations are known, let us confine ourselves to the solution of the plane wave for the time being.

Longitudinal waves can occur elastic media. The analogies between electromagnetism and elastomechanics have been intensely discussed, and it was a great surprise when it was discovered that electromagnetic waves could not be longitudinal, only transverse.

This result is due to the fact that electromagnetic waves must satisfy not only the wave equation (like elastomechanical waves) but also the maxwell equations. The following plane and sphere waves are solutions to the homogenous wave equations:

$$\vec{E} = \vec{E}_0 \cdot e^{i\cdot(\omega\cdot t - \vec{k}\cdot\vec{r})} \quad \vec{E} = \frac{\vec{E}_0 \cdot e^{i\cdot(\omega\cdot t - \vec{k}\cdot\vec{r})}}{r} \tag{90}$$

$$\vec{B} = \vec{B}_0 \cdot e^{i\cdot(\omega\cdot t - \vec{k}\cdot\vec{r})} \quad \vec{B} = \frac{\vec{B}_0 \cdot e^{i\cdot(\omega\cdot t - \vec{k}\cdot\vec{r})}}{r} \tag{91}$$

$$\Lambda = \Lambda_0 \cdot e^{i\left(\omega\cdot t - \vec{k}\cdot\vec{r}\right)} \quad \Lambda = \frac{\Lambda_0 \cdot e^{i\left(\omega\cdot t - \vec{k}\cdot\vec{r}\right)}}{r} \tag{92}$$

$$\vec{k} = \frac{\omega}{c} \cdot \hat{k} \tag{93}$$

, where

- $\vec{E}_0$, $\vec{B}_0$, $\Lambda_0$the wave amplitudes
- ω the circular frequency
- $\vec{k}$ the wavenumber vector (points in the direction of the normal of the plane wave)
- $\hat{k}$ the plane/sphere wave normal vector

In the next step, I derive the relationship between the electric field strength and the magneto scalar potential. Let’s start from equation (25) and assume a vacuum, and let $u = \omega \cdot t - \vec{k} \cdot \vec{r}$ the phase of the plane wave.

$$\vec{\nabla} \cdot \vec{E} = \sum_{j=1}^{3} \frac{\partial E_j}{\partial x_j} = \sum_{j=1}^{3} \frac{dE_j}{du} \cdot \frac{\partial u}{\partial x_j} = -\frac{\omega}{c} \sum_{j=1}^{3} \frac{dE_j}{\omega dt} \cdot \hat{k}_j = -\frac{1}{c} \cdot \frac{\partial}{\partial t}\left(\vec{E} \cdot \hat{k}\right) \tag{94}$$

Then we arrive at the following relation:

$$\frac{1}{c} \cdot \hat{k} \cdot \vec{E} = \Lambda \tag{95}$$

In this case, the electrical waves can be considered longitudinal. The possible existence of such a wave may be subject to experimentation, so the theory can be tested (van Vlaenderen & Waser, Generalisation of classical electrodynamics to admit a scalar field and longitudinal waves, 2001). The electric field strength vector included in the modified Maxwell equations is in some cases transversely (T) and in other cases longitudinally (L) polarized. This leads to inconsistency in notation. In the future, I will call attention to this and distinguish between the two.

In the next step, let's see what shape the time-dependent scalar and vector potential will have. Take equations (17) and (18) and substitute them into equation (25). Finally, we form the (-1) times of equation (25). Then we get the following familiar differential equation for φ:

$$\frac{1}{c^2} \cdot \frac{\partial^2 \varphi}{\partial t^2} - \nabla^2 \varphi = \frac{\rho}{\varepsilon_0} \tag{96}$$

Following classical electrodynamics, we already know the shape of φ scalar potential:

$$\varphi(\vec{r},t) = \frac{1}{4\cdot\pi\cdot\varepsilon_0}\int_V \frac{\rho(\vec{r}',t')}{|\vec{R}|}dV' \tag{97}$$

Take equations (17), (18) and (19), then substitute them into equation (29). Finally, let's form the (-1) times of equation (29). Then we get the following familiar differential equation for $\vec{A}$:

$$\frac{1}{c^2}\cdot\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} = \mu_0\cdot\vec{J} \tag{98}$$

Following classical electrodynamics, we already know the shape of $\vec{A}$ vector potential:

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\cdot\pi}\int_V \frac{\vec{J}(\vec{r}',t')}{|\vec{R}|}dV' \tag{99}$$

The retarded time coordinate t' can be expressed as follows:

$$t' = t - \frac{|\vec{R}|}{c} \tag{100}$$

Note that I did not use the Lorenz gauge.

Now let's turn to the time-dependent magneto scalar potential, which can be defined in the form of equation (17). For this we can use the definition (97) of the time-dependent scalar potential and the definition (99) of the time-dependent vector potential.

$$\Lambda(\vec{r},t) = \frac{\mu_0}{4\cdot\pi}\int_V \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}',t')}{|\vec{R}|}dV' + \frac{\mu_0}{4\cdot\pi}\int_V \frac{\frac{\partial\rho(\vec{r}',t')}{\partial t'}}{|\vec{R}|}dV' \tag{101}$$

The definition of the magneto scalar potential (17) and the general solution of the inhomogeneous wave equation for Λ lead to the same problem as we encountered with the static magneto scalar field. Using the same reasoning, the final form of the time-dependent magneto scalar potential is none other than the above equation. In a system here the charge density and the current density change over time, the method of separating the time-dependent and the space-dependent parts is often used. Assumed a periodic time dependence:

$$\rho(\vec{r},t) = \rho(\vec{r})\cdot e^{-i\omega t} \tag{102}$$

$$\vec{J}(\vec{r},t) = \vec{J}(\vec{r})\cdot e^{-i\omega t} \tag{103}$$

Let us assume that potentials and fields have similar time dependences. Then the space-dependent part of the scalar potential changes to the following form:

$$\varphi(\vec{r}) = \frac{1}{4\cdot\pi\cdot\varepsilon_0}\int_V \rho(\vec{r}')\cdot\frac{e^{i\cdot k\cdot|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dV' \tag{104}$$

The space-dependent part of the vector potential changes to the following shape:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4 \cdot \pi} \int_V \vec{J}(\vec{r}') \cdot \frac{e^{i \cdot k \cdot |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} dV' \tag{105}$$

After reviewing the time-dependent scalar and vector potential, in this section, in a special but very important approximation, I specifically define the multipole radiation created by the monopole and the dipole moments. The multiple radiation is a theoretical toolkit that perfectly characterizes the electromagnetic of the gravitational radiation from time-varying sources. The techniques used to study the multipole radiation are somewhat similar to those used to study static sources. However, there are significant differences in the details of the analysis because time-dependent fields behave differently than static ones. The field from the multipolar momentum depends on both the distance from the origin and the angular direction of an evaluation point relative to the coordinate system. Depending on the size of the source, the wavelength of the radiation, and the distance from the origin, three zones are distinguished: near field, middle field and far field. In the near field, the distance from the source is much smaller than the wavelength (i.e. $\lambda \gg r$). Then the field behaves quasi-statically and $k \cdot r \ll 1$. In the middle field, the distance from the source is proportional to the wavelength (i.e. $\lambda \approx r$). We don't talk much about the middle field, because it is very complex. In the far field, the distance from the source is much larger than the wavelength. ($i.e. \lambda \ll r$). Let's stick to the far field radiation zone. Then we can use the following approximation:

$$\frac{e^{i \cdot k \cdot |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \approx \frac{e^{i \cdot k \cdot r}}{r} \cdot \left(1 + \left(\frac{1}{r} - ik\right) \cdot \hat{r} \cdot \vec{r}'\right) \tag{106}$$

In classical electrodynamics, the amount of charge cannot change over time, the field created by the monopole member must be static. However, equation (87) clearly states that the charges transform into a magneto scalar field and vice versa. But this does not mean that the charge is not a conserved quantity globally, but only locally. According to equation (26) there are no magnetic monopoles, therefore $\vec{\nabla} \cdot \vec{A}_M(\vec{r}, t) = 0$. Therefore, the monopole term will result in zero.

In the dipole approximation, I would be satisfied with the second term of the series expansion. Therefore, the monopole term will result in zero.

$$\vec{\mathrm{A}}_D = \frac{\mu_0}{4\pi} \cdot \frac{e^{i \cdot k \cdot r}}{r} \cdot \left(\frac{1}{r} - ik\right) \cdot \vec{m} \times \hat{r} \tag{107}$$

$$\varphi_D(\vec{r}) = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \frac{e^{i \cdot k \cdot r}}{r} \cdot \left(\frac{1}{r} - ik\right) \cdot \hat{r} \int_V \rho(\vec{r}') \cdot \vec{r}' dV' = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \frac{e^{i \cdot k \cdot r}}{r} \cdot \left(\frac{1}{r} - ik\right) \cdot \hat{r} \cdot \vec{p} \tag{108}$$

, where $\vec{p}$ is the electric dipole moment vector. In order to obtain the time-dependent magneto scalar potential of the dipole, all we have to do is proceed according to equation (17).

$$\vec{\nabla}\cdot\vec{\mathrm{A}}_D=\frac{\mu_0}{4\pi}\cdot e^{-i\cdot\omega\cdot t}\cdot\frac{e^{i\cdot k\cdot r}}{r}\cdot\left(\frac{1}{r}-ik\right)\cdot(\vec{\nabla}\times\vec{\mathrm{m}})\cdot\hat{r} \tag{109}$$

$$\frac{\partial\varphi_D(\vec{r},t)}{\partial t}=-i\cdot\omega\cdot\frac{1}{4\cdot\pi\cdot\varepsilon_0}\cdot\frac{e^{i\cdot k\cdot r-i\omega t}}{r}\cdot\left(\frac{1}{r}-ik\right)\cdot\hat{r}\cdot\vec{p} \tag{110}$$

$$\begin{gathered}\Lambda_D(\vec{r},t)=\vec{\nabla}\cdot\vec{A}_D(\vec{r},t)+\frac{1}{c^2}\cdot\frac{\partial\varphi_D(\vec{r},t)}{\partial t}=\\ \frac{\mu_0}{4\pi}\cdot\frac{e^{i\cdot k\cdot r-i\omega t}}{r}\cdot\left(\frac{1}{r}-ik\right)\cdot\left[(\vec{\nabla}\times\vec{\mathrm{m}})-i\cdot\omega\cdot\vec{p}\right]\cdot\hat{r}\end{gathered} \tag{111}$$

Now let’s take the dipole approximation of equation (101).

$$\begin{aligned}\Lambda_D(\vec{r},\mathrm{t})=\frac{\mu_0}{4\cdot\pi}\cdot\frac{e^{i\cdot k\cdot r-i\cdot\omega\cdot t}}{r}\cdot\left(\frac{1}{r}-i\cdot k\right)\cdot\hat{r}\int\vec{r}'\cdot\left(\vec{\nabla}'\vec{J}(\vec{r}',t')+\partial\rho(\vec{r}',t')/\partial t'\right)dV'\\ =\frac{\mu_0}{4\cdot\pi}\cdot\frac{e^{i\cdot k\cdot r-i\cdot\omega\cdot t}}{r}\cdot\left(\frac{1}{r}-i\cdot k\right)\cdot\hat{r}\int\vec{r}'\cdot I(\vec{r}',t')\,dV'=\\ \frac{\mu_0}{4\cdot\pi}\cdot\frac{e^{i\cdot k\cdot r-i\cdot\omega\cdot t}}{r}\cdot\left(\frac{1}{r}-i\cdot k\right)\cdot\hat{r}\cdot\vec{P}\end{aligned} \tag{112}$$

, where:

- $I(\vec{r}',t')$ is the charge source/sink per unit volume
- $\vec{P}$ is the dipole moment of the charge source/sink

We can see that there are no undesirable elements in it anymore. When we have a charge source/sink in case of time-dependent magneto scalar field means $\vec{\nabla}\cdot\vec{J}\neq 0$ and $\partial\rho/\partial t\neq 0$.

10. The Poynting theorem

In the field of electrodynamics Poynting's theorem expresses the energy conservation of an electromagnetic field developed by British physicist John Henry Poynting. Poynting's theorem is analogous to the work of classical mechanics and is mathematically similar to the continuity equation. Let's start with Newton's second axiom and suppose that all the forces acting on a given point charge are exactly the Lorentz force. Let's introduce the material density $\rho_m$and the charge density ρ. Use the equation $\vec{J} = \rho \cdot \vec{v}$ to create a relationship between current density and density. Multiply both sides of the equation by $\vec{v}$. Since $\left(\vec{v} \times \vec{B}\right) \cdot \vec{v} = 0$, we get the following equation:

$$\frac{d}{dt}\left(\frac{1}{2} \cdot \rho_m \cdot \vec{v}^2\right) = \vec{J} \cdot \vec{E} \tag{113}$$

The total work performed by the fields for a continuous charge and current distribution in a finite volume:

$$\int_V \vec{J} \cdot \vec{E} dV \tag{114}$$

This power characterizes the conversion of electromagnetic energy into mechanical or thermal energy. In the other pan in the balance must therefore have the same rate of decrease in the energy of the electromagnetic field. In order to write the explicit form of the law, we express it in another form based on the Maxwell equations. $\vec{J}$ can be eliminated using equation (29).

$$\begin{gathered}\int_V -\vec{J} \cdot \vec{E} dV = \\ \int_V \left(-\frac{1}{\mu_0} \cdot \vec{E} \cdot \vec{\nabla} \times \vec{B} + \frac{1}{\mu_0} \cdot \vec{E} \cdot \vec{\nabla} \Lambda + \varepsilon_0 \cdot \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}\right) dV \\ = \int_V \left(-\frac{1}{\mu_0} \cdot \vec{E} \cdot \vec{\nabla} \times \vec{B} + \frac{1}{\mu_0} \cdot \vec{E} \cdot \vec{\nabla} \Lambda + \frac{d}{dt}\left(\frac{1}{2} \cdot \varepsilon_0 \cdot \vec{E}^2\right)\right) dV\end{gathered} \tag{115}$$

Multiply equation (25) by Λ/μ_0 and take the volumetric integral on both sides.

$$\int_V (\Lambda \cdot \rho \cdot c^2) dV = \int_V \left(\frac{1}{\mu_0} \cdot \Lambda \cdot \vec{\nabla} \cdot \vec{E} + \frac{d}{dt}\left(\frac{1}{2} \cdot \frac{1}{\mu_0} \cdot \Lambda^2\right)\right) dV \tag{116}$$

Multiply equation (28) by $\vec{B}/\mu_0$ and take the volumetric integral on both sides.

$$-\int_V \left(\frac{d}{dt}\left(\frac{1}{2} \cdot \frac{1}{\mu_0} \cdot \vec{B}^2\right)\right) dV = \int_V \left(\frac{1}{\mu_0} \cdot \vec{B} \cdot \vec{\nabla} \times \vec{E}\right) dV \tag{117}$$

Add equations (115), (116), (117), substitute equation (113) and use the identities below:

$$\vec{\nabla} \cdot \left(\vec{E} \times \vec{B}\right) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}) \tag{118}$$

$$\vec{\nabla} \cdot \left(\Lambda \cdot \vec{E}\right) = \left(\vec{\nabla}\Lambda\right) \cdot \vec{E} + \Lambda \cdot \vec{\nabla} \cdot \vec{E} \tag{119}$$

Then we come to the following relation:

$$\int_V \left(\frac{\vec{\nabla} \cdot (\vec{E} \times \vec{B} + \Lambda \cdot \vec{E})}{\mu_0}\right) dV +$$
$$\int_V \left(+\frac{d}{dt}\left(\frac{1}{2} \cdot \rho_m \cdot \vec{v}^2 + \frac{1}{2} \cdot \varepsilon_0 \cdot \vec{E}^2 + \frac{1}{2} \cdot \frac{1}{\mu_0} \cdot \vec{B}^2 + \frac{1}{2} \cdot \frac{1}{\mu_0} \Lambda^2\right)\right) dV = \tag{120}$$
$$\int_V (\Lambda \cdot \rho \cdot c^2) dV$$

In terms of the result (Arbab, Modified electrodynamics for London's superconductivity, 2017), came to a similar conclusion with the difference that I did not use the generalized gauge transformation he invented. Let u_{EMS} be the total energy density, $\vec{S}_{EMS}$the Poynting vector describing the energy flow, w_{EMS} the rate at which the field does the work on the charged particles. Since the volume V can be chosen as desired, the above relation can be transformed into a differential equation:

$$\vec{\nabla} \cdot \vec{S}_{EMS} + \frac{du_{EMS}}{dt} + w_{EMS} = 0 \tag{121}$$

The Poynting vector is the directed energy flow of the electromagnetic field (the energy transfer is per unit area and time). It can be seen that the direction of energy propagation does not coincide with the direction of propagation of the waves. This phenomenon is typical of anisotropic media, however, it is solely the work of Λ and can also occur in a vacuum. Notice that Λ has an energy density, i.e. it can be considered as a real physical object. Furthermore, if Λ=0, we get back the well-known energy conservation for electromagnetic radiation. The physical meaning of the integral and differential pointing theorems is as follows: the rate of the energy transfer (per unit volume) from a region of space is equal to the rate of work done on the charge distribution, plus the energy flow leaving the region. A few lines above I came to the conclusion (here), that the direction of propagation of energy is not the same as the direction of propagation of waves. $\vec{E}$ belonging to the second term of the Poynting vector is longitudinally polarized, for which equation (95) can be used. If considered

$$\vec{S}_{EMS} = \frac{\vec{E} \times \vec{B}}{\mu_0} + \frac{1}{\mu_0} \cdot \frac{1}{c} \cdot \hat{k} \cdot \vec{E} \cdot \vec{E} \tag{122}$$

I would like to point out again that this designation of electric field strength vector is inconsistent. The electric field strength in the first term of the Poynting vector is transverse. It will become clear later (here) that the second term is longitudinal. We use the appropriate notations (T and L). The relationship between $\vec{B}$ and $\vec{E}$ can be deduced from equation (28):

$$\vec{B}_T = \frac{1}{c} \cdot \hat{k} \times \vec{E}_T \tag{123}$$

Using equations (123) and (95), the following conclusion can be reached:

$$\begin{aligned} \vec{S}_{EMS} &= \frac{1}{\mu_0} \cdot \frac{1}{c} \cdot \vec{E}_T \times \left(\hat{k} \times \vec{E}_T\right) + \frac{1}{\mu_0} \cdot \frac{1}{c} \cdot \hat{k} \cdot \vec{E}_L \cdot \vec{E}_L \\ &= \frac{1}{\mu_0} \cdot \frac{1}{c} \cdot \hat{k} \cdot \left(\left|\vec{E}_T\right|^2 + \left|\vec{E}_L\right|^2\right) \end{aligned} \tag{124}$$

I do not make an equal sign between $\left|\vec{E}_T\right|^2$ and $\left|\vec{E}_L\right|^2$ for now.

11.The momentum theorem

The momentum of a charged substance in an external electromagnetic field is not constant, just as energy is not. Since the field exerts a force on the charges, the momentum of the charges in the field is not constant. The impulse taken up by the classical case charges is also the same as the decrease of the state determiner that can be assigned to a field, and with this state determiner, considering the field and the interaction of the charges as the impulse of the field, the joint impulse remains. Greek indices values 1,2,3.

As a first step, multiply $\varepsilon_0 \cdot \vec{E}$ by cross product with equation (28).

$$\varepsilon_0 \cdot \vec{E} \times \vec{\nabla} \times \vec{E} = -\varepsilon_0 \cdot \vec{E} \times \frac{\partial \vec{B}}{\partial t} \tag{125}$$

As the next step, multiply equation (29) by cross product with $-\vec{B}/\mu_0$.

$$\left(\vec{\nabla} \times \vec{B}\right) \times \left(-\frac{\vec{B}}{\mu_0}\right) = \vec{J} \times \left(-\vec{B}\right) - \frac{1}{\mu_0} \cdot \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \frac{1}{\mu_0} \cdot \vec{\nabla}\Lambda \times \vec{B} \tag{126}$$

The next step is to multiply equation (25) with $-\varepsilon_0 \cdot \vec{E}$.

$$-\varepsilon_0 \cdot \vec{E} \cdot \vec{\nabla} \cdot \vec{E} = -\vec{E} \cdot \rho + \varepsilon_0 \cdot \vec{E} \cdot \frac{\partial \Lambda}{\partial t} \tag{127}$$

The next step is to multiply equation (26) with $-\vec{B}/\mu_0$.

$$-\frac{\vec{B}}{\mu_0} \cdot \vec{\nabla} \cdot \vec{B} = 0 \tag{128}$$

As the next step, we add the equations obtained during each step.

$$\begin{gathered}\varepsilon_0 \cdot \vec{E} \times \vec{\nabla} \times \vec{E} - \frac{1}{\mu_0} \cdot \left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} - \varepsilon_0 \cdot \vec{E} \cdot \vec{\nabla} \cdot \vec{E} - \frac{1}{\mu_0} \cdot \vec{B} \cdot \vec{\nabla} \cdot \vec{B} = \\ -\varepsilon_0 \cdot \vec{E} \times \frac{\partial \vec{B}}{\partial t} - \vec{J} \times \vec{B} - \frac{1}{\mu_0} \cdot \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \frac{1}{\mu_0} \cdot \vec{\nabla}\Lambda \times \vec{B} - \vec{E} \cdot \rho + \varepsilon_0 \cdot \vec{E} \cdot \frac{\partial \Lambda}{\partial t}\end{gathered} \tag{129}$$

As a next step, we use the following identities.

$$\frac{\partial}{\partial t}\left(\vec{E} \cdot \Lambda\right) = \frac{\partial \vec{E}}{\partial t} \cdot \Lambda + \vec{E} \cdot \frac{\partial \Lambda}{\partial t} \tag{130}$$

$$\vec{\nabla} \times \left(\vec{B} \cdot \Lambda\right) = \left(\vec{\nabla}\Lambda\right) \times \vec{B} + \Lambda \cdot \vec{\nabla} \times \vec{B} \tag{131}$$

$$\frac{\partial}{\partial t}\left(\vec{E} \times \vec{B}\right) - \vec{E} \times \frac{\partial \vec{B}}{\partial t} \tag{132}$$

$$\vec{E} \times \vec{\nabla} \times \vec{E} = \frac{1}{2} \cdot \left(\vec{E}\right)^2 - \left(\vec{E} \cdot \vec{\nabla}\right) \cdot \vec{E} \tag{133}$$

$$\vec{B} \times \vec{\nabla} \times \vec{B} = \frac{1}{2} \cdot \left(\vec{B}\right)^2 - \left(\vec{B} \cdot \vec{\nabla}\right) \cdot \vec{B} \tag{134}$$

Finally, after some rearranging, we get that:

$$\rho \cdot \vec{E} + \vec{J} \times \vec{B} - \Lambda \cdot \vec{J} + \frac{1}{\mu_0} \cdot \frac{1}{c^2} \cdot \frac{\partial}{\partial t}\left(\vec{E} \times \vec{B} - \vec{E} \cdot \Lambda\right) =$$
$$\vec{\nabla} \cdot \left[\varepsilon_0 \cdot \left(\vec{E} \cdot \vec{E} - \frac{E^2}{2} \cdot \underline{I}\right) + \frac{1}{\mu_0} \cdot \left(\vec{B} \cdot \vec{B} - \frac{B^2}{2} \cdot \underline{I}\right) + \frac{\Lambda^2}{2 \cdot \mu_0} \cdot \underline{I}\right] - \frac{1}{\mu_0} \cdot \vec{\nabla} \times \left(\vec{B} \cdot \Lambda\right) \tag{135}$$

, where:

- $\underline{I}$ is the identity matrix

However, an inconsistent feature is the last term, because, by writing it in index notation:

$$\vec{\nabla} \times \left(\vec{B} \cdot \Lambda\right)\Big|_\alpha = \vec{\nabla} \times \left(\Lambda \cdot \vec{\nabla} \times \vec{A}\right)\Big|_\alpha = \frac{\partial}{\partial x^\beta}\left[\Lambda \cdot \left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right)\right] \tag{136}$$

We see that it is the divergence of an antisymmetric tensor, which would, which would thus lead to the non-conservation of angular-momentum in a closed system (Landau & Lifshitz, The classical theory of fields, 1971).

Another inconsistency is due to the difference in sign of the Poynting vector in equation (120) relative to that in term $\vec{E} \cdot \Lambda$ inside the time derivative. This means that the field momentum and the field energy flow have opposite directions.

I would like to point out again that this designation of electric field strength vector is inconsistent. The electric field strength in the first term of the Poynting vector is transverse. It will become clear later (here) that the second term is longitudinal.

12. The constitutive relations in vacuum

To obtain Maxwell's constitutive equations, the relationships between $\vec{D}$ and $\vec{E}$, furthermore $\vec{B}$ and $\vec{H}$ must be determined. In addition, a relationship between Λ and it's pair can be determined by examining the dimensions of the physical quantities. First, we declare the units of measurement of the physical quantities that occurs.

Notation	Physical quantity	SI unit of measure
$\vec{J}$	Current density	$\frac{A}{m^2}$
ρ	Electrical charge density	$\frac{C}{m^3}$
Q	Electrical charge	$A \cdot s = C$
$\vec{B}$	Magnetic induction	$\frac{V \cdot s}{m^2} = T$
$\vec{H}$	Magnetic field strength	$\frac{A}{m}$
$\vec{D}$	Displacement field strength	$\frac{A \cdot s}{m^2} = \frac{Q}{m^2}$
$\vec{E}$	Electric field strength	$\frac{V}{m}$

Table 1

The next step is to declare the (original) Maxwell equations containing $\vec{D}$ and $\vec{H}$.

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{137}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{138}$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \tag{139}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{140}$$

With the constitutive equations we can determine the material's response to the electromagnetic radiation. In real life, they are rarely easy to declare, on the other hand, for small energy values, the relationship is linear with a good approximation. Furthermore, note that the unit of Λ, like $\vec{B}$, is Tesla [T]. By comparing the original and the modified Maxwell equations, we can obtain the constitutive equations. Since vacuum is now assumed and there is no polarization or magnetization, the relative permittivity and permeability will not play a role.

$$\vec{D} = \varepsilon_0 \cdot \vec{E} \tag{141}$$

$$\vec{B} = \mu_0 \cdot \vec{H} \tag{142}$$

$$\Lambda = \mu_0 \cdot \zeta \tag{143}$$

According to equation (143), the magneto scalar radiation interacts with the material. Therefore a receiver antenna can be built in order to intercept this kind of radiation.

13. The Lagrangian density

In this chapter, I derive the modified Maxwell equations from the Lagrangian within the framework of covariant electrodynamics with the restriction that I do not use the Lorenz gauge. I take a signature (+,-,-,-) for coordinates (x^0, x^1, x^2, x^3), which, for the case of Minkowski metric are $(x^0 = c \cdot t$, and $(x^\alpha$ the spatial three-dimensional Cartesian coordinates, with Greek indices taking values 1,2,3, and Latin indices values 0,1,2,3. The (negative) determinant of the metric tensor is denoted by η, and the invariant four-dimensional volume element $\sqrt{-\eta} dx^0 dx^1 dx^2 dx^3 = \sqrt{-\eta} d\Omega$,

$$x^a = (c \cdot t, \vec{r}) \;\&\; x_a = (c \cdot t, -\vec{r}) \tag{144}$$

$$\partial^a = \frac{\partial}{\partial x_a} = \left(\frac{1}{c} \cdot \frac{\partial}{\partial t}, -\vec{\nabla}\right) \;\&\; \partial_a = \frac{\partial}{\partial x^a} = \left(\frac{1}{c} \cdot \frac{\partial}{\partial t}, \vec{\nabla}\right) \tag{145}$$

$$J^a = (c \cdot \rho, \vec{j}) \tag{146}$$

$$A^a = \left(\frac{\varphi}{c}, \vec{A}\right) \;\&\; A_a = \left(\frac{\varphi}{c}, -\vec{A}\right) \tag{147}$$

$$\eta^{ab} = \eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{148}$$

$$F_{ab} = \frac{\mathfrak{D} A_b}{\mathfrak{D} x^a} - \frac{\mathfrak{D} A_a}{\mathfrak{D} x^b} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} \tag{149}$$

$$F_{ab} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{150}$$

$$F^{ab} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix} \tag{151}$$

$$\frac{\mathfrak{D} A^d}{\mathfrak{D} x^d} = \frac{\mathfrak{D} A_d}{\mathfrak{D} x_d} = \frac{1}{\sqrt{-\eta}} \cdot \frac{\partial}{\partial x^d}\left(\sqrt{-\eta} \cdot A^m\right) = \frac{1}{\sqrt{-\eta}} \cdot \frac{\partial}{\partial x^d}\left(\sqrt{-\eta} \cdot A_c \cdot \eta^{cd}\right) \tag{152}$$

$$\frac{\mathfrak{D}}{\mathfrak{D} x_d} = \eta^{cd} \cdot \frac{\mathfrak{D}}{\mathfrak{D} x^c} \tag{153}$$

$$L^{A-B} = L^M + L' = -\frac{1}{4 \cdot \mu_0} \cdot F_{ab} \cdot F^{ab} - \frac{1}{2 \cdot \mu_0} \cdot \left(\frac{\mathfrak{D}A^d}{\mathfrak{D}x^d}\right)^2 \quad (154)$$

$$S^{A-B} = \frac{1}{c} \int [L^{AB} - J^a \cdot A_a] \sqrt{-\eta} d\Omega \quad (155)$$

, where:

- x^a and x_a are the contravariant and covariant form of the four-vector
- ∂^a and ∂_a are the contravariant and covariant form of the four-partial derivative
- $\mathfrak{D}/\mathfrak{D}x^a$ represents the covariant derivative
- A^a and A_a are the contravariant and covariant form of the four-potential
- η^{ab} is the metric tensor
- F^{ab} and F_{ab} are the contravariant and covariant form of the electromagnetic tensor
- L^{A-B} is the Aharonov-Bohm Lagrangian
- L^M is the Maxwell Lagrangian
- L' is the "other" Lagrangian
- S^{A-B} is the Aharonov-Bohm action

By minimizing the action, we obtain the Euler-Lagrange equation, from which we can obtain the modified versions of Maxwell's equations (137) and (140).

$$\delta S^{A-B} = \frac{1}{c} \int \left[\frac{1}{\mu_0} \cdot \frac{\mathfrak{D}F_{ab}}{\mathfrak{D}x_b} - \frac{1}{\mu_0} \cdot \frac{\mathfrak{D}}{\mathfrak{D}x^a}\left(\frac{\mathfrak{D}A^d}{\mathfrak{D}x^d}\right) + J_a\right] \delta A^a \sqrt{-\eta} d\Omega \quad (156)$$

$$\frac{\mathfrak{D}F_{ab}}{\mathfrak{D}x_b} = \frac{\mathfrak{D}}{\mathfrak{D}x^a}\left(\frac{\mathfrak{D}A^d}{\mathfrak{D}x^d}\right) - \mu_0 \cdot J_a = \frac{\mathfrak{D}\Lambda}{\mathfrak{D}x^a} \quad (157)$$

$$\Lambda = \frac{\mathfrak{D}A^d}{\mathfrak{D}x^d} \quad (158)$$

$$\frac{\mathfrak{D}F_{ab}}{\mathfrak{D}x_b} = \frac{\mathfrak{D}^2 A_b}{\mathfrak{D}x^a \mathfrak{D}x_b} - \frac{\mathfrak{D}^2 A_a}{\mathfrak{D}x^b \mathfrak{D}x_b} \quad (159)$$

$$\frac{\mathfrak{D}^2 A_a}{\mathfrak{D}x^b \mathfrak{D}x_b} = \mu_0 \cdot J_a \quad (160)$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \frac{\partial \Lambda}{\partial t} \quad (161)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \cdot \vec{J} + \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\Lambda \quad (162)$$

The other two Maxwell equations can be derived through the Bianchi identity.

$$\frac{\mathfrak{D}F_{ab}}{\mathfrak{D}x^c} + \frac{\mathfrak{D}F_{ca}}{\mathfrak{D}x^b} + \frac{\mathfrak{D}F_{bc}}{\mathfrak{D}x^a} = \frac{\partial F_{ab}}{\partial x^c} + \frac{\partial F_{ca}}{\partial x^b} + \frac{\partial F_{bc}}{\partial x^a} = 0 \quad (163)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (164)$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \quad (165)$$

The Lagrangian density must meet several criteria to adequately describe the properties of the electromagnetic interaction. The Maxwell equations constructed considering the Lorenz gauge are called Maxwell equations. The interaction it creates is called the electro-magnetic (EM) interaction. The Maxwell equations constructed from Aharonov-Bohm Lagrangian are called modified Maxwell equations. The interaction it creates is called the electro-magnetic-scalar (EMS) interaction.

The first such criterion is the number of polarization degrees of freedom (PDF). For the full Lagrangian density of Aharonov-Bohm, if Lorenz gauge is chosen, it describes the well-known transverse electromagnetic interaction. The electric field strength vector $\vec{E}$ and the magnetic induction $\vec{B}$ basically have 3 PDF each. So far, there are a total of 6 PDF. We know that $\vec{k} \cdot \vec{E} = 0$ and $\vec{k} \cdot \vec{B} = 0$, which allows 1-1 PDF for $\vec{E}$ and $\vec{B}$-separately. We also know that $\vec{E} \cdot \vec{B} = 0$, but this does not change the number of PDF. The time component in the contravariant and covariant forms of the four vector is not considered a PDF. So the EM interaction has 2 PDF. Experimental results confirm this finding. As the next step, we examine what the situation is in the case of cancelling the Lorenz gauge. The electric field strength vector $\vec{E}$ and the magnetic induction $\vec{B}$ basically have 3 PDF each. Since Λ is a scalar function, it has 1 PDF. So far, there are a total of 7 PDF. We know that $\vec{k} \cdot \vec{E} = 0$, $\vec{k} \cdot \vec{B} = 0$ and $\vec{k} \cdot \vec{E} = \Lambda$, which allows 1 polarization PDF for $\vec{B}$ and 2 for $\vec{E}$. I would like to point out again that this designation of electric field strength vector is inconsistent. The electric field strength vector cannot be transverse and longitudinal at the same time. Λ still has 1 degree of polarization freedom. So the EMS interaction has 4 PDF.

The second criterion is the range of interaction. The range of the EM interaction is infinite, because the particles mediating the interaction have no mass. Experimental results confirm this finding. The range of the EMS interaction is also infinite, since the mass will automatically be equal to zero if if γ=1 and k=0 in equation (1).

The third criterion is invariance to the Lorenz gauge. This criterion is fulfilled in the case of the EM interaction, but not in the case of the interaction described by the modified Maxwell equations, since we chose to omit the Lorenz gauge.

The fourth criterion is invariance to the Lorentz transformation. The Lagrangian density must be invariant to the Lorentz transformation. If the Lagrangian density consists of several components, the components must be Lorentz invariant individually, as well as their sum. It can be shown that the Lorentz invariant quantities of the first component of the total Lagrangian density are: $2 \cdot \left(\vec{B}^2 - \frac{\vec{E}^2}{c^2}\right)$ and $-\frac{4}{c} \cdot \vec{B} \cdot \vec{E}$. It can be shown that the Lorentz invariant quantity of the second component of the total Lagrangian density is: Λ^2. Thus, the EM and EMS interactions are also invariant to the Lorentz transformation.

14. The electromagnetic stress-energy tensor and the conservation laws

In order to derive a consistent electromagnetic stress-energy tensor and energy momentum conservation laws, (Minotti & Modanese, Quantum Uncertainty and Energy Flux in Extended Electrodynamics, 2021) – referring as Minotti et al. - took the advantage of the expression of the Aharonov-Bohm Lagrangian in a general four-dimensional metric. This allows the electromagnetic stress-energy tensor of the fields, T_{ab}^{AB}, to be evaluated as (Landau & Lifshitz, The classical theory of fields, 1971):

$$\frac{1}{2}\sqrt{-\eta}T_{ab}^{AB} = \frac{\partial}{\partial\eta^{ab}}\left(\sqrt{-\eta}L^{AB}\right) - \frac{\partial}{\partial x^{c}}\left[\frac{\partial}{\partial(\partial\eta^{ab}/\partial x^{c})}\left(\sqrt{-\eta}L^{AB}\right)\right] \tag{166}$$

$$T_{ab}^{AB} = T_{ab}^{M} + T'_{ab} \tag{167}$$

$$T_{ab}^{M} = -\frac{1}{\mu_0}\cdot\left(F_{ac}\cdot F_{bd}\cdot\eta^{cd} - \frac{1}{4}F_{cd}\cdot F^{cd}\cdot\eta_{ab}\right) \tag{168}$$

$$T'_{ab} = \frac{1}{\mu_0}\cdot\left[A_a\cdot\frac{\partial\Lambda}{\partial x^{b}} + A_b\cdot\frac{\partial\Lambda}{\partial x^{a}} - \left(\frac{\Lambda^2}{2} + A^{e}\cdot\frac{\partial\Lambda}{\partial x^{r}}\right)\cdot\eta_{ab}\right] \tag{169}$$

From now Minotti et al. could specialize the evaluations in the metric of interest, Minkowski metric, and determine the energy and momentum laws by evaluation of the divergence of the energy tensor.

$$\frac{\partial T_{ab}^{AB}}{\partial x_b} = -F_{ab}\cdot J^{b} + A_a\cdot\frac{\partial J_d}{\partial x_b} \tag{170}$$

If one considers the fields interacting with matter, the latter described by an energy-tensor T_{ab}^{matter} , energy-momentum conservation requires that

$$\frac{\partial}{\partial x_b}\left(T_{ab}^{AB} + T_{ab}^{matter}\right) = 0 \tag{171}$$

In terms of three-dimensional vectors, the power of the fields on matter (power lost by the fields) is

$$w_{EMS} = \vec{J}\cdot\vec{E} - I\cdot\varphi \tag{172}$$

$$I = \left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) \tag{173}$$

While the Lorentz force per unit volume on matter is

$$\vec{f}_{EMS} = \rho\cdot\vec{E} + \vec{J}\times\vec{B} - \mathrm{I}\cdot\vec{A} \tag{174}$$

An interesting thing to note is that the potentials have a direct effect on matter when local conservation of charge is not fulfilled. Having obtained a symmetric tensor, no more problems with conservation of total angular momentum exist. Besides, the proportionality of (specific) energy flow and momentum of the fields is automatically satisfied (no more problems with the difference in signs of the scalar parts).

Minotti et al. calculated the energy density for the fields and the energy flow. Thus we have the energy conservation law.

$$u_{EMS} = \frac{1}{\mu_0} \cdot \left(\frac{\vec{E}^2}{2 \cdot c^2} + \frac{\vec{B}^2}{2} + \frac{\varphi}{c} \cdot \frac{\partial \Lambda}{\partial t} - \vec{A} \cdot \vec{\nabla} \Lambda - \frac{\Lambda^2}{2} \right) \tag{175}$$

$$\vec{S}_{EMS} = \frac{1}{\mu_0} \cdot \left(\vec{E} \times \vec{B} - \varphi \cdot \vec{\nabla} \Lambda + \vec{A} \cdot \frac{\partial \Lambda}{\partial t} \right) \tag{176}$$

$$\vec{\nabla} \cdot \vec{S}_{EMS} + \frac{du_{EMS}}{dt} + w = 0 \tag{177}$$

Minotti et al. calculated the Aharonov-Bohm stress tensor. Thus we have the momentum conservation law.

$$\begin{aligned} T_{\alpha\beta}^{AB} = -\frac{1}{\mu_0} \cdot \Bigg[\frac{1}{c^2} \cdot \left(E_\alpha \cdot E_\beta - \frac{|\vec{E}|^2}{2} \cdot \delta_{\alpha\beta} \right) + B_\alpha \cdot B_\beta - \frac{|\vec{B}|^2}{2} \cdot \delta_{\alpha\beta} + A_\alpha \cdot \frac{\partial \Lambda}{\partial x_\beta} \\ + A_\beta \cdot \frac{\partial \Lambda}{\partial x_\alpha} - \left(\frac{\Lambda^2}{2} + \frac{\varphi}{c^2} \cdot \frac{\partial \Lambda}{\partial t} + \vec{A} \cdot \vec{\nabla} \Lambda \right) \cdot \delta_{\alpha\beta} \Bigg] \end{aligned} \tag{178}$$

In this way, the components of the field momentum density vector $\vec{g}_{EMS}$ are:

$$g_\alpha = \frac{1}{\mu_0 \cdot c^2} \cdot \left(\vec{E} \times \vec{B} - \varphi \cdot \vec{\nabla} \Lambda + \vec{A} \cdot \frac{\partial \Lambda}{\partial t} \right)_\alpha \tag{179}$$

so that $\vec{g}_{EMS} = \vec{S}_{EMS}/c^2$, as it must. The three-dimensional symmetric tensor $T_{\alpha\beta}^{AB}$corresponds to the field tensor, le tus call it $\sigma_{\alpha\beta}$($\overleftrightarrow{\sigma}_{EMS}$in covariant representation), so that the momentum conservation is written as:

$$\frac{\partial \vec{g}_{EMS}}{\partial t} + \vec{\nabla} \cdot \overleftrightarrow{\sigma}_{EMS} + \vec{f}_{EMS} = 0 \tag{180}$$

It is interesting that the previously derived energy (121) and momentum (135) conservation laws are expressed purely in terms of the fields themselves and not the potentials. However these are mathematically correct, but does not have the correct interpretation in terms of energy density, energy flow, power and momentum over matter.

15. The invariance to the Lorentz transformation

In this chapter, I prove that the modified Maxwell equations are invariant to the Lorentz transformation. Assume two coordinate systems: K and K'. Let's assume that the system K' moves in a straight line with speed $\vec{v}$ in the x direction with respect to K and is not acted upon by a force or field. Then the two systems form an inertial system, i.e. K and K' are mechanically indistinguishable from each other. Furthermore, we use the fact that the speed of light in a vacuum is at the end of every relevant system.

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{181}$$

It can be shown that the coordinates are transformed as follows:

$$x = \gamma \cdot (x' + v \cdot t') \tag{182}$$
$$y' = y' \tag{183}$$
$$z' = z \tag{184}$$
$$t = \gamma \cdot \left(t' + \frac{v \cdot x'}{c^2}\right) \tag{185}$$

It can be shown that the partial derivatives transform as follows:

$$\frac{\partial}{\partial x} = \gamma \cdot \left(\frac{\partial}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial}{\partial t'}\right) \tag{186}$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \tag{187}$$
$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \tag{188}$$
$$\frac{\partial}{\partial t} = \gamma \cdot \left(\frac{\partial}{\partial t'} - v \cdot \frac{\partial}{\partial x'}\right) \tag{189}$$

It can be shown that the potentials are transformed as follows:

$$\varphi = \gamma \cdot (\varphi' + A'_x \cdot v) \tag{190}$$
$$A_x = \gamma \cdot \left(\frac{v}{c^2} \cdot \varphi' + A'_x\right) \tag{191}$$
$$A_y = A'_y \tag{192}$$
$$A_z = A'_z \tag{193}$$

As a first step, we determine the transformation of the magneto scalar potential Λ using the definition according to equation (17).

$$\Lambda = \frac{1}{c^2} \cdot \frac{\partial \varphi}{\partial t} + \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{194}$$

$$\Lambda = \frac{\gamma}{c^2} \cdot \left(\frac{\partial \varphi}{\partial t'} - v \cdot \frac{\partial \varphi}{\partial x'}\right) + \cdot \gamma \left(\frac{\partial A_x}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial A_x}{\partial t'}\right) + \frac{\partial A_y}{\partial y'} + \frac{\partial A_z}{\partial z'} \tag{195}$$

$$\Lambda = \frac{1}{c^2} \cdot \frac{\partial \varphi'}{\partial t'} + \frac{\partial A'_x}{\partial x'} + \frac{\partial A'_y}{\partial y'} + \frac{\partial A'_z}{\partial z'} \tag{196}$$

$$\Lambda = \Lambda' \tag{197}$$

It can be shown that the electric field strength components $\vec{\mathrm{E}}$ are transformed as follows:

$$E_x = E_x{}' \tag{198}$$
$$E_y = \gamma \cdot \left(E_y' + v \cdot B_z{}'\right) \tag{199}$$
$$E_z = \gamma \cdot \left(E_z' - v \cdot B_y{}'\right) \tag{200}$$

It can be shown that the magnetic induction components$\vec{\mathrm{B}}$ are transformed as follows:

$$B_x = B_x{}' \tag{201}$$
$$B_y = \gamma \cdot \left(B_y' - \frac{v}{c^2} \cdot E_z{}'\right) \tag{202}$$
$$B_z = \gamma \cdot \left(B_z' + \frac{v}{c^2} \cdot E_y{}'\right) \tag{203}$$

It can be shown that the current density components $\vec{\mathrm{J}}$ and the charge density ρ are transformed as follows:

$$J_x = \gamma \cdot (J_x' + \rho' \cdot v) \tag{204}$$
$$J_y = J_y{}' \tag{205}$$
$$J_z = J_z{}' \tag{206}$$
$$\rho = \gamma \cdot \left(\rho' + \frac{v}{c^2} \cdot J_x'\right) \tag{207}$$

As a second step, let's perform the transformation on equation (25):

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0} - \frac{\partial \Lambda}{\partial t} \tag{208}$$

$$\gamma \cdot \left(\frac{\partial E_x}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial E_x}{\partial t'}\right) + \frac{\partial E_y}{\partial y'} + \frac{\partial E_z}{\partial z'} = \\ \frac{1}{\varepsilon_0} \cdot \gamma \cdot \left(\rho' + \frac{v}{c^2} \cdot J'_x\right) - \gamma \cdot \left(\frac{\partial \Lambda}{\partial t'} - v \cdot \frac{\partial \Lambda}{\partial x'}\right) \tag{209}$$

$$\gamma \cdot \left(\frac{\partial E_x{}'}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial E_x{}'}{\partial t'}\right) + \gamma \cdot \left(\frac{\partial E_y{}'}{\partial y'} + v \cdot \frac{\partial B_z{}'}{\partial y'}\right) \\ + \gamma \cdot \left(\frac{\partial E_z{}'}{\partial z'} - v \cdot \frac{\partial B_y{}'}{\partial z'}\right) = \\ \frac{1}{\varepsilon_0} \cdot \gamma \cdot \left(\rho' + \frac{v}{c^2} \cdot J'_x\right) - \gamma \cdot \left(\frac{\partial \Lambda'}{\partial t'} - v \cdot \frac{\partial \Lambda'}{\partial x'}\right) \tag{210}$$

$$\frac{\partial E_x{}'}{\partial x'} + \frac{\partial E_y{}'}{\partial y'} + \frac{\partial E_z{}'}{\partial z'} = \tag{211}$$

$$= \frac{\rho'}{\varepsilon_0} - \frac{\partial \Lambda'}{\partial t'} + v \cdot \left(\frac{\partial B_y{}'}{\partial z'} - \frac{\partial B_z{}'}{\partial y'} + \frac{1}{c^2} \cdot \frac{\partial E_x{}'}{\partial t'} + \mu_0 \cdot J_x' + \frac{\partial \Lambda'}{\partial x'} \right)$$

The bracketed part of equation (211) is the x component of equation (29) set to 0. As a third step, perform the transformation on the x component of equation (29):

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 \cdot J_x + \frac{1}{c^2} \cdot \frac{\partial E_x}{\partial t} + \frac{\partial \Lambda}{\partial x} \tag{212}$$

$$\frac{\partial B_z}{\partial y'} - \frac{\partial B_y}{\partial z'} = \mu_0 \cdot J_x + \frac{1}{c^2} \cdot \gamma \cdot \left(\frac{\partial E_x}{\partial t'} - v \cdot \frac{\partial E_x}{\partial x'} \right) + \gamma \cdot \left(\frac{\partial \Lambda}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial \Lambda}{\partial t'} \right) \tag{213}$$

$$\gamma \cdot \left(\frac{\partial B_z{}'}{\partial y'} + \frac{v}{c^2} \cdot \frac{\partial E_y{}'}{\partial y'} \right) - \gamma \cdot \left(\frac{\partial B_y{}'}{\partial z'} - \frac{v}{c^2} \cdot \frac{\partial E_z{}'}{\partial z'} \right) = \mu_0 \cdot \gamma \cdot (J_x' + \rho' \cdot v) + \frac{1}{c^2} \cdot \gamma \cdot \left(\frac{\partial E_x{}'}{\partial t'} - v \cdot \frac{\partial E_x{}'}{\partial x'} \right) + \gamma \cdot \left(\frac{\partial \Lambda'}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial \Lambda'}{\partial t'} \right) \tag{214}$$

$$\frac{\partial B_z{}'}{\partial y'} - \frac{\partial B_y{}'}{\partial z'} = \mu_0 \cdot J_x' + \frac{1}{c^2} \cdot \frac{\partial E_x{}'}{\partial t'} + + \frac{1}{c} \cdot \frac{\partial E_x{}'}{\partial x'} + v \cdot \left(-\frac{1}{c^2} \cdot \frac{\partial E_x{}'}{\partial x'} - \frac{1}{c^2} \cdot \frac{\partial E_y{}'}{\partial y'} - \frac{1}{c^2} \cdot \frac{\partial E_z{}'}{\partial z'} + \mu_0 \cdot \rho' - \frac{1}{c^2} \cdot \frac{\partial \Lambda'}{\partial t'} \right) \tag{215}$$

The bracketed part of equation (215) is the x component of equation (25) set to 0. As a fourth step, perform the transformation on the y component of equation (29):

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 \cdot J_y + \frac{1}{c^2} \cdot \frac{\partial E_y}{\partial t} + \frac{\partial \Lambda}{\partial y} \tag{216}$$

$$\frac{\partial B_x}{\partial z'} - \gamma \cdot \left(\frac{\partial B_z}{\partial x'} - \frac{v}{c^2} \cdot \frac{\partial B_z}{\partial t'} \right) = \mu_0 \cdot J_y + \frac{1}{c^2} \cdot \gamma \cdot \left(\frac{\partial E_y}{\partial t'} - v \cdot \frac{\partial E_y}{\partial x'} \right) + \frac{\partial \Lambda}{\partial y'} \tag{217}$$

$$\frac{\partial B_x{}'}{\partial z'} - \gamma^2 \cdot \frac{\partial}{\partial x'} \left(B_z' + \frac{v}{c^2} \cdot E_y{}' \right) + \gamma^2 \cdot \frac{v}{c^2} \cdot \frac{\partial}{\partial t'} \left(B_z' + \frac{v}{c^2} \cdot E_y{}' \right) = \mu_0 \cdot J_y' + \frac{1}{c^2} \cdot \gamma^2 \cdot \frac{\partial}{\partial t'} \left(E_y' + v \cdot B_z{}' \right) - \gamma^2 \cdot \frac{v}{c^2} \cdot \frac{\partial}{\partial x'} \left(E_y{}' + v \cdot B_z' \right) + \frac{\partial \Lambda'}{\partial y'} \tag{218}$$

$$\frac{\partial B_x{}'}{\partial z'} - \frac{\partial B_z{}'}{\partial x'} \cdot \gamma^2 \cdot \left(1 - \frac{v^2}{c^2} \right) = \mu_0 \cdot J_y' + \frac{\gamma^2}{c^2} \cdot \left(1 - \frac{v^2}{c^2} \right) \cdot \frac{\partial E_y{}'}{\partial t'} + \frac{\partial \Lambda'}{\partial y'} \tag{219}$$

$$\frac{\partial B_x{}'}{\partial z'} - \frac{\partial B_z{}'}{\partial x'} = \mu_0 \cdot J_y' + \frac{1}{c^2} \cdot \frac{\partial E_y{}'}{\partial t'} + \frac{\partial \Lambda'}{\partial y'} \tag{220}$$

As a fifth step, perform the transformation on the z component of equation (29):

$$\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=\mu_0\cdot J_z+\frac{1}{c^2}\cdot\frac{\partial E_z}{\partial t}+\frac{\partial \Lambda}{\partial z} \tag{221}$$

$$\gamma\cdot\left(\frac{\partial B_y}{\partial x'}-\frac{v}{c^2}\cdot\frac{\partial B_y}{\partial t'}\right)-\frac{\partial B_x}{\partial y'}=\mu_0\cdot J_z+\frac{1}{c^2}\cdot\gamma\cdot\left(\frac{\partial E_z}{\partial t'}-v\cdot\frac{\partial E_z}{\partial x'}\right)+\frac{\partial \Lambda}{\partial z'} \tag{222}$$

$$\gamma^2\cdot\frac{\partial}{\partial x'}\left(B_y'-\frac{v}{c^2}\cdot E_z{}'\right)-\frac{v}{c^2}\cdot\gamma^2\cdot\frac{\partial}{\partial t'}\left(B_y'-\frac{v}{c^2}\cdot E_z{}'\right)-\frac{\partial B_x{}'}{\partial y'}=\mu_0\cdot J_z'+\frac{1}{c^2}\cdot\gamma^2\cdot\frac{\partial}{\partial t'}\left(E_z'-v\cdot B_y{}'\right)-\frac{v}{c^2}\cdot\gamma^2\cdot\frac{\partial}{\partial x'}\left(E_z'-v\cdot B_y{}'\right)+\frac{\partial \Lambda'}{\partial z'} \tag{223}$$

$$\frac{\partial B_y{}'}{\partial x'}\cdot\gamma^2\cdot\left(1-\frac{v^2}{c^2}\right)-\frac{\partial B_x{}'}{\partial y'}=\mu_0\cdot J_y'+\frac{\gamma^2}{c^2}\cdot\left(1-\frac{v^2}{c^2}\right)\cdot\frac{\partial E_z{}'}{\partial t'}+\frac{\partial \Lambda'}{\partial z'} \tag{224}$$

$$\frac{\partial B_y{}'}{\partial x'}-\frac{\partial B_x{}'}{\partial y'}=\mu_0\cdot J_z'+\frac{1}{c^2}\cdot\frac{\partial E_z{}'}{\partial t'}+\frac{\partial \Lambda'}{\partial z'} \tag{225}$$

It can be seen that equations (25) and (29) work in the K' system in the same way as in the K system. Equations (26) and (28) are also invariant to the Lorentz transformation, I leave the proof to the reader. Taking into account the invariance to the Lorentz transformation and the homogeneous version of equation (87), we can conclude that magneto scalar waves propagate at the speed of light relative to all reference systems. Let the plane wave normal vector $\hat{k}$ point in the x direction. Since the magnitude of $\hat{k}$ is 1, $\hat{k}$ can be $(1{,}0{,}0)$, or $(0{,}1{,}0)$, or $(0{,}0{,}1)$. It can be shown that the transformation of Maxwell's equations from one coordinate system to another selects $\hat{k}=(1{,}0{,}0)$, thus $\mathit{\Lambda}=E_x/c$. I would like to point out again that this notation of the electric field strength vector is inconsistent.

16. The curl free vector potential and its consequences

In this chapter, I present the potential and consequences of the curl-free vector. The Helmholtz decomposition uniquely decomposes any three-vector into longitudinal (L) and transverse (T) components (Griffiths, 2007). For example, the $\vec{J}$ current density takes the following general form:

$$\vec{J} = \vec{J}_L + \vec{J}_T = \vec{\nabla}\delta + \vec{\nabla} \times \vec{j} \tag{226}$$

, where $\vec{\nabla}\delta$ is the longitudinal component, $\vec{\nabla} \times \vec{j}$ is the transversal component. The $\vec{B}$ magnetic induction takes the following general form:

$$\vec{B} = \vec{B}_L + \vec{B}_T = \vec{\nabla} \times \left(\vec{A}_L + \vec{A}_T\right) = \vec{\nabla} \times \vec{\nabla}\sigma + \vec{\nabla} \times \vec{\nabla} \times \vec{a} \tag{227}$$

The rotation of the longitudinal member is automatically eliminated, the rotation of the transverse member is as follows:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{a} = \vec{\nabla}\vec{\nabla} \cdot \vec{a} - \nabla^2\vec{a}. \tag{228}$$

This means that the magnetic induction $\vec{B}$ has only a transverse component, ie: $\vec{B} = \vec{B}_T$. C. Monstein and J.P. Wesley, as well as Lee M. Hively and Andrew S. Loebl assume in their experiments that the vector potential is curl-free. This means that $\vec{B} = \vec{\nabla} \times \vec{A} = 0$. Then the vector potential can be defined as follows: $\vec{A} = \vec{\nabla}\alpha$. This means that the vector potential and its longitudinal component can be written in the same way as the gradient of a potential: $\vec{A} \to \vec{A}_L$. Here, the arrow means that the vector potential is transformed. Since the definition of the vector potential has changed, the electric field strength vector $\vec{E}$ built from it and the definition of the magneto scalar potential Λ will also change: $\vec{E} \to \vec{E}_L$ and $\Lambda \to \Lambda_L$.. We will see later that the definition of the current density vector J ⃗ will also change as follows: $\vec{J} \to \vec{J}_L$. If all this is true, not only the modified Maxwell equations need to be modified, but perhaps everything else as well. This does not mean that electrodynamics is bad, but that the consequences of the curl-free vector potential were taken into account and had to be adapted. Hereinafter, the Maxwell equations constructed from longitudinally transformed quantities are called longitudinal Maxwell equations. The interaction it creates is called the electro-scalar (ES) interaction.

16.1. The longitudinal Maxwell equations

As a first step, I introduce the longitudinal Maxwell equations.

$$\vec{\nabla}\cdot\vec{E}_L=\frac{\rho}{\varepsilon_0}-\frac{\partial\Lambda_L}{\partial t} \tag{229}$$

$$\vec{\nabla}\times\vec{E}_L=0 \tag{230}$$

$$\mu_0\cdot\vec{J}_L+\frac{1}{c^2}\cdot\frac{\partial\vec{E}_L}{\partial t}+\vec{\nabla}\Lambda_L=0 \tag{231}$$

$$\Lambda_L=\frac{1}{c^2}\cdot\frac{\partial\varphi}{\partial t}+\vec{\nabla}\cdot\vec{A}_L=\frac{1}{c^2}\cdot\frac{\partial\varphi}{\partial t}+\vec{\nabla}\cdot\vec{\nabla}\alpha=\frac{1}{c^2}\cdot\frac{\partial\varphi}{\partial t}+\nabla^2\alpha \tag{232}$$

$$\vec{E}_L=-\frac{\partial\vec{A}_L}{\partial t}-\vec{\nabla}\varphi=-\frac{\partial\vec{\nabla}\alpha}{\partial t}-\vec{\nabla}\varphi \tag{233}$$

This gives us the longitudinal Maxwell equations. Since $\vec{\nabla}\times\vec{E}_L=0$, therefore $\vec{E}_L$ can take the following form: $\vec{E}_L=-\vec{\nabla}\beta$. Then the scalar potentials can be written in the following form:

$$-\vec{\nabla}\beta=-\frac{\partial\vec{\nabla}\alpha}{\partial t}-\vec{\nabla}\varphi \tag{234}$$

Let's integrate both sides of the above equation and let the integration constant be zero.

$$-\beta=-\frac{\partial\alpha}{\partial t}-\varphi \tag{235}$$

It can be seen that both the vector potential and the electric field strength vector can be written as the gradient of a potential. So far, I have dealt with the differential form of the longitudinal Maxwell equations. In the next step, I define their integral form. Let's start from equations (229), (230), (231).

Take the volume integral of both sides of equation (229). We apply the Gauss theorem. Then we arrive at the following equation.

$$\oint_S\vec{E}_L\cdot\vec{n}dS=\frac{1}{\varepsilon_0}\int_V\rho dV-\frac{\partial}{\partial t}\int_V\Lambda_L dV \tag{236}$$

Take the surface integral of both sides of equation (231). We apply Stokes' theorem. Then we arrive at the following equation.

$$\oint_C\vec{E}_L d\vec{r}=0 \tag{237}$$

Since $\vec{E}_L$ can be written as the gradient of a potential (multiplied by (-1)) according to equation 156.) and the integral of $\vec{E}_L$over the closed curve is equal to zero, it can be said that $\vec{E}_L$is conservative. Only the start and end points matter. The condition also includes the fact that $\vec{E}_L$ must be uniquely connected. In my opinion, this is accomplished.

Take the curve integral of both sides of equation (231). We apply the gradient theorem. Then we arrive at the following equation.

$$\mu_0 \int_C \vec{J}_L \vec{n} \mathrm{d}\mathrm{r} + \frac{1}{c^2} \cdot \frac{\partial}{\partial t} \int_C \vec{E}_L \cdot \vec{n} \mathrm{d}\mathrm{r} + \oint_t \left(\vec{\nabla} \Lambda_L\right) \cdot \vec{v} dt = 0 \tag{238}$$

With this, I derived the integral form of the longitudinal Maxwell equations.

Let's continue the investigation with equation (231). Let's take the integral along a closed curve of both sides of equation (231). Let's move from left to right. In the case of the first term, we can apply the Stokes theorem and the statement made in (here). I derived the current density. It turns out that the first term will be zero. For the second term, we can apply equation (237), according to which the second term will also be zero. Only the third member remains. Multiply both sides by (-1).

$$-\oint_C \vec{\nabla} \Lambda_L d\vec{r} = 0 \tag{239}$$

Let $\vec{G} = -\vec{\nabla} \Lambda_L$. Since the vector potential $\vec{G}$ can be written as the gradient of a scalar potential Λ_L (multiplied by (-1)) and the integral over the closed curve of the vector potential $\vec{G}$ is equal to zero, it can be said that $\vec{G}$is conservative. Only the start and end points matter. The condition also includes the fact that $\vec{G}$ must be uniquely connected. In my opinion, this is accomplished.

16.2. The longitudinal continuity equation

In this chapter, I derive the longitudinal continuity equation. (here) I start from the longitudinal Maxwell equations and check whether the result will be the same as in (here) with quaternions. Also, don't forget about the curl-free potential.

Derive equation (229) with respect to time, form the divergence of both sides of equation (231) and use equation (24). Finally, relations (229) and (231) are added together.

$$\frac{1}{c^2} \cdot \frac{\partial^2 \Lambda_L}{\partial t^2} - \nabla^2 \Lambda_L = \mu_0 \cdot \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}_L\right) \tag{240}$$

Equation (240) has a double meaning. On the one hand, charges can be sources and sinks of magneto scalar waves, and on the other hand, magneto scalar waves can be sources and sinks of charges. According to the equation, the charge is a locally non-conserving quantity.

We form the time derivative of equation (231). We form the rotation of equation (230). We form the gradient of both sides of the equation (229). Substitute equation (230) into equation (229). Add equation (229) to equation (231). Finally, we use relation (24).

$$\frac{1}{c^2}\cdot\frac{\partial^2\vec{E}_L}{\partial t^2}-\nabla^2\vec{E}_L=-\frac{1}{\varepsilon_0}\cdot\left(\frac{1}{c^2}\cdot\frac{\partial\vec{J}_L}{\partial t}+\vec{\nabla}\rho\right) \qquad (241)$$

Assume that equations (240) and (241) have no source. Then we get back the conservation equations (46) and (47) obtained by the quaternions.

Finally, let the third equation follow. Divide both sides of equation (231) by μ_0. Take the curl of equation (231). Using equation (230) and the identity $\vec{\nabla}\times\vec{\nabla}\Lambda_L=0$, we will get that the longitudinal current density is curl-free.

$$\vec{\nabla}\times\vec{J}_L=0 \qquad (242)$$

This equation is the same as equation (48) obtained with quaternions. This means that $\vec{J}_L$ can be written as the gradient of a potential: $\vec{J}_L=\vec{\nabla}\kappa$.

16.3. The static magneto scalar field

When examining the static magneto scalar field (taking Helmholtz theory into account), as a first step, if we omit the time derivative component of equation (231), integrate both sides of the equation and omit the integration constant, we arrive at the following relationship:

$$\Lambda(\vec{r})=-\mu_0\cdot\int\ \vec{J}_L(\vec{r})d\mathrm{r} \qquad (243)$$

Since $\vec{J}_L=\vec{\nabla}\kappa$, therefore, equation (243) takes the following form:

$$\Lambda=-\mu_0\cdot\kappa \qquad (244)$$

Since the spatial quantities remain constant in time, we omit the time derivative components of equations (231) and (232) and form the gradient of both sides of equation (232) and use the $\vec{\nabla}\times\vec{\nabla}\times\vec{A}_L=\vec{\nabla}\vec{\nabla}\cdot\vec{A}_L-\nabla^2\vec{A}_L$ identity. Since $\vec{\nabla}\times\vec{A}_L=0$, therefore $\vec{\nabla}\vec{\nabla}\cdot\vec{A}_L=\nabla^2\vec{A}_L$. Substituting this into equation (231), we arrive at the following relationship:

$$\vec{\nabla}\Lambda=\vec{\nabla}\vec{\nabla}\cdot\vec{A}_L=\nabla^2\vec{A}_L=-\mu_0\cdot\vec{J}_L \qquad (245)$$

Then we get equation (69) for the longitudinal quantities. Since $\vec{J}_L=\vec{\nabla}\kappa$ and $\vec{A}_L=\vec{\nabla}\alpha$, substituting these into (245), integrating both sides and choosing the integration constant as 0, we arrive at the following relationship:

$$-\nabla^2\alpha = \mu_0 \cdot \kappa \tag{246}$$

The shape of the scalar potential α takes the following form:

$$\alpha(\vec{r}) = \frac{\mu_0}{4 \cdot \pi} \int_V \frac{\kappa(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' \tag{247}$$

As a next step, I prove that $\vec{A}_L$ indeed curl-free. Let's take the gradient of equation (247)

$$\vec{\nabla}\alpha(\vec{r}) = \frac{\mu_0}{4 \cdot \pi} \int_V \frac{\vec{\nabla}\kappa(\vec{r}')}{|\vec{r} - \vec{r}'|} dV' + \frac{\mu_0}{4 \cdot \pi} \int_V \kappa(\vec{r}') \cdot \vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) dV' \tag{248}$$

Let's take the rotation of equation (248).

$$\begin{gathered} \vec{\nabla} \times \vec{\nabla}\alpha(\vec{r}) = \\ \frac{\mu_0}{4 \cdot \pi} \int_V \frac{\vec{\nabla} \times \left(\vec{\nabla}\kappa(\vec{r}')\right)}{\left|\vec{r} - \vec{r}^{'}\right|} dV^{'} \\ -\frac{\mu_0}{4 \cdot \pi} \int_V \vec{\nabla}\kappa(\vec{r}') \times \vec{\nabla}\left(\frac{1}{\left|\vec{r} - \vec{r}^{'}\right|}\right) dV' \\ +\frac{\mu_0}{4 \cdot \pi} \int_V \kappa(\vec{r}') \cdot \vec{\nabla} \times \vec{\nabla}\left(\frac{1}{|\vec{r} - \vec{r}'|}\right) dV' \end{gathered} \tag{249}$$

$$-\frac{\mu_0}{4 \cdot \pi} \int_V \vec{\nabla}\left(\frac{1}{\left|\vec{r} - \vec{r}^{'}\right|}\right) \times \vec{\nabla}\kappa(\vec{r}') dV' = 0$$

16.4. The magneto scalar waves

In the case of magneto scalar waves (taking into account the curl-free vector potential), the first step is to derive the inhomogeneous wave equation for Λ_L. Derive equation (229) with respect to time, form the divergence of both sides of equation (231) and use equation (24). Finally, relations (229) and (231) are added together.

$$\frac{1}{c^2} \cdot \frac{\partial^2 \Lambda_L}{\partial t^2} - \nabla^2 \Lambda_L = \mu_0 \cdot \left(\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}_L\right) \tag{250}$$

As the next step, I derive the inhomogeneous wave equation for $\vec{E}_L$.

We form the time derivative of equation (231). We form the rotation of equation (230). We form the gradient of both sides of the equation (229). Substitute equation (230) into equation (229). Add equation (229) to equation (231). Finally, we use relation (24).

$$\frac{1}{c^2} \cdot \frac{\partial^2 \vec{E}_L}{\partial t^2} - \nabla^2 \vec{E}_L = -\frac{1}{\varepsilon_0} \cdot \left(\frac{1}{c^2} \cdot \frac{\partial \vec{J}_L}{\partial t} + \vec{\nabla}\rho\right) \tag{251}$$

As the next step, I derive the inhomogeneous wave equation for φ . Take equations (232) and (233) and substitute them into equation (229). Finally, we form the equation (-1) times (229).

$$\frac{1}{c^2}\cdot\frac{\partial^2\varphi}{\partial t^2}-\nabla^2\varphi=\frac{\rho}{\varepsilon_0} \tag{252}$$

As the next step, I derive the inhomogeneous wave equation for $\vec{A}_L$. Let's take equations (232), (233) and then substitute them into equation (231). Finally, we form the equation (-1) times (231).

$$\frac{1}{c^2}\cdot\frac{\partial^2\vec{A}_L}{\partial t^2}-\nabla^2\vec{A}_L=\mu_0\cdot\vec{J}_L \tag{253}$$

Since $\vec{J}_L=\vec{\nabla}\kappa$ and $\vec{A}_L=\vec{\nabla}\alpha$, substituting these into equation (253), integrating both sides and choosing the integration constant as 0, we arrive at the following relationship:

$$\frac{1}{c^2}\cdot\frac{\partial^2\alpha}{\partial t^2}-\nabla^2\alpha=\mu_0\cdot\kappa \tag{254}$$

The shape of the scalar potential α takes the following form:

$$\alpha(\vec{r},t)=\frac{\mu_0}{4\cdot\pi}\int_V\frac{\kappa(\vec{r}',t')}{|\vec{r}-\vec{r}'|}dV' \tag{255}$$

16.5. The longitudinal Lagrangian density

In this chapter, I derive the longitudinal Lagrangian density.

$$L=L(\vartheta_i,\dot{\vartheta}_i,\vec{\nabla}\vartheta_i) \tag{256}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial(\dot{\vartheta}_i)}\right)+\vec{\nabla}\cdot\left(\frac{\partial L}{\partial(\vec{\nabla}\vartheta_i)}\right)-\frac{\partial L}{\partial\vartheta_i}=0 \tag{257}$$

$$\begin{array}{lll} \vartheta_1=A_1(x_1x_2,x_3,t) & \dot{\vartheta}_1=\dot{A}_1 & \vec{\nabla}\vartheta_1=\vec{\nabla}A_1 \\ \vartheta_2=A_2(x_1x_2,x_3,t) & \dot{\vartheta}_2=\dot{A}_2 & \vec{\nabla}\vartheta_2=\vec{\nabla}A_2 \\ \vartheta_3=A_3(x_1x_2,x_3,t) & \dot{\vartheta}_3=\dot{A}_3 & \vec{\nabla}\vartheta_3=\vec{\nabla}A_3 \\ \vartheta_4=\varphi(x_1x_2,x_3,t) & \dot{\vartheta}_4=\dot{\varphi} & \vec{\nabla}\vartheta_4=\vec{\nabla}\varphi \end{array} \tag{258}$$

Substitute equations (232) and (233) into equation (229). Multiply both sides of the equation by (-1).

$$\frac{\partial}{\partial t}\left(-\frac{1}{c^2}\cdot\frac{\partial\varphi}{\partial t}\right)+\vec{\nabla}\cdot(\vec{\nabla}\varphi)-\left(-\frac{\rho}{\varepsilon_0}\right)=0 \tag{259}$$

Let i=4 and $\vartheta_4=\varphi$. Substitute it into the Euler-Lagrange equation (257). Let's compare it with equation (259). Then we get the following derivatives.

$$\frac{\partial L}{\partial(\dot{\varphi})} = -\frac{1}{c^2} \cdot \frac{\partial \varphi}{\partial t} \qquad \frac{\partial L}{\partial(\vec{\nabla}\varphi)} = \vec{\nabla}\varphi \qquad \frac{\partial L}{\partial \varphi} = -\frac{\rho}{\varepsilon_0} \tag{260}$$

The components of the Lagrangian density of the equation (260) are determined by integration from left to right according to $\dot{\varphi}$, $\vec{\nabla}\varphi$ and φ.

$$L_{a_1} = -\frac{1}{2} \cdot \frac{1}{c^2} \cdot (\dot{\varphi})^2 \qquad L_{a_2} = \frac{1}{2} \cdot \left(\vec{\nabla}\varphi\right)^2 \qquad L_{a_3} = -\frac{\rho \cdot \varphi}{\varepsilon_0} \tag{261}$$

The individual Lagrangian density components are added together.

$$L_a = L_{a_1} + L_{a_2} + L_{a_2} = -\frac{1}{2} \cdot \frac{1}{c^2} \cdot (\dot{\varphi})^2 + \frac{1}{2} \cdot \left(\vec{\nabla}\varphi\right)^2 - \frac{\rho \cdot \varphi}{\varepsilon_0} \tag{262}$$

Substitute equations (232) and (233) into equation (231). Multiply both sides of the equation by (-1) and c^2.

190.)

$$\frac{\partial}{\partial t}\left(\frac{\partial \vec{A}_L}{\partial t}\right) + \vec{\nabla}\left(-c^2 \cdot \vec{\nabla} \cdot \vec{A}_L\right) - \frac{\vec{J}_L}{\varepsilon_0} = 0 \tag{263}$$

Let i=j=1,2,3 and $\vartheta_j = A_j$. Substitute it into the Euler-Lagrange equation (257). Let's compare it with equation (263). Then we get the following derivatives.

$$\frac{\partial L}{\partial(\dot{A}_j)} = \frac{\partial A_j}{\partial t} \qquad \frac{\partial L}{\partial(\vec{\nabla}A_j)} = -c^2 \cdot \vec{\nabla}A_j \qquad \frac{\partial L}{\partial A_j} = \frac{J_j}{\varepsilon_0} \tag{264}$$

The components of the Lagrangian density of the equation (264) are determined by integration from left to right according to $\dot{A}_j$, $\vec{\nabla}A_j$ and A_j.

$$L_{b_1} = \frac{1}{2} \cdot \left(\dot{A}_j\right)^2 \qquad L_{b_2} = -\frac{1}{2} \cdot c^2\left(\vec{\nabla}A_j\right)^2 \qquad L_{b_3} = \frac{A_j \cdot J_j}{\varepsilon_0} \tag{265}$$

The individual Lagrangian density components are added together.

$$L_b = L_{b_1} + L_b + L_{b_2} = \frac{1}{2} \cdot \left(\dot{A}_j\right)^2 - \frac{1}{2} \cdot c^2\left(\vec{\nabla}A_j\right)^2 + \frac{A_j \cdot J_j}{\varepsilon_0} \tag{266}$$

Add the Lagrange density components (262) and (266) and rearrange the equation. Compress the A_j components into $\vec{A}_L$.

$$\begin{gathered} L = L_a + L_b = \\ -\frac{1}{2} \cdot \frac{1}{c^2} \cdot (\dot{\varphi})^2 + \frac{1}{2} \cdot \left(\vec{\nabla}\varphi\right)^2 - \frac{\rho \cdot \varphi}{\varepsilon_0} \\ + \frac{1}{2} \cdot \left(\dot{\vec{A}}_L\right)^2 - \frac{1}{2} \cdot c^2 \cdot \left(\vec{\nabla} \cdot \vec{A}_L\right)^2 + \frac{\vec{A}_L \cdot \vec{J}_L}{\varepsilon_0} = \\ \frac{1}{2} \cdot \left(-\vec{\nabla}\varphi - \dot{\vec{A}}_L\right)^2 - \vec{\nabla}\varphi \cdot \dot{\vec{A}}_L - \frac{1}{2} \cdot c^2 \cdot \left(\frac{1}{c^2} \cdot \dot{\varphi} + \vec{\nabla} \cdot \vec{A}_L\right)^2 + \dot{\varphi} \cdot \vec{\nabla} \cdot \vec{A}_L \\ + \frac{\vec{J}_L \cdot \vec{A}_L}{\varepsilon_0} - \frac{\rho \cdot \varphi}{\varepsilon_0} \end{gathered} \tag{267}$$

Since the Lagrangian density must have an energy density dimension, L must be multiplied by ε_0.

$$L_{wave} = \varepsilon_0 \cdot L = \frac{1}{2} \cdot \varepsilon_0 \cdot \vec{E_L}^2 - \frac{1}{2} \cdot \frac{1}{\mu_0} \cdot \Lambda_L^2 \\ +\varepsilon_0 \cdot \left(\dot{\varphi} \cdot \vec{\nabla} \cdot \vec{A}_L - \vec{\nabla}\varphi \cdot \dot{\vec{A}}_L\right) + \left(\vec{J}_L \cdot \vec{A}_L - \rho \cdot \varphi\right) \tag{268}$$

In equation (268), the first two terms form the Lagrangian density of free field, the other two terms form the Lagrangian density of the interaction. When I substitute the longitudinal electric field strength and the magneto scalar potential into the longitudinal Maxwell equations by definition, we get the wave equations for the scalar and vector potential. This Lagrangian density does not give us the longitudinal Maxwell equations, only the wave equations.

I announce without derivation that, if we subtract the first new interaction term from the L_{wave} and substitute it into the Euler-Lagrange equation (257), we will get the longitudinal Maxwell equations (229) and (231) separately for φ and $\vec{A}_L$.

$$L_{longitudinal} = L_{wave} - \varepsilon_0 \cdot \left(\dot{\varphi} \cdot \vec{\nabla} \cdot \vec{A}_L - \vec{\nabla}\varphi \cdot \dot{\vec{A}}_L\right) \tag{269}$$

This gives us the longitudinal Lagrangian density.

As a next step, I define the longitudinal Lagrangian density using a covariant formalism. The magnetic induction components of the electromagnetic tensor can be replaced by zeros, since the $\vec{A}_L$ vector potential is curl-free. The definition of the electromagnetic tensor remains the same.

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & 0 & 0 \\ -\frac{E_y}{c} & 0 & 0 & 0 \\ -\frac{E_z}{c} & 0 & 0 & 0 \end{pmatrix} \tag{270}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & 0 & 0 \\ \frac{E_y}{c} & 0 & 0 & 0 \\ \frac{E_z}{c} & 0 & 0 & 0 \end{pmatrix} \tag{271}$$

After that, write down the longitudinal Lagrange density. Without further ado, I announce that by minimizing the effect, we get the Euler-Lagrange equation from which we get back the longitudinal Maxwell equations (229) and (231). The longitudinal Maxwell equation (230) can be derived by means of the Bianchi identity I leave the proof to the reader.

As the next step, we examine how many PDF are allowed by the interaction described by the longitudinal Maxwell equations. The longitudinal electric field strength vector $\vec{E}_L$ has 3 pdf. Since Λ_L is a scalar function, it has 1 PDF. So far, there are a total of 4 PDF. We know that $\vec{k} \cdot \vec{E}_L = \Lambda_L$, which allows 1 PDF for $\vec{E}_L$. Λ still has 1 PDF. So the ES interaction has 2 PDF.

As the next step, we examine the range of the ES interaction. The range of the ES interaction is infinite, since the mass will automatically be equal to zero if γ=1 and k=0 in equation (1).

As the next step, we examine whether the longitudinal Maxwell equations are invariant to the Lorenz gauge. In the case of the ES interaction, it is not fulfilled since we have chosen to leave out the Lorenz gauge.

As the next step, we examine whether the longitudinal Maxwell equations are invariant to the Lorentz transformation. It can be shown that the Lorentz invariant quantity of the first component of the longitudinal Lagrange density is as follows: $-\frac{\vec{E}_L{}^2}{c^2}$. It can be shown that the Lorentz invariant quantity of the second component of the longitudinal Lagrangian density is as follows: $\Lambda_L{}^2$. Therefore, the interaction constructed from the longitudinal Maxwell equations is also invariant to the Lorentz transformation.

17. Practical considerations for an SLW antenna

In this chapter, I present a possible experimental implementation that brings us one step closer to understanding magneto scalar waves (scalar longitudinal wave, slw), (Hively & Loebl, Classical and extended electrodynamics, 2019), (Reed & Hively, 2020). In the case of electromagnetic radiation, we use the Lorenz measure. In the case of electro-scalar radiation, the curl-free vector potential is used.

Electromagnetic radiation	Electro-scalar radiation
$\vec{\nabla} \times \vec{E} \neq 0$ $\vec{B} \neq 0$ $\Lambda = 0$	$\vec{\nabla} \times \vec{E}_L = 0$ $\vec{B} = 0$ $\Lambda_L \neq 0$

Table 2

How can one build one and what requirements must it meet?

An antenna of this type consists of two spiral conductors connected in the middle of the coil, causing currents in opposite directions in adjacent turns. These conductors are indicated by solid (702) and dashed lines. The magnetic field is canceled by currents in opposite directions. In the case of time-varying current strength, classical electromagnetic radiation cannot be generated either. This creates the conditions for gradient-driven current density.

This coil (non-inductive bifilar coil) is a two-dimensional monopole antenna where

- the inductance is zero (due to the opposite electric currents)
- the capacity is zero (adjacent threads have the same electric charge density)
- the charge density and current density show spatial periodicity along the length of the conductor
- the coil length must be less than λ/10 (United States Patent No. US9306527B1, 2016).

Another such antenna consists of two parts. The outer conductor (204) is electrically connected to the top of the skirt balun (206) with a length (λ/4) that is 206 (0 degrees) in current from the bottom (inner surface) to the top (inner surface) (206) (90 degrees) and back (204) (180 degrees). The 180 degree phase shift eliminates the return current along 204, forming a monopole antenna. Essentially all of the current goes to charge and discharge the center conductor (202) without creating any bias current. The resulting current density is curl-free, which is a necessary condition for generating magneto scalar and longitudinal electric waves. An eddy-free current density does not create a magnetic field (United States Patent No. US9306527B1, 2016).

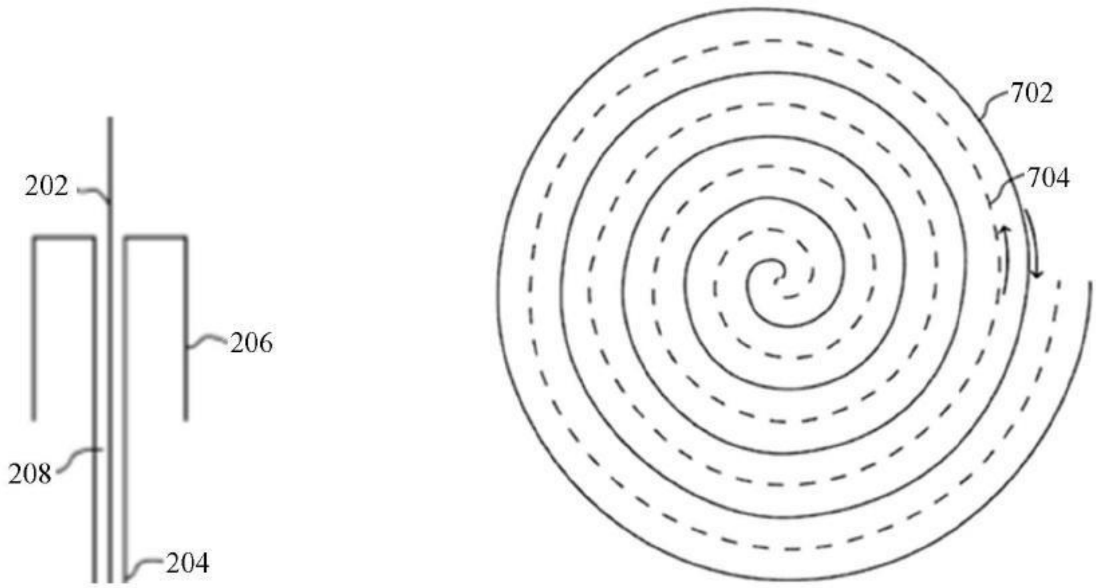

Figure 2 (Hively, 2016)

My problem with the explanation for the measurement arrangement on the right of Figure 2 is that, according to the argument, the two parallel conductors separately create the magnetic field around them. Magnetic fields cancel each other out. So, the magnetic fields are already created. Since the magnetic fields have cancelled each other out, another field cannot appear, because that would contradict the principle of conservation of energy. Since magnetic fields are created, this means that the electromagnetic interaction, which prefers conservation of energy, is making its way. The other thing is that electric and magnetic waves create each other, the two phenomena are one and the same, one cannot exist without the other. From this we can draw the conclusion that the requirement of a curl-free potential does not hold. In the case of the electro-scalar interaction, energy is a non-conservative quantity. Conditions must be created in advance for this type of interaction to make its way.

In the measurement arrangement on the left of Figure 2, if the current is reflected from the upper part of the outer conductor with a phase shift of 180 degrees and extinguishes the upward current (more precisely, a standing wave is generated). In my opinion, the inner conductor can still create a magnetic field around itself. Also, I don't understand how the displacement current would be extinguished. However, there are no measurement results on the conservation (or damage) of charge and energy, at least they do not mention it. The question may arise in us, whether we can change the nature of the electromagnetic interaction just by properly designed experimental conditions?

Based on what has been described so far, we do not have clear evidence of the existence of electro-scalar radiation (in macroscopic scale).

18. Application

In this chapter, I present a possible application. In relativistic quantum field theory with local interactions, charge is locally conserved. However, when we detach ourselves a little bit from the paradigm of interactions between individual particles, some problems typical of quantum mechanics appear. Quantum mechanics is wider than quantum field theory. But in fact, some useful quantum mechanical models exist, which are explicitly non-local. More generally, non-local field theory is a wide subject. Non-local models can originate from coarse graining effective approximations of local models, or they can be conceived even at a more fundamental level than local theories (Kegeles & Oriti, 2016). One of these models is the fractional Schrödinger equation (Laskin, 2002), (Lenzi, et al., 2008).

$$i \cdot \hbar \cdot \frac{\partial \psi}{\partial t} = \left(\frac{\vec{p}^{\alpha}}{2 \cdot m} + V\right) \cdot \psi \quad 1 < \alpha < 2 \tag{272}$$

In (Wei, 2016) it can be seen that its continuity equation contains an anomalous source term, called extra-current. The fractional Schrödinger equation has recently found novel important applications in optics, in particular concerning the behavior of wavepackets in a harmonic potential (Zhang, et al., 2015). Another well-known non-local wave equation of the Schrödinger kind is the equation, where the non-local interaction potential.

$$\begin{gathered} i \cdot \hbar \cdot \frac{\partial \psi(x,t)}{\partial t} = \\ -\frac{\hbar^2}{2 \cdot m} \cdot \frac{\partial^2}{\partial x^2} \psi(x,t) + \int_0^t d\tau \int_{-\infty}^{+\infty} dy \, U(x-y, t-\tau) \cdot \psi(y,\tau) \end{gathered} \tag{273}$$

Its solutions have been studied in (Lenzi, et al., 2008), (Sandev, Petreska, & Lenzi, Time-dependent Schrödinger-like equation with nonlocal term, 2014), (Sandev, Petreska, & Lenzi, Effective potential from the generalized time-dependent Schrödinger equation, 2016). Its continuity equation contains an anomalous source term, called extra-current. Possible applications include the dissipative quantum transport processes in quantum dots (Baraff, 1998), (Ferry & Barker, 1999). The Gorkov equation for the superconducting order parameter also has a non-local form.

$$\mathrm{F(x)} = \int d^3 y K(x,y) F(y) \tag{274}$$

Its continuity equation contains an anomalous source term, called extra-current. The Gorkov equation has been used to analyze the proximity effect in superconductors (the diffusion of the order parameter into an adjacent normal material), especially for those cases where the microscopic BCS approach is not applicable. This equation reduces to a local Ginzburg Landau equation only under certain assumptions, which has a locally conserved current. It is not known yet whether the full non-local equation leads to an extra-current, because the form of the kernel K varies considerably and depends on several microscopic assumptions. Nevertheless, this equation deserves further analysis because it can describe Josephson junctions. For some condensed matter systems, admitting discontinuities in the probability current is the cause of the violation of local conservation. This leads us to ask a logical theoretical question: how should one compute the electromagnetic field of a microscopic current that is not locally conserved? In such cases, the Aharonov-Bohm theory offers a logically consistent approach. It is known that the Maxwell equations are only compatible with a locally conserved current. In four-dimensional form, they are written as $\partial_{\mu} F^{\mu\nu} = j^{\nu}$. Since the tensor $F^{\mu\nu}$ is antisymmetric, it follows as a necessary condition that $\partial_\mu j^\mu = 0$. This is the covariant form of the continuity equation. For quantum systems lacking a locally conserved current, one can employ an extension of Maxwell's equations: $\partial_\mu F^{\mu\nu} = j^\nu + i^\nu$. According to (Modanese G. , 2017) , i^ν is a secondary, additional current, which is obtained by applying the non-local operator $\partial^{-2} = (\partial^\alpha \partial_\alpha)^{-2}$ to the term which breaks the local conservation:

$$i^\nu = -\partial^\nu \partial^{-2}(\partial_\gamma j^\gamma) \tag{275}$$

Because of the antisymmetry of $F^{\mu\nu}$, the field equations automatically imply that $\partial_\nu(j^\nu + i^\nu) = 0$. The extended Maxwell equations have an important censorship property, which imply that measurements of the field strength made with test particles cannot reveal any possible local non-conservation of the source. The apparent conservation is due to the secondary source. Typical electromagnetic signatures of the failure of local conservation are at high frequency the generation of a longitudinal electric radiation field, and at low frequency a small effect of "missing" magnetic field (Minotti & Modanese, Electromagnetic Signatures of Possible Charge Anomalies in Tunneling, 2022). There is a possible way of formulating phenomenological models for example in (Minotti & Modanese, Electromagnetic Signatures of Possible Charge Anomalies in Tunneling, 2022), which we refer here as the γ model. This model can conceive based on a single parameter, γ. It is considered a candidate to account for non-local conservation of charge in material media is the tunnel effect of electrons. In (Minotti & Modanese, Electromagnetic Signatures of Possible Charge Anomalies in Tunneling, 2022) the authors are interested into tunneling phenomena having relatively long range and involving macroscopic wavefunctions and coherent matter, for which it makes sense to evaluate the classical electromagnetic field generated by the tunneling particles. The tunneling effect is one of the most typical features of quantum mechanics, and a clear demonstration of the wavelike behavior of matter at a microscopic scale. Tunneling processes involving macroscopic wavefunctions in superconductors give rise to the Josephson effect (Josephson, 1962). In physics, the Josephson effect is a phenomenon that occurs when two superconductors are placed in proximity, with some barrier or restriction between them. The Josephson effect produces a current, known as a supercurrent, that flows continuously without any voltage applied, across a device known as a Josephson junction (JJ). These consist of two or more superconductors coupled by a weak link. The weak link can be a thin insulating barrier (known as a superconductor–insulator–superconductor junction, or S-I-S), a short section of non-superconducting metal (S-N-S), or a physical constriction that weakens the superconductivity at the point of contact (S-c-S) (Josephson effect, 2023). In most cases the field is not detectable, because the number of tunneling particles is small, and their current too. The γ model starts with the assumption that when current circulates in a conductor part of it is due to resonant tunneling of bound electrons close to the Fermi level, and that this tunneling can be interpreted as a discontinuous transport of those electrons across the classically forbidden zones. Minotti et al showed that the effective/averaged consequence of the microscopic discontinuities is to generate an extra source moment $\delta\vec{P}$ is proportional to the average macroscopic density $\vec{J}$ by the follong way: $\delta\vec{P} = \gamma \cdot \vec{J} \cdot \delta V$. The coefficient γ is in turn proportional to the number of discontinuities per unit volume, to their average length and average current affected. Considering that it is expected that $\gamma \ll 1$, $\vec{J}$ using Maxwell equations, and then evaluate the anomalous effects due to the dipolar moments of the extra sources by adding up the effect of each elementary dipole:

$$\Lambda(\vec{r},\mathrm{t}) = \frac{\mu_0 \cdot \gamma}{4 \cdot \pi} \int_V \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \cdot \vec{J}(\vec{r}',\mathrm{t}') dV' \tag{276}$$

$$\vec{E}_L(\vec{r},\mathrm{t}) = \frac{\mu_0 \cdot \gamma}{4 \cdot \pi} \int_V \frac{\vec{J}(\vec{r}',\mathrm{t}')}{|\vec{r} - \vec{r}'|} \cdot dV' \tag{277}$$

This method can be applied to the resonating cavity with a mode $TM_0 10$ and allows to compute the anomalous longitudinal electric field $\vec{E}_L$ in the far field region along the axis of the cavity. The explicit expressions for the extra-current and current density found in this work (Modanese G. , Time in quantum mechanics and the local non conservation of the probability current, 2018) serve as guidance for non-local quantum sources for extended Maxwell equations. More realistic sources could be obtained, as mentioned, from the Gorkov equation.

19. Quantum mechanical aspect

In this chapter, I present the quantum mechanical aspect of the Aharonov-Bohm electrodynamics.

19.1. The possible sources of the Aharonov-Bohm electrodynamics

The non-local time-dependent Schödinger equation in 1D of Lenzi et al. (Lenzi, et al., 2008) has the following form:

$$i \cdot \hbar \cdot \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2 \cdot m} \cdot \frac{\partial^2}{\partial x^2} \psi(x,t) + \int_0^t d\tau \int_{-\infty}^{+\infty} dy\, U(x-y, t-\tau) \cdot \psi(y,\tau) \tag{278}$$

, where:

- U is a non-local interaction kernel
- $\hbar$ is the reduced Planck constant
- m is mass of the particle

$$\psi(y,t) = \frac{1}{\sqrt{2 \cdot \pi \cdot \hbar}} \int_{-\infty}^{+\infty} dp e^{\frac{i \cdot p \cdot y}{\hbar}} \psi(p,t) \tag{279}$$

$$U(x-y, t-\tau) = \frac{U_0}{\sqrt{2 \cdot \pi \cdot \hbar}} \int_0^t d\tau \delta(t-\tau) \int_{-\infty}^{+\infty} dv e^{\frac{i \cdot v \cdot (x-y)}{\hbar}} \cdot |v|^\alpha \tag{280}$$

, where:

- U_0 is the constant of the potential
- α is the fractional constant, $1 < \alpha < 2$

$$i \cdot \hbar \cdot \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2 \cdot m} \cdot \frac{\partial^2}{\partial x^2} \psi(x,t) + \frac{U_0}{2 \cdot \pi \cdot \hbar} \int_0^t d\tau \delta(t-\tau) \int_{-\infty}^{+\infty} dy \int_{-\infty}^{\infty} dv e^{\frac{i \cdot v \cdot (x-y)}{\hbar}} \cdot |v|^\alpha \int_{-\infty}^{+\infty} dp e^{\frac{i \cdot p \cdot y}{\hbar}} \psi(p,t) \tag{281}$$

$$\int_{-\infty}^{+\infty} dy e^{\frac{i \cdot y \cdot (p-x)}{\hbar}} = 2 \cdot \pi \cdot \delta\left(\frac{p-v}{\hbar}\right) = 2 \cdot \pi \cdot \hbar \cdot \delta(p-v) \tag{282}$$

$$v \to p \tag{283}$$

$$i \cdot \hbar \cdot \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2 \cdot m} \cdot \frac{\partial^2}{\partial x^2} \psi(x,t) + U_0 \int_{-\infty}^{+\infty} dp e^{\frac{i \cdot p \cdot x}{\hbar}} \cdot |p|^\alpha \cdot \psi(p,t) \tag{284}$$

Can we apply some sort of fractional derivative in order to replace $|p|^\alpha$? Well, the answer is not so simple. It turns out the special derivative what we are looking for is already invedted and it is the fractional quantum Riesz derivative. With 3D generalization:

$$(-\hbar^2 \Delta)^{\frac{\alpha}{2}} \psi(\vec{r},t) = \frac{1}{(2 \cdot \pi \cdot \hbar)^3} \int_{-\infty}^{+\infty} d^3\vec{p} e^{\frac{i \cdot \vec{p} \cdot \vec{r}}{\hbar}} \cdot |\vec{p}|^\alpha \cdot \psi(\vec{p},t) \tag{285}$$

We can generalize this operator with the Fourier transform:

$$(-\hbar^2 \Delta)^{\frac{\alpha}{2}} \psi(\vec{r},t) = \mathcal{F}^{-1}\left\{\left|\vec{k}\right|^\alpha \psi\left(\vec{k},t\right)\right\} \tag{286}$$

$$\psi(\vec{k},t) = \mathcal{F}\{\psi(\vec{k},t)\} = \int_{-\infty}^{+\infty} d^3\vec{r} e^{\frac{i\cdot\vec{k}\cdot\vec{r}}{\hbar}} \cdot \psi(\vec{r},t) \tag{287}$$

Let's use the fractional quantum Riesz derivative on the time-dependent Schödinger equation. Nothe that this operator is also non-local.

$$i\cdot\hbar\cdot\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2\cdot m}\cdot\frac{\partial^2}{\partial x^2}\psi(x,t) + U_0\cdot\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)^{\frac{\alpha}{2}}\psi(x,t) \tag{288}$$

The effect of the non-local kernel on the wave function is the fractional quantum Riesz derivative of the wave function. Let's apply the Fourier transform on both sides of the above equation.

$$i\cdot\hbar\cdot\frac{\partial\psi(k,t)}{\partial t} = \frac{\hbar^2}{2\cdot m}\cdot k^2\cdot\psi(k,t) + U_0\cdot\hbar^\alpha\cdot|k|^\alpha\cdot\psi(k,t) \tag{289}$$

Let's apply the Laplace transform on both sides of the above equation.

$$i\cdot\hbar\cdot[s\cdot\psi(k,s) - \psi(k,0)] = \frac{\hbar^2}{2\cdot m}\cdot k^2\cdot\psi(k,s) + U_0\cdot\hbar^\alpha\cdot|k|^\alpha\cdot\psi(k,s) \tag{290}$$

Let's arrange the above equation.

$$\psi(k,s) = \frac{\psi(k,0)}{s + \frac{i\cdot\hbar}{2\cdot m}\cdot k^2 + \frac{i\cdot U_0\cdot\hbar^\alpha}{\hbar}\cdot|k|^\alpha} \tag{291}$$

Let's apply the inverse Laplace transform on both sides of the above equation. The $\psi(k,t)$ according to Modanese et al. (Modanese G. , Time in quantum mechanics and the local non conservation of the probability current, 2018) has the following form:

$$\psi(k,t) = \psi(k,0)\cdot e^{\left[-\frac{i\cdot t}{\hbar}\cdot\left(\frac{\hbar^2\cdot k^2}{2\cdot m} + U_0\cdot|k|^\beta\right)\right]} \tag{292}$$

$$\psi(k,0) = \frac{1}{\sqrt{2\cdot\pi\cdot\sigma_k^2}}\cdot e^{-\frac{k^2}{2\cdot\sigma_k^2}} \tag{293}$$

, where σ_k^2 is a constant. The continuity equation for the equation above is found in (Lenzi, et al., 2008) to be following, where the argument of ψ, when missing, is understood to be (x,t):

$$\frac{\partial}{\partial t}(\psi^*\psi) + \frac{i\cdot\hbar}{2\cdot m}\frac{\partial}{\partial x}\left(\psi^*\cdot\frac{\partial\psi}{\partial t} - \psi\cdot\frac{\partial\psi^*}{\partial t}\right) = \\ -\frac{i}{\hbar}\int_0^t d\tau\int_{-\infty}^{+\infty} dy\, U(x-y,t-\tau)\cdot[\psi(y,\tau)\cdot\psi^*(x,t) - \psi(x,t)\cdot\psi^*(y,\tau)] \tag{294}$$

The term with the double integral is what we have called "Extra current I"., where U_0 is a constant.

19.1.1. The analytical solution for $\alpha = 3/2$

Here comes the analytical solution for $\alpha = 3/2$.

19.1.2. The analytical solution for $\alpha = 2$

For the parameter choices $\alpha = 2$, $\hbar = m = U_0 = 1$. After much consideration and modification the extra current I has the following form:

$$I(x,t) = -\frac{i\cdot\pi}{(2\cdot\pi)^3\cdot\hbar}\int_{-\infty}^{+\infty} dk \int_{-\infty}^{+\infty} dp \cdot \left[\psi^*(k,t)\cdot\psi(p,\tau)\cdot|p|^\alpha\cdot\ U_0\cdot e^{i\cdot(p-k)\cdot x} - c.c.\right] \tag{295}$$

An analytical result for the extra current can be obtained by a first-order perturbative expansion in the coupling parameter U_0.

$$I^{(1)}(x,t) = A\cdot B\cdot C \tag{296}$$

$$A = -\frac{i\cdot\pi\cdot\ U_0}{(2\cdot\pi)^3\cdot\hbar}\cdot\frac{1}{\sqrt{2\cdot\pi}}\cdot\frac{1}{\sqrt{2\cdot\pi}} \tag{297}$$

$$B = \int_{-\infty}^{+\infty} e^{\frac{i}{2}\cdot t\cdot k^2}\cdot e^{-\frac{k^2}{2}}\cdot e^{-i\cdot k\cdot x}\, dk \tag{298}$$

$$C = \int_{-\infty}^{+\infty} e^{-\frac{i}{2}\cdot t\cdot p^2}\cdot e^{-\frac{p^2}{2}}\cdot e^{i\cdot p\cdot x}\cdot|p|^\beta dp \tag{299}$$

In the next step I numerically performed the two trasforms for t=1 and I plotted the real and imaginary part of the integrals as well as the super current. After that I numerically calculated for t=1 the probability density as well as the real and imaginary part of the wavefunction. The source code can be found (here). The analytical solution for the Schödinger equation in case of the nonlocal potential (with $\alpha = 2$):

$$\psi(x,t) = \frac{1}{\sqrt{1+i\cdot 2\cdot c\cdot\sigma_k^2}}\cdot e^{-\frac{x^2\cdot\sigma_k^2}{2}\cdot\left(\frac{1-i\cdot 2\cdot c\cdot\sigma_k^2}{1+4\cdot c^2\cdot\sigma_k^4}\right)} \tag{300}$$

$$c = \frac{t}{\hbar}\cdot\left(\frac{\hbar^2}{2\cdot m} + U_0\right) \tag{301}$$

The probability density ρ and the probability current j:

$$\rho = \psi\cdot\psi^* = \frac{1}{\sqrt{1+4\cdot c^2\cdot\sigma_k^4}}\cdot e^{-\frac{x^2\cdot\sigma_k^2}{1+4\cdot c^2\cdot\sigma_k^4}} \tag{302}$$

$$j = \frac{\hbar}{m}\cdot\Im\left\{\psi^*\cdot\frac{\partial\psi}{\partial x}\right\} = \frac{\hbar}{m}\cdot\frac{2\cdot c\cdot\sigma_k^4}{\left(1+4\cdot c^2\cdot\sigma_k^4\right)^{3/2}}\cdot x\cdot e^{-\frac{x^2\cdot\sigma_k^2}{1+4\cdot c^2\cdot\sigma_k^4}} \tag{303}$$

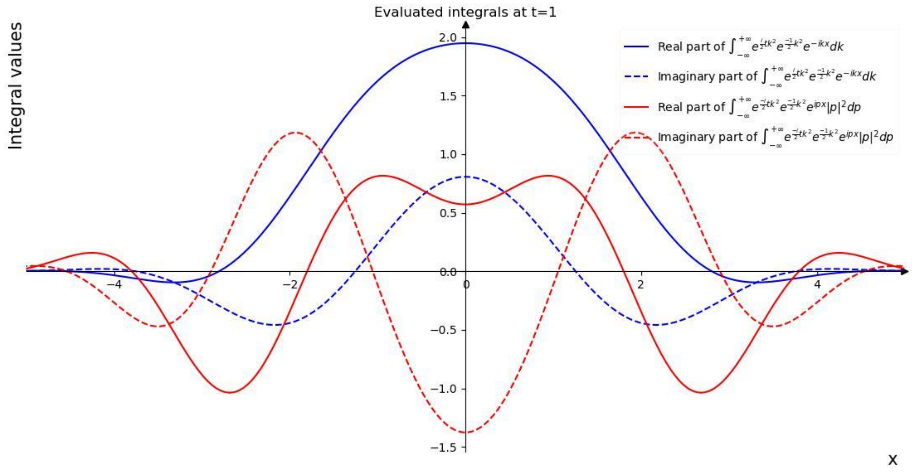

Figure 3 The real and imaginary part of the integrals one by one at t=1

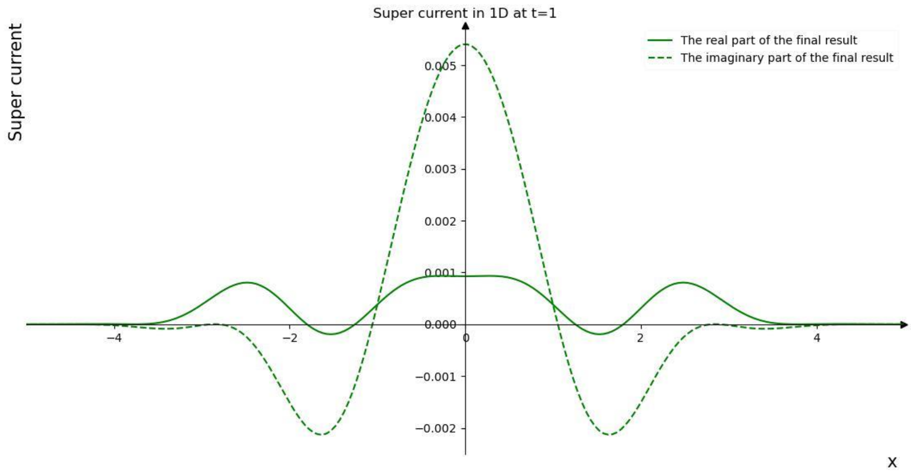

Figure 4 The super current in 1D at t=1

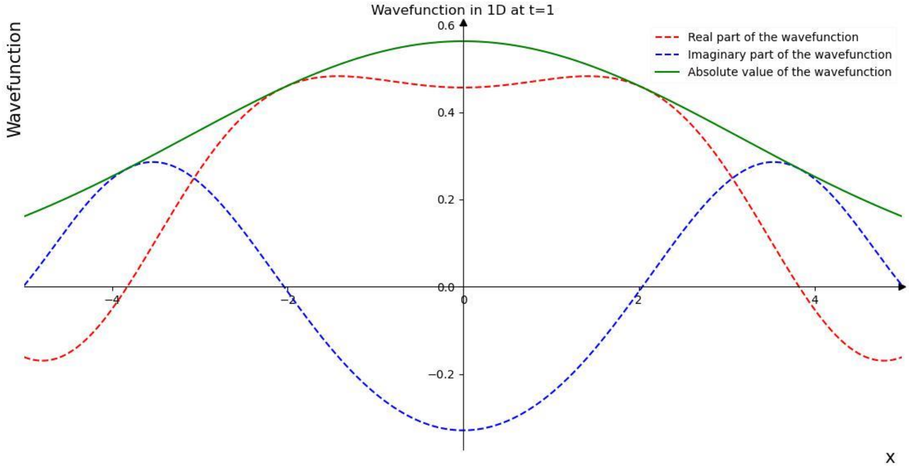

Figure 5 Wavefunction in 1D at t=1

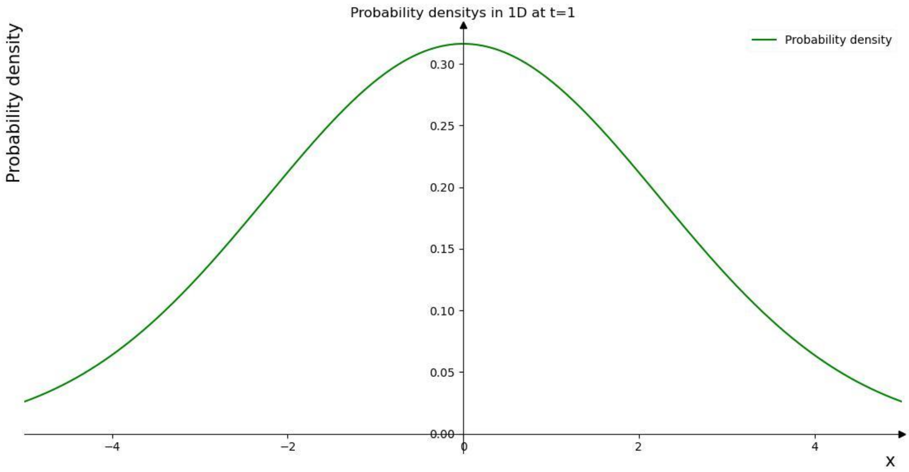

Figure 6 Probability density in 1D at t=1

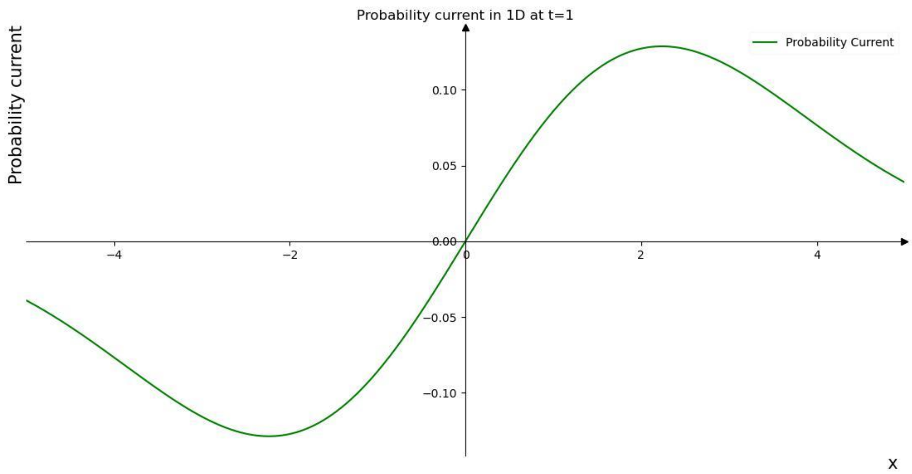

Figure 7 Probability current in 1D at t=1

19.2. Mathematical preliminaries for the numerical approach

In this section, the mathematical definitions of different kind of fractional derivatives introduced this paper (Podlubny, 1999). We will also dive into the Gamma function introduced this paper (Townsend, 2015).

19.2.1. The Gamma fuction

The Gamma function will allow us better understanding the base of the different forms of the fractional derivatives. The Gamma function takes the following form:

$$\Gamma(x) = \int_{a}^{\infty} e^{-t} \cdot t^{x-1}\, dt \tag{304}$$

One basic property of this function that will become of use later, is:

$$\Gamma(x+1) = x \cdot \Gamma(x) \tag{305}$$

Using this fact and letting $x \in \mathbb{N}$ we can have the following form:

$$\Gamma(x+1) = x \cdot \Gamma(x) = x \cdot (x-1) \cdot \Gamma(x-1) = \cdots = x! \tag{306}$$

With this we can say that if we let $x \in \mathbb{R}^{+}$ then:

$$\Gamma(x+1) = x! \tag{307}$$

For any value of x. This will allow us to extend our notation of rractional to positive real valued numbers. So in the future if we need to replace something contaion a non-integer valued factorial, we can replace it with the Gamma function.

19.2.2. The fractional derivatives

The left and right Riemann-Liouville derivatives with order $\alpha > 0$ of the function f(x), $x \in [a, b]$ are defined as:

$$ {}_{a}^{RL}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{a}^{x} (x-s)^{m-\alpha-1} \cdot f(s)\, ds \tag{308}$$

$$ {}_{x}^{RL}D_{b}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t}^{b} (s-x)^{m-\alpha-1} \cdot f(s)\, ds \tag{309}$$

$$ m = ceil(\alpha) = \lceil \alpha \rceil \quad m-1 < \alpha < m \tag{310}$$

The left and right Caputo derivatives with order $\alpha > 0$ of the function f(x), $x \in [a, b]$ are defined as:

$$ {}_{a}^{C}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} (x-s)^{m-\alpha-1} \cdot f^{(m)}(s)\, ds \tag{311}$$

$$ {}_{x}^{C}D_{b}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{t}^{b} (s-x)^{m-\alpha-1} \cdot f^{(m)}(s)\, ds \tag{312}$$

$$ m = ceil(\alpha) = \lceil \alpha \rceil \quad m-1 < \alpha < m \tag{313}$$

Although the definitions of the Riemann-Liouville and the Caputo derivatives cannot be assumed equal, they do have the following relationship:

$$ {}_{a}^{RL}D_{x}^{\alpha} f(x) = {}_{a}^{C}D_{x}^{\alpha} f(x) + \sum_{k=0}^{m-1} \frac{f^{(k)}(a) \cdot (t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} \tag{314}$$

$$ m = ceil(\alpha) = \lceil \alpha \rceil \quad m-1 < \alpha < m \tag{315}$$

, where:

- $\frac{d^m}{dt^m}$ is the m-th derivative
- $f^{(m)}$ and $f^{(k)}$ is the m-th and the k-th derivative of f respectively

The left and right Grünwald-Letnikov derivatives with order $\alpha > 0$ of the function f(x), $x \in [a, b]$ are defined as:

$$ \begin{aligned} {}_{a}^{GL}D_{x}^{\alpha} f(x) = {} & \frac{1}{\Gamma(m-\alpha+1)} \int_{a}^{x} (x-s)^{m-\alpha} \cdot f^{(m+1)}(s)\, ds \\ & + \sum_{k=0}^{m} \frac{f^{(k)}(a) \cdot (x-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} \end{aligned} \tag{316}$$

$$ \begin{aligned} {}_{x}^{GL}D_{b}^{\alpha} f(x) = {} & \frac{1}{\Gamma(m-\alpha+1)} \int_{t}^{b} (s-x)^{m-\alpha} \cdot f^{(m+1)}(s)\, ds \\ & + \sum_{k=0}^{m} \frac{f^{(k)}(b) \cdot (x-b)^{k-\alpha}}{\Gamma(k+1-\alpha)} \end{aligned} \tag{317}$$

$$ m = ceil(\alpha) = \lceil \alpha \rceil \quad m-1 < \alpha < m \tag{318}$$

Let suppose that the function f(x) is m-1 times continuously differentiable in the interval $[a, b]$ and that $f^{(m)}(x)$ is integrable in $[a, b]$. Then for every α $(m-1 < \alpha < m)$ the Riemann-Liouville derivative exists and coincides with the Grünwald-Letnikov derivative.

19.2.3. The fractional Laplacian operator

In the study of the fractional differential equations, there often occurs a symmetric fractional generalization of the fractional Laplacian operator, which is called the Riesz operator. This kind of operator can be defined with either with Riemann-Liouville and Caputo derivatives (named Riesz and Riesz-Caputo operators respectively). The Riesz fractional operator for $m-1<\alpha<m$ on the finite interval $[0<x<L]$ is defined as (Samko, Kilbas, & Marichev, 1993):

$$\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}f(x)=-\frac{\left[{}_{0}^{RL}D_{t}^{\alpha}f(x)+{}_{t}^{RL}D_{L}^{\alpha}f(x)\right]}{2\cdot\cos\left(\frac{\alpha\cdot\pi}{2}\right)} \tag{319}$$

For a function f(t) defined on the infinite domain $[-\infty<x<\infty]$, the following equality holds (for $m-1<\alpha<m$):

$$-(-\Delta)^{\frac{\alpha}{2}}f(x)=-\frac{\left[{}_{-\infty}^{RL}D_{x}^{\alpha}f(x)+{}_{x}^{RL}D_{\infty}^{\alpha}f(x)\right]}{2\cdot\cos\left(\frac{\alpha\cdot\pi}{2}\right)}=\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}f(x) \tag{320}$$

19.3. The numerical approach

In this section I approach the fractional Laplacian operator numerically using L2 approximation method introduced this paper (Yang, Liu, & Turner, 2010). The relationship between the Riemann-Liouville and Grünwald-Letnikov definitions allows the use of the Riemann-Lioville definition during the problem formulation, and then the Grünwald-Letnikov definition for obtaining the numerical solution. The left and right Grünwald-Letnikov derivatives with order $1<\alpha<2$ of the function $\psi(x), x\in[0,L]$ are defined as:

$$\begin{aligned}{}_{0}^{GL}D_{x}^{\alpha}\psi(x)&=\frac{\psi(0)\cdot x^{-\alpha}}{\Gamma(1-\alpha)}+\frac{\psi'^{(0)}\cdot x^{1-\alpha}}{\Gamma(2-\alpha)}\\&+\frac{1}{\Gamma(2-\alpha)}\int_{0}^{x}(x-s)^{1-\alpha}\cdot\psi^{(2)}(s)\,ds\end{aligned} \tag{321}$$

$$\begin{aligned}{}_{x}^{GL}D_{L}^{\alpha}\psi(x)&=\frac{\psi(L)\cdot(L-x)^{-\alpha}}{\Gamma(1-\alpha)}+\frac{\psi'^{(0)}\cdot(L-x)^{1-\alpha}}{\Gamma(2-\alpha)}\\&+\frac{1}{\Gamma(2-\alpha)}\int_{x}^{L}(s-x)^{1-\alpha}\cdot\psi^{(2)}(s)\,ds\end{aligned} \tag{322}$$

$$n=ceil(\alpha)=\lceil\alpha\rceil\quad n-1<\alpha<n \tag{323}$$

Assume that the spatial domain is $[0,L]$. The mesh is N equal intervals of $h=L/N$ and $x_l=l\cdot h$ for $0\le l\le N$. Assume that the temporal domain is $[0,T]$. The mesh is M equal intervals of $k=T/M$ and $t_i=i\cdot k$ for $0\le i\le M$. Let ψ_l^i be the numerical approximation of $\psi(x,t)$ at point $(l \cdot h, i \cdot k)$. I discretized the second order space derivative by second-order accurate central difference formula as follows:

$$\Delta\psi(x_l, t_i) \approx \frac{\psi_{l+1}^i - 2 \cdot \psi_l^i + \psi_{l-1}^i}{h^2} \tag{324}$$

I discretized the first order time derivative by first-order forward difference formula as follows:

$$\frac{\partial\psi(x_l, t_i)}{\partial t} \approx \frac{\psi_l^{i+1} - \psi_l^i}{k} \tag{325}$$

The second term of equation (321) can be approximated by:

$$\frac{\psi'(0) \cdot x^{1-\alpha}}{\Gamma(2-\alpha)} \approx \frac{h^{-\alpha}}{\Gamma(2-\alpha) \cdot l^{\alpha-1}} \cdot \left(\psi_1^i - \psi_0^i\right) \tag{326}$$

The third term of equation (321) can be approximated by:

$$\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)} \int_0^x (x-s)^{1-\alpha} \cdot \psi^{(2)}(s)\, ds = \frac{1}{\Gamma(2-\alpha)} \int_0^x s^{1-\alpha} \cdot \psi^{(2)}(x-s)\, ds \\
&= \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{l-1} \int_{j \cdot h}^{(j+1) \cdot h} s^{1-\alpha} \cdot \psi^{(2)}(x-s)\, ds \approx \frac{1}{\Gamma(2-\alpha)} \sum_{j=0}^{l-1} \\
&\frac{\psi^i(x-(j-1)\cdot h) - 2 \cdot \psi^i(x - j \cdot h) + \psi^i(x-(j+1)\cdot h)}{h^2} \\
&\cdot \int_{j \cdot h}^{(j+1)\cdot h} s^{1-\alpha}\, ds \\
&= \frac{h^{-\alpha}}{\Gamma(3-\alpha)} \sum_{j=0}^{l-1} \left(\psi_{l-1+1}^i - 2 \cdot \psi_{l-j}^i + \psi_{l-j-1}^i\right) \cdot \left[(j+1)^{2-\alpha} - j^{2-\alpha}\right]
\end{aligned} \tag{327}$$

, where:

- $\psi^i(x - j \cdot h) = \psi_{l-j}^i$
- $\psi^i(x - (j-1) \cdot h) = \psi_{l-j+1}^i$
- $\psi^i(x - (j+1) \cdot h) = \psi_{l-j-1}^i$

Hence we obtain an approximation of the left-handed fractional derivative (321) as:

$$
{}_{0}^{GL}D_x^{\alpha}\psi(x_l,t_i) \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\left\{\frac{(1-\alpha)\cdot(2-\alpha)\psi_0^i}{l^{\alpha}} + \frac{(2-\alpha)\cdot\left(\psi_1^i-\psi_0^i\right)}{l^{\alpha-1}} + \sum_{j=0}^{l-1}\left(\psi_{l-j+1}^i - 2\cdot\psi_{l-j}^i + \psi_{l-j-1}^i\right)\cdot\left[(j+1)^{2-\alpha}-j^{2-\alpha}\right]\right\} \tag{328}
$$

Similarly, we can derive an approximation of the right-handed fractional derivative (322) as:

$$
{}_{x}^{GL}D_L^{\alpha}\psi(x_l,t_i) \approx \frac{h^{-\alpha}}{\Gamma(3-\alpha)}\left\{\frac{(1-\alpha)\cdot(2-\alpha)\psi_N^i}{(N-l)^{\alpha}} + \frac{(2-\alpha)\cdot\left(\psi_N^i-\psi_{N-1}^i\right)}{(N-l)^{\alpha-1}} + \sum_{j=0}^{N-l-1}\left(\psi_{l+j-1}^i - 2\cdot\psi_{l+j}^i + \psi_{l+j+1}^i\right)\cdot\left[(j+1)^{2-\alpha}-j^{2-\alpha}\right]\right\} \tag{329}
$$

, where:

- $l = 1, \cdots, N-1$
- $i = 1, \cdots, M-1$

Therefore using the fractional Grünwald-Letnikov definitions (321) and (322), together with the numerical approximations (324), (325), (328) and (329), equation (288) can be cast into the following PDE:

$$
i\cdot\hbar\cdot\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\cdot m}\cdot\frac{\partial^2}{\partial x^2}\psi - \hbar^{\alpha}\cdot U_0\cdot\frac{\left[{}_{0}^{RL}D_x^{\alpha}\psi + {}_{x}^{RL}D_L^{\alpha}\psi\right]}{2\cdot\cos\left(\frac{\alpha\cdot\pi}{2}\right)} \tag{330}
$$

Before solving this differential equation, we have to specify the initial and boundary conditions. Let's inverse Fourier transform equation (293):

$$
\psi(x,0) = \mathcal{F}^{-1}\{\psi(k,0)\} = \frac{1}{\sqrt{2\cdot\pi\cdot\sigma_k^2}}\int_{-\infty}^{\infty} e^{-\frac{k^2}{2\cdot\sigma_k^2}}\cdot e^{-i\cdot k\cdot x}dk = e^{-\frac{x^2\cdot\sigma_k^2}{2}} \tag{331}
$$

Now let's formulate the initial and boundary conditions:

- $\psi(x,0) = \psi_0(x)$ implies $\psi_l^0 = \psi_0(x_l)$; $l = 1, \cdots, N-1$
- $\psi(0,t) = 1$ implies $\psi_0^i = 1$; $i = 1, \cdots, M-1$
- $\psi(L,t) = 0$ implies $\psi_N^i = 0$; $i = 1, \cdots, M-1$

The source code can be found (here).

20. Experimental test

Let us consider a possible experimental observation of the missing field effect for stationary currents. Modanese et. al. discussed in (Modanese G. , Design of a test for the electromagnetic coupling of non-local wavefunctions, 2019) if in a linear conductor there are regions in which a violation of continuity occurs and a fraction $|\vec{J}_{secondary}|/|\vec{J}_{physical}|$ of the current does not flow as "physical current $\rho \cdot \vec{v}$" but as "secondary current $\vec{\nabla}\Lambda$". To detect the missing field, Modanese et. al. proposed to measure at a fixed distance r the field of a normal conductor carrying a current I, and then the field of the anomalous conductor carrying the same current. A differential measurement with three wires was devised, in order to minimize errors. The scheme was especially suited for brief current pulses and for the case when the supposed anomalous conductor is a superconductor, which can be driven in/out a normal state by changing its temperature (Minotti & Modanese, Are Current Discontinuities in Molecular Devices Experimentally Observable?, 2021).

21. Conclusion

In the early 2000s, there was a minor flare-up in the history of physics due to the article by C. Monstein and J.P. Wesley (C. Monstein, 2002), when they claimed to have experimentally supported the existence of longitudinal electromagnetic waves. Later, following K. Rębilas (Rębilas, On the origin of "longitudinal electrodynamic waves", 2008), it became clear that the experimental results can be explained within the framework of classical electrodynamics. The existence of the magneto scalar field and its consequences can only be proven or disproved by a series of well-designed experiments. Still, we cannot avoid the fact that the relations (69) and (98) derived with the modified Maxwell equations were obtained without using gauge theory. The theory of modified Maxwell's equations makes predictions such as the local violation of charge conservation.

22. Appendix

22.1. symbolic1.py

```
from sympy import *
import time
import matplotlib.pyplot as plt

start = time.time()
k, x, p = symbols('k x p')
t=1
beta = 2
constant = (-I*pi)/((2*pi)**3)*(1/sqrt(2*pi))*(1/sqrt(2*pi))
constant = simplify(constant)
final_result = True

def my_constant(x):
   if(x):
      return constant
   else:
      return 1

def my_text(x):
   if(x):
      return "Super current"
   else:
      return "Integral values"

expression1 = my_constant(final_result)*exp((I*t/2-1/2)*k**2)*exp(-I*k*x)
result1 = integrate(expression1,(k,-oo,oo))
expression2 = exp((-I*t/2-1/2)*p**2)*exp(I*p*x)*abs(p**beta)
result2 = integrate(expression2,(p,-oo,oo))
result1 = simplify(result1)
result1_im = im(result1)
result1_re = re(result1)
result2 = simplify(result2)
result2_im = im(result2)
result2_re = re(result2)

print("Time to compute: " + str(time.time() - start)  + " seconds")

plots1=plotting.plot(result1_re, adaptive=False, nb_of_points=400,
show=False)
plots2=plotting.plot(result1_im, adaptive=False, nb_of_points=400,
show=False)
```

```
plots3=plotting.plot(result2_re, adaptive=False, nb_of_points=400,
show=False)
plots4=plotting.plot(result2_im, adaptive=False, nb_of_points=400,
show=False)

x1 = []
y1 = []
x2 = []
y2 = []
x3 = []
y3 = []
x4 = []
y4 = []
x5 = []
y5 = []
x6 = []
y6 = []
for plot in plots1:
    pts = plot.get_points()
    x1.append(pts[0])
    y1.append(pts[1])
for plot in plots2:
    pts = plot.get_points()
    x2.append(pts[0])
    y2.append(pts[1])
for plot in plots3:
    pts = plot.get_points()
    x3.append(pts[0])
    y3.append(pts[1])
for plot in plots4:
    pts = plot.get_points()
    x4.append(pts[0])
    y4.append(pts[1])
i=0
for element in y1[0]:
    y5.append(element*(y3[0][i]))
    i=i+1
x5 = x1[0]
i=0
for element in y2[0]:
    y6.append(element*(y4[0][i]))
    i=i+1
x6 = x3[0]

fig, ax = plt.subplots()
```

```
ax.plot(1, 0, ">k", transform=ax.get_yaxis_transform(), clip_on=False)
ax.plot(0, 1, "^k", transform=ax.get_xaxis_transform(), clip_on=False)

ax.spines['left'].set_position(('data', 0.0))
ax.spines['bottom'].set_position(('data', 0.0))
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')

ax.set_xlim([-5, 5])

ax.annotate(my_text(final_result), xy=(0, 1), xytext=(-15,2), ha='left',
va='top', xycoords='axes fraction', textcoords='offset points',
rotation=90, fontsize = 15)
ax.annotate('x', xy=(0.98, 0), ha='left', va='top', xycoords='axes
fraction', fontsize = 15)

if(final_result):
   ax.plot(x5, y5,"g", label='The real part of the final result')
   ax.plot(x6, y6,"g--", label='The imaginary part of the final result')
   plt.title("Super current in 1D at t=1")
else:
   ax.plot(x1[0], y1[0],"b", label=r'Real part of $\int_{-
\infty}^{+\infty} e^{\frac{i}{2}tk^2} e^{\frac{-1}{2}k^2} e^{-ikx} dk$')
   ax.plot(x2[0], y2[0],"b--", label=r'Imaginary part of $\int_{-
\infty}^{+\infty} e^{\frac{i}{2}tk^2} e^{\frac{-1}{2}k^2} e^{-ikx} dk$')
   ax.plot(x3[0], y3[0],"r", label=r'Real part of $\int_{-
\infty}^{+\infty} e^{\frac{-i}{2}tk^2} e^{\frac{-1}{2}k^2} e^{ipx}
|p|^{2} dp$')
   ax.plot(x4[0], y4[0],"r--", label=r'Imaginary part of $\int_{-
\infty}^{+\infty} e^{\frac{-i}{2}tk^2} e^{\frac{-1}{2}k^2} e^{ipx}
|p|^{2} dp$')
   plt.title("Evaluated integrals at t=1")

ax.legend(loc="upper right", fontsize=10, fancybox=True, framealpha=0.1)
plt.show()
```

22.2. symbolic2.py

```
from sympy import *
import time
import matplotlib.pyplot as plt

start = time.time()
k, x= symbols('k x')

t=1
constant = 1/sqrt(2*pi)
n = 400
density_bool = False

def my_text(x):
   if(x):
      return "Probability density"
   else:
      return "Wavefunction"

expression = constant*exp((-I*t*3/2-1/2)*k**2)*exp(I*k*x)
result = integrate(expression,(k,-oo,oo))
result = simplify(result)
result_im = im(result)
result_re = re(result)

print("Time to compute: " + str(time.time() - start)  + " seconds")

plots1=plotting.plot(result_re, adaptive=False, nb_of_points=n,
show=False)
plots2=plotting.plot(result_im, adaptive=False, nb_of_points=n,
show=False)

x1 = []
y1 = []
x2 = []
y2 = []
wf_abs = []
density = []

for plot in plots1:
pts = plot.get_points()
x1.append(pts[0])
y1.append(pts[1])
for plot in plots2:
   pts = plot.get_points()
```

```
    x2.append(pts[0])
    y2.append(pts[1])
for i in range(n):
    element1 = sqrt((y1[0][i])**2 + y2[0][i]**2)
    element2 = (y1[0][i])**2 + (y2[0][i])**2
    wf_abs.append(element1)
    density.append(element2)
x = x2[0]

fig, ax = plt.subplots()
ax.plot(1, 0, ">k", transform=ax.get_yaxis_transform(), clip_on=False)
ax.plot(0, 1, "^k", transform=ax.get_xaxis_transform(), clip_on=False)
ax.spines['left'].set_position(('data', 0.0))
ax.spines['bottom'].set_position(('data', 0.0))
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
ax.set_xlim([-5, 5])
ax.annotate(my_text(density_bool), xy=(0, 1), xytext=(-15,2), ha='left',
va='top', xycoords='axes fraction', textcoords='offset points',
rotation=90, fontsize = 15)
ax.annotate('x', xy=(0.98, 0), ha='left', va='top', xycoords='axes
fraction', fontsize = 15)

if(density_bool):
    ax.plot(x, density,"g", label='Probability density')
    plt.title("Probability densitys in 1D at t=1")
else:
    ax.plot(x1[0], y1[0],"r--", label='Real part of the wavefunction')
    ax.plot(x2[0], y2[0],"b--", label='Imaginary part of the
wavefunction')
    ax.plot(x, wf_abs,"g", label='Absolute value of the wavefunction')
    plt.title("Wavefunction in 1D at t=1")

ax.legend(loc="upper right", fontsize=10, fancybox=True, framealpha=0.1)
plt.show()
```

22.3. probability current.py

```
import numpy as np
from sympy import *
import matplotlib.pyplot as plt

x = np.linspace(-5,5,400)
def y(x):
    return (3/(10**(3/2)))*x*exp(-x**2/10)
probability_current = []
for element in x:
    probability_current.append(y(element))

fig, ax = plt.subplots()
ax.plot(1, 0, ">k", transform=ax.get_yaxis_transform(), clip_on=False)
ax.plot(0, 1, "^k", transform=ax.get_xaxis_transform(), clip_on=False)
ax.spines['left'].set_position(('data', 0.0))
ax.spines['bottom'].set_position(('data', 0.0))
ax.spines['right'].set_color('none')
ax.spines['top'].set_color('none')
ax.set_xlim([-5, 5])
ax.annotate("Probability current", xy=(0, 1), xytext=(-15,2), ha='left',
va='top', xycoords='axes fraction', textcoords='offset points',
rotation=90, fontsize = 15)
ax.annotate('x', xy=(0.98, 0), ha='left', va='top', xycoords='axes
fraction', fontsize = 15)
ax.plot(x, probability_current,"g", label='Probability Current')
plt.title("Probability current in 1D at t=1")
ax.legend(loc="upper right", fontsize=10, fancybox=True, framealpha=0.1)
plt.show()
```

22.4. numerical solution.py

```
import numpy as np
import matplotlib.pyplot as plt
import time

# temporal mesh
T_min = 0
T_max = 5
T = abs(T_max - T_min)
k = 0.001
M = int(round(T / k))
t = np.linspace(T_min,T_max,M+1)

# spatial mesh
```

```python
L_min = 0
L_max = 5
L = abs(L_max - L_min)
h = 0.001
N = int(round(L / h))
x = np.linspace(L_min,L_max,N+1)

# initialization: row -> time (M number) ; column -> space (N number)
PSI = np.zeros((M + 1, N + 1))

# initial condition
sigma = 1
def PSI_0(x):
  return np.exp(-((x**2*sigma**2)/2))

for i in range(0,N+1):
    PSI[0][i] = PSI_0(x[i])

# boundary conditions
for i in range(0,M+1):
    PSI[i][0] = 1
    PSI[i][N] = 0

#plt.plot(x,PSI[0])
#plt.show()

# here comes the code
start = time.time()
print("Time to compute: " + str(time.time() - start)  + " seconds")
```

23. References

Popper, K. R. (1972). *Objective Knowledge: An Evolutionary Approach.* Oxford, England: Oxford University Press.

Aharonov, Y., & Bohm, D. (1963). Further Discussion of the Role of Electromagnetic Potentials in the Quantum Theory. *Physical Review, 130*(4), 1625-1632.

Al-Khalili, J. (2015). The birth of the electric machines: a commentary on Faraday (1832) 'Experimental researches in electricity'. *Philosophical Transactions of the Royal Society A, 373*(2039).

Arbab, A. I. (2017). Extended electrodynamics and its consequences. *Modern Physics Letters B, 31*(9).

Arbab, A. I. (2017). *Modified electrodynamics for London's superconductivity.* Retrieved from https://www.researchgate.net/profile/A-Arbab/publication/320331402_Modified_electrodynamics_for_London%27s_superconductivity/links/59de3306aca27247d7942604/Modified-electrodynamics-for-Londons-superconductivity.pdf

Arbab, A. I. (2020). *Do we need to modify Maxwell's equations?* Retrieved from https://arxiv.org/abs/1801.01534

Arbab, A. I., & Satti, Z. A. (2009). On the Generalized Maxwell Equations and Their Prediction of ElectroscalarWave. *Progress in Physics, 2*, 8-13.

Baraff, G. A. (1998). Model for the effect of finite phase-coherence length on resonant transmission and capture by quantum wells. *Physical Review B, 58*(20), 13799.

Changli, L. (2015). *Comments on Overdetermination of Maxwell's Equations*. Retrieved from vixra.org: https://vixra.org/pdf/1503.0063v5.pdf

Dubrovskii, V. A. (1985). An elastic model of the physical vacuum. *Akademiia Nauk SSSR, Doklady, 282*(1), 83-88.

Ferry, D. K., & Barker, J. R. (1999). Complex potentials, dissipative processes, and general quantum transport. *Applied Physics Letters, 74*(4), 582–584.

Griffiths, D. J. (2007). *Introduction to Electrodynamics.* New Delhi: Prentice-Hall of India,.

Heaviside, O. (2007). *Electromagnetic Theory.* New York: Cosimo Classics.

Hively, L. (2016). *United States Patent No. US9306527B1.*

Hively, L., & Loebl, A. (2019). Classical and extended electrodynamics. *Physics Essays*, 112-126(15).

Hong, I. K., & Kim, C. S. (2019). Quaternion Electromagnetism and the Relation with 2-Spinor Formalism. *Universe, 5*(6).

Jackson, J. D. (1999). *Classical electrodynamics.* John Wiley and Sons.

Josephson effect. (2023). Retrieved from Wikipedia: https://en.wikipedia.org/wiki/Josephson_effect#cite_note-possibleNewEffects-1

Josephson, B. D. (1962). Possible new effects in superconductive tunnelling. *Physics Letters, 1*(7), 251-253.

Kegeles, A., & Oriti, D. (2016). Generalized conservation laws in non-local field theories. *Journal of Physics A: Mathematical and Theoretical, 49*(13), 135401.

Landau, L. D., & Lifshitz, E. M. (1971). *The classical theory of fields.* Pergamon Press.

Landau, L. D., Pitaevskii, L. P., Kosevich, A. M., & Lifshitz, E. M. (1986). *Theory of Elasticity.* Butterworth-Heinemann.

Laskin, N. (2002). Fractional Schrödinger equation. *Physical Review E, 66*(5), 056108.

Lenzi, E. K., de Oliveira, B. F., Astrath, N. G., Malacarne, L. C., Mendes, R. S., Baesso, M. L., & Evangelista, L. R. (2008). Fractional approach, quantum statistics, and non-crystalline solids at very low temperatures. *The European Physical Journal B, 62*(2), 155–158.

Lorenz, L. (1867). On the identity of the vibrations of light with electrical currents. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, 34*(230), 287-301.

Love, A. E. (1927). *A treatise on the mathematical theory of elasticity.* Cambridge: Cambridge University Press.

Maxwell, J. C. (1865). A dynamical theory of the electromagnetic field. *Philosophical Transactions of the Royal Society, 155*, 459-512.

Minotti, F., & Modanese, G. (2021). Are Current Discontinuities in Molecular Devices Experimentally Observable? *Symmetry, 13*(4), 691.

Minotti, F., & Modanese, G. (2021). Quantum Uncertainty and Energy Flux in Extended Electrodynamics. *Quantum Reports, 3*(4), 703-723.

Minotti, F., & Modanese, G. (2022). Electromagnetic Signatures of Possible Charge Anomalies in Tunneling. *Quantum Reports, 4*(3), 277-295.

Minotti, F., & Modanese, G. (2023). *Gauge waves generation and detection in Aharonov-Bohm*. Retrieved from arXiv: https://arxiv.org/abs/2310.13968

Modanese, G. (2017). Generalized Maxwell equations and charge conservation censorship. *Modern Physics Letters B, 31*(6), 1750052.

Modanese, G. (2018). Time in quantum mechanics and the local non conservation of the probability current. *Mathematics, 6*(9), 155.

Modanese, G. (2019). Design of a test for the electromagnetic coupling of non-local wavefunctions. *Results in Physics, 12*, 1056-1061.

Monstein, C., & Wesley, P. J. (2002). Observation of scalar longitudinal electrodynamic waves. *Europhysics Letters*, 514-520.

Okun, L. B. (1989). Comments on Testing Charge Conservation and Pauli Exclusion Principle. *Comments on Nuclear and Particle Physics, 19*(3), 99-116.

Podgainy, D. V., & Zaimidoroga, O. A. (2010). *Nonrelativistic theory of electroscalar field and Maxwell electrodynamics*. Retrieved from arXiv: https://arxiv.org/abs/1005.3130

Podlubny, I. (1999). *Fractional Differential Equations.* New York: Academic Press.

Rębilas, K. (2008). On the origin of "longitudinal electrodynamic waves". *Europhysics Letters*, 60007(1–5).

Reed, D., & Hively, L. (2020). Implications of Gauge-Free Extended Electrodynamics. *Symmetry*.

Samko, S. G., Kilbas, A. A., & Marichev, O. I. (1993). *Fractional Integrals and Derivatives: Theory and Applications.* Amsterdam: Gordon and Breach.

Sandev, T., Petreska, I., & Lenzi, E. K. (2014). Time-dependent Schrödinger-like equation with nonlocal term. *Journal of Mathematical Physics, 55*(9), 092105.

Sandev, T., Petreska, I., & Lenzi, E. K. (2016). Effective potential from the generalized time-dependent Schrödinger equation. *Mathematics, 4*(4), 59.

Spirichev, Y. A. (2017). *About Longitudinal Waves of an Electromagnetic Field*. Retrieved from viXra: https://vixra.org/pdf/1710.0125v1.pdf

Stueckelberg action. (2022). Retrieved from Wikipedia: https://en.wikipedia.org/wiki/Stueckelberg_action

Stueckelberg, E. (1938). DieWechselwirkungskräfte in der Elektrodynamik und in der Feldtheorie der Kräfte. *Helv. Phys. Acta*, 225-244.

Townsend, S. (2015). *Numnerical methods in fractional calculus (thesis)*. Retrieved from Cal Poly Pomona: https://broncoscholar.library.cpp.edu/bitstream/handle/10211.3/160926/TownsendSean_Thesis2015.pdf?sequence=1

van Vlaenderen, K., & Waser, A. (2001). Generalisation of classical electrodynamics to admit a scalar field and longitudinal waves. *Hadronic Journal, 24*, 609.

Waser, A. (2015). *Quaternions in Classical Electrodynamics*. Retrieved from ResearchGate: https://www.researchgate.net/publication/228356615_Quaternions_in_electrodynamics

Wei, Y. (2016). Comment on "Fractional quantum mechanics" and "Fractional Schrödinger equation". *Physical Review E, 93*(6), 066103.

Yang, Q., Liu, F., & Turner, I. (2010). Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. *Applied Mathematical Modelling, 34*(1), 200-218.

Zhang, Y., Liu, X., Belić, M. R., Zhong, W., Zhang, Y., & Xiao, M. (2015). Propagation Dynamics of a Light Beam in a Fractional Schrödinger Equation. *Physical Review Letters, 115*(18), 180403.