Programming Assignment: Orthogonal Matching Pursuit (OMP)

In this programming assignment, we will implement and study the performance of the Orthogonal Matching Pursuit (OMP) algorithm for recovering sparse signals. Please read the paper “Greed is good: algorithmic results for sparse approximation" by Joel Tropp to learn about OMP (available on piazza). Consider the measurement model

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}$$

where $\mathbf{y}$ is the measurement, $\mathbf{A} \in \mathbb{R}^{M \times N}$ is the measurement matrix, and $\mathbf{n}$ is the additive noise. Here, $\mathbf{x} \in \mathbb{R}^N$ is the unknown signal (to be estimated) with $s \ll N$ non-zero elements. The indices of the non-zero entries of $\mathbf{x}$ (also known as the support of $\mathbf{x}$) is denoted by $\mathcal{S} = \{i, x_i \neq 0\}$, with $|\mathcal{S}| = s$.

(a) Performance Metrics: Let $\hat{\mathbf{x}}$ be the estimate of $\mathbf{x}$ obtained from OMP. To measure the performance of OMP, we consider the Normalized Error defined as

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_2 / \|\mathbf{x}\|_2$$

The average Normalized Error is obtained by averaging the Normazlized Error over 2000 Monte Carlo runs.

(b) Experimental setup:

1. Generate $\mathbf{A}$ as a random matrix with independent and identically distributed entries drawn from the standard normal distribution. Normalize the columns of $\mathbf{A}$.

2. Generate the sparse vector $\mathbf{x}$ with random support of cardinality s (i.e. s indices are generated uniformly at random from integers 1 to N), and non-zero entries drawn as uniform random variables in the range $[-10, -1] \cup [1, 10]$.

3. The entries of noise $\mathbf{n}$ are drawn independently from the normal distribution with standard deviation σ and zero mean.

4. For each cardinality $s \in [1, s_{max}]$, the average Normalized Error should be computed by repeating step 1) to step 3) 2000 times and averaging the results over these 2000 Monte Carlo runs.

(c) Noiseless case: $(\mathbf{n} = \mathbf{0})$

Implement OMP (you may stop the OMP iterations once $\|\mathbf{y} - \mathbf{A}\mathbf{x}^{(k)}\|_2$ is close to 0) and evaluate its performance. Calculate the probability of Exact Support Recovery (i.e. the fraction of runs when $\hat{\mathcal{S}} = \mathcal{S}$) by averaging over 2000 random realizations of $\mathbf{A}$, as a function of M and s_{max} (for different fixed values of N). For each N, the probability of exact support recovery is a two dimensional plot (function of M and s_{max}) and you can display it as an image. The resulting plot is called the “noiseless phase transition" plot, and it shows how many measurements (M) are needed for OMP to successfully recover the sparse signal, as a function of s_{max}. Do you observe a sharp transition region where the probability quickly transitions from a large value (close to 1) to a very small value (close to 0)? Generate different phase transition plots for the following values of N: 20, 50 and 100. Regenerate phase transition plots for average Normalized Error (instead of probability of successful recovery). Comment on both kinds of plots.

(d) Noisy case: $(\mathbf{n} \neq \mathbf{0})$

1. Assume that sparsity s is known. Implement OMP (terminate the algorithm after first s columns of $\mathbf{A}$ are selected). Generate "noisy phase transition" plots (for fixed N and σ) where success is defined as the event that the Normalized Error is less than 10^{-3}. Repeat the experiment for two values of σ (one small and one large) and choose N as 20, 50 and 100. Comment on the results.

2. Assume the sparsity s is NOT known, but $\|\mathbf{n}\|_2$ is known. Implement OMP where you may stop the OMP iterations once $\|\mathbf{y} - \mathbf{A}\mathbf{x}^{(k)}\|_2 \leq \|\mathbf{n}\|_2$). Generate phase transition plots. Comment on the results.

3. Design a numerical experiment to test OMP on real images. Describe your approach in detail about how you generate the measurement model, and comment on the quality of reconstructed images as you vary the number of measurements. What is the maximum compression (i.e the ratio of M/N) that still leads to (visually) satisfactory reconstruction? Show the reconstructed image for different values of M to justify your answer.