Representing Planar graphs on a Rotating Sphere

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***Abstract*—A stereographic projection is a particular mapping (function) that projects a sphere onto a plane. The projection is defined on the entire sphere, except at one point called the projection point. The mapping when defined is smooth and bijective. It is conformal, i.e. it preserves angles at which curves meet. Although this projection has some compromises, it is a great tool with applications in diverse fields as complex analysis, cartography, geology and photography. Any planar graph can be represented on a sphere.**

***Keywords*—Stereograhpic Projection, Geometry, Sphere, Projection, Plane, Geography, Graphical projection**

I. Introduction

In geometry, the stereographic projection is a particular mapping (function) that projects a sphere onto a plane. The projection is defined on the entire sphere, except at one point: the projection point. Where it is defined, the mapping is smooth and bijective. It is conformal, meaning that it preserves angles at which curves meet. It is neither isometric nor area-preserving: that is, it preserves neither distances nor the areas of figures. [5]

Intuitively, then, the stereographic projection is a way of picturing the sphere as the plane, with some inevitable compromises. Because the sphere and the plane appear in many areas of mathematics and its applications, so does the stereographic projection; it finds use in diverse fields including complex analysis, cartography, geology, and photography. In practice, the projection is carried out by computer or by hand using a special kind of graph paper called a stereographic net, shortened to stereonet, or Wulff net.Inverse stereographic projection involves projecting a planar graph onto a sphere. Methods to do the same have been discussed in the following paper.

II. Definitions

A) Stereographic Projection :-
For any sphere, the stereographic projection is defined for a point. Thus, we would precisely say that the stereograhpic projection maps a boundary point of the sphere to a point on the plane. In other words, the **domain** of this mapping is the 3D space of points on sphere, and the **codomain** is the plane on which they projection has to be made.
Mathematically, if S is the boundary of a unit sphere is defined in 3D Cartesian space,

$$S = (x, y, z)|x^2 + y^2 + z^2 = 1$$

and P is a plane, then the domain of the function which will serve as the stereographic projection is S and the co-domain is P.
There are different types of formulations of this projection, all which vary by the choice of the plane on which the projection is to be done. Mostly, the plane chosen is either the equatorial plane or tangent to one of the poles. [2] The diagram shows several points: $O(0, 0, 0)$, $N(0, 0, a)$, $S(s, 0, 0)$, $R(r, 0, 0)$, $T(-r, 0, 0)$, $A(a, 0, 0)$. $A, R, S,$ and T are (stereographic) projections of A', R', S' and T', respectively.

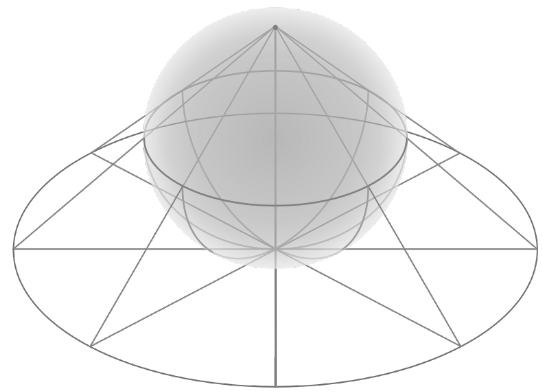

Fig. 1. Stereographic Projection

B) Planar Graph :-
In graph theory, a planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

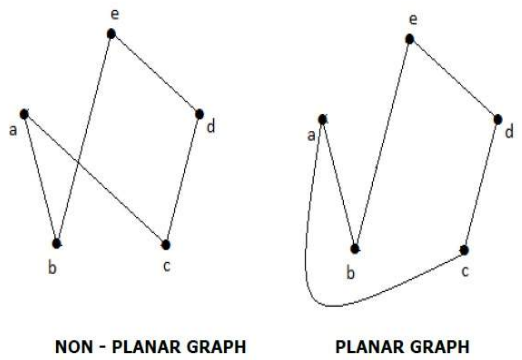

Fig. 2. Examples of planar and non-planar graphs

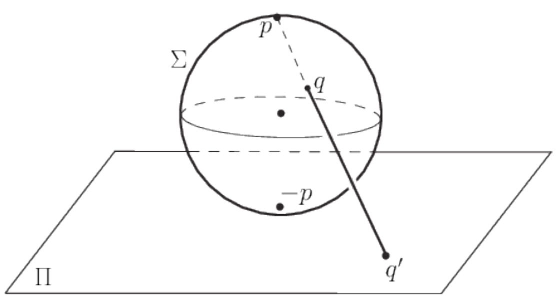

Fig. 3. Stereographic Projection of a point as seen in 3d

C) Regions :-
A plane representation of a graph divides the plane into regions. A region is characterised by the set of edges forming its boundary.

D) Infinite Region :-
The portion of the plane lying outside the graph embedded in a plane is called the infinite or the unbounded region in a graph.

III. Formulations based on choice of plane

This work describes three formulations based on the choice of the projection plane.

A. Equatorial plane

The unit sphere in three-dimensional space $R3$ is the set of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$. Let $N = (0, 0, 1)$ be the "north pole", and let M be the rest of the sphere. The plane $z = 0$ runs through the center of the sphere; the "equator" is the intersection of the sphere with this plane.

For any point P on M, there is a unique line through N and P, and this line intersects the plane $z = 0$ in exactly one point P'. Define the stereographic projection of P to be this point P' in the plane. In Cartesian

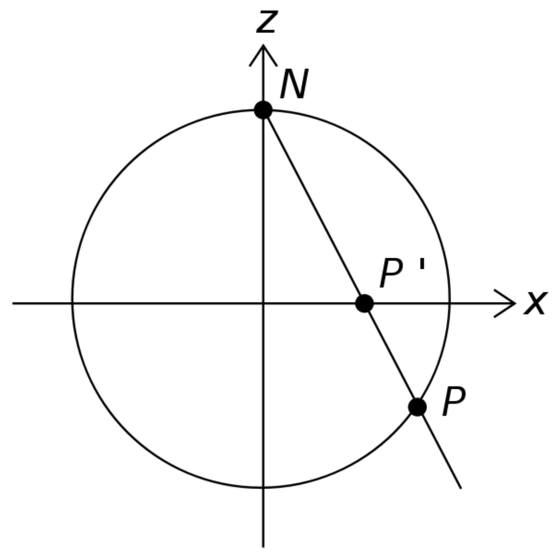

Fig. 4. Stereographic Projection on equatorial plane

coordinates (x, y, z) on the sphere and (X, Y) on the plane, the projection and its inverse are given by the formulas

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right),$$

$$(x, y, z) = \left(\frac{2X}{1+X^2+Y^2}, \frac{2Y}{1+X^2+Y^2}, \frac{-1+X^2+Y^2}{1+X^2+Y^2}\right).$$

In spherical coordinates (φ, θ) on the sphere (with φ the zenith angle, $0 \leq \varphi \leq \pi$, and θ the azimuth, $(0 \leq theta \leq 2\pi)$ and polar coordinates (R, θ) on the plane, the projection and its inverse are

$$(R, \Theta) = \left(\frac{\sin\varphi}{1-\cos\varphi}, \theta\right) = \left(\cot\frac{\varphi}{2}, \theta\right),$$

$$(\varphi, \theta) = \left(2\arctan\frac{1}{R}, \Theta\right).$$

B. Polar plane

Some notations and requirements take a plane tangent to one of the poles of the sphere as described in the figure below. [7]

The stereographic projection from the north pole (0, 0, 1) onto the plane z = -1, which is tangent to the unit sphere at the south pole (0, 0, -1). The values X and Y produced by this projection are exactly twice those produced by the equatorial projection described in the preceding section. For example, this projection sends the equator to the circle of radius 2 centered at the origin. While the equatorial projection produces no infinitesimal area distortion along the equator, this pole-tangent projection instead produces no infinitesimal area distortion at the south pole.

In this case, the formula becomes

$$(x, y, z) \to (\xi, \eta) = \left(\frac{x}{\frac{1}{2} - z}, \frac{y}{\frac{1}{2} - z} \right),$$

$$(\xi, \eta) \to (x, y, z) = \left(\frac{\xi}{1 + \xi^2 + \eta^2}, \frac{\eta}{1 + \xi^2 + \eta^2}, \frac{-1 + \xi^2 + \eta^2}{2 + 2\xi^2 + 2\eta^2} \right).$$

C. General plane

In general, one can define a stereographic projection from any point Q on the sphere onto any plane E such that

- E is perpendicular to the diameter through Q, and
- E does not contain Q.

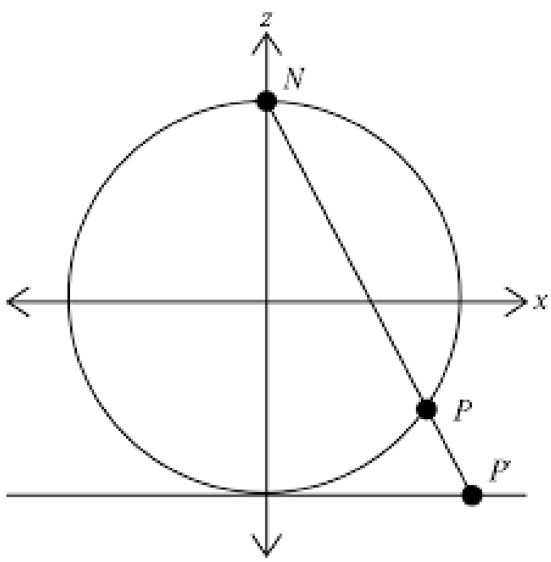

Fig. 5. Stereographic Projection on a polar plane

As long as E meets these conditions, then for any point P other than Q the line through P and Q meets E in exactly one point P', which is defined to be the stereographic projection of P onto E.

IV. Methods

Every Planar Graph can be represented as a sphere. This is known as inverse stereographic projection. The primary purpose of representing a graph as a sphere is to eliminate the distinction between the finite and the infinite regions of a plane. The method followed to embed a planar graph on a sphere is as follows :-

- Place the sphere on the plane and call the point of contact the south pole(SP).

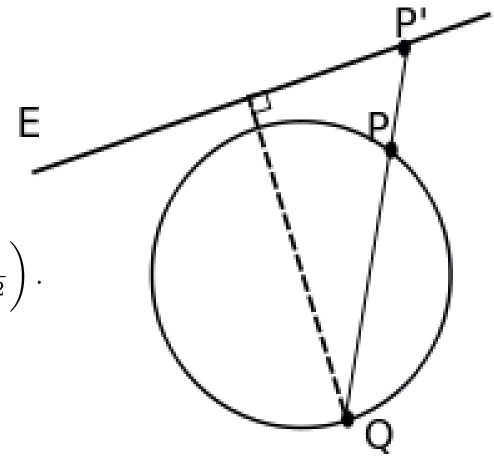

Fig. 6. Stereographic Projection on a general plane

- At point SP, draw a straight line perpendicular to the plane and let the point where this line intersects the surface of the sphere be called north pole(NP).
- Draw a straight line joining any point on the planar graph to the north pole.The corresponding point of intersection on the surface of the sphere will be the point on the sphere corresponding to a particular point on the planar graph.
- Repeat steps 2 and 3 for all points on the sphere.
- The infinite region of the graph maps to the point NP on the sphere.

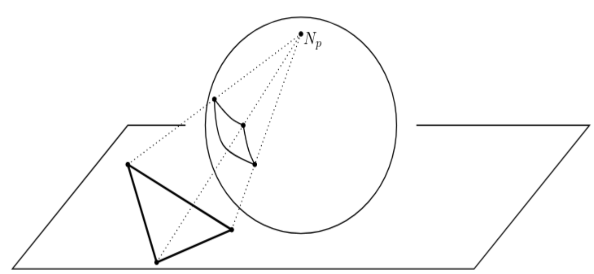

Fig. 7. Embedding a planar graph on a sphere

V. Consequence of Rotating a Sphere

The whole purpose of embedding a graph on a sphere was to eliminate the distinction between the finite and infinite regions of a graph. A graph can have multiple embedding on a plane and corresponding to each embedding it can have different finnite and infinite regions. The point NP on the sphere corresponds to the infinte

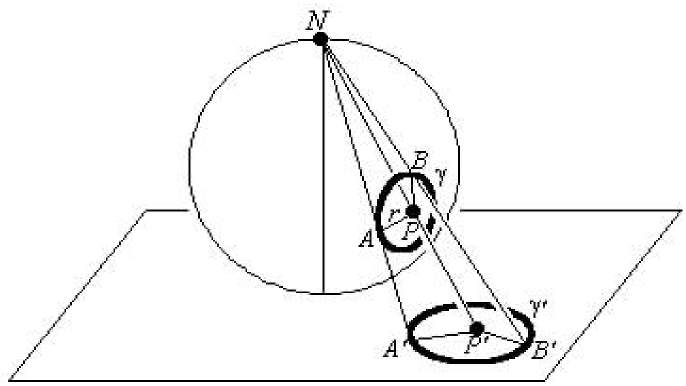

Fig. 8. Embedding a planar graph on a sphere

region of the planar graph.
We can simply rotate the sphere to and make any region correspond to the infinite region. Thus, any finite region can be an infinite region and any infinite region can be represented as a finite region on a rotating sphere.
It should be noted that the embeddings of a planar graph are not distinct if they can be made to coincide by simply rotating one sphere with respect to the other.

VI. Implementation

We have used python to write the code required for the above implementation. The graph vertex co-ordinates are taken as input from the user. Following this we find the stereographic projection of the graph on a sphere of radius 1 with the center at 0,0,1 in the xyz co-ordinate system. We iteratively rotate the sphere by an angle of 5degrees about the x-axis till the south pole(SP) becomes the north pole(NP). At every rotation, we find the stereographic projection of the graph on the sphere.We have used matplotlib to plot the graph and the sphere.

VII. Applications

A. Cartography

The fundamental problem of cartography is that no map from the sphere to the plane can accurately represent both angles and areas. In general, area-preserving map projections are preferred for statistical applications, while angle-preserving (conformal) map projections are preferred for navigation. [1]

Stereographic projection falls into the second category. When the projection is centered at the Earth's north or south pole, it has additional desirable properties: It sends meridians to rays emanating from the origin and parallels to circles centered at the origin.

Fig. 9. Stereographic projection of the world north of 30° S. 15° graticule.

B. Photography

Some fisheye lenses use a stereographic projection to capture a wide-angle view. Compared to more traditional fisheye lenses which use an equal-area projection, areas close to the edge retain their shape, and straight lines are less curved. However, stereographic fisheye lenses are typically more expensive to manufacture. Image remapping software, such as Panotools, allows the automatic remapping of photos from an equal-area fisheye to a stereographic projection. [8]

Fig. 10. Image captured using Fisheye lens [6]

A lot more applications can be done in various fields varying from crystallography to geology, planetary science, mathematics, complex analysis. The basic idea of sphere to plane can also be employed for circle to line, which is in fact the way of presentation in this entire work.

VIII. Code Snippets

```
def getNewCoordinates(point,Np):
    px,py,pz = point
    x1,y1,z1 = Np

    #writing x and y in terms of x = a1*z+b1 and y = a2 * z + b2
    temp = (x1 - px) / (z1 - pz)
    a1 = temp
    b1 = x1 - ( temp * z1)

    temp = ( y1 - py ) / ( z1 - pz)
    a2 = temp
    b2 = y1 - (temp * z1)

    alpha = (a1 ** 2) + (a2 ** 2) + 1
    beta = 2*((a1*b1)+(a2*b2)- R )
    c = (b1 * b1) * (b2 * b2)
    a = alpha
    b = beta
    #print(a,b)
    d = (b * b) - (4 * a * c)
    zsol1 = (-b-cmath.sqrt(d))/(2*a)
    zsol2 = (-b+cmath.sqrt(d))/(2*a)
    za = zsol1.real
    zb = zsol2.real
    znew = 0
    if((za - z1) ** 2 > (zb - z1) ** 2):
        znew = za
    else:
        znew = zb
    xnew = a1 * znew + b1
    ynew = a1 * znew + b2
    return np.asarray([xnew,ynew,znew])
```

Fig. 11. Equations involved in the computation

```
def getNewNorthPole(dt):
    pi = math.pi
    radian = (pi/180.0)*dt
    sin = math.sin(radian)
    cos = math.cos(radian)
    x = 0
    y = R * sin
    z = R * cos + R
    newNp = (x,y,z)
    return newNp
```

Fig. 12. Function calculating the new north pole on rotation of the sphere

IX. Results

Analysis of Algorithm :-

The solving of the quadratic equation to find out the stereographic projections of a single point takes s $O(1)$time. The total number of points that need to be projected depends on the number of Vertices. Hence complexity of one iteration of dtheta is $O(V)$. The total number of iterations may depend on the interval of the angle increment. Therefore, the total complexity would be $O((180/dtheta) * V)$.

The figures given below give us the projection of the planar graph on the sphere for different degrees of rotation of the sphere,The sphere is rotated about the x-axis.

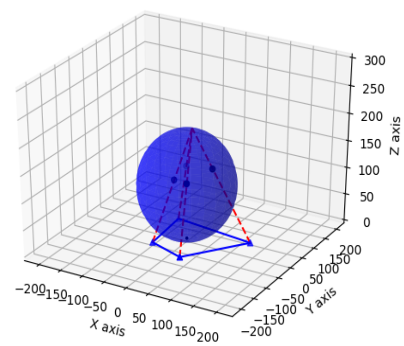

Fig. 13. RESULT 1

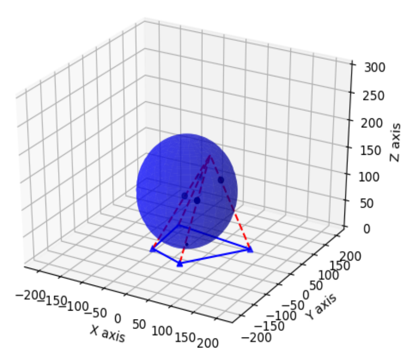

Fig. 14. RESULT 2

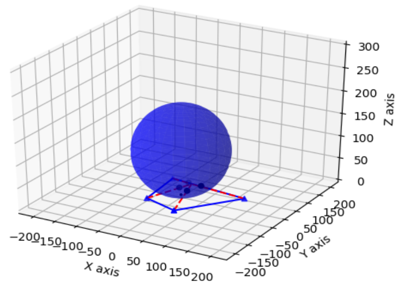

Fig. 15. RESULT 3

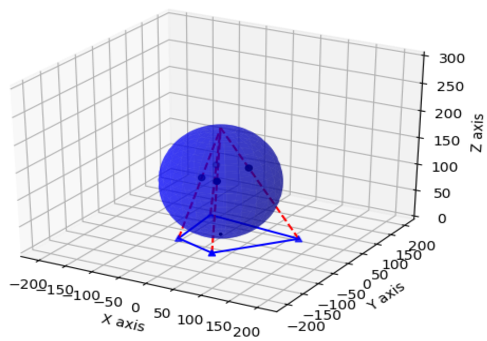

Fig. 16. RESULT 4

X. Conclusion

Any planar graph can be represented as a sphere and rotating the sphere has the effects shown in the paper.

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