Varignon's Theorem and the Midpoint Quadrilateral

A well-known result called Varignon's Theorem states that if you connect the midpoints of the sides of any quadrilateral in order, you get a parallelogram whose area is exactly half the area of the original quadrilateral.

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1. State the Setup

Let ABCD be any quadrilateral. Denote:
- E as the midpoint of AB,
- F as the midpoint of BC,
- G as the midpoint of CD,
- H as the midpoint of DA.

We want to show that the quadrilateral EFGH is a parallelogram with area (1/2) of the area of ABCD.

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2. Show EFGH is a Parallelogram

- Since E and F are midpoints, EF is a mid-segment of triangle ABC, meaning EF || AC.
- Similarly, HG is a mid-segment in triangle ADC, so HG || AC as well.
- Therefore, EF || HG.
- Using similar reasoning, EH || FG.

Thus, EFGH is a parallelogram.

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3. Show the Area is Half

Method A: Decompose into Triangles

1. Draw diagonal AC in quadrilateral ABCD. This splits ABCD into two triangles: ABC and ADC.
2. In triangle ABC, the segment EF is a mid-segment. It divides the triangle into two equal areas.
3. Similarly, HG divides triangle ADC into two equal areas.
4. Thus, the parallelogram EFGH has exactly half the area of ABCD.

Method B: Vector Approach

1. Assign coordinates to A, B, C, D as vectors.
2. Compute midpoints E, F, G, H.
3. Using vector cross products, verify that Area(EFGH) = (1/2) Area(ABCD).

Key Takeaways:

- **Varignon's Theorem**: Connecting the midpoints of consecutive sides of any quadrilateral produces a parallelogram.
- **Half-Area Property**: That parallelogram has exactly half the area of the original quadrilateral.