

Performance-based regularization in mean-CVaR portfolio optimization

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Abstract We introduce performance-based regularization (PBR), a new approach to addressing estimation risk in data-driven optimization, to mean-CVaR portfolio optimization. We assume the available log-return data is iid, and detail the approach for two cases: nonparametric and parametric (the log-return distribution belongs in the elliptical family). The nonparametric PBR method penalizes portfolios with large variability in mean and CVaR estimations. The parametric PBR method solves the empirical Markowitz problem instead of the empirical mean-CVaR problem, as the solutions of the Markowitz and mean-CVaR problems are equivalent when the log-return distribution is elliptical. We derive the asymptotic behavior of the nonparametric PBR solution, which leads to insight into the effect of penalization, and justification of the parametric PBR method. We also show via simulations that the PBR methods produce efficient frontiers that are, on average, closer to the population efficient frontier than the empirical approach to the mean-CVaR problem, with less variability.

Keywords: performance-based regularization; portfolio optimization; Conditional Value-at-Risk; coherent risk measure; portfolio estimation error

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1 Introduction

In recent years, there has been a growing interest in Conditional Value-at-Risk (CVaR) as a financial risk measure. This interest is based on two key advantages of CVaR over Value-at-Risk (VaR), the risk measure of choice in the financial industry over the last twenty years. Firstly, $CVaR(\beta)$, the conditional expectation of losses in the top $100(1 - \beta)\%$ ($\beta = 0.95, 0.99$ are typical values used in industry), is more informative about the tail end of the loss distribution than $VaR(\beta)$, which is only the *threshold* for losses in the top $100(1 - \beta)\%$. Secondly, CVaR satisfies the four coherence axioms of Artzner, Delbaen, Eber and Heath (1999) [Acerbi and Tasche (2001)], whereas VaR fails the subadditivity requirement.

Portfolio optimization with CVaR as a risk measure is first studied by Rockafellar and Uryasev (2000), who show that empirical CVaR minimization can be formulated as a linear program. Subsequent works include CVaR optimization for a portfolio of credit instruments [Andersson, Mausser, Rosen and Uryasev (2001)] and derivatives [Alexander, Coleman and Li (2006)], and portfolio optimization based on extensions of CVaR [Mansini, Ogryczak and Speranza (2007)]. However, most discussions of CVaR in portfolio optimization to date are concerned with formulation and tractability of the problem, and assume full knowledge of the distribution of the portfolio loss. In practice, one cannot ignore the fact that the loss distribution is not known and must be estimated from historical data, constructed from expert knowledge, or a combination of both. Naive estimation of the loss distribution can pose serious problems — Lim, Shanthikumar and Vahn (2011) demonstrates how fragile the solution to the empirical mean-CVaR problem is, even in the ideal situation of having iid Gaussian log-return data.

The issue of estimation errors in portfolio optimization is not, however, new knowledge. The estimation issue for the classical Markowitz (mean-variance) problem has been raised as early as 1980 [Jobson and Korkie (1980)]. There have since been many suggestions for mitigating this issue for the Markowitz problem; two main approaches are *robust optimization* [Goldfarb and Iyengar (2003)] and *what we call “standard regularization”* [Chopra (1993), Frost and Savarino (1988), Jagannathan and Ma (2003), DeMiguel, Garlappi, Nogales and Uppal (2009)]. The robust optimization approach is to take the source of uncertainty (e.g. the asset log-returns, or its distribution), specify an uncertainty set about the source, and minimize the worst-case return-risk problem over this uncertainty set. The standard regularization approach is to solve the empirical mean-variance problem, but with a constraint on the size of the solution, as measured by L_2 or a more generalized norm. The term “regularization” is adopted from statistics and machine learning, where it refers to controlling for the size of the decision variable for better out-of-sample performance [Hastie, Tibshirani and Friedman (2009)]. Both robust optimization and standard regularization approaches have been studied for the mean-CVaR problem; Gotoh and Shinozaki (2010) and Zhu and Fukushima (2009) show implementations of the robust optimization approach when the source of uncertainty is, respectively, the log-return vector and the log-return distribution, and Gotoh and Takeda (2010) demonstrates implementation of standard regularization.

In this paper, we propose performance-based regularization (PBR), a new approach to addressing estimation risk in data-driven optimization, and illustrate this method for the mean-CVaR portfolio optimization problem. We demonstrate PBR for two situations: *the investor has non-parametric or parametric (specifically, the elliptical family of distributions describe the log-returns) information on the log-returns.*

The nonparametric PBR method penalizes portfolios with large variability in mean and CVaR estimations. Specifically, we penalize the sample variances of the mean and CVaR estimators. The resulting problem is a combinatorial optimization problem, however we show that its convex relaxation, a quadratically-constrained quadratic program, is tight. The problem can be interpreted as a

chance-constrained program that picks portfolios for which approximate probabilities of deviations of the mean and CVaR estimations from their true values are constrained.

The parametric PBR method solves the empirical Markowitz problem instead of the empirical mean-CVaR problem if the underlying log-return distribution is in the elliptical family (which includes Gaussian and t distributions). This is based on the observation that CVaR of a portfolio is a weighted sum of the portfolio mean and the portfolio variance if the log-return distribution is in the elliptical family, resulting in the equivalence of the population efficient frontiers¹ of the Markowitz and mean-CVaR problems. As we are striving to reach the population frontier with greater stability, it makes intuitive sense to use the empirical Markowitz solution in lieu of the empirical mean-CVaR solution for this model.

The PBR methods are anticipated to enhance the performance by yielding solutions that are, on average, closer to achieving the original objective (minimize the *true* CVaR subject to *true* return equal to some level). As such, the PBR approach is fundamentally different from robust optimization, in that robust optimization deals with the source of uncertainty to minimize the worst-case performance, whereas PBR deals with the performance uncertainty to increase the average performance. Comparing to the statistics/machine learning literature, PBR for the nonparametric case can be seen as an extension of standard regularization, in that nonparametric PBR also constrains the decision variable, however does so indirectly through penalizing the variability of mean and CVaR estimations.

Details of the nonparametric PBR method can be found in Sec. 3.1 and the parametric PBR method in Sec. 3.2. In Sec. 4, we provide theoretical results for the PBR methods after deriving the Central Limit Theorem for the nonparametric PBR solution. In Sec. 5, we evaluate the PBR methods against the straight-forward approach of solving the empirical mean-CVaR problem for three different log-return models via simulation experiments. We find that on average, the sample efficient frontiers of the PBR solutions are closer to the population efficient frontier than those of the straight-forward approach.

2 Mean-CVaR portfolio optimization

Notations. Throughout the paper, we denote convergence in probability by \xrightarrow{P} and in distribution by \Rightarrow . The notation $X \stackrel{d}{=} Y$ for two random variables X and Y means they have the same distribution, and the symbol $X \sim \mathcal{D}$ is used to indicate that the random variable X follows some standard distribution \mathcal{D} .

2.1 Setup

An investor is to choose a portfolio $w \in \mathbb{R}^p$ on p different assets. Her wealth is normalized to 1, so $w^\top 1_p = 1$, where 1_p denotes $p \times 1$ vector of ones. The log-returns of the p assets is denoted by X , a $p \times 1$ random vector, which follows some absolutely continuous distribution F with twice continuously differentiable pdf and finite mean μ and covariance Σ . The investor wants to pick a portfolio that minimizes the CVaR of the portfolio loss at level $100(1 - \beta)\%$, for some $\beta \in (0.5, 1)$, while reaching an expected return R . That is, she wants to solve the following problem:

$$\begin{aligned} w_0 = \underset{w}{\operatorname{argmin}} \quad & \operatorname{CVaR}(-w^\top X; \beta) \\ \text{s.t.} \quad & w^\top \mu = R \\ & w^\top 1_p = 1, \end{aligned} \tag{CVaR-pop}$$

¹By “population” we mean having a perfect market knowledge.

where

$$CVaR(-w^\top X; \beta) := \min_{\alpha} \alpha + \frac{1}{1-\beta} \mathbb{E}(-w^\top X - \alpha)^+, \quad (1)$$

as in Rockafellar and Uryasev (2000).

In reality, the investor does not know the distribution F . We assume the investor observes n iid realizations of asset returns, $\mathbf{X} = [X_1, \dots, X_n] \in \mathbb{R}^{p \times n}$. Then the most straight-forward thing is to solve the following problem, where plugged-in estimators replace the true CVaR and return values:

$$\begin{aligned} \hat{w}_n = \underset{w}{\operatorname{argmin}} \quad & \widehat{CVaR}_n(-w^\top \mathbf{X}; \beta) \\ \text{s.t.} \quad & w^\top \hat{\mu}_n = R \\ & w^\top \mathbf{1}_p = 1 \end{aligned} \quad (\text{CVaR-emp})$$

where

$$\widehat{CVaR}_n(-w^\top \mathbf{X}; \beta) := \min_{\alpha \in \mathbb{R}} \alpha + \frac{1}{n(1-\beta)} \sum_{i=1}^n (-w^\top X_i - \alpha)^+, \quad (2)$$

is a sample average estimator for $CVaR(-w^\top X; \beta)$ and $\hat{\mu}_n = n^{-1} \sum_{i=1}^n X_i$ is the sample mean of the observed asset log-returns.

2.2 Estimation risk of the empirical solution

Asymptotically, as the number of observations n goes to infinity (with p constant), \hat{w}_n converges in probability to w_0 [see Sec. 4.2 for details]. In practice, however, the investor has a limited number of relevant observations. If, for example, there are $n = 250$ iid daily observations, and the investor wishes to control the top 5% of the losses, then there are only $250 \times 0.05 = 12.5$ points to estimate the portfolio CVaR at level $\beta = 0.95$. For stock log-returns, $n = 250$ iid daily observations is rather optimistic; there is ample empirical evidence that suggests daily log-returns are non-stationary over this period of time [McNeil, Frey and Embrechts (2005)]. Even for time scales with more evidence for stationarity (e.g. bi-weekly/monthly), the stationarity tends to last for no more than 5 years [McNeil et al. (2005)].

As a result, solving (CVaR-emp) using real data results in highly unreliable solutions. Let us illustrate this point, assuming an ideal market scenario. There are $p = 10$ stocks, with daily returns following a Gaussian distribution²: $X \sim \mathcal{N}(\mu_{sim}, \Sigma_{sim})$, and the investor has $n = 250$ iid observations of X . In the following, we conduct an experiment similar to those found in Lim et al. (2011), to evaluate the performance and reliability of solving (CVaR-emp) under this ideal scenario. Briefly, the experimental procedure is as follows:

- Simulate 250 historical observations from $\mathcal{N}(\mu_{sim}, \Sigma_{sim})$.
- Solve (CVaR-emp) with $\beta = 0.95$ and some return level R to find an instance of \hat{w}_n .
- Plot the realized return $\hat{w}_n^\top \mu$ versus realized risk $CVaR(-\hat{w}_n^\top X; \beta)$; this corresponds to one grey point in Fig. (1).
- Repeat for different values of R to obtain a sample efficient frontier.
- Repeat many times to get a distribution of the sample efficient frontier.

²the parameters are the sample mean and covariance matrix of data from 500 daily returns of 10 different US stocks from Jan 2009– Jan 2011

The result of the experiment is summarized in Fig. (1). The green curve corresponds to the population efficient frontier. Each of the grey dots corresponds to a solution instance of (CVaR-emp). There are two noteworthy observations: the solutions \hat{w}_n are sub-optimal, and they are highly variable. For instance, for a daily return of 0.1%, the CVaR ranges from 1.3% to 4%.

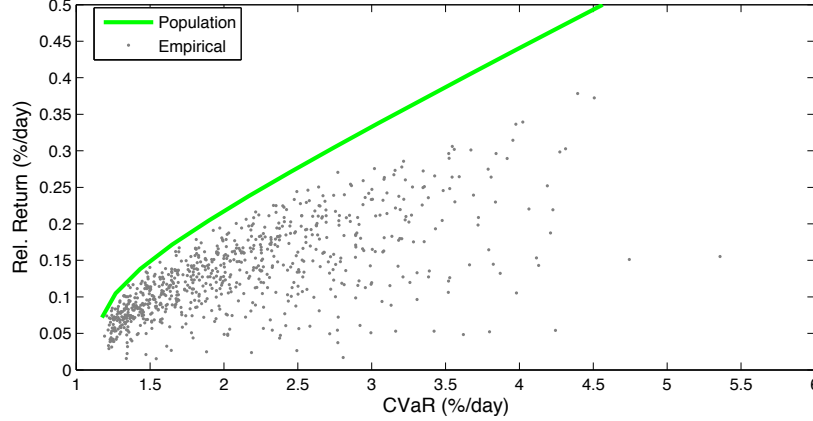


Figure 1: Distribution of realized daily return (%) vs. daily risk (%) of empirical solution \hat{w} . Green line represent the population frontier, i.e. the efficient frontier corresponding to solving (CVaR-pop).

In the following section, we introduce *performance-based regularization* (PBR) as an approach to improve upon (CVaR-emp). The PBR approach is so-called because its goal is to improve upon \hat{w}_n in terms of its performance, i.e. closeness to the population efficient frontier, ideally with less variability. We describe PBR for two cases: the investor has nonparametric or parametric knowledge of the market.

3 Performance-based regularization

3.1 Nonparametric case

In the nonparametric case, we assume the asset log-returns X follows some distribution P with finite mean μ and covariance Σ , and the investor has n iid observations: $\mathbf{X} = [X_1, \dots, X_n] \in \mathbb{R}^{p \times n}$. The nonparametric PBR approach to (CVaR-pop) is to solve the following problem:

$$\begin{aligned} \min_w \quad & \widehat{CVaR}_n(-w^\top \mathbf{X}; \beta) \\ \text{s.t.} \quad & w^\top \hat{\mu}_n = R \\ & w^\top \mathbf{1}_p = 1 \\ & P_1(w) \leq U_1 \\ & P_2(w) \leq U_2 \end{aligned} \tag{3}$$

where P_1 and P_2 are penalty functions that characterize the uncertainty associated with $w^\top \hat{\mu}_n$ and $\widehat{CVaR}_n(-w^\top \mathbf{X}; \beta)$ respectively. The idea is to penalize decisions w for which the uncertainty about the true values $w^\top \mu$ and $CVaR(-w^\top X; \beta)$ is large.

What, then, are appropriate penalty functions? Recall that we are trying to find solutions that yield efficient frontiers that are closer to the population efficient frontier, ideally with smaller variability. Thus the variances of $w^\top \hat{\mu}_n$ and $\widehat{CVaR}_n(-w^\top \mathbf{X}; \beta)$ make appropriate penalty functions,

as they characterize the deviation from the respective population values. The variance of $w^\top \hat{\mu}_n$ is given by

$$\text{Var}(w^\top \hat{\mu}_n) = \frac{1}{n^2} \sum_{i=1}^n \text{Var}(w^\top X_i) = \frac{1}{n} w^\top \Sigma w,$$

and the variance of $\widehat{CVaR}_n(-w^\top \mathbf{X}; \beta)$ is approximately equal to $\gamma_0^2/n(1-\beta)^2 = \text{Var}[\max(-w^\top X - \alpha_\beta)]$, where

$$\alpha_\beta = \inf\{\alpha : P(-w^\top X \geq \alpha) \leq 1 - \beta\}$$

is the Value-at-Risk (VaR) of the portfolio w at level β , due to the following lemma.

Lemma 1. Suppose $\mathbf{X} = [X_1, \dots, X_n] \stackrel{iid}{\sim} F$, where F is absolutely continuous with twice continuously differentiable pdf. Then

$$\frac{\sqrt{n}(1-\beta)}{\gamma_0} (\widehat{CVaR}_n(-w^\top \mathbf{X}; \beta) - CVaR(-w^\top X; \beta)) \Rightarrow \mathcal{N}(0, 1). \quad (4)$$

Proof. See Appendix A. □

Of course, we do not know the true variances, so we contend with *sample* variances of the estimators $w^\top \hat{\mu}_n$ and $\widehat{CVaR}_n(-w^\top \mathbf{X}; \beta)$. That is, we consider the following penalty functions:

$$\begin{aligned} P_1(w) &= \frac{1}{n} w^\top \hat{\Sigma}_n w, \text{ where } \hat{\Sigma}_n = \text{Cov}(\mathbf{X}), \\ P_2(w) &= \frac{1}{n(1-\beta)^2} z^\top \Omega_n z, \text{ where} \\ \Omega_n &= \frac{1}{n-1} [I_n - n^{-1} \mathbf{1}_n \mathbf{1}_n^\top], \text{ } I_n = n \times n \text{ identity matrix, and} \\ z_i &= \max(0, -w^\top X_i - \alpha) \text{ for } i = 1, \dots, n. \end{aligned}$$

For the rest of this paper, we investigate the nonparametric PBR method with sample variance penalty functions. Of course, this is just one particular choice, and it opens up the question of how different penalty functions affect the solution performance, and whether there are such things as “optimal” penalty functions. These are difficult questions worthy of further research, and we do not investigate them in this paper. Nevertheless, we derive the asymptotic behavior of the solution of nonparametric PBR method in Sec. 4, which gives us some insight into i) how one could compare the effects of different penalty functions and ii) the first-order effect of many typical penalty functions.

The nonparametric PBR method with sample variance of return and CVaR estimators as penalties is:

$$\begin{aligned} (\hat{\alpha}_n^v, \hat{w}_n^v, \hat{z}_n^v) &= \underset{\alpha, w, z}{\text{argmin}} \quad \alpha + \frac{1}{n(1-\beta)} \sum_{i=1}^n z_i \\ &\quad s.t. \quad w^\top \hat{\mu}_n = R \\ &\quad \quad w^\top \mathbf{1}_p = 1 \\ &\quad \quad \frac{1}{n} w^\top \hat{\Sigma}_n w \leq U_1 \\ &\quad \quad \frac{1}{n(1-\beta)^2} z^\top \Omega_n z \leq U_2 \\ &\quad \quad z_i = \max(0, -w^\top X_i - \alpha), \quad i = 1, \dots, n. \end{aligned} \quad (\text{CVaR-pen})$$

At first glance, (CVaR-pen) is a combinatorial optimization problem due to the cutoff variables z_i , $i = 1, \dots, n$. However, it turns out that the convex relaxation of (CVaR-pen), a quadratically-constrained quadratic program (QCQP), is tight, thus we can solve (CVaR-pen) efficiently. Before stating the result, let us first introduce the convex relaxation of (CVaR-pen):

$$\begin{aligned}
\min_{\alpha, w, z} \quad & \alpha + \frac{1}{n(1-\beta)} \sum_{i=1}^n z_i \\
\text{s.t.} \quad & w^\top \hat{\mu}_n = R \quad (\nu_1) \\
& w^\top 1_p = 1 \quad (\nu_2) \\
& \frac{1}{n} w^\top \hat{\Sigma}_n w \leq U_1 \quad (\lambda_1) \\
& \frac{1}{n(1-\beta)^2} z^\top \Omega_n z \leq U_2 \quad (\lambda_2) \\
& z_i \geq 0 \quad i = 1, \dots, n \quad (\eta_1) \\
& z_i \geq -w^\top X_i - \alpha, \quad i = 1, \dots, n \quad (\eta_2)
\end{aligned} \tag{CVaR-relax}$$

and its dual (where the dual variables correspond to the primal constraints as indicated above):

$$\begin{aligned}
\max_{\nu_1, \nu_2, \lambda_1, \lambda_2, \eta_1, \eta_2} \quad & g(\nu_1, \nu_2, \eta_1, \eta_2, \lambda_1, \lambda_2) \\
\text{s.t.} \quad & \eta_2^\top 1_n = 1 \\
& \lambda_1 \geq 0, \lambda_2 \geq 0 \\
& \eta_1 \geq 0, \eta_2 \geq 0
\end{aligned} \tag{CVaR-relax-d}$$

where

$$\begin{aligned}
g(\nu_1, \nu_2, \lambda_1, \lambda_2, \eta_1, \eta_2) = & -\frac{n}{2\lambda_1} (\nu_1 \hat{\mu}_n + \nu_2 1_p - X \eta_2)^\top \hat{\Sigma}_n^{-1} (\nu_1 \hat{\mu}_n + \nu_2 1_p - X \eta_2) \\
& - \frac{n(1-\beta)^2}{2\lambda_2} (\eta_1 + \eta_2)^\top \Omega_n^\dagger (\eta_1 + \eta_2) + R\nu_1 + \nu_2 - U_1 \lambda_1 - U_2 \lambda_2,
\end{aligned}$$

and Ω_n^\dagger is the Moore-Penrose pseudo inverse of the singular matrix Ω_n .

We now show (CVaR-pen) can be solved efficiently by its convex relaxation:

Theorem 1. *Let $(\alpha^*, w^*, z^*, \lambda_1^*, \lambda_2^*, \eta_1^*, \eta_2^*)$ be the primal-dual optimal point of (CVaR-relax) and (CVaR-relax-d). If $\eta_2^* \neq 1_n/n$, then (α^*, w^*, z^*) is an optimal point of (CVaR-pen). Otherwise, if $\eta_2^* = 1_n/n$, we can find the optimal solution to (CVaR-relax) by solving (CVaR-relax-d) with an additional constraint $\eta_1^\top 1_n \geq \delta$, where δ is a constant $0 < \delta \ll 1$.*

Proof. See Appendix B. □

Remark 1 – Bias introduced by penalty functions.

Note that if the penalties induce active constraints (i.e. U_1, U_2 are small enough), \hat{w}_n^v does not converge to w_0 as $n \rightarrow \infty$, i.e. the penalty constraints introduce bias. This is not a problem, however, because we are concerned with finite sample performance, not asymptotic consistency. In Sec. 5, we see that the bias introduced by the penalized solution is actually in the direction that improves performance in the return-risk space.

Remark 2 – Interpretation as chance-programming.

Both $w^\top \hat{\mu}_n$ and $\widehat{CVaR}_n(-w^\top X; \beta)$ are asymptotically normally distributed, so constraining their variances results in the reduction of the corresponding confidence intervals at some fixed level ϵ .

Hence penalizing their variances can be interpreted as **chance-programming** [Charnes, Cooper and Symonds (1958)]. Analytically, the chance constraint on $|w^\top \hat{\mu}_n - w^\top \mu|$ can be transformed to a penalty constraint in the following manner:

$$\begin{aligned} P\left(|w^\top \hat{\mu}_n - w^\top \mu| \leq t\right) &\geq 1 - \epsilon \\ \approx 2\Phi\left(\frac{t}{\sqrt{w^\top \Sigma w/n}}\right) - 1 &\geq 1 - \epsilon \text{ for large } n \\ \iff \frac{1}{n}w^\top \Sigma w &\leq \left(\frac{t}{\Phi^{-1}(1 - \epsilon/2)}\right)^2. \end{aligned}$$

That is, for a fixed level ϵ , there is a one-to-one mapping between the parameter U_1 of the penalty constraint $w^\top \hat{\Sigma}_n w/n \leq U_1$ and the parameter t of the chance constraint. The (asymptotic) variance penalty on $\widehat{CVaR}_n(-w^\top X; \beta)$ has a similar interpretation as a chance constraint.

However, the penalty method can be interpreted as chance programming only if we choose the variance of the respective estimators as the penalty functions. Although we focus on the sample variance penalty function in this paper, **we assert that the penalty method need not be restricted to this particular choice.**

3.2 Parametric case

In the parametric case, we assume the asset log-returns follow an elliptical distribution; i.e. the level sets of the distribution density function form ellipsoids. An elliptical distribution has a stochastic representation as follows [see Anderson (1958) or Muirhead (1982)]:

$$X \stackrel{d}{=} \mu + Y\Sigma^{1/2}U \quad (5)$$

where μ is the mean vector, U is a $p \times 1$ random vector uniformly distributed on the p -dimensional sphere of radius 1 (i.e. $U \stackrel{d}{=} Z_p/\|Z_p\|_2$, $Z_p \sim \mathcal{N}(0, I_p)$), and Y is a non-negative random variable independent of U . A special case is the Gaussian model: choosing $Y = \chi_p$, we get $X \sim \mathcal{N}(\mu, \Sigma)$. The elliptical family of distributions can thus be thought of as a generalization of the Gaussian family, and may be more reasonable for financial modeling because the non-random mixing of covariances can capture non-trivial tail dependence and heavier tails [McNeil et al. (2005)]. In particular, t -distributions also belong in the elliptical family.

The parametric PBR method is to solve the empirical Markowitz problem instead of (CVaR-emp) if X belongs in the elliptical family:

$$\begin{aligned} \hat{w}_n^M = \underset{w}{\operatorname{argmin}} \quad & w^\top \hat{\Sigma}_n w \\ \text{s.t.} \quad & w^\top \hat{\mu}_n = R \\ & w^\top 1_p = 1. \end{aligned} \quad (\text{Mark-emp})$$

The method is based on Lemma 2, which shows that the solutions of (CVaR-pop) and the population Markowitz problem [which is the same as (Mark-emp) except with (Σ, μ) replacing $(\hat{\Sigma}_n, \hat{\mu}_n)$] are equivalent if X is elliptically distributed. Lemma 2 is an extension of results mentioned elsewhere [Rockafellar and Uryasev (2000), De Giorgi (2002)] that show the equivalence of the solutions of (CVaR-pop) and the population Markowitz problem when X is Gaussian. However, to our knowledge, the implication that we can solve (Mark-emp) in lieu of (CVaR-emp) to obtain a better-performing solution has not been asserted.

Lemma 2. Suppose $X \sim \text{Ellip}(\mu, \Sigma, Y)$ as in (5) and $Y > 0$. Then the solution of the population mean-CVaR problem (CVaR-pop) and the population Markowitz problem are equivalent.

Proof. The proof is straightforward: we show $\text{CVaR}(-w^\top X; \beta)$ is a weighted sum of the portfolio mean $w^\top \mu$ and portfolio std $\sqrt{w^\top \Sigma w}$.

First, the portfolio loss is:

$$L(w) := -w^\top X \stackrel{d}{=} -w^\top \mu + Y v^\top U \sqrt{w^\top \Sigma w},$$

where $v^\top = w^\top \Sigma^{1/2} / \sqrt{w^\top \Sigma w}$, with $\|v\|_2 = 1$. Before we compute $\text{CVaR}(-w^\top X; \beta) = \text{CVaR}(L(w); \beta)$, we need to compute α_β , the VaR of $L(w)$ at level β [equivalently, the $(1 - \beta)$ -quantile of $L(w)$]. Since $L(w)$ is a continuous random variable, $\alpha_\beta = F_{L(w)}^{-1}(1 - \beta)$, where $F_{L(w)}^{-1}$ is the inverse cdf of $L(w)$. Now

$$F_{L(w)}(x) = P(L(w) \leq x) = P\left(Y v^\top U \geq \frac{-x - w^\top \mu}{\sqrt{w^\top \Sigma w}}\right),$$

so to compute α_β , we need the distribution of $Y v^\top U$. Since v has norm 1, $v^\top Z_p \stackrel{d}{=} Z_1$, where $Z_1 \sim \mathcal{N}(0, 1)$, and since $U \stackrel{d}{=} Z_p / \|Z_p\|_2$,

$$v^\top U \stackrel{d}{=} \frac{Z_1}{\sqrt{Z_1^2 + \chi_{p-1}^2}},$$

where χ_{p-1}^2 is independent of Z_1 . Thus $(v^\top U)^2 \sim \text{Beta}(1/2, (p-1)/2)$, and by the symmetry of the normal, we have

$$P(Y v^\top U \geq x) = P(Y I(1/2) \sqrt{B} \geq x),$$

where $B \sim \text{Beta}(1/2, (p-1)/2)$ and $I(1/2) \sim \text{Bernoulli}(1/2)$, independent of the rest. This quantity clearly does not depend on our choice of w , hence the solution to the equation

$$F_{L(w)}(x) = 1 - \beta$$

is given by

$$\alpha_\beta = -w^\top \mu + q(1 - \beta; Y I(1/2) \sqrt{B}) \sqrt{w^\top \Sigma w},$$

where q is a function that does not depend on w , and is unique since $L(w)$ is a continuous random variable.

Thus CVaR at level β is given by

$$\begin{aligned} \text{CVaR}(L(w); \beta) &= \frac{1}{1 - \beta} \mathbb{E}[L(w) I(L(w) \geq \alpha_\beta)] \\ &= -w^\top \mu + G(1 - \beta; Y I(1/2) \sqrt{B}) \sqrt{w^\top \Sigma w}, \end{aligned} \tag{6}$$

where G does not depend on w . Hence minimizing $\text{CVaR}(L(w); \beta)$ subject to $w^\top \mu = R$ and $w^\top 1_p = 1$ is equivalent to minimizing $w^\top \Sigma w$ subject to the same constraints, which is precisely the population Markowitz problem. \square

4 Theory

We have thus far introduced nonparametric and parametric PBR methods to improve upon the empirical mean-CVaR problem (CVaR-emp). While we evaluate these methods in Sec. 5 via simulation experiments, it is still desirable to obtain some theoretical understanding of \hat{w}_n , \hat{w}_n^v and \hat{w}_n^M .

The solution to the empirical Markowitz problem \hat{w}_n^M has an explicit form and its asymptotic behavior has been studied elsewhere [for $X \sim \mathcal{N}(\mu, \Sigma)$, see Jobson and Korkie (1980), and for $X \sim \text{Elliptical}$, see El Karoui (2009)]. So we focus on deriving the asymptotic behavior of \hat{w}_n and \hat{w}_n^v — specifically, we show that they follow the Central Limit Theorem (CLT). **Application of the delta method** from classical statistics [see for example, Chapter 3 of Van der Vaart (2000)] then allows us to conclude that the corresponding sample efficient frontiers also follow the CLT. From these results, we can get some insight into the effect of the penalty functions in the nonparametric PBR method, and (indirectly) justify the parametric PBR method when the log-returns are Gaussian.

Notations. In this section, we make use of stochastic little-o and big-O notations: for a given sequence of random variables R_n , $X_n = o_P(R_n)$ means $X_n = Y_n R_n$ where $Y_n \xrightarrow{P} 0$, and $X_n = O_P(R_n)$ means $X_n = Y_n R_n$ where $Y_n = O_P(1)$, i.e. for every $\varepsilon > 0$ there exists a constant M such that $\sup_n P(|Y_n| > M) < \varepsilon$.

Measurability Issues. We also encounter quantities that may not be measurable (e.g. supremum over uncountable families of measurable functions). We note that whenever the “probability” of such quantities are written down, we actually mean the outer probability. For further details, see Appendix C of Pollard (1984).

4.1 Preliminaries

The quantities \hat{w}_n and \hat{w}_n^v are solutions to non-trivial optimization problems so they cannot be written down analytically, and it seems characterizing their asymptotic distributions would be difficult. However, we are not at a complete loss. In statistics, an M-estimator³ is an estimator that minimizes an empirical function of the type

$$\theta \mapsto M_n(\theta) := \frac{1}{n} \sum_{i=1}^n m_\theta(X_i), \quad (7)$$

where X_1, \dots, X_n are iid observations, over some parameter space Θ . The solution $\hat{\theta}_n$ is then a reasonable estimator of the minimizer θ_0 of the true mean $M(\theta) = \mathbb{E}[m_\theta(X_1)]$. It is well-known that $\hat{\theta}_n$ obeys the Central Limit Theorem (i.e. is asymptotically normally distributed) under some regularity conditions. Intuitively, assuming θ is one-dimensional and M_n is sufficiently smooth, the CLT result is based on Taylor expansion of the first-order condition $dM_n(\hat{\theta}_n)/d\theta = 0$ about θ_0 :

$$0 = \frac{dM_n(\hat{\theta}_n)}{d\theta} = \frac{dM_n(\theta_0)}{d\theta} + (\hat{\theta}_n - \theta_0) \frac{d^2 M_n(\theta_0)}{d\theta^2} + O_P(|\hat{\theta}_n - \theta_0|^2).$$

³“M” stands for Minimization (or Maximization). For readers unfamiliar with M-estimation, maximum likelihood estimation falls in this category.

Under reasonable assumptions that $d^2 M_n(\theta_0)/d\theta^2$ obeys the Weak Law of Large Numbers and $\hat{\theta}_n$ is a consistent estimator of θ_0 (i.e. $|\hat{\theta}_n - \theta_0| \xrightarrow{P} 0$), we have

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = -\frac{1}{\mathbb{E}\left[\frac{d^2 M_n(\theta_0)}{d\theta^2}\right]} \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{dm_{\theta_0}(X_i)}{d\theta} + o_P(1),$$

with the latter expression obeying the standard CLT as it is a normalized sum of iid random variables.

So we ask, can we transform (CVaR-emp) and (CVaR-pen) to a problem for which we can use the M-estimation results?

The first step towards transforming (CVaR-emp) and (CVaR-pen) is to make them into constraint-free optimization problems. This is achievable, albeit with some thoughts, and we defer the details to Sec. 4.2. Next, we need to show \hat{w}_n and \hat{w}_n^v are consistent, i.e. they converge in probability to the corresponding population solutions. The proof of consistency is also provided in Sec. 4.2.

Once (CVaR-emp) is transformed to a global optimization problem, it is equivalent to an M-estimation problem in that the objective is a sample average of iid random variables of the form Eq. (7). Thus we conclude \hat{w}_n is asymptotically normally distributed with mean w_0 and covariance matrix Σ_{w_0} , which we can compute.

However, (CVaR-pen) after transformation into a global problem is not quite an M-estimation problem, because, after some algebra, the objective is of the form (see Sec. 4.2 for details):

$$\theta \mapsto M_n(\theta) = \frac{1}{n(n-1)} \sum_{i \neq j} m_\theta^U(X_i, X_j), \quad (8)$$

where $m^U(\cdot, \cdot)$ is a permutation-symmetric function, and the sum is over all possible pairs (i, j) for $1 \leq i, j \leq n$, resulting in a sample average of identically distributed but non-independent terms.

For fixed θ , statistics of the form Eq. (8) are known as U-statistics, and we believe the solution \hat{w}_n^v is still well-behaved because U-statistics can be decomposed into a term of the form $M_n^1(\theta) = \sum_{i=1}^n m_\theta^1(X_i)$ (known as its Hajék projection or first term in its Hoeffding decomposition; see Hoeffding (1948)) and a remainder which converges to zero in probability at rate \sqrt{n} . Thus we intuit that the asymptotic behavior of \hat{w}_n^v is equivalent to the minimizer of $M_n^1(\theta)$, the latter for which we can apply the standard M-estimation result. We make this intuition rigorous in Sec. 4.3. In Sec. 4.4, we provide details of the asymptotic distributions of \hat{w}_n and \hat{w}_n^v when $X \sim \mathcal{N}(\mu, \Sigma)$, and provide a justification of the parametric PBR method.

4.2 Consistency of \hat{w}_n and \hat{w}_n^v

In this subsection we show consistency of $\hat{w}_n^v = \hat{w}_n^v(\lambda_1, \lambda_2)$. The result goes through for \hat{w}_n by setting $\lambda_1 = \lambda_2 = 0$.

4.2.1 Transformation into global optimization

The penalized CVaR portfolio optimization problem with dualized mean and sample/asymptotic variance penalty constraints is

$$\begin{aligned} \min_{(\alpha, w) \in \mathbb{R} \times \mathbb{R}^p} \quad & M_n(\alpha, w; \lambda_1, \lambda_2) \\ \text{s.t.} \quad & w^\top 1_p = 1, \end{aligned} \quad (\text{CVaR-dual})$$

where

$$M_n(\theta; \lambda_1, \lambda_2) = \frac{1}{n} \sum_{i=1}^n m_\theta(X_i) + \frac{\lambda_1}{n} w^\top \hat{\Sigma}_n w + \frac{\lambda_2}{n-1} \sum_{i=1}^n \left(z_\theta(X_i) - \frac{1}{n} \sum_{j=1}^n z_\theta(X_j) \right)^2, \quad (9)$$

$$m_\theta(x) = \alpha + \frac{1}{1-\beta} z_\theta(x) - \lambda_0 w^\top x, \quad (10)$$

and $\lambda_0 > 0$, $\lambda_1, \lambda_2 \geq 0$ are pre-determined constants.

We dualize the mean constraint $w' \hat{\mu}_n = R$ because it makes the analysis of the corresponding solution much easier. While dualizing the mean constraint adds a sample average of iid terms to the objective, leaving it as a constraint results in a solution that has a non-trivial dependence on the underlying randomness.

Now eliminating the non-random constraint $w^\top 1_p = 1$ is straight-forward; one possible way is to re-parameterize w as $w = w_1 + Lv$, where $L = [0_{(p-1) \times 1}, I_{(p-1) \times (p-1)}]^\top$, $v = [w_2, \dots, w_p]^\top$ and $w_1 = [1 - v^\top 1_{(p-1)}, 0_{1 \times (p-1)}]^\top$. The transformed problem is thus

$$\min_{\theta \in \mathbb{R}^p} M_n(\theta; \lambda_1, \lambda_2), \quad (11)$$

where $\theta = (\alpha, v) \in \mathbb{R} \times \mathbb{R}^{p-1}$ is free of constraints, and the corresponding population problem is

$$\min_{\theta \in \mathbb{R}^p} M(\theta; \lambda_1, \lambda_2) = \mathbb{E}[M_n(\theta; \lambda_1, \lambda_2)]. \quad (12)$$

In what follows, we assume $M(\theta; \lambda_1, \lambda_2)$ has a unique minimizer $\theta_0(\lambda_1, \lambda_2)$. We also let $\hat{\theta}_n(\lambda_1, \lambda_2)$ be a near-minimizer of $M_n(\theta; \lambda_1, \lambda_2)$, i.e.

$$M_n(\hat{\theta}_n; \lambda_1, \lambda_2) < \inf_{\theta \in \mathbb{R}^p} M_n(\theta; \lambda_1, \lambda_2) + o_P(1). \quad (13)$$

4.2.2 Transformation of the objective to a U-statistic

Let $\theta = (\alpha, v) \in \mathbb{R} \times \mathbb{R}^{p-1}$ and $z_\theta(x) := (-x^\top(w_1 + Lv) - \alpha)^+$. With simple algebra, we can re-write the objective Eq. (9) as a U-statistic:

$$M_n(\theta; \lambda_1, \lambda_2) = \frac{1}{\binom{n}{2}} \sum_{\substack{1 \leq i, j \leq n \\ i \neq j}} m_{(\theta; \lambda_1, \lambda_2)}^U(X_i, X_j), \quad (14)$$

where

$$m_{(\theta; \lambda_1, \lambda_2)}^U(x_i, x_j) := \frac{1}{2} [m_\theta(x_i) + m_\theta(x_j)] + \frac{\lambda_1}{2} [(w_1 + Lv)^\top (x_i - x_j)]^2 + \frac{\lambda_2}{2} (z_\theta(x_i) - z_\theta(x_j))^2. \quad (15)$$

4.2.3 Consistency of $\hat{\theta}_n(\lambda_1, \lambda_2)$

Let us now prove consistency of $\hat{\theta}_n(\lambda_1, \lambda_2)$ for fixed $\lambda_1, \lambda_2 \geq 0$. The intuition behind the proof is as follows: if $M(\theta; \lambda_1, \lambda_2)$ is well-behaved such that for every $\varepsilon > 0$ there exists $\eta > 0$ such that $\|\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2)\|_2 > \varepsilon \implies M(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2) > \eta$, then consistency follows from showing that the probability of the event $\{M(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2) > \eta\}$ goes to zero for all $\varepsilon > 0$. In the proof, we show that $0 \leq M(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2) \leq -(M_n(\hat{\theta}_n; \lambda_1, \lambda_2) -$

$M(\hat{\theta}_n; \lambda_1, \lambda_2) + o_P(1)$, hence the result follows by proving Uniform Law of Large Numbers (ULLN) for $M_n(\theta; \lambda_1, \lambda_2)$:

$$\sup_{\theta \in \mathbb{R}^p} |M_n(\theta; \lambda_1, \lambda_2) - M(\theta; \lambda_1, \lambda_2)| \xrightarrow{P} 0. \quad (16)$$

ULLN has been extensively studied in the statistics and empirical processes literature and one of the standard approaches to showing ULLN is through bracketing numbers. Given two functions l, u , the bracket $[l, u]$ is the set of all functions g with $l \leq g \leq u$. An ε -bracket in $L_r(P)$ is a bracket $[l, u]$ with $\mathbb{E}_P(u - l)^r < \varepsilon^r$, and the bracketing number $N_{[\cdot]}(\varepsilon, \mathcal{F}, L_r(P))$ is the minimum number of ε -brackets needed to cover \mathcal{F} . Having a finite bracketing number $N_{[\cdot]}(\varepsilon, \mathcal{F}, L_r(P)) < \infty$ for every $\varepsilon > 0$ means one can find a finite approximation to \mathcal{F} with ε -accuracy for all $\varepsilon > 0$, and ULLN holds for such \mathcal{F} [Theorem 19.4 Van der Vaart (2000)].

There are certainly known sufficient conditions for finite bracketing numbers. For our problem, if we can replace \mathbb{R}^p with a compact set, we can show F is a Lipschitz class of functions (defined in the next paragraph), which is known to have finite $N_{[\cdot]}(\varepsilon, \mathcal{F}, L_r(P))$ for every $\varepsilon > 0$. Now for all practical purposes, we need only consider a compact subset of Θ , $[-K, K]^p$ where K is appropriately large enough, because the elements of $\theta = (\alpha, v)$ are only meaningful if bounded in size (α is the Value-at-Risk of the portfolio $w = w_1 + Lv$). Hence for the rest of this section we assume a K exists such that $\hat{\theta}_n \in [-K, K]^p$ for all n and $\theta_0 \in [-K, K]^p$.

Definition 1 (Lipschitz class). Consider a class of measurable functions $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$, $f_\theta : \mathcal{X} \rightarrow \mathbb{R}$, under some probability measure P . We say \mathcal{F} is a Lipschitz class about $\theta_0 \in \Theta$ if $\theta \mapsto f_\theta(x)$ is differentiable at θ_0 for P -almost every x with derivative $\dot{f}_{\theta_0}(x)$ and such that, for every θ_1 and θ_2 in a neighborhood of θ_0 , there exists a measurable function \dot{f} with $\mathbb{E}[\dot{f}^2(X_1)] < \infty$ such that

$$|f_{\theta_1}(x) - f_{\theta_2}(x)| \leq \dot{f}(x) \|\theta_1 - \theta_2\|_2.$$

Example 19.7 of Van der Vaart (2000) shows that if $\mathcal{F} = \{f_\theta : \theta \in \Theta\}$ is a class of measurable functions with bounded $\Theta \subset \mathbb{R}^d$ and \mathcal{F} is Lipschitz about $\theta_0 \in \Theta$ then for every $0 < \varepsilon < \text{diam}(\Theta)$, there exists C such that

$$N_{[\cdot]}(\varepsilon \sqrt{\mathbb{E}(|\dot{f}(X)|^2)}, \mathcal{F}, L_2(P)) \leq C \left(\frac{\text{diam}(\Theta)}{\varepsilon} \right)^d, \quad (17)$$

i.e. has a finite bracketing number for all $\varepsilon > 0$. This result is needed in proving consistency in the following.

Theorem 2. For fixed $\lambda_1, \lambda_2 \geq 0$, let $\hat{\theta}_n(\lambda_1, \lambda_2)$ be a near-minimizer of $M_n(\theta; \lambda_1, \lambda_2)$ as in Eq. (13), and let $\theta_0(\lambda_1, \lambda_2)$ be the unique minimizer of $M(\theta; \lambda_1, \lambda_2)$. Also let

$$\mathcal{F}_1 = \{m_\theta : \theta \in [-K, K]^p\}, \quad \mathcal{F}_2 = \{m_{(\theta; \lambda_1, \lambda_2)}^U : \theta \in [-K, K]^p\},$$

where m_θ and $m_{(\theta; \lambda_1, \lambda_2)}^U$ are defined in Eqs. (10) and (15). Suppose the following:

Assumption 1. $\theta \mapsto M(\theta; \lambda_1, \lambda_2)$ is continuous and $\liminf_{|\theta| \rightarrow \pm\infty} M(\theta; \lambda_1, \lambda_2) > M(\theta_0; \lambda_1, \lambda_2)$.

Assumption 2. X_1, \dots, X_n are iid continuous random vectors with finite fourth moment.

Then

$$\|\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2)\|_2 \xrightarrow{P} 0.$$

Proof. See Appendix C.1. □

4.3 Central Limit Theorem for $\hat{\theta}_n(\lambda_1, \lambda_2)$

We are now ready to show the CLT for $\hat{\theta}_n(\lambda_1, \lambda_2)$. The CLT for $\hat{\theta}_n(0, 0)$ is a straightforward application of known M-estimation results for Lipschitz class of objective functions [e.g. Theorem 5.23 of Van der Vaart (2000)].

The CLT for $\hat{\theta}_n(\lambda_1, \lambda_2)$ when λ_1, λ_2 are not both zero does not follow straight-forwardly from M-estimation results because $M_n(\theta; \lambda_1, \lambda_2)$ is a sample average of identically distributed but non-independent terms. However, statistics of the form $M_n(\theta; \lambda_1, \lambda_2)$ are known as U-statistics, and we can decompose them into a sum of iid random variables and a component which is $o_P(1/\sqrt{n})$ [Hoeffding (1948)]:

$$M_n(\theta; \lambda_1, \lambda_2) = \frac{1}{n} \sum_{i=1}^n m_{(\theta; \lambda_1, \lambda_2)}^1(X_i) + E_n(\theta; \lambda_1, \lambda_2), \quad (18)$$

where $m_{(\theta; \lambda_1, \lambda_2)}^1(X_i) = 2\mathbb{E}_{X_j}[m_{(\theta; \lambda_1, \lambda_2)}^U(X_i, X_j)] - \mathbb{E}_{X_1, X_2}[m_{(\theta; \lambda_1, \lambda_2)}^U(X_1, X_2)]$ and $E_n(\theta; \lambda_1, \lambda_2) = o_P(1/\sqrt{n})$. Hence we suspect $|R_n^U(\hat{\theta}_n; \lambda_1, \lambda_2)| \xrightarrow{P} 0$, where

$$R_n^U(\theta; \lambda_1, \lambda_2) = \sqrt{n}(\theta - \theta_0) - [\nabla_{\theta_0}^2 \mathbb{E} m_{(\theta; \lambda_1, \lambda_2)}^1(X_i)]^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n m_{(\theta; \lambda_1, \lambda_2)}^1(X_i).$$

Now $\hat{\theta}_n$ changes with every n so we need uniform probabilistic convergence of $R_n^U(\theta; \lambda_1, \lambda_2)$, and implicitly of $E_n(\theta; \lambda_1, \lambda_2)$. For this we need to show convergence of particular stochastic processes; an empirical process and a U-process.

Definition 2. Let X_1, \dots, X_n be iid random vectors from \mathcal{X} . For a measurable function $f : \mathcal{X} \rightarrow \mathbb{R}$, the empirical process at f is

$$\mathbb{G}_n f := \frac{1}{\sqrt{n}} \sum_{i=1}^n [f(X_i) - \mathbb{E} f(X_1)],$$

and for a measurable function $g : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, the U-process at g is

$$\mathbb{U}_n g := \frac{\sqrt{n}}{\binom{n}{2}} \sum_{i \neq j} [g(X_i, X_j) - \mathbb{E}_{X_1, X_2} g(X_1, X_2)].$$

To show convergence of quantities such as $\sup_{t \in T} |X_n(t)|$ for some stochastic process $\{X_n(t) : t \in T\}$, we need to introduce the notion of weak convergence of stochastic processes. If $X_n(\cdot, \omega)$ is a bounded function for every $\omega \in \Omega$, then we can consider $X_n(\cdot, \omega)$ to be a point in the function space $\ell^\infty(T)$, the space of bounded functions on T which is equipped with the supremum norm. Hence, showing the convergence of $\sup_{t \in T} |X_n(t)|$ is equivalent to showing weak convergence of X_n in this function space.

Definition 3 (Weak convergence of a stochastic process). A sequence of $X_n : \Omega_n \mapsto \ell^\infty(T)$ converges weakly to a tight random element ⁴ X iff both of the following conditions hold:

1. *Finite approximation:* the sequence $(X_n(t_1), \dots, X_n(t_k))$ converges in distribution in \mathbb{R}^k for every finite set of points t_1, \dots, t_k in T .

⁴A random element is a generalization of a random variable. Let (Ω, \mathcal{G}, P) be a probability space and \mathbb{D} a metric space. Then the \mathcal{G} -measurable map $X : \Omega \mapsto \mathbb{D}$ is called a random element.

2. *Maximal inequality:* for every $\varepsilon, \eta > 0$ there exists a partition of T into finitely many sets T_1, \dots, T_k such that

$$\limsup_{n \rightarrow \infty} P \left[\sup_i \sup_{s, t \in T_i} |X_n(s) - X_n(t)| \geq \varepsilon \right] \leq \eta.$$

The point at the end of this is, as taking the supremum is a continuous map in the topology of $\ell^\infty(T)$, weak convergence of $X_n(\cdot)$ to $X(\cdot)$ would allow us to conclude $\sup_{t \in T} |X_n(t) - X(t)| \rightarrow 0$.

Regarding empirical processes, we say a class of measurable functions \mathcal{F} is *P-Donsker* if $\{\mathbb{G}_n f : f \in \mathcal{F}\}$ converges weakly to a tight random element in $\ell^\infty(\mathcal{F})$. This property is related to the bracketing numbers introduced in Sec. 4.2: a class \mathcal{F} is P-Donsker if $\varepsilon \log[N_{[\cdot]}(\varepsilon, \mathcal{F}, L_2(P))] \rightarrow 0$ as $\varepsilon \rightarrow 0$ [due to Donsker; see Theorem 19.5 of Van der Vaart (2000)]. Many sufficient conditions for the weak convergence of $\{\mathbb{U}_n f : f \in \mathcal{F}\}$ are provided in Arcones and Gine (1993), and we make use of one in our proof of CLT for $\hat{\theta}_n(\lambda_1, \lambda_2)$ below.

Theorem 3. Fix $\lambda_1, \lambda_2 \geq 0$, λ_1, λ_2 not both zero and assume the same setting as Theorem 2. Also let

$$\dot{m}_{(\theta_0; \lambda_1, \lambda_2)}^U(x) = \nabla_\theta m_{(\theta_0; \lambda_1, \lambda_2)}^U(x)|_{\theta=\theta_0(\lambda_1, \lambda_2)}, \text{ for } x \in \mathbb{R}^p,$$

and further assume

Assumption 3. $\mathbb{E}_{X_1, X_2}[m_{(\theta_0; \lambda_1, \lambda_2)}^U(X_1, X_2)^2] < \infty$.

Assumption 4. $\theta \mapsto M(\theta; \lambda_1, \lambda_2)$ admits a second-order Taylor expansion at its point of minimum $\theta_0(\lambda_1, \lambda_2)$ with nonsingular symmetric second derivative matrix $V_{\theta_0(\lambda_1, \lambda_2)}$. Then

$$\sqrt{n}(\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2)) = -V_{\theta_0(\lambda_1, \lambda_2)}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{m}_{(\theta_0; \lambda_1, \lambda_2)}^1(X_i) + o_p(1)$$

where

$$\dot{m}_{(\theta; \lambda_1, \lambda_2)}^1(X_i) = 2\mathbb{E}_{X_2}[\dot{m}_{(\theta; \lambda_1, \lambda_2)}^U(X_1, X_2)] - \mathbb{E}_{X_1, X_2}[\dot{m}_{(\theta; \lambda_1, \lambda_2)}^U(X_1, X_2)]$$

is the first-order term in the Hoeffding decomposition of $M_n(\theta; \lambda_1, \lambda_2)$.

Remark – Implication on the choice of penalty functions.

We have just shown that asymptotically, the sample variance penalty functions affect the solution performance only through its Hajék projection. This observation can generalize to many typical penalty functions (e.g. different statistics of mean and CVaR estimators), and as such, the implication is that of all possible penalty functions to consider, one may focus on a subclass of functions that can be expressed as a sample average of iid terms.

Corollary 1. Assume the same setting as Theorem 3. Then

$$\sqrt{n}(\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2)) \Rightarrow \mathcal{N}(0, \Sigma_{\theta_0}(\lambda_1, \lambda_2)), \quad (19)$$

where $\Sigma_{\theta_0}(\lambda_1, \lambda_2) = A^{-1}BA^{-1}$,

$$\begin{aligned} A = A_{\theta_0}(\lambda_1, \lambda_2) &= \nabla_{\theta}^2 \mathbb{E}[m_{(\theta; \lambda_1, \lambda_2)}^1(X_1)] \Big|_{\theta=\theta_0} \\ &= \nabla_{\theta}^2 \left[\frac{1}{1-\beta} \mathbb{E}z_{\theta}(X_1) + \lambda_1 w^{\top} \Sigma w + \lambda_2 \text{Var}(z_{\theta}(X_1)) \right] \Big|_{\theta=\theta_0} \\ B = B_{\theta_0}(\lambda_1, \lambda_2) &= \mathbb{E}[\nabla_{\theta_0} m_{(\theta; \lambda_1, \lambda_2)}^1(X_1) \nabla_{\theta_0} m_{(\theta; \lambda_1, \lambda_2)}^1(X_1)^{\top}] \end{aligned}$$

where

$$\begin{aligned} &\nabla_{\theta} m_{(\theta; \lambda_1, \lambda_2)}^1(x) \\ &= \begin{bmatrix} 1 - \frac{1}{1-\beta} \mathbb{I} + 2\lambda_2 \mathbb{E}[(z_{\theta}(X) - \mathbb{E}z_{\theta}(X))(-\mathbb{I} + \mathbb{E}\mathbb{I})] \\ -\frac{1}{1-\beta} L^{\top} X \mathbb{I} - \lambda_0 L^{\top} X + 2\lambda_1 L^{\top} (X - \mu)(X - \mu)^{\top} w + 2\lambda_2 \mathbb{E}[(z_{\theta}(X) - \mathbb{E}z_{\theta}(X))(-L^{\top} X \mathbb{I} + \mathbb{E}L^{\top} X \mathbb{I})] \end{bmatrix}, \end{aligned}$$

and $\mathbb{I} = \mathbb{I}(z_{\theta}(X) \geq 0)$.

Remarks.

1. For asymptotics of $\hat{w}_n(\lambda_1, \lambda_2)$ we have

$$\sqrt{n}(\hat{w}_n(\lambda_1, \lambda_2) - w_0(\lambda_1, \lambda_2)) \Rightarrow \mathcal{N}(0, \Sigma_{w_0}(\lambda_1, \lambda_2)), \quad (20)$$

where $\Sigma_{w_0}(\lambda_1, \lambda_2) = (0_p \ L) \Sigma_{\theta_0}(\lambda_1, \lambda_2) (0_p \ L)^{\top}$.

2. Setting $\lambda_1, \lambda_2 = 0$, we get back the unpenalized mean-CVaR problem.

3. Asymptotic distribution of the efficient frontier.

With Eq. (20), we can state the distribution of the true efficient frontier — that is, the distribution of $\hat{w}_n(\lambda_1, \lambda_2)^{\top} \mu$ and $g(\hat{w}_n(\lambda_1, \lambda_2)) := \text{CVaR}(-\hat{w}_n(\lambda_1, \lambda_2)^{\top} X_{n+1}; \beta)$, where $X_{n+1} \sim F$, independent of $X_1 \dots, X_n$. For the portfolio mean, we have

$$\sqrt{n}(\hat{w}_n(\lambda_1, \lambda_2)^{\top} \mu - w_0(\lambda_1, \lambda_2)^{\top} \mu) \Rightarrow \mathcal{N}(0, \mu^{\top} \Sigma_{w_0}((\lambda_1, \lambda_2)) \mu)$$

and for the true CVaR, by the delta Method

$$\sqrt{n}(g(\hat{w}_n(\lambda_1, \lambda_2)) - g(w_0(\lambda_1, \lambda_2))) \Rightarrow \mathcal{N}\left\{0, g'(w_0(\lambda_1, \lambda_2))^{\top} \Sigma_{w_0}(\lambda_1, \lambda_2) g'(w_0(\lambda_1, \lambda_2))\right\}. \quad (21)$$

The asymptotic distribution of $g(\hat{w}_n(\lambda_1, \lambda_2))$ clearly depends on the distribution of the assets X . In the case when $X \sim \text{Ellip}(\mu, \Sigma, Y)$, $g(w) = -w^{\top} \mu + G\sqrt{w^{\top} \Sigma w}$ according to our previous calculations in Eq. (6). Hence

$$\sqrt{n}(g(\hat{w}_n) - g(w_0)) \Rightarrow \mathcal{N}\left(0, \left(-\mu + G \frac{\Sigma w_0}{\sqrt{w_0^{\top} \Sigma w_0}}\right)^{\top} \Sigma_{w_0} \left(-\mu + G \frac{\Sigma w_0}{\sqrt{w_0^{\top} \Sigma w_0}}\right)\right). \quad (22)$$

4.4 Example. Asymptotic analysis for $X \sim \mathcal{N}(\mu, \Sigma)$

In the following, we provide the detailed computation of $\Sigma_{\theta_0}(0, 0)$ for the unpenalized solution $\hat{\theta}_n(0, 0)$ when $X \sim \mathcal{N}(\mu, \Sigma)$.

Lemma 3. Suppose $X \sim \mathcal{N}(\mu, \Sigma)$. Then

$$\begin{aligned} z_{\theta_0}(X) &= -w_0^\top X - \alpha_0 \sim \sigma_0 \mathcal{N}(-\Phi^{-1}(\beta), 1), \text{ and} \\ p_0 &= f_{-w_0^\top X}(0) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{1}{2\sigma_0^2}(\Phi^{-1}(\beta))^2\right\}, \end{aligned}$$

where $\sigma_0 = \sqrt{w_0^\top \Sigma w_0}$. Then $\Sigma_{\theta_0}(0, 0) = A_0^{-1} B_0 A_0^{-1}$, where A_0, B_0 are symmetric matrices with

$$\begin{aligned} A_0(1, 1) &= \frac{p_0}{1 - \beta} \\ A_0(j, l) &= \frac{p_0}{(1 - \beta)} \mathbb{E}[L_j^\top X L_l^\top X | z_{\theta_0}(X) = 0] \quad \text{for } 2 \leq j, l \leq p \\ A_0(1, j) &= -\frac{p_0}{(1 - \beta)} \mathbb{E}[L_j^\top X | z_{\theta_0}(X) = 0] \quad \text{for } 2 \leq j \leq p, \end{aligned}$$

where L_j is the j -th column of L , and

$$\begin{aligned} B_0(1, 1) &= \frac{\beta}{1 - \beta} \\ B_0(j, l) &= \lambda_0^2 (L_j^\top \Sigma L_l + L_j^\top \mu L_l^\top \mu) + \frac{1}{(1 - \beta)} \left(\frac{1}{1 - \beta} + 2\lambda_0 \right) \mathbb{E}[L_j^\top X L_l^\top X \mathbb{I}(z_{\theta_0}(X) \geq 0)] \quad \text{for } 2 \leq j, l \leq p \\ B_0(1, j) &= 0 \quad \text{for } 2 \leq j \leq p. \end{aligned}$$

Proof. This is a straight-forward application of Corollary 1 for the case $X \sim \mathcal{N}(\mu, \Sigma)$. \square

Let us now compare the asymptotic results derived above with simulations with finite number of observations. Consider 5 assets, a range of observations ($n = 250, 500, 1000, 2000$) and $X \sim \mathcal{N}(\mu_{sim}, \Sigma_{sim})$, where the model parameters are the same as the model parameters of the first five assets used in Sec. 2.2. For simulations, we solve the mean-CVaR problem with dualized mean constraint:

$$\begin{aligned} \min_w \quad & \widehat{CVaR}_n(-w^\top X; \beta) - \lambda_0 w^\top \hat{\mu}_n \\ \text{s.t.} \quad & w^\top 1_p = 1, \end{aligned}$$

and follow steps similar to Sec. 2.2.

In Fig. 2, we summarize the empirical frontiers by plotting their averages and indicating $\pm 1/2$ standard deviation error bars, in both true mean (vertical) and true risk estimations (horizontal) in grey. The population frontier is also plotted, and is shown in green, and the theoretical $\pm 1/2$ standard deviations of mean and risk estimations are juxtaposed with the empirical error bars in red. We make a couple of observations:

1. With increasing n , the theoretical error bars approach the simulated ones, as expected.
2. The theory seems to better predict the mean estimation error (vertical) better than the risk estimation error (horizontal). With finite n , the mean estimation error, which is computed using Eq. (21), depends only on one approximate quantity $\Sigma_{w_0}(0, 0)$, whereas the risk estimation error, computed using Eq. (22), depends on $\Sigma_{w_0}(0, 0)$ and w_0 . Although \hat{w}_n is a consistent estimator of w_0 asymptotically, with finite n the difference does play a role, as shown by the relative inaccuracy of the horizontal error bars compared to the vertical ones. The finite sample bias also explains the gap in the positions of the population and simulated efficient frontiers.

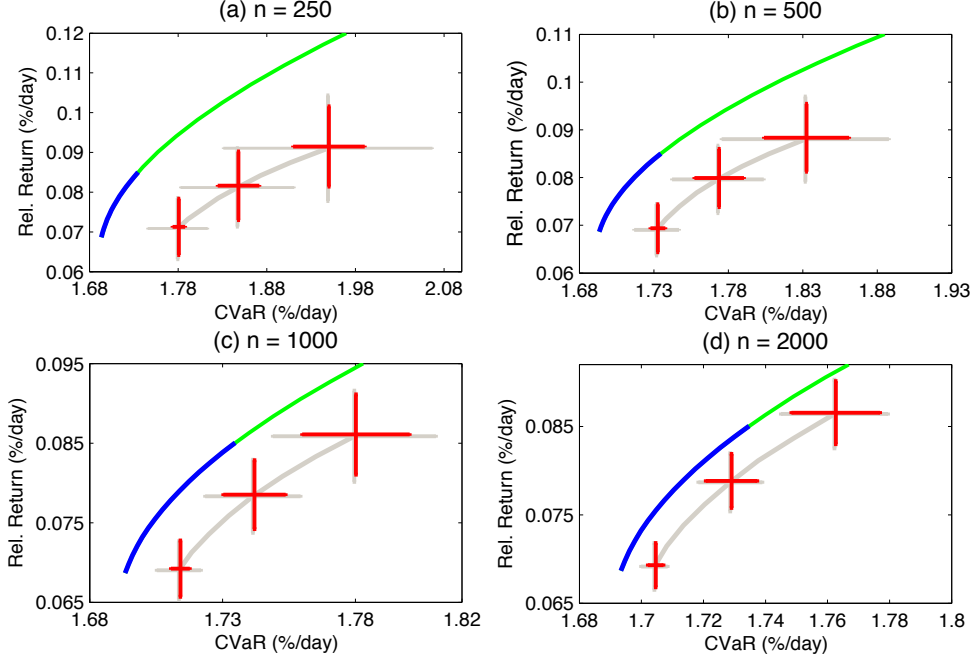


Figure 2: Comparison of theoretical (red) and simulated (grey) distributions of the empirical efficient frontier when $X \sim \mathcal{N}(\mu_{sim}, \Sigma_{sim})$ for increasing number of observations $n = [250, 500, 1000, 2000]$. The error bars indicate $\pm 1/2$ std variabilities in the mean and CVaR. Green is the population efficient frontier, and blue indicates the portion that corresponds to the return range considered for the simulations. Observe that the asymptotic variance calculated theoretically (red bars) approach the simulated variance (grey bars) with increasing n .

Let us now derive asymptotic properties of the penalized solution $\hat{\theta}_n(\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \geq 0$, when $X \sim \mathcal{N}(\mu, \Sigma)$. First, we show that when $X \sim \mathcal{N}(\mu, \Sigma)$, penalizing variance of CVaR estimation is redundant if one penalizes the sample variance of the mean.

Lemma 4. Suppose $X \sim \mathcal{N}(\mu, \Sigma)$ and let $z_\theta(X) = -\alpha - w^\top X$. Then $z_\theta(X) \sim \mathcal{N}(\mu_1, \sigma_1^2)$ where $\mu_1 = -\sigma_1 \Phi^{-1}(\beta)$, $\sigma_1^2 = w^\top \Sigma w$, and

$$\text{Var}[\max(z_\theta(X), 0)] = C(\beta) \sigma_1^2,$$

where $C(\beta)$ is a constant that only depends on β . Thus penalizing the sample variance of CVaR via $P_2(w) = \widehat{\text{Var}}_n[z_{\theta(w)}(X), 0] \leq U_2$ is redundant if one penalizes the sample variance of the mean via $P_1(w) = w^\top \hat{\Sigma}_n w = \hat{\sigma}_{1,n}^2 \leq U_1$.

Proof. Straight-forward calculations show

$$\text{Var}[\max(z_\theta(X), 0)] = \{([\Phi^{-1}(\beta)]^2 + 1)(1 - \beta) - 3\Phi^{-1}(\beta)f_{Z_0}[\Phi^{-1}(\beta)]\} \sigma_1^2,$$

where f_{Z_0} is the pdf of the standard normal random variable Z_0 . \square

The implication now is that when $X \sim \mathcal{N}(\mu, \Sigma)$, we need only consider $\lambda_1 \geq 0, \lambda_2 = 0$ to characterize the asymptotic properties of the penalized solution, which we describe below.

Lemma 5. Suppose $X \sim \mathcal{N}(\mu, \Sigma)$. Then

$$\Sigma_{\theta_0}(\lambda_1, 0) = A_1^{-1} B_1 A_1^{-1},$$

where A_1, B_1 are symmetric matrices with

$$\begin{aligned} A_1(1, 1) &= A_0(1, 1) \\ A_1(j, l) &= A_0(j, l) + \lambda_1 L_j^\top \Sigma L_l \quad \text{for } 2 \leq j, l \leq p \\ A_1(1, j) &= A_0(1, j) \quad \text{for } 2 \leq j \leq p \end{aligned}$$

where L_j is the j -th column of L , and

$$\begin{aligned} B_1(1, 1) &= B_0(1, 1) \\ B_1(j, l) &= B_0(j, l) + \lambda_1 \mathbb{E}[b_{0,j} b_{1,l} + b_{0,l} b_{1,j} + \lambda_1 b_{1,j} b_{1,l}] \quad \text{for } 2 \leq j, l \leq p \\ B_1(1, j) &= B_0(1, j) + \lambda_1 \mathbb{E}[b_{0,1} b_{1,j}] \quad \text{for } 2 \leq j \leq p \end{aligned}$$

where for $2 \leq j, l \leq p$,

$$\begin{aligned} \mathbb{E}[b_{0,j} b_{1,l}] &= -\frac{2}{1-\beta} \mathbb{E}[L_j^\top X L_l^\top (X - \mu) w^\top (X - \mu) \mathbb{I}(z_{\theta_0}(X) \geq 0)] - 2\lambda_0 L_j^\top \mu L_l^\top \Sigma w \\ \mathbb{E}[b_{1,j} b_{1,l}] &= 4\mathbb{E}[L_j^\top (X - \mu)(X - \mu)^\top L_l w^\top (X - \mu)(X - \mu)^\top w] \\ \mathbb{E}[b_{0,1} b_{1,l}] &= 2L_l \Sigma w - \frac{2}{1-\beta} \mathbb{E}[L_l^\top (X - \mu) w^\top (X - \mu) \mathbb{I}(z_{\theta_0}(X) \geq 0)]. \end{aligned}$$

Proof. This is a straight-forward application of Corollary 1 for the case $X \sim \mathcal{N}(\mu, \Sigma)$. \square

Remark – Justification of the parametric PBR method.

The nonparametric PBR method with only a penalty on the mean estimation is a linear combination of the empirical mean-CVaR problem (CVaR-emp) and the empirical Markowitz problem (Mark-emp) because the penalty is precisely the portfolio variance estimate $w^\top \hat{\Sigma}_n w$. In particular, this single-penalty problem approaches (Mark-emp) with increasing λ_1 . In Figure 3, we plot 1 std of $w_n^v(\lambda_1, 0)^\top \mu$ and $CVaR(-w_n^v(\lambda_1, 0)^\top X; \beta)$ for the single-penalty problem as λ_1 is increased, for different values of λ_0 , computed using Lemma 5. Observe that the asymptotic standard deviations for both portfolio mean and CVaR decrease with increasing λ_1 , uniformly in λ_0 . Given that both solutions to (CVaR-emp) and (Mark-emp) converge to the population solution w_0 , the asymptotic theory deems the empirical Markowitz solution superior.

5 Numerical results

In this section, we present simulation results to evaluate the nonparametric and parametric PBR methods presented in Sec. 3 against the straight-forward approach (CVaR-emp). We consider $p = 10$ assets and three distributional models for the asset log-returns: X is multivariate Gaussian, elliptical and mixture of multivariate Gaussian and negative exponential. For each model, we follow the procedure outlined in Section 2 to construct sample efficient frontiers corresponding to (CVaR-emp), (CVaR-pen) and (Mark-emp).

One question that arises while solving (CVaR-pen) is how one chooses the penalty terms U_1

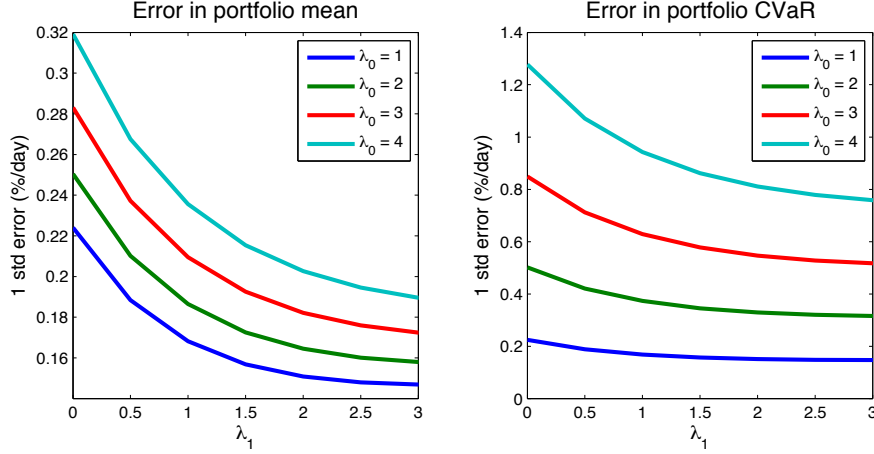


Figure 3: 1 asymptotic std of the portfolio mean and CVaR for the single-penalty problem as λ_1 is increased when $X \sim \mathcal{N}(\mu, \Sigma)$, for different values of λ_0 .

and U_2 in the constraints

$$\begin{aligned} \frac{1}{n} w^\top \hat{\Sigma}_n w &\leq U_1 \\ \frac{1}{n(1-\beta)^2} z^\top \Omega_n z &\leq U_2. \end{aligned}$$

If U_1, U_2 are too small, the problem becomes infeasible, whereas if they are too large, the penalization does not have any effect. It is sensible to choose U_1, U_2 as a proportion of $\hat{w}_n^\top \hat{\Sigma}_n \hat{w}_n / n$ and $\hat{z}_n^\top \Omega_n \hat{z}_n / (n(1-\beta)^2)$ respectively, where (\hat{w}_n, \hat{z}_n) is the solution to the unpenalized problem (CVaR-emp). We denote the proportions r_1 and r_2 respectively. In practice, one would perform cross-validation to find values of $(r_1, r_2) \in [0, 1] \times [0, 1]$ that maximize out-of-sample performance.

5.1 Gaussian/elliptical models

Here we consider

$$X \sim \mu_{sim} + \lambda \mathcal{N}(0, \Sigma_{sim})$$

where λ is as in (5), with $\lambda = 1$ for a Gaussian model and $\lambda \sim \Gamma(3, 0.5)$ for an elliptical model. The parameters μ_{sim} and Σ_{sim} are the same as those used in Sec. 2.2. We plot the histograms for 100,000 sample returns for an equally-weighted portfolio $w = 1_p/p$ under the Gaussian and elliptical models in Fig. (4).

We summarize the simulation results in Fig. (5), where $(r_1, r_2) = (0.92, 1)$ for both the Gaussian and elliptical models [recall that the second penalty is redundant due to Lemma 2]. Notice that for both models, the empirical Markowitz efficient frontier dominates the penalized efficient frontier which in turn dominates the empirical mean-CVaR efficient frontier, in both *position* of the average of the simulated frontiers and *variability*, as indicated by the vertical and horizontal error bars.

For the Gaussian case, $r_1 = 0.92$ was just feasible in that further reduction in this value led to most instances of the problem being infeasible. From Fig. (5b), we can see that this is because the penalized solutions are approaching the empirical Markowitz solutions with this choice of r_1 as the average simulated efficient frontiers of penalized (grey) and empirical Markowitz (blue) solutions are close. For the elliptical model, $r_1 = 0.92$ could be further reduced with the resulting penalized

efficient frontier approaching the empirical Markowitz efficient frontier. In summary, the empirical Markowitz solutions perform uniformly better than both the original and penalized mean-CVaR solutions, with the penalized efficient frontier nearing the empirical Markowitz efficient frontier with decreasing r_1 .

5.2 Mixture model

Let us now consider returns being driven by a mixture of multivariate normal and negative exponential distributions, such that with a small probability, all assets undergo a perfectly correlated exponential-tail loss. Formally,

$$X \sim (1 - I(q))N(\mu_{sim}, \Sigma_{sim}) + I(q)(Y1_p + f), \quad (23)$$

where $(\mu_{sim}, \Sigma_{sim})$ are parameters with the same value as in the Gaussian/elliptical models, $I(q) \sim \text{Bernoulli}(q)$, and $f = [f_1, \dots, f_p]^\top$ is a $p \times 1$ vector of constants, and Y is a negative exponential random variable with density

$$P(Y = y) = \begin{cases} \lambda e^{\lambda y}, & \text{if } y \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In our simulations, we consider $q = 0.05$, $f_i = \mu_i - \sqrt{\Sigma_{ii}}$ for $i = 1, \dots, p$ and $\lambda = 1$. The histogram for 100,000 sample returns of an equally-weighted portfolio under this mixture model is shown in Fig. (4a).

We summarize the simulation results in Fig. (4b), where $(r_1, r_2) = (0.5, 0.5)$. In this case, the penalized efficient frontiers perform better on average than the efficient frontiers generated by the other two methods. The empirical Markowitz efficient frontiers do not seem to perform any better than the original efficient frontiers on average, which is not surprising because the empirical Markowitz solution is only intended for X having an elliptical distribution.

6 Conclusion

We investigate Performance-Based Regularization as a method to reduce estimation risk in empirical mean-CVaR portfolio optimization. The nonparametric PBR method solves the empirical mean-CVaR problem with penalties on the uncertainties in mean and CVaR estimations. The parametric PBR method solves the empirical Markowitz problem instead if the underlying model is elliptically distributed. Both theoretical analysis and simulation experiments show the PBR methods improve upon the naive approach to data-driven mean-CVaR portfolio optimization.

From a larger perspective, the PBR approach is a new and promising way of dealing with estimation risk and introducing robustness to data-driven optimization, and is not restricted to the mean-CVaR problem. We leave investigating PBR in a general problem context for future work.

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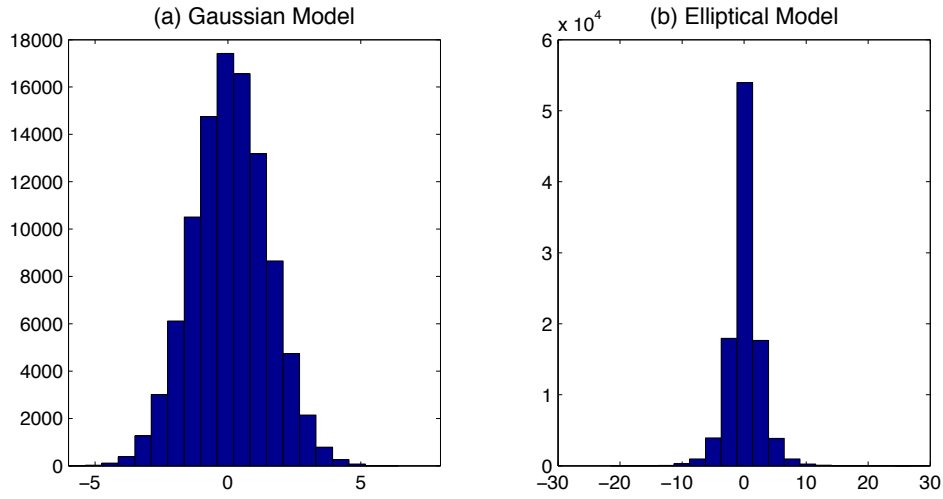


Figure 4: Distribution of equally weighted portfolio under (a) Gaussian and (b) elliptical model

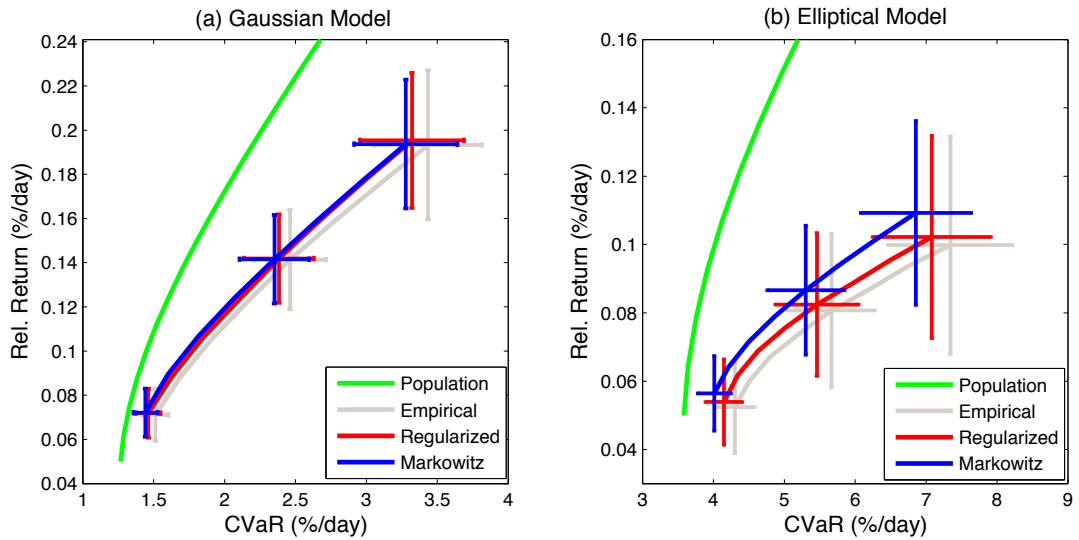


Figure 5: Average of population risk vs return for solutions to (CVaR-emp) in grey, (CVaR-pen) in red and (Mark-emp) in blue under (a) Gaussian model and (b) elliptical model. Green curve denotes the population efficient frontier. Horizontal and vertical lines show $\pm 1/2$ std error.

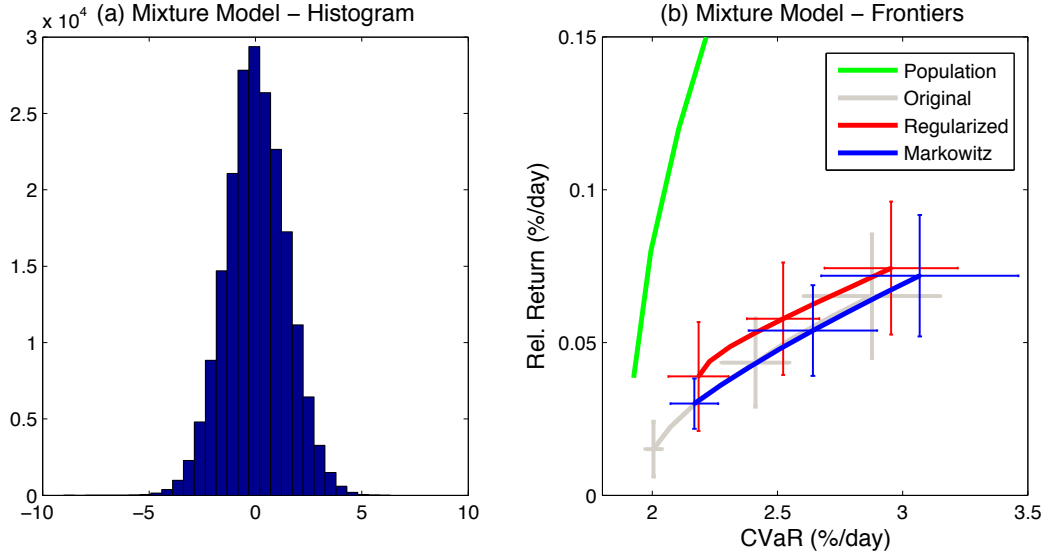


Figure 6: (a) Distribution of returns for an equally weighted portfolio under the mixture model. (b) Average of population risk vs return for solutions to (CVaR-emp) in grey, (CVaR-pen) in red and (Mark-emp) in blue under the mixture model. Green curve denotes the population efficient frontier. Horizontal and vertical lines show $\pm 1/2$ std error.

A Asymptotics of the CVaR estimator

Setting. Let $\mathbf{L} = [L_1, \dots, L_n]$ be n iid observations (of portfolio losses) from a distribution F which is absolutely continuous, has a twice continuously differentiable pdf and a finite second moment.

In this section, we prove the asymptotic distribution of the estimator $\widehat{CVaR}_n(L; \beta)$ introduced in Eq. (2) of Sec. 2.1. First, we define a closely related CVaR estimator:

Definition 4 (Type 1 CVaR estimator.). For $\beta \in (0.5, 1)$, we define Type 1 CVaR estimator to be

$$\widehat{CV1}_n(\mathbf{L}; \beta) := \min_{\alpha \in \mathbb{R}} (1 - \varepsilon_n)\alpha + \frac{1}{n - \lceil n\beta \rceil + 1} \sum_{i=1}^n (L_i - \alpha)^+,$$

where ε_n is some constant satisfying $0 < \varepsilon_n < (n - \lceil n\beta \rceil + 1)^{-1}$, $\sqrt{n}\varepsilon_n \xrightarrow{P} 0$.

Now consider the following CVaR estimator, expressed without the minimization:

Definition 5 (Type 2 CVaR estimator.). For $\beta \in (0.5, 1)$, we define Type 2 CVaR estimator to be

$$\widehat{CV2}_n(\mathbf{L}; \beta) := \frac{1}{n - \lceil n\beta \rceil + 1} \sum_{i=1}^n L_i 1(L_i \geq \hat{\alpha}_n(\beta)),$$

where $\hat{\alpha}_n(\beta) := L_{(\lceil n\beta \rceil)}$, the $\lceil n\beta \rceil$ -th order statistic of the sample L_1, \dots, L_n .

Type 2 CVaR estimator is asymptotically normally distributed [Chen (2008)]. In the remainder of this section, we show that $\widehat{CV2}_n(\mathbf{L}; \beta)$ is asymptotically equivalent to $\widehat{CV1}_n(\mathbf{L}; \beta)$, which is in turn asymptotically equivalent to $\widehat{CVaR}_n(\mathbf{L}; \beta)$. We then conclude $\widehat{CVaR}_n(\mathbf{L}; \beta)$ is also asymptotically normal, converging to the same asymptotic distribution as $\widehat{CV2}_n(\mathbf{L}; \beta)$.

Proposition 1. *The solution $\alpha^* = L_{(\lceil n\beta \rceil)}$ is unique to the one-dimensional optimization problem*

$$\min_{\alpha \in \mathbb{R}} \left\{ G_n(\alpha) := (1 - \varepsilon_n)\alpha + \frac{1}{n - \lceil n\beta \rceil + 1} \sum_{i=1}^n (L_i - \alpha)^+ \right\},$$

where ε_n is some constant satisfying $0 < \varepsilon_n < (n - \lceil n\beta \rceil + 1)^{-1}$.

Proof. The expression to be minimized is a piecewise linear convex function with nodes at L_1, \dots, L_n . We show that $G_n(\alpha)$ has gradients of opposite signs about a single point, $L_{(\lceil n\beta \rceil)}$, hence this point must be the unique optimal solution. Now consider, for $m \in \{-\lceil n\beta \rceil + 1, \dots, n - \lceil n\beta \rceil\}$:

$$\begin{aligned} \Delta(m) &= G_n(L_{(\lceil n\beta \rceil + m + 1)}) - G_n(L_{(\lceil n\beta \rceil + m)}) \\ &= (1 - \varepsilon_n)(L_{(\lceil n\beta \rceil + m + 1)} - L_{(\lceil n\beta \rceil + m)}) - \frac{1}{n - \lceil n\beta \rceil + 1} A, \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{i=1}^n [(L_i - L_{(\lceil n\beta \rceil + m + 1)})^+ - (L_i - L_{(\lceil n\beta \rceil + m)})^+] \\ &= (n - \lceil n\beta \rceil - m)(L_{(\lceil n\beta \rceil + m + 1)} - L_{(\lceil n\beta \rceil + m)}). \end{aligned}$$

Thus

$$\Delta(m) = (L_{(\lceil n\beta \rceil + m + 1)} - L_{(\lceil n\beta \rceil + m)}) \left((1 - \varepsilon_n) - \frac{n - \lceil n\beta \rceil - m}{n - \lceil n\beta \rceil + 1} \right).$$

Now $\Delta(0) > 0$ since $(L_{(\lceil n\beta \rceil + 1)} - L_{(\lceil n\beta \rceil)}) > 0$ and $(1 - \varepsilon_n) > (n - \lceil n\beta \rceil)(n - \lceil n\beta \rceil + 1)^{-1}$ by the restriction on ε_n , and $\Delta(-1) < 0$ since $(L_{(\lceil n\beta \rceil)} - L_{(\lceil n\beta \rceil - 1)}) > 0$ and $(1 - \varepsilon_n) < 1$ again by the choice of ε_n . Thus $G_n(\alpha)$ has a unique minimum at $\alpha^* = L_{(\lceil n\beta \rceil)}$. \square

Remark. Note if $\varepsilon_n = 0$, then multiple solutions occur because $\Delta(-1) = 0$.

Corollary 2. *Type 1 and Type 2 CVaR estimators are related by*

$$\widehat{CV2}_n(\mathbf{L}; \beta) = \widehat{CV1}_n(\mathbf{L}; \beta) + \varepsilon_n L_{(\lceil n\beta \rceil)}.$$

Proof. Rewriting Type 2 CVaR estimator:

$$\begin{aligned} \widehat{CV2}_n(\mathbf{L}; \beta) &= \frac{1}{n - \lceil n\beta \rceil + 1} \sum_{i=1}^n L_i \mathbf{1}(L_i \geq L_{(\lceil n\beta \rceil)}) \\ &= L_{(\lceil n\beta \rceil)} + \frac{1}{n - \lceil n\beta \rceil + 1} \sum_{i=1}^n (L_i - L_{(\lceil n\beta \rceil)}) \mathbf{1}(L_i \geq L_{(\lceil n\beta \rceil)}) \\ &= \widehat{CV1}_n(\mathbf{L}; \beta) + \varepsilon_n L_{(\lceil n\beta \rceil)}, \end{aligned}$$

where the final equality is due to Proposition 1. \square

We now show asymptotic normality of $\widehat{CV1}_n(\mathbf{L}; \beta)$.

Lemma 6. *Type 1 CVaR estimator is asymptotically normal as follows:*

$$\frac{\sqrt{n}(1 - \beta)}{\gamma_0} \left(\widehat{CV1}_n(\mathbf{L}; \beta) - CVaR(L_1; \beta) \right) \Rightarrow \mathcal{N}(0, 1), \quad (24)$$

where $\gamma_0^2 = \text{Variance}[(L_1 - \alpha_\beta)1(L_1 \geq \alpha_\beta)]$, and $\alpha_\beta = \inf\{\alpha : P(L_1 \geq \alpha) \leq 1 - \beta\}$, Value-at-Risk of the random loss L_1 at level β .

Proof. Asymptotic normality for Type 2 CVaR estimator is proven in Chen (2008), and the result is immediate from invoking Slutsky's lemma on Corollary 2 and the assumption $\sqrt{n}\varepsilon_n \rightarrow 0$. \square

A.1 Proof of Lemma 1

The asymptotic distribution of $\widehat{CVaR}_n(\mathbf{L}; \beta)$ is the same as $\widehat{CV1}_n(\mathbf{L}; \beta)$ because

$$\sqrt{n}|\widehat{CVaR}_n(\mathbf{L}; \beta) - \widehat{CV1}_n(\mathbf{L}; \beta)| = o_P(1).$$

B Proof of Theorem 1

Lemma 7. Consider the optimization problem

$$\begin{aligned} \min_{z \in \mathbb{R}^n} \quad & z^\top \mathbf{1}_n \\ \text{s.t.} \quad & z_i \geq 0 \quad \forall i \\ & z_i \geq c_i \quad \forall i \\ & z^\top \Omega_n z \leq f \end{aligned} \tag{25}$$

where $c_i > 0 \forall i$, $f > 0$, $\Omega_n = (n-1)^{-1}(\mathbf{I}_n - n^{-1}\mathbf{1}_n\mathbf{1}_n^\top)$, the sample covariance operator. Suppose (25) is feasible with an optimal solution (x^*, z^*) . Let $S_1(z) := \{1 \leq i \leq n : z_i = 0\}$, $S_2(z) := \{1 \leq i \leq n : z_i = c_i\}$ and $V(z) := S_1^c \cap S_2^c$ (i.e. $V(z)$ is the set of indices for which $z_i > \max(0, c_i)$). Then the optimal solution z^* falls into one of two cases: either $S_1(z^*) \neq \emptyset$ and $V(z^*) = \emptyset$, or $S_1(z^*) = \emptyset$ and $V(z^*) \neq \emptyset$.

Proof. The problem (25) is a convex optimization problem because Ω_n is a positive semidefinite matrix. The problem is also strictly feasible, since $z_0 = 2\max_i\{c_i\}\mathbf{1}_n$ is a strictly feasible point: clearly, $z_{0,i} > \max\{0, c_i\} \forall i$ and $z_0^\top \Omega_n z_0 = 0 < f$ as $\mathbf{1}_n$ is orthogonal to Ω_n . Thus Slater's condition for strong duality holds, and we can derive properties of the optimal solution by examining KKT conditions.

The Lagrangian is

$$\mathcal{L}(z, \eta_1, \eta_2, \lambda) = \lambda z^\top \Omega_n z + (1_n - \eta_1 - \eta_2)^\top z + \eta_2^\top c - \lambda f$$

The KKT conditions are

- Primal feasibility
- Dual feasibility: $\eta_1^*, \eta_2^* \geq 0$ component-wise and $\lambda^* \geq 0$
- Complementary slackness:

$$z_i^* \eta_{1,i}^* = 0 \quad \forall i, \quad (z_i^* - c_i) \eta_{2,i}^* = 0 \quad \forall i \quad \text{and} \quad \lambda^*[(z^*)^\top \Omega_n z^* - f] = 0$$
- First Order Condition:

$$\nabla_{z^*} \mathcal{L} = 2\lambda \Omega_n z^* + (1_n - \eta_1^* - \eta_2^*) = 0 \tag{26a}$$

By substituting for Ω_n , (26a) can be written as

$$\frac{2\lambda}{n-1} \left(z^* - \frac{1}{n} (1_n^\top z^*) 1_n \right) = -1_n + \eta_1^* + \eta_2^*. \quad (27)$$

Suppose $S_1(z^*) \neq \emptyset$ at the optimal primal-dual point $(z^*, \eta_1^*, \eta_2^*, \lambda^*)$. Then $\exists i_0 \in S_1(z^*)$ such that $z_{i_0}^* = 0$. The i_0 -th component of (27) gives

$$-\frac{2\lambda^*}{n(n-1)} (1_n^\top z^*) = -1 + \eta_{1,i_0}^* + \eta_{2,i_0}^*. \quad (28)$$

Now suppose $V(z^*) \neq \emptyset$ at the optimal primal-dual point $(z^*, \eta_1^*, \eta_2^*, \lambda^*)$. Then $\exists j_0 \in V(z^*)$ such that $z_{j_0}^* > \max(0, c_i)$, $\eta_{1,j_0}^* = 0$ and $\eta_{2,j_0}^* = 0$. The j_0 -th component of (27) gives

$$\frac{2\lambda^*}{n-1} \left(z_{j_0}^* - \frac{1}{n} (1_n^\top z^*) \right) = -1, \quad (29)$$

which also implies $\lambda^* > 0$.

Now suppose $S_1(z^*)$ and $V(z^*)$ are both nonempty. Combining (28) and (29), we arrive at the necessary condition

$$\frac{2\lambda^*}{n-1} z_{j_0}^* = -\eta_{1,i_0}^* - \eta_{2,i_0}^*.$$

which is clearly a contradiction since $LHS > 0$ whereas $RHS \leq 0$. Hence $S_1(z^*)$ and $V(z^*)$ cannot both be nonempty. \square

B.1 Proof of Theorem 1

Proof. Clearly, (CVaR-relax) is a relaxation of (CVaR-pen): the components of the variable z in (CVaR-relax) are relaxations of $\max(0, -w^\top X_i - \alpha)$. Thus the two problem formulations are equivalent if at optimum, $z_i = \max(0, -w^\top X_i - \alpha) \forall i = 1, \dots, n$ for (CVaR-relax).

Let $(\alpha^*, w^*, z^*, \nu_1^*, \nu_2^*, \eta_1^*, \eta_2^*, \lambda_1^*, \lambda_2^*)$ be the primal-dual optimal point for (CVaR-relax) and (CVaR-relax-d). Our aim is to show that $V(z^*)$, the set of indices for which $z_i^* > \max(0, -w^\top X_i - \alpha)$, is empty. Suppose the contrary. Then by Lemma 7, $S_1(z^*)$, the set of indices for which $z_i^* = 0$, is empty. This means $z_i^* > 0 \forall i$ and $\eta_{1,i}^* = 0 \forall i$ by complementary slackness.

Now consider the sub-problem for a fixed η_2 in the dual problem (CVaR-relax-d):

$$\max_{\eta_1: \eta_1 \geq 0} -(\eta_1 + \eta_2) \Omega_n^\dagger (\eta_1 + \eta_2). \quad (30)$$

As 1_n is orthogonal to Ω_n^\dagger , and Ω_n^\dagger is positive semidefinite, the optimal solution is of the form $\eta_1 = a 1_n - \eta_2$, where a is any constant such that $a \geq \max_i(\eta_{2,i})$, with a corresponding optimal objective 0. Hence, bearing in mind the constraints $\eta_2 \geq 0$ and $\eta_2^\top 1_n = 1$ in (CVaR-relax-d), $\eta_1 = 0$ is one of the optimal solutions iff $\eta_2^* = 1_n/n$. Thus if $\eta_2^* \neq 1_n/n$, we get a contradiction. Otherwise, we can force the dual problem to find a solution with $\eta_1 \neq 0$ by introducing an additional constraint $\eta_1^\top 1_n \geq \delta$ for some constant $0 < \delta \ll 1$. \square

C Details of asymptotic theory

C.1 Proof of Theorem 2

Proof. By uniqueness of $\theta_0(\lambda_1, \lambda_2)$ and Assumption 1 (and compactness arguments), for every $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\|\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2)\|_2 > \varepsilon \implies M(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2) > \eta.$$

Thus if we can show the probability of the event $\{M(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2) > \eta\}$ goes to zero for every $\varepsilon > 0$, then we have consistency.

We also have

$$M_n(\hat{\theta}_n; \lambda_1, \lambda_2) \leq M_n(\theta_0; \lambda_1, \lambda_2) + o_P(1) = M(\theta_0; \lambda_1, \lambda_2) + o_P(1), \quad (\star)$$

the first inequality because $\hat{\theta}_n(\lambda_1, \lambda_2)$ is a near-minimizer of M_n , and the second equality by the Weak Law of Large Numbers (WLLN) on $M_n(\theta_0; \lambda_1, \lambda_2)$.

Thus

$$\begin{aligned} 0 &\leq M(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2) \\ &= [M(\hat{\theta}_n; \lambda_1, \lambda_2) - M_n(\hat{\theta}_n; \lambda_1, \lambda_2)] + [M_n(\hat{\theta}_n; \lambda_1, \lambda_2) - M_n(\theta_0; \lambda_1, \lambda_2)] + [M_n(\theta_0; \lambda_1, \lambda_2) - M(\theta_0; \lambda_1, \lambda_2)] \\ &\leq M(\hat{\theta}_n; \lambda_1, \lambda_2) - M_n(\hat{\theta}_n; \lambda_1, \lambda_2) + o_P(1), \end{aligned}$$

because the second term in $[\]$ is $o_P(1)$ by (\star) , and the last term in $[\]$ is $o_P(1)$ by WLLN. We are left to prove $|M_n(\hat{\theta}_n; \lambda_1, \lambda_2) - M(\hat{\theta}_n; \lambda_1, \lambda_2)| \xrightarrow{P} 0$. At first glance, one may consider invoking the WLLN again. However, as $\hat{\theta}_n(\lambda_1, \lambda_2)$ is a random sequence of vectors that changes for every n , we cannot apply the WLLN which is a pointwise result (i.e. for each fixed $\theta \in \Theta$), and we need to appeal to the stronger ULLN.

Case I: $\lambda_1 = \lambda_2 = 0$. To show ULLN for the original objective, we show that \mathcal{F}_1 is a Lipschitz class of functions, hence $N_{[\]}(\varepsilon, \mathcal{F}_1, L_r(P))$ for every $\varepsilon > 0$. Now $\theta \mapsto m_\theta(x) = \alpha + (1 - \beta)^{-1}(-\alpha - w_0^\top x - v^\top L^\top x)^+$ is clearly differentiable at θ_0 for all $x \in \mathbb{R}^p$. Furthermore,

$$\nabla_\theta m_\theta(x) = \begin{bmatrix} -1 \\ -L^\top x \end{bmatrix} I(x),$$

where $I(x) := \mathbb{I}(-\alpha - w_0^\top x - v^\top L^\top x \geq 0)$, hence

$$\dot{m}(x) := \max(1, \|L^\top x\|_\infty) \quad (31)$$

is an upper bound on $\|\nabla_\theta m_\theta(x)\|_\infty$ and is independent of θ . Thus $|m_{\theta_1}(x) - m_{\theta_2}(x)| \leq \dot{m}(x) \|\theta_1 - \theta_2\|_2$ for all $\theta_1, \theta_2 \in [-K, K]^{1+p}$, and together with Assumption 2 (here a weaker assumption that X has finite second moment suffices), \mathcal{F}_1 is a Lipschitz class.

Case II: $\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_1, \lambda_2$ not both zero. Corollary 3.5 in Arcones and Gine (1993) says that ULLN also holds for the penalized objective if $N_{[\]}(\varepsilon, \mathcal{F}_2, L_2(P \times P)) < \infty$ for every $\varepsilon > 0$. Let us now show that \mathcal{F}_2 is also a Lipschitz class of functions. Again, it is clear that

$$\theta \mapsto m_{(\theta; \lambda_1, \lambda_2)}^U(x_1, x_2) = \frac{1}{2} [m_\theta(x_1) + m_\theta(x_2)] + \frac{\lambda_1}{2} [(w_1 + Lv)^\top (x_1 - x_2)]^2 + \frac{\lambda_2}{2} (z_\theta(x_1) - z_\theta(x_2))^2$$

is differentiable at θ_0 for all $(x_1, x_2) \in \mathbb{R}^p \times \mathbb{R}^p$. Also for all $\theta \in [-K, K]^{1+p}$,

$$\begin{aligned}
\nabla_\theta \frac{\lambda_1}{2} [(w_1 + Lv)^\top (x_1 - x_2)]^2 &= \lambda_1 (x_1 - x_2)(x_1 - x_2)^\top (w_1 + Lv) \\
\implies \|\nabla_\theta \frac{\lambda_1}{2} [(w_1 + Lv)^\top (x_1 - x_2)]^2\|_\infty &\leq \lambda_1 \|x_1 - x_2\|_\infty^2 \|w_1 + Lv\|_\infty \leq \lambda_1 C(K) \|x_1 - x_2\|_\infty^2 \\
&\quad \text{for some constant } C(K) \text{ dependent on } K, \text{ and} \\
\nabla_\theta \frac{\lambda_2}{2} (z_\theta(x_1) - z_\theta(x_2))^2 &= \lambda_2 (z_\theta(x_1) - z_\theta(x_2)) \begin{bmatrix} -I(x_1) + I(x_2) \\ -L^\top x_1 I(x_1) + L^\top x_2 I(x_2) \end{bmatrix}, \text{ and} \\
|z_\theta(x_1)| &= |-(\alpha - w_0^\top x_1 - v^\top L^\top x_1)^+| \\
&\leq |\alpha - w_0^\top x_1 - v^\top L^\top x_1| \leq K + |w_0^\top x_1| + K|e^\top x_1| \\
\implies \|\nabla_\theta \frac{\lambda_2}{2} (z_\theta(x_1) - z_\theta(x_2))^2\|_\infty &\leq \lambda_2 |z_\theta(x_1) - z_\theta(x_2)| (\dot{m}(x_1) + \dot{m}(x_2)) \\
&\quad \dot{m} \text{ as defined in Eq. (31)} \\
&\leq \lambda_2 C'(K) (\|x_1\|_\infty + \|x_2\|_\infty) (\dot{m}(x_1) + \dot{m}(x_2)), \\
&\quad \text{for some constant } C'(K) \text{ dependent on } K,
\end{aligned}$$

hence

$$\dot{m}_{(\lambda_1, \lambda_2)}^U(x_1, x_2) := \frac{1}{2} [\dot{m}(x_1) + \dot{m}(x_2)] + \lambda_1 C(K) \|x_1 - x_2\|_\infty^2 + \lambda_2 C'(K) (\|x_1\|_\infty + \|x_2\|_\infty) (\dot{m}(x_1) + \dot{m}(x_2)) \quad (32)$$

is an upper bound on $\|\nabla_\theta m_{(\theta; \lambda_1, \lambda_2)}^U(x_1, x_2)\|_\infty$ that is independent of θ . Thus

$$|m_{(\theta_1; \lambda_1, \lambda_2)}^U(x_1, x_2) - m_{(\theta_2; \lambda_1, \lambda_2)}^U(x_1, x_2)| \leq \dot{m}_{(\lambda_1, \lambda_2)}^U(x_1, x_2) \|\theta_1 - \theta_2\|_2,$$

and together with Assumption 2, \mathcal{F}_2 is a Lipschitz class. \square

C.2 Proof of Theorem 3

In what follows, we suppress the dependence on λ_1, λ_2 for notational convenience.

Proof. The proof parallels the proof of Theorem 5.23 of Van der Vaart (2000). Let us assume for now that

1. For every given random sequence h_n that is bounded in probability,

$$\mathbb{U}_n[\sqrt{n}(m_{\theta_0 + h_n/\sqrt{n}}^U - m_{\theta_0}^U) - h_n^\top \dot{m}_{\theta_0}^U] \xrightarrow{P} 0, \quad (*)$$

and

2. $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_P(1)$.

Since $\theta \mapsto M(\theta)$ is twice-differentiable, and $\nabla_\theta M(\theta)|_{\theta=\theta_0} = 0$ by first-order condition, we can rewrite Eq. (*) to get

$$\begin{aligned}
n \binom{n}{2}^{-1} \sum_{i \neq j} [m_{\theta_0 + h_n/\sqrt{n}}^U(X_i, X_j) - m_{\theta_0}^U(X_i, X_j)] &= \frac{1}{2} h_n^\top V_{\theta_0} h_n + h_n^\top \mathbb{U}_n[\dot{m}_{\theta_0}^U] + o_p(1) \\
&= \frac{1}{2} h_n^\top V_{\theta_0} h_n + h_n^\top \mathbb{G}_n[\dot{m}_{\theta_0}^1] + o_p(1),
\end{aligned}$$

where we use the fact, from Hoeffding decomposition,

$$\begin{aligned}\mathbb{U}_n[\dot{m}_{\theta_0}^U] &= \frac{\sqrt{n}}{\binom{n}{2}} \sum_{i \neq j} [\dot{m}_{\theta_0}^U(X_i, X_j) - \mathbb{E}_{X_1, X_2}[\dot{m}_{\theta_0}^U(X_1, X_2)]] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n [\dot{m}_{\theta_0}^1(X_i) - \mathbb{E}\dot{m}_{\theta_0}^1(X_1)] + o_p(1) = \mathbb{G}_n[\dot{m}_{\theta_0}^1] + o_p(1),\end{aligned}$$

with \dot{m}_{θ}^1 as in the statement of the theorem.

The above statement is valid for both $\hat{h}_n = \sqrt{n}(\hat{\theta}_n - \theta_0)$ and for $\tilde{h}_n = -V_{\theta_0}^{-1}\mathbb{G}_n\dot{m}_{\theta_0}^1$. Upon substitution, we obtain

$$\begin{aligned}n \binom{n}{2}^{-1} \sum_{i \neq j} [m_{\theta_0 + \hat{h}_n/\sqrt{n}}^U(X_i, X_j) - m_{\theta_0}^U(X_i, X_j)] &= \frac{1}{2} \hat{h}_n^\top V_{\theta_0} \hat{h}_n + \hat{h}_n^\top \mathbb{G}_n[\dot{m}_{\theta_0}^1] + o_p(1) \\ &\leq n \binom{n}{2}^{-1} \sum_{i \neq j} [m_{\theta_0 + \tilde{h}_n/\sqrt{n}}^U(X_i, X_j) - m_{\theta_0}^U(X_i, X_j)] = -\frac{1}{2} \mathbb{G}_n[\dot{m}_{\theta_0}^1]^\top V_{\theta_0}^{-1} \mathbb{G}_n[\dot{m}_{\theta_0}^1] + o_p(1)\end{aligned}$$

where the inequality is from the definition of $\hat{\theta}_n = \theta_0 + \hat{h}_n/\sqrt{n}$ as a near-minimizer.

Taking the difference and completing the square, we get

$$\frac{1}{2} (\hat{h}_n + V_{\theta_0}^{-1} \mathbb{G}_n \dot{m}_{\theta_0}^1)^\top V_{\theta_0} (\hat{h}_n + V_{\theta_0}^{-1} \mathbb{G}_n \dot{m}_{\theta_0}^1) + o_p(1) \leq 0,$$

and because V_{θ_0} is nonsingular, the quadratic form on the left must converge to zero in probability. The same must be true for $\|\hat{h}_n + V_{\theta_0}^{-1} \mathbb{G}_n \dot{m}_{\theta_0}^1\|_2$.

To complete the proof, we need to show (*) and $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_P(1)$ hold.

Proof of (*).

Let $f_h := \sqrt{n}(m_{\theta_0+h/\sqrt{n}}^U - m_{\theta_0}^U) - h^\top \dot{m}_{\theta_0}^U$. As we are considering only sequences h_n that are bounded in probability, it suffices to show $\sup_{h: \|h\|_2 \leq 1} |\mathbb{U}_n[f_h]|$ goes to zero in probability. Again by Hoeffding decomposition, for any given random sequence h_n that is bounded in probability, $\mathbb{U}_n[f_{h_n}] = \mathbb{G}_n[f_{h_n}^1] + E_n(h_n)$, where f_h^1 is the first term in the Hoeffding decomposition of $\mathbb{U}_n[f_h]$ given by

$$\begin{aligned}f_h^1 &= \sqrt{n}(m_{\theta_0+h/\sqrt{n}}^1 - m_{\theta_0}^1) - h^\top \dot{m}_{\theta_0}^1, \\ m_{\theta}^1(x_1) &= 2\mathbb{E}_{X_2}[m_{\theta}^U(x_1, X_2)] - \mathbb{E}_{X_1, X_2}[m_{\theta}^U(X_1, X_2)],\end{aligned}$$

and \dot{m}_{θ}^1 as defined in the statement of the theorem. According to Lemma 19.31 in Van der Vaart (2000), if $\mathcal{F}_2' := \{m_{\theta}^1 : \theta \in [-K, K]^{1+p}\}$ is a Lipschitz class of functions,

$$\sup_{h: \|h\|_2 \leq 1} |\mathbb{G}_n[f_{h_n}^1]| \xrightarrow{P} 0.$$

Now by Assumption 2 that X_i 's are iid continuous random vectors with finite fourth moment,

$\theta \mapsto m_\theta^1(x)$ is differentiable at θ_0 for all $x \in \mathbb{R}$. Further, by triangle inequality,

$$\begin{aligned} |m_{\theta_1}^1(x) - m_{\theta_2}^1(x)| &\leq 2\mathbb{E}_{X_2}|m_{\theta_1}^U(x, X_2) - m_{\theta_2}^U(x, X_2)| + \mathbb{E}_{X_1, X_2}|m_{\theta_1}^U(X_1, X_2) - m_{\theta_2}^U(X_1, X_2)| \\ &\leq m^1(x)\|\theta_1 - \theta_2\|_2, \end{aligned}$$

where $m^1(x) = (2\mathbb{E}_{X_2}|\dot{m}^U(x, X_2)| + \mathbb{E}_{X_1, X_2}|\dot{m}^U(X_1, X_2)|)$, \dot{m}^U as in Eq. (32). Since X_i 's have finite fourth moment, $\mathbb{E}[m^1(X_1)^2] < \infty$ and thus \mathcal{F}'_2 is a Lipschitz class.

Now we are left to show $\sup_{h: \|h\|_2 \leq 1} |E_n(h)| \xrightarrow{P} 0$. Let $\mathcal{F}_h := \{f_h : \|h\|_2 \leq 1\}$. According to Theorem 4.6 of Arcones and Gine (1993), $\sup_{h: \|h\|_2 \leq 1} |E_n(h)| \xrightarrow{P} 0$ if \mathcal{F}_h has a finite, integrable envelope function and both \mathcal{F}_h and $\mathcal{F}'_h := \{f'_h : \|h\|_2 \leq 1\}$ are Lipschitz classes about $h = 0$. \mathcal{F}_h has a finite, integrable envelope function $F(x_1, x_2) = \dot{m}^U(x_1, x_2) + \|\dot{m}_{\theta_0}(x_1, x_2)\|_2 < \infty$ due to Assumption 2 and the Lipschitz property of m_θ^U :

$$\begin{aligned} |f_h| &\leq |\sqrt{n}(m_{\theta_0+h/\sqrt{n}}^U - m_{\theta_0}^U) - h^\top \dot{m}_{\theta_0}| \\ &\leq (\dot{m}^U + \|\dot{m}_{\theta_0}\|_2)\|h\|_2. \end{aligned}$$

It is now straight-forward to check that \mathcal{F}_h is a Lipschitz class about $h = 0$, and \mathcal{F}'_h also, because it inherits the key properties from \mathcal{F}_h .

Proof of $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_P(1)$.

The proof of $\sqrt{n}(\hat{\theta}_n(0, 0) - \theta_0(0, 0)) = O_p(1)$ can be found in Theorem 5.52 and Corollary 5.53 of Van der Vaart (2000), and is a standard M-estimation result. In essence, Theorem 5.52 shows that, under some regularity conditions, $P(\sqrt{n}\|\hat{\theta}_n(0, 0) - \theta_0(0, 0)\|_2 > \alpha)$ can be bounded by $P(|\mathbb{G}_n[m_\theta]| > \alpha') = P(\sqrt{n}|M_n(\theta) - M(\theta)| > \alpha')$, which is shown to go to zero via some maximal inequalities. Corollary 5.53 shows that the Lipschitz condition on $\{m_\theta : \theta \in [-K, K]^{1+p}\}$ is sufficient to satisfy the regularity conditions of the theorem.

We can extend Theorem 5.52 to show $\sqrt{n}(\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2))$, $\lambda_1, \lambda_2 \geq 0$ not both zero, by bounding $P(\sqrt{n}\|\hat{\theta}_n(\lambda_1, \lambda_2) - \theta_0(\lambda_1, \lambda_2)\|_2 > \alpha)$ by

$$P(|\mathbb{U}_n[m_\theta^U]| > \alpha') \leq P(|\mathbb{G}_n[m_\theta^1]| + |E'_n(\theta)| > \alpha'),$$

where E'_n is the remainder term after first-order projection of the U-process $\mathbb{U}_n[m_\theta^U]$. It remains to show that for every sufficiently small $\delta > 0$,

$$\sup_{\theta: \|\theta - \theta_0\|_2 < \delta} |E'_n(\theta)| \xrightarrow{P} 0, \quad (33)$$

which can be proven using the same reasoning for $\sup_{h: \|h\|_2 \leq 1} |E_n(h)| \xrightarrow{P} 0$ in the proof of (*). \square

C.3 Computation of key statistics

Given the distribution for X , both $A_0 = A_{\theta_0}(0, 0)$ and $B_0 = B_{\theta_0}(0, 0)$ are computable. The lemma below computes the key quantities that constitute A_0 and B_0 when $X \sim \mathcal{N}(\mu, \Sigma)$.

Lemma 8. Suppose $X \sim \mathcal{N}(\mu, \Sigma)$, and $z_\theta(X) = -\alpha - w^\top X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, where $\mu_1 = -\sigma_1 \Phi^{-1}(\beta)$

and $\sigma_1^2 = w^\top \Sigma w$. Then

$$p_0 = P(z_\theta(X) = 0) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left(-\frac{1}{2}\Phi^{-1}(\beta)^2\right) \quad (34)$$

$$\mathbb{E}[\max(z_\theta(X), 0)] = \frac{\sigma_1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\Phi^{-1}(\beta)^2\right) - \sigma_1(1 - \beta)\Phi^{-1}(\beta) \quad (35)$$

$$\mathbb{E}[L_j^\top X \mathbb{I}(Z_1 \geq 0)] = (1 - \beta)(L_j^\top \mu - \Phi^{-1}(\beta)\frac{L_j^\top \Sigma w}{\sigma_1}) - \frac{L_j^\top \Sigma w}{\sigma_1^2} \mathbb{E}[\max(z_\theta(X), 0)] \quad (36)$$

$$\mathbb{E}[L_j^\top X | Z_1 = 0] = L_j^\top \mu - \Phi^{-1}(\beta)\frac{L_j^\top \Sigma w}{\sigma_1} \quad (37)$$

$$\begin{aligned} \mathbb{E}[L_j^\top X L_l^\top X \mathbb{I}(Z_1 \geq 0)] &= \frac{1}{4}(g(\mu_1, (L_j + L_l)^\top \mu, \sigma_1, \sigma_2, -(L_j + L_l)^\top \Sigma w_1) \\ &\quad - g(\mu_1, (L_j - L_l)^\top \mu, \sigma_1, \sigma_2, -(L_j - L_l)^\top \Sigma w_1)) \end{aligned} \quad (38)$$

$$\begin{aligned} \mathbb{E}[L_j^\top X L_l^\top X | Z_1 = 0] &= \frac{1}{4}(h(\mu_1, (L_j + L_l)^\top \mu, \sigma_1, \sigma_2, -(L_j + L_l)^\top \Sigma w_1) \\ &\quad - h(\mu_1, (L_j - L_l)^\top \mu, \sigma_1, \sigma_2, -(L_j - L_l)^\top \Sigma w_1)) \end{aligned} \quad (39)$$

where

$$g(\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}) = (1 - \beta) [\mu_2^2 + \sigma_2^2] + p_0 \sigma_{12} \left[-\Phi^{-1}(\beta) \frac{\sigma_{12}}{\sigma_1} + 2\mu_2 \right]$$

$$h(\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}) = (\mu_2 + \frac{\sigma_{12}}{\sigma_1} \Phi^{-1}(\beta))^2 + \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}.$$

Proof. We use the fact that if $Z_1 \sim \mathcal{N}(\mu_1, \sigma_1)$ and $Z_2 \sim \mathcal{N}(\mu_2, \sigma_2)$,

$$Z_2 | Z_1 = \mathcal{N}(\mu_2 + \sigma_{12}/\sigma_1^2(Z_1 - \mu_1), \sigma_2^2 - \sigma_{12}^2/\sigma_1^2), \quad (40)$$

where $\sigma_{12} = \text{Cov}(Z_1, Z_2)$.

• **Terms involving only $L_j^\top X$.**

Note that from (40), $\mathbb{E}[Z_2 | Z_1 = 0] = \mu_2 - \frac{\sigma_{12}}{\sigma_1^2} \mu_1$. Let $Z_2 = L_j^\top X$, and recall that $\mathbb{E}(L_j^\top X) = L_j^\top \mu$ and $\mathbb{E}(Z_1) = -\sigma_1 \Phi^{-1}(\beta)$. Also, note that $\sigma_{12} = -L_j^\top \Sigma w$. After some algebra, we get (37).

Since we know the distribution of $Z_2 | Z_1$, we have

$$\begin{aligned} \mathbb{E}[Z_2 \mathbb{I}(Z_1 \geq 0)] &= \mathbb{E}[\mathbb{I}(Z_1 \geq 0)(\mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(Z_1 - \mu_1))] \\ &= (1 - \beta)(\mu_2 - \frac{\sigma_{12}}{\sigma_1^2} \mu_1) + \frac{\sigma_{12}}{\sigma_1^2} \mathbb{E}[Z_1 \mathbb{I}(Z_1 \geq 0)] \\ &= (1 - \beta)(L_j^\top \mu - \Phi^{-1}(\beta)\frac{L_j^\top \Sigma w}{\sigma_1}) - \frac{L_j^\top \Sigma w}{\sigma_1^2} \mathbb{E}[\max(Z_1, 0)] \end{aligned}$$

• **Terms involving $L_j^\top X L_l^\top X$.**

To compute $\mathbb{E}[L_j^\top X L_l^\top X \mathbb{I}(Z_1 \geq 0)]$ and $\mathbb{E}[L_j^\top X L_l^\top X | Z_1 = 0]$, first note that

$$\mathbb{E}[L_j^\top X L_l^\top X \mathbb{I}(Z_1 \geq 0)] = \frac{1}{4} \mathbb{E} \left[[(L_j^\top X + L_l^\top X)^2 - (L_j^\top X - L_l^\top X)^2] \mathbb{I}(Z_1 \geq 0) \right].$$

and similarly

$$\mathbb{E}[L_j^\top X L_l^\top X | Z_1 = 0] = \frac{1}{4} \mathbb{E}[(L_j^\top X + L_l^\top X)^2 - (L_j^\top X - L_l^\top X)^2 | Z_1 = 0].$$

Hence it is sufficient to first find expressions for $\mathbb{E}[Z_2^2 \mathbb{I}(Z_1 \geq 0)]$ and $\mathbb{E}[Z_2^2 | Z_1 = 0]$ for some normal Z_2 , then apply the resulting formulae to $Z_2 = (L_j \pm L_l)^\top X$. This results in $\mu_2 = (L_j \pm L_l)^\top \mu$, $\sigma_{12} = -(L_j \pm L_l)^\top \Sigma w$ and $\sigma_2^2 = (L_j \pm L_l)^\top \Sigma (L_j \pm L_l)$.

From tower property and the conditional distribution of $Z_2 | Z_1$,

$$\mathbb{E}[Z_2^2 \mathbb{I}(Z_1 \geq 0)] = \mathbb{E}[\mathbb{I}(Z_1 \geq 0) \left[(\mu_2 + \frac{\sigma_{12}}{\sigma_1^2} (Z_1 - \mu_1))^2 + \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \right]].$$

By simple computations,

$$\begin{aligned} \mathbb{E}[(Z_1 - \mu_1) \mathbb{I}_{Z_1 \geq 0}] &= \frac{\sigma_1}{\sqrt{2\pi}} \exp(-\mu_1^2/(2\sigma_1^2)) = \sigma_1^2 f_{Z_1}(0) = \sigma_1^2 p_0, \text{ and} \\ \mathbb{E}[(Z_1 - \mu_1)^2 \mathbb{I}_{Z_1 \geq 0}] &= \sigma_1^2 (\mu_1 p_0 + (1 - \beta)). \end{aligned}$$

Now $\mu_1/\sigma_1 = -\Phi^{-1}(\beta)$, and

$$\begin{aligned} \mathbb{E}[Z_2^2 \mathbb{I}_{Z_1 \geq 0}] &= (1 - \beta) [\mu_2^2 + \sigma_2^2] + p_0 \left[\mu_1 \frac{\sigma_{12}^2}{\sigma_1^2} + 2\sigma_{12}\mu_2 \right] \\ &= (1 - \beta) [\mu_2^2 + \sigma_2^2] + p_0 \sigma_{12} \left[-\Phi^{-1}(\beta) \frac{\sigma_{12}}{\sigma_1} + 2\mu_2 \right] \\ &:= g(\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12}) \end{aligned}$$

Similarly,

$$\mathbb{E}[Z_2^2 | Z_1 = 0] = (\mu_2 - \frac{\sigma_{12}}{\sigma_1^2} \mu_1)^2 + \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} = (\mu_2 + \frac{\sigma_{12}}{\sigma_1} \Phi^{-1}(\beta))^2 + \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} := h(\mu_1, \mu_2, \sigma_1, \sigma_2, \sigma_{12})$$

□

References

- Acerbi, C., and D. Tasche (2001) ‘On the coherence of expected shortfall.’ *Journal of Banking and Finance* 26, 1487–1503
- Alexander, S., T.F. Coleman, and Y. Li (2006) ‘Minimizing CVaR and VaR for a portfolio of derivatives.’ *Journal of Banking & Finance* 30(2), 583–605
- Anderson, T.W. (1958) *An introduction to multivariate statistical analysis* (John Wiley & Sons)
- Andersson, F., H. Mausser, D. Rosen, and S. Uryasev (2001) ‘Credit risk optimization with conditional value-at-risk criterion.’ *Mathematical Programming* 89(2), 273–291
- Arcones, M.A., and E. Gine (1993) ‘Limit theorems for u-processes.’ *The Annals of Probability* 21(3), 1494–1542
- Artzner, P., F. Delbaen, J.M. Eber, and D. Heath (1999) ‘Coherent Measures of Risk.’ *Mathematical Finance* 9(3), 203–228

- Charnes, A., W. W. Cooper, and G. H. Symonds (1958) ‘Cost horizons and certainty equivalents: An approach to stochastic programming of heating oil.’ *Management Science* 4(3), pp. 235–263
- Chen, S.X. (2008) ‘Nonparametric estimation of expected shortfall.’ *Journal of Financial Econometrics* 6(1), 87
- Chopra, V.K. (1993) ‘Improving Optimization.’ *The Journal of Investing* 2(3), 51–59
- De Giorgi, E.G. (2002) ‘A Note on Portfolio Selection under Various Risk Measures.’ *SSRN eLibrary*
- DeMiguel, V., L. Garlappi, F.J. Nogales, and R. Uppal (2009) ‘A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms.’ *Management Science* 55(5), 798
- El Karoui, N. (2009) ‘High-dimensionality effects in the markowitz problem and other quadratic programs with linear equality constraints: risk underestimation.’ Technical Report, Citeseer
- Frost, P.A., and J.E. Savarino (1988) ‘For better performance: Constrain portfolio weights.’ *The Journal of Portfolio Management* 15(1), 29–34
- Goldfarb, D., and G. Iyengar (2003) ‘Robust Portfolio Selection Problems.’ *Mathematics of Operations Research* 28(1), 1–38
- Gotoh, J., and K. Shinozaki (2010) ‘A Robust Portfolio Technique for Mitigating the Fragility of CVaR Minimization.’ Technical Report, Dept. of Industrial and Systems Engineering, Chuo University
- Gotoh, Jun-ya, and Akiko Takeda (2010) ‘On the role of norm constraints in portfolio selection.’ *Computational Management Science* 8, 323–353
- Hastie, T., R. Tibshirani, and J. Friedman (2009) *The Elements of Statistical Learning: Data Mining, Inference, and Prediction.*, 2 ed. (Springer-Verlag)
- Hoeffding, W. (1948) ‘A class of statistics with asymptotically normal distribution.’ *The Annals of Mathematical Statistics* pp. 293–325
- Jagannathan, R., and T. Ma (2003) ‘Risk reduction in large portfolios: Why imposing the wrong constraints helps.’ *The Journal of Finance* 58(4), 1651–1684
- Jobson, J.D., and B. Korkie (1980) ‘Estimation for Markowitz Efficient Portfolios.’ *Journal of the American Statistical Association* 75(371), 544–554
- Lim, A.E.B., J.G. Shanthikumar, and G.-Y. Vahn (2011) ‘Conditional Value-at-Risk in portfolio optimization: coherent but fragile.’ *Operations Research Letters* 39(3), 163 – 171
- Mansini, Renata, Włodzimierz Ogryczak, and M. Speranza (2007) ‘Conditional value at risk and related linear programming models for portfolio optimization.’ *Annals of Operations Research* 152, 227–256. 10.1007/s10479-006-0142-4
- McNeil, A.J., R. Frey, and P. Embrechts (2005) *Quantitative risk management: Concepts, techniques and tools* (Princeton Univ Pr)
- Muirhead, R.J. (1982) *Aspects of multivariate statistical theory* (Wiley Online Library)

- Pollard, D. (1984) *Convergence of stochastic processes* (Springer)
- Rockafellar, R.T., and S. Uryasev (2000) ‘Optimization of conditional value-at-risk.’ *Journal of Risk* 2, 21–41
- Van der Vaart, A.W. (2000) *Asymptotic statistics* (Cambridge Univ Press)
- Zhu, S., and M. Fukushima (2009) ‘Worst-case conditional value-at-risk with application to robust portfolio management.’ *Operations research* 57(5), 1155–1168