Separation Logic

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Outline

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Brief recap: reasoning about code

- Program semantics described by logical conditions satisfied by language constructs
- Classical model, first put forward by Robert W. Floyd and Tony Hoare

Floyd-Hoare Logic in 1 slide

$$\{P\}S\{Q\}$$

- P : pre-conditions
- S : statement
- Q : post conditions

Partial correctness: **If the inital state fullfils pre-conditions and the statement terminates**, the final state satisfies the post conditions.

Total correctness: **If the initial state fullfils the pre-conditions** then the statement terminates and the final state satisfies the post-conditions.

Limitations

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Global view of state becomes a burden when introducing pointers(think of pointer aliasing..)

Motivating example

```c
void deletetree(struct node *root){
    if(root != NULL){
    struct node *left = root->l;
    struct node *right = root->r;
    deletetree(left);
    deletetree(right);
    free(root);
    }
}
```

How can we prove memory safety?

First attempt

Specification

$\{h: \ tree(t, h)\}$
$deletetree(t)$
$\{h' : true\}$

Proof

First attempt

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$$\{h: \ tree(t,h)\}$$
$$deletetree(t)$$
$$\{h': true\}$$

Proof

$$\{h : h[t] = [l, r]$$
$$\wedge\, tree(l, h)$$
$$\wedge\, tree(r, h)$$
$$\wedge\, t, l, r \ distinct\}$$

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$$\{h' : true\}$$

We can't prove safety of tree(r,h)!

First attempt

Specification

$\{h: \; tree(t, h)\}$
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$\{h' : \forall p, h'[p] = h[p]$
if p is not in the tree $\}$

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$deletetree(l)$

How can we be sure that deletetree(l)
does not modify tree(r,h)? We should say that in tree(t,h)...

Outline

The model

Ints $\triangleq \{\cdots, -1, 0, 1, \cdots\}$

Atoms, Locations $\subseteq Ints$

Stores $\triangleq$ Variables $\rightharpoonup_{fin}$ Ints

Variables $\triangleq \{x, y, \cdots\}$

Locations $\cap$ Atoms $= \{\}$, nil $\in$ Atoms

Heaps $\triangleq$ Locations $\rightharpoonup_{fin}$ Ints

States $\triangleq$ Stores $\times$ Heaps

$[\![E]\!]_s \in Ints$, $[\![B]\!]_s \in \{\text{true}, \text{false}\}$

$h \in Heaps$, $h[E] \in Ints$

The language

Expressions:

$$E, F, G := x, y, \cdots \mid 0 \mid 1 \mid E + F \mid E \times F \mid E - F$$

$$B := false \mid B \Rightarrow B \mid E = F \mid E < F \mid isatom?(E) \mid isloc?(E)$$

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Assertions:

$$\begin{aligned} P, Q, R ::= {} & B \mid E \mapsto F && \textit{Atomic Formulae} \\ & \mid false \mid P \Rightarrow Q \mid \forall x.P && \textit{Classical Logic} \\ & \mid emp \mid P * Q \mid P \mathrel{-\!\!*} Q && \textit{Spatial Connectives} \end{aligned}$$

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$\neg P = P \Rightarrow False$
$true = \neg(false)$
$P \vee Q = \neg(P) \Rightarrow Q$

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$$\neg P = P \Rightarrow False$$
$$true = \neg(false)$$
$$P \vee Q = \neg(P) \Rightarrow Q$$

$$P \wedge Q = \neg(\neg P \vee \neg Q)$$
$$\exists x.P = \neg \forall x. \neg P$$

Some notation

1. $dom(h)$ and $dom(s)$ denote the domain of definition for $h \in Heaps$ and $s \in Stores$, respectively
2. $h\#h' \to dom(h) \cap dom(h') = \emptyset$
3. $h * h'$ is the union of disjoint heaps
4. $(f|i \mapsto j)$ represent the partial function that behaves like f except that i goes to j.

$$E \mapsto F_0, \cdots, F_n \triangleq (E \mapsto F_0) * \cdots * (E + n \mapsto F_n)$$

$$E \doteq F \triangleq (E = F) \wedge emp$$

$$E \mapsto - \triangleq \exists y.E \mapsto y$$

Semantics

For store s and heap h

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$$s, h \models B \; iff \; [\![B]\!]_s = true$$

$$s, h \models E \mapsto F \; iff \; \{[\![E]\!]_s\} = dom(h) \; and \; h([\![E]\!]_s) = [\![F]\!]_s$$

$$s, h \models false \quad never$$

$$s, h \models P \Rightarrow Q \; iff \; if \; s, h \models P \; then \; s, h \models Q$$

$$s, h \models \forall x.P \; iff \; \forall v \in Ints.[s \mid x \mapsto v], h \models P$$

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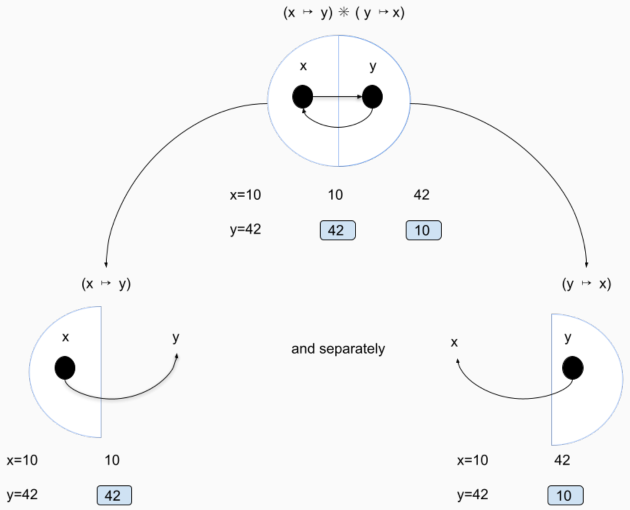
$$s, h \models \forall x.P \ iff \ \forall v \in Ints.[s \mid x \mapsto v], h \models P$$

$$s, h \models emp \ iff \ h = [] \ is \ the \ empty \ heap$$

$$s, h \models P * Q \ iff \ \exists h_0, h_1.h_0 \# h_1, \ h_0 * h_1 = h, \ s, h_0 \models P \ and \ s, h_1 \models Q$$

$$s, h \models P -\!\!* \ Q \ iff \ \forall h'. \ if \ h' \# h \ and \ s, h' \models P \ then \ s, h * h' \models Q$$

Visual example



Core system

Proof rules in seapartion logic are divide in:

1. Axioms for basic mutation commands $\rightarrow$ *Small axioms*
2. Inference rules for modular reasoing $\rightarrow$ *Structural rules*

Small axioms

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$\{x \doteq m\}\mathbf{x} := \mathbf{cons}(\mathbf{E_1}, \cdots, \mathbf{E_k})\{x \mapsto E_1[m/x], \cdots, E_k[m/x]\}$ ("Allocate memory")

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$\{x \doteq n\}\mathbf{x} := \mathbf{E}\{x \doteq (E[n/x])\}$

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("Allocate memory")

$\{x \doteq n\}\mathbf{x} := \mathbf{E}\{x \doteq (E[n/x])\}$

$\{E \mapsto n \wedge x = m\}\mathbf{x} := [\mathbf{E}]\{x = n \wedge E[m/x] \mapsto n\}$ ("Load")

Structural rules

Frame Rule

$$\frac{\{P\}C\{Q\}}{\{P * \color{red}{frame}\}C\{Q * \color{red}{frame}\}}\ Mod(C) \cap Free(frame) = \emptyset$$

Structural rules

Frame Rule

$$\frac{\{P\}C\{Q\}}{\{P * \color{red}{frame}\color{black}\}C\{Q * \color{red}{frame}\color{black}\}} \; Mod(C) \cap Free(frame) = \emptyset$$

Auxiliary variable elimination

$$\frac{\{P\}C\{Q\}}{\{\exists x.P\}C\{\exists x.Q\}} \; x \notin Free(C)$$

Structural rules

Variable substitution

$$\frac{\{P\}C\{Q\}}{(\{P\}C\{Q\})[E_1/x_1, \cdots E_k/x_k]}$$

$$x_i \ free \ and \ if \ x_i \in Mod(C) \ then \ E_i \ is \ not \ free \ in \ any \ E_j$$

Structural rules

Variable substitution

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$$x_i \ free\ and\ if\ x_i \in Mod(C)\ then\ E_i\ is\ not\ free\ in\ any\ E_j$$

Rule of consequence

$$\frac{P \Rightarrow P' \quad \{P\}C\{Q\} \quad Q \Rightarrow Q'}{\{P\}C\{Q'\}}$$

Derived laws

The structural rules can be used to obtain more convenient derived laws.
As an example, we can simplify the rule for memory allocation by assuming $x \notin Free(E_1, \cdots, E_k)$.

$$\{emp\}x := cons(E_1, \cdots, E_k)\{x \mapsto E_1, \cdots, E_k\}$$

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$$\{emp\}x := cons(E_1, \cdots, E_k)\{x \mapsto E_1, \cdots, E_k\}$$

The core system can also be extended with the usual Hoare rules

$$\frac{\{P \wedge B\}C\{Q\} \quad \{P \wedge \neg B\}C\{Q\}}{\{P\}if\ B\ then\ C\ else\ C'\{Q\}}$$

Revisiting the tree example

```
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    free(root);
    }
}
```

Specification:

{tree(root)} deletetree(root) {emp}

$$\text{tree(root)} = if\ root == 0\ then\ emp$$
$$else\ \exists xy.root \mapsto [l : x, r : y] * tree(x) * tree(y)$$

Proof:

$$\{root \mapsto [l : left, r : right] * tree(left) * tree(red)\}$$

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$deletetree(left);$

Proof:

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$deletetree(left);$

$\{root \mapsto [l : left, r : right] * emp * tree(red)\}$

Proof:

$\{root \mapsto [l : left, r : right] * tree(left) * tree(red)\}$

$deletetree(left);$

$\{root \mapsto [l : left, r : right] * emp * tree(red)\}$

$deletetree(right);$

Proof:

$\{root \mapsto [l : left, r : right] * tree(left) * tree(red)\}$

$deletetree(left);$

$\{root \mapsto [l : left, r : right] * emp * tree(red)\}$

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Proof:

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$\{root \mapsto [l : left, r : right] * emp * emp\}$

$free(root);$

Proof:

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$\{root \mapsto [l : left, r : right] * emp * emp\}$

$free(root);$

$\{emp * emp * emp\}$
$\{emp\}$

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Proofs in SL are very simple by design, but automation is needed to scale the analysis to large programs.
To fully automate proofs we need a way to infer pre and post conditions from bare code.

In SL this is solved with **bi-abduction**

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To fully automate proofs we need a way to infer pre and post conditions from bare code.

In SL this is solved with **bi-abduction**

$$(A * ?antiframe \vdash B * ?frame)$$

Can we find a pair of frame and antiframe that make the entailment valid?

Biabduction in action

Suppose we have the code

(closeResource(r1);closeResource(r2))

A human would say that we can execute closeResource(r1) only if we have $r1 \mapsto open$.
The equivalent biabduction question is

$$(emp * ?antiframe \vdash r1 \mapsto open * ?frame)$$

Biabduction in action

$$(emp * \color{red}{?antiframe} \vdash r1 \mapsto open * \color{blue}{?frame})$$

Biabduction in action

$$(emp * ?antiframe \vdash r1 \mapsto open * ?frame)$$

$$(antiframe = r1 \mapsto open),\ (frame = emp)$$

Biabduction in action

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$$(emp * r1 \mapsto open \vdash r1 \mapsto open * emp)$$
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Biabduction in action

$$(emp * ?antiframe \vdash r1 \mapsto open * ?frame)$$

$$(antiframe = r1 \mapsto open),\ (frame = emp)$$

$$(emp * r1 \mapsto open \vdash r1 \mapsto open * emp)$$
$$(r1 \mapsto open \vdash r1 \mapsto open)$$

$$\{r1 \mapsto open\}$$
$$(closeResource(r1))$$
$$\{r1 \mapsto closed\}$$

Biabduction in action

$(emp * ?antiframe \vdash r1 \mapsto open * ?frame)$

$(antiframe = r1 \mapsto open),\ (frame = emp)$

$(emp * r1 \mapsto open \vdash r1 \mapsto open * emp)$
$(r1 \mapsto open \vdash r1 \mapsto open)$

$\{r1 \mapsto open\}$
$(closeResource(r1))$
$\{r1 \mapsto closed\}$

Let's now consider $closeResource(r2)$

Biabduction in action

$$(r1 \mapsto closed * ?antiframe \vdash r2 \mapsto open * ?frame)$$

Biabduction in action

$$(r1 \mapsto closed * ?antiframe \vdash r2 \mapsto open * ?frame)$$

$$(antiframe = r2 \mapsto open),\ (frame = r1 \mapsto closed)$$

Biabduction in action

$$(r1 \mapsto closed * ?antiframe \vdash r2 \mapsto open * ?frame)$$

$$(antiframe = r2 \mapsto open), \ (frame = r1 \mapsto closed)$$

$(\{r1 \mapsto open * r2 \mapsto open\})$
$(closeResource(r1))$
$(\{r1 \mapsto closed * r2 \mapsto open\})$
$(closeResource(r2))$
$\{r1 \mapsto closed, \ r2 \mapsto closed\}$

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