

UNIVERSIDADE DE SÃO PAULO

Instituto de Física - Departamento de Física Matemática

Topological Entanglement Entropy in Abelian Higher Gauge Theories

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Entropia Topológica de Emaranhamento em Teorias de Higher Gauge Abelianas

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Abstract

We compute topological entanglement entropy for a large set of lattice models in d -dimensions. It is well known that many such quantum systems can be constructed out of lattice gauge models. For dimensionality higher than 2 there are generalizations going beyond gauge theories. They are called higher gauge theories and rely on higher-order generalizations of groups. Our main concern is a large class of d -dimensional quantum systems derived from Abelian higher gauge theories. In this work, we calculate the bipartition entanglement entropy for this class of models. Our formalism allows us to do most of the calculation for arbitrary dimension d . We show that the entanglement entropy S_A in a sub-region A is proportional to $\log(GSD_{\tilde{A}})$, where $GSD_{\tilde{A}}$ is the ground state degeneracy of a particular restriction of the full model to A . When A has the topology of a d -dimensional ball, the $GSD_{\tilde{A}}$ counts the number of edge states. In this case, S_A scales with the area of the $(d - 1)$ -dimensional boundary of A . The precise formula for the entropy we obtain is in agreement with entanglement calculations for known topological models.

Resumo

Nós calculamos a entropia de emaranhamento topológica para um grande conjunto de modelos em dimensão d . Sabe-se que muitos sistemas quânticos podem ser construídos a partir de teorias de gauge na rede. Em dimensões maiores a 2 existem generalizações além das teorias de gauge. Chamadas *higher gauge theories*, estas são baseadas em generalizações de ordem superior do conceito de grupo. O nosso objeto de estudo é um conjunto grande de modelos d -dimensionais, que são obtidos a partir de teorias Abelianas de *higher gauge*. Neste trabalho, calculamos a entropia de emaranhamento para dito conjunto de modelos. O nosso formalismo permite fazer a maior parte do cálculo para dimensão arbitrária d . Mostramos que a entropia de emaranhamento S_A , em uma sub-região A do sistema, é proporcional à $\log(GSD_{\tilde{A}})$, onde $GSD_{\tilde{A}}$ é a degenerescência do estado fundamental de uma restrição particular do modelo na região A . Quando A tem a topologia de uma bola de dimensão d , a quantidade $GSD_{\tilde{A}}$ conta o número de estados de borda. Neste caso, S_A escala com a área da borda $(d - 1)$ -dimensional de A . O resultado exato da entropia que obtemos está em concordância com os resultados conhecidos na literatura.

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Chapter 1

Introduction

On our daily experience we can easily distinguish between different phases of matter by mere naked-eye observation. For instance, the gas, solid and liquid phases of water are part of our quotidian and we can clearly differentiate and even describe to some level of detail these many phases. There are, however, more intricate *states of matter* that can occur in different situations and scales. Take plasmas, Bose-Einstein condensates, quantum spin liquids or superfluids as examples.

States of matter are, in general, classified into phases that can be connected to each other via phase transitions. This can be seen as follows: different states of matter are distinguished by their internal structure (order). Take the example of a solid at some finite temperature where the atoms are arranged in a regular (or almost regular) pattern depending on its constituents, their interactions and the external conditions such as pressure and temperature. If we choose to vary some of these conditions, say, the temperature, eventually the *crystal order* will be destroyed and the solid will suffer a transition into a *liquid phase* where the motion of the atoms is now less correlated. If the temperature continues to be raised the system will again go through a transition into a very disordered phase, namely, it will become a gas, where the motion of an atom hardly depends on the motion of the other constituents.

Thus, phases of matter in principle could be classified by means of phase transitions, under this scheme different states or phases of matter have different internal structure, or *order*. The properties of a material are mainly determined by how its constituents are organized rather than the specifics of the constituents themselves. Phases, in this sense, can be understood as emergent since the several properties a material can have, depend on the several ways the constituents are organized, or ordered. Different orders lead to different materials or phases which in turn lead to different properties of the material. One key step in order to develop a general theory that could ultimately classify these phases of matter was the realization that the internal orders of a system are related to the *symmetries* of its elementary constituents. This means that if two phases of matter are different is because they differ on their symmetries. As a material undergoes a phase transition the internal symmetries of the system change. This is the fundamental idea in what is known as the *Ginzburg-Landau* theory of phase transitions [6, 7, 8] that was originally developed to describe the transition to a superconductor phase of matter by means of a local order parameter and its fluctuations. For a long time it was thought that this theory could describe all phases of matter and their phase transitions.

However, the discovery of the *Fractional Quantum Hall Effect* (FQHE) by Tsui et al. [9] revealed that Landau's scheme, based on local order parameters, was not enough to have an exhaustive classification of phases of matter. The conceptual novelty brought by the discovery of FQHE states is that they appear to exhibit internal orders or “patterns” that do not have any relation with any kind of local symmetries (or the breaking of them) and thus cannot be described by the usual Ginzburg-Landau symmetry-breaking scheme. It is this global motion pattern that corresponds to the *topological order* in FQHE states [10], in this sense, the FQH states stand as the first observation of a topological phase of matter [11, 12]. The word *topological* in this context comes from the fact that there are no local order parameters able to distinguish between different phases, this is, global parameters must be considered instead. One example of such a global quantity is the ground state degeneracy (GSD) that depends on the underlying topology where the system is realized [10, 13, 14, 15]. Another example of such phases can be found in the study of Spin Liquids [16, 17, 18, 19]. In particular the so called *chiral spin liquids* [20], studied with the intention to explain high T_c superconductivity. For some time people tried to use the symmetry-breaking scheme in order to characterize the chiral spin liquid. Wen [14] realized that there are several different chiral spin liquids with the exact same symmetry. So, definitely, the characterization must take this into consideration, symmetry is not enough.

1.1 Topological Order

Although there is no precise definition of topological order, a notion of *topological order* can be given through some measurable quantities, which in turn are related to topological invariants. We already mentioned one of such quantities, namely, the *ground state degeneracy* (GSD), as in the FQHE case. But the features of topologically ordered systems are not restricted to the topology dependent ground state degeneracy. The excitations of such systems exhibit characteristic properties such as the fractionalization of the charge and anyonic statistics. In particular, for the FQHE states, which arise in systems whose constituents are electrons each one with charge e , the excitations carry a charge that is a fraction of e . This is closely related to the degeneracy as it can be shown that fractionalization implies the degeneracy of the ground state [21, 22]. Another feature involving the quasi particle excitations of such systems is that they obey exotic statistics. In 3 spatial dimensions it is known that the quantum states of identical particles behave either as bosons or fermions under the exchange of a pair, even though this fact may seem simple it is fundamental for the understanding of nature, and it is at the root of the classification of elements as we know them since the Pauli exclusion principle holds only for fermions. Nevertheless, in two dimensional systems, such as the FQHE states, there are new possibilities for quantum statistics that interpolate continuously between those of bosons and fermions. Under an exchange of two quasi-particles the quantum state can acquire an overall phase $e^{i\theta}$, where the special cases $\theta = 0, \pi$ correspond to the bosonic and fermionic statistics respectively. The *statistical angle* θ can take different values, and the particles obeying these generalized statistics are called *anyons* [23, 24, 25, 26, 27, 28].

Topological order is understood as a property of the quantum states of a system. Specially the ground states, as one of the most robust observables is the GSD. It is an emergent phenomenon since it depends mostly on how the elementary constituents of

the quantum system organize between themselves. In this sense, topological order is related to the entanglement in wavefunctions. Topological order is said to be intrinsic whenever there is *Long Range Entanglement* (LRE). This can be understood by looking at the microscopics of the system. Topological order is a property of a local quantum system with total Hilbert space being a tensor product of local ones, $\mathcal{H} = \bigotimes_i \mathcal{H}_i$. A *product state* is an state of the form $|\Psi\rangle = \bigotimes_i |\Psi_i\rangle$, where $|\Psi_i\rangle \in \mathcal{H}_i$. An state is said to be topologically ordered if it cannot be deformed into a product state without going through a phase transition that might have to do with a change in the underlying topology. Such quantum states are said to be *long range entangled*. On the contrary, gapped quantum states that can be deformed into a product state are said to have *short range entanglement*, which when symmetry is unbroken, give rise to nontrivial phases, called *symmetry protected topological* (SPT) phases, such as the Haldane phase [29]. The mechanism here prevents the deformation of the state into a trivial state by the symmetry.

The above suggests that the nature of *entanglement* in a quantum state is by itself a signature of topological order. In fact, entanglement entropy turns out to be a good measure of the presence of topological order in a quantum state [30, 31], we refer the reader to [32] for a detailed discussion about the relation between topological entanglement entropy and topological order in 2 spatial dimensions. For two dimensional phases, such as the FQHE states, it is shown that the scaling of the entanglement entropy has a constant term correction [33, 34] often called *topological entanglement entropy*, this is examined in detail for the simplest example in [35, 36, 37, 38] and in §2.3 example 2.3.2, for instance.

Hence there is a clear need for a general theory of topological phases and consequently the need for a mathematical framework that could ultimately characterize and classify these topological phases of matter. In the past years there has been a major interest on the study of these phases of matter via a detailed analysis of exactly solvable lattice models that exhibit the features of having *topological order*. The simplest example is the so called **Toric Code** model (cf. §2.2) introduced by A. Kitaev in [1] which is constructed as a many body interacting system defined over a 2-dimensional lattice. It exhibits the features of a topologically order system as its ground state is 4-fold degenerate when the lattice is embedded on the surface of a Torus, hence part of its name. The degeneracy is protected from local perturbations that come as the elementary excitations of the model. These elementary excited states can be interpreted as quasi-particle anyonic excitations located at the vertices and faces of the lattice, they display bosonic statistics when braided among themselves while composite excitations show fermionic statistics. The model can be interpreted as a particular lattice gauge theory [39] where the gauge group is the abelian \mathbb{Z}_2 group. Furthermore, for any finite group G , in [1] Kitaev introduces a more general class of models called *Quantum Double Model* (QDM) defined through a Hamiltonian that is written as a sum of mutually commuting projectors [40, 41, 42, 43] (cf. §2.1), which are Lattice Gauge Theories with group G in the Hamiltonian formalism. The elementary excitations of this models are anyons whose fusion and braiding properties depend on the specific choice of the group G giving rise to the possibility of having non-abelian anyons that can be used to implement a fault-tolerant quantum computation process [44, 45, 46, 47], where unitary transformations are obtained by the braiding of anyons and the final measurement is performed by the joining of pairs of excitations.

In general, two dimensional topological phases have been exhaustively studied and are rather well understood; A large class of 2D topological orders are identified by the

systematic construction of the so called **string-net** models [2, 48] where it is shown that each topological phase is associated to particular tensor category called *fusion category*, the QDM being a subclass of these models as shown in [49]. Exactly solvable models coming from gauge theories, such as string-nets and QDMs appear to be very efficient when characterizing topological order, at least in two spatial dimensions. In the sense that they realize all types of anyonic excitations. The same cannot be said for higher dimensional topological phases, since much less is known about them. For example, the 3D versions of the QDM are not enough to account for all topological phases in 3D. This is, in part, due to the difficulties of understanding higher dimensional topological quantum field theories. Nevertheless, in recent years there have been serious improvements in the level of understanding of such theories. Two main trends can be identified for the study of higher dimensional topological phases: On one hand there is an increasing interest in lattice models that come from higher dimensional analogues of gauge theories [3, 50, 51, 52, 53, 54], which in general we call *higher gauge theories*, approach that we ascribe for this work. On the other hand, the proposal of the so called *fracton* phases [55, 56, 57, 58, 59, 60, 61, 62], arises from a slightly different strategy for the study of higher dimensional topological phases, more precisely, the *gauging* procedure [60, 63, 64] allows the construction of rather intricate lattice models with remarkable new properties. Among these properties, in [65, 66] the *topological entanglement entropy* of several fracton models is explicitly calculated, showing that they indeed appear as new quantum phases.

With respect to the approach we follow in the study of higher topological phases, in [50] a class of Topological Quantum Field Theories involving 1-form and 2-form gauge fields has been studied using 2-groups instead of the usual notion of 1-groups. Consequently, the existence of gapped phases of matter that are protected by a 2-group instead of a 1-group symmetry was proposed. Moreover, in [52], and independently in [67, 68], a Hamiltonian formulation of the Yetter's homotopy 2-type TQFT [69] was constructed with the aim of understanding $(3+1)$ topological phases of matter. Also, it is worth mentioning the works of [70], where bosonic lattice realizations of SPT phases with higher form symmetry are presented, and [71, 72], where their connection to fault-tolerant logical gates in topological quantum codes is discussed. The common feature in all above attempts to unravel higher dimensional phases are founded in the idea of more general kind of gauge symmetry, a *higher gauge symmetry* and the use of some notion of *n-groups* to implement such higher symmetry on actual quantum states.

In [3] we take a first step along the lines of *higher gauge symmetries*, this is, we construct and study a class of models that could live in arbitrary dimensions and go beyond usual gauge theories. These theories involve higher gauge fields and symmetries in all possible dimensions. The notion of a gauge group is replaced by a more general mathematical object, namely, a chain complex of abelian groups, which plays the role of a *n-group*. As a consequence, the notion of *gauge configuration* is replaced by the notion of *maps between two chain complexes*. In this mathematical framework is quite natural to construct lattice Hamiltonian formulations of higher gauge theories on arbitrary dimensions. The class of Hamiltonian models obtained from this picture are shown to have a degenerate ground state subspace whose basis elements are in one-to-one correspondence with special cohomology classes [73]. Furthermore, this formalism allows to explicitly show that the ground state degeneracy (GSD) is a topological invariant, for which we give a closed formula in terms of the order of the 0-th cohomology group with coefficients in the

chain complex of abelian groups. Moreover, due to a theorem by Brown [73], the GSD can be understood as exhibiting contributions from each dimension, therefore exhibiting the different intrinsic *topological orders* involved.

In this thesis, we study the entanglement entropy of the Abelian higher gauge theories defined in [3], all at once. This can be achieved using the language of homological algebra, and the result we obtain for the entanglement entropy relates this quantity to the ground state degeneracy of a model restricted to a subregion of the system, in a very particular way as we will precisely show. This result seems to be in agreement with the holographic principle, moreover, it also provides the corresponding topological correction whenever topological order is detected.

The way the entanglement entropy is obtained, in essence, relies on the fact that the lattice models are constructed as *stabilizer codes* [74]. The entropy calculation, is similar in spirit to the one in [35, 75, 76, 77, 78]. The goal of this work is to show that the same strategy allows to readily extract the entanglement entropy of any Abelian higher gauge theory (in the sense of [3]), exhibiting the topological constant term in the presence of topological order.

The calculation of entanglement entropy starts out by defining a density matrix from the (possibly degenerate) ground states, $|\psi\rangle$, of a stabilizer Hamiltonian. This is, define $\rho := |\psi\rangle\langle\psi|$. Then consider a bipartition of the system into a sub-region A and the complement (B), such distinction allows to obtain the *reduced density matrix*, $\rho_A = \text{Tr}_B(\rho)$ by performing a partial trace over sub-region B . The entanglement entropy is then defined as the *von Neumann* entropy of the reduced density matrix, namely,

$$S_A := -\text{Tr}(\rho_A \log \rho_A). \quad (1.1)$$

In a gapped phase, the entanglement entropy is expected to satisfy an *area law* as the leading term; The topological information is contained in subleading terms and, in general, it is not easy to extract. Several prescriptions [33, 34] were constructed in order to extract the topological correction to the entanglement entropy in two dimensional gapped systems. These prescriptions have been generalized [79, 80] for $d = 3$ and consequently used to successfully obtain the entanglement entropy of paradigmatic fracton models [65, 66].

1.2 About this Manuscript

The objective of this manuscript is two fold. First, we intend to give a concise summary of the main research activities concerning the Ph.D. project of Juan Pablo Ibieta Jimenez at the DFMA-IFUSP under the supervision of Prof. Dr. Paulo Teotônio Sobrinho. We also think this work could be seen as a pedagogical treatise of Abelian Higher Gauge Theories.

This manuscript is organized as follows, we begin the discussion with Chapter 2 where we review the most simple class of 2D topological models in the lattice, the *Quantum Double Models* (QDMs). These models are based on a discrete group G and they can be thought of as the Hamiltonian version of lattice gauge theories with gauge group G . The local degrees of freedom are labeled by group elements in G . The dynamics of the model is determined by a Hamiltonian operator that acts on the entire many-body Hilbert space \mathcal{H} . The ground state subspace $\mathcal{H}_0 \subset \mathcal{H}$ has topological nature and we show this using projection operator that we call *ground state projector*. This operator becomes very

important later on this thesis as we will see on the subsequent chapters. In particular, when the gauge group is the Abelian \mathbb{Z}_2 we recover the *Toric Code* model that was originally introduced by Kitaev in [1]. We exhibit the topological nature of the models using this example. The last part of Chapter 2 we introduce the quantity of main interest for this thesis, namely, the *entanglement entropy* [34] of quantum states. We choose to do this by exhibiting two explicit calculations, a simple two-qubit system and the Toric Code. In the latter case, we obtain the known result [35, 36, 75, 76] for the entanglement entropy of the ground states. This is, an area law together with a constant term called *topological entanglement entropy* [33, 34].

Chapter 3 introduces quantum models that are interpreted as the Hamiltonian version of Higher Gauge Theories in their Abelian version. We start by casting the QDMs of 2 as gauge theories in 2D which we call 1-gauge theories in reference to having the degrees of freedom at links (1-dimensional objects). We then introduce two more 2D quantum models, called 0,1-gauge and 1,2-gauge theories, while the former considers degrees of freedom at both vertices (0-dimensional) and links (1-dimensional) of the lattice; whereas the latter considers degrees of freedom at links (1-d) and faces (2-d), hence the name for these two theories. A 3D model coined 1,2,3-gauge theory is then introduced to show that the generalization is carried over higher dimensions as well, in all the above cases we give specific examples (we choose the gauge groups) and show the topological properties encoded in the ground state subspace of this models. We end the chapter by defining the most general class of models, which we call *Abelian Higher Gauge Theories* and that are the main object of study in Chapter 4. Chapter 3 should be regarded as a pedagogical introduction of the kind of theories we will treat in this work.

On chapter 4 we study the topological properties of the ground state subspace of Abelian higher gauge theories in general. To do that we write the models using the language of Homological Algebra. This chapter is based on [3]. For this reason, the exposition encountered in chapter 3 can be complemented by looking at [3, 81]. The construction considers an underlying manifold, X , of arbitrary dimension and that could hold arbitrary gauge fields living on vertices, links, faces, volumes, etc. of the lattice. The model consists in a geometric part which we choose to be a simplicial chain complex $C(X)$ and an algebraic part, that plays the role of the gauge group. This generalized gauge group is the Abelian version of an n -group and it is given by an abstract chain complex of Abelian groups. The important concepts of gauge transformations, gauge configurations and holonomies are encoded in Abelian groups $\text{hom}(C, G)^p$ for $p = -1, 0, 1$, respectively, and we show that the small sequence,

$$\text{hom}(C, G)^{-1} \xrightarrow{\delta^{-1}} \text{hom}(C, G)^0 \xrightarrow{\delta^0} \text{hom}(C, G)^1,$$

is all we require to define an (Abelian) higher gauge theory in its most general form. The ground states of the models are shown to be characterized by the 0-th cohomology group $\mathcal{H}^0(C; G)$ which in turn decomposes into usual cohomology groups through a theorem by R. Brown [73]. We end chapter 4 with four illustrative and detailed examples corresponding all the examples introduced in chapter 3.

In chapter 5 we show the main result of this work. The entanglement entropy S_A of the models in chapters 3 and 4. We show how the language of Homological Algebra is very convenient to perform the calculation of the *entanglement entropy* which follows practically the same scheme shown in the example 2.3.2 of §2.3. Thus, we show that the

entanglement entropy of an (Abelian) higher gauge theory is related to the ground state degeneracy of a restricted version of the model. Moreover, this ground state degeneracy essentially counts the number of edge states, which in turn reveals the nature of entanglement in such states. We show how the area law plus the topological correction term are obtained by calculating S_A for particular examples, this is, we show that entanglement entropy is indeed a good probe for the presence of topological order in quantum states. Remarkably, all ground states of (Abelian) higher gauge theories appear to follow the area law for entanglement entropy.

We finish the manuscript with concluding remarks and outlook for future research that can be thought of as extensions of this work. I would like to end this introductory chapter by mentioning that this work, which is mainly focused on the models developed in [3], is a small part of the result of a joint research work that includes the efforts of the research group lead by Prof. Dr. Paulo Teotônio Sobrinho and integrated by the Ph.D. students Marzia Petrucci, myself and former members of the group.

Chapter 2

Quantum Double Models and Topological Entanglement Entropy

We start the Chapter with a simple but in-detail review of a class of two dimensional models known as *Quantum Double Models* [1, 82, 83, 84, 85]. They are quantum models defined by many-body quantum Hamiltonians acting on the global Hilbert space \mathcal{H} . The local degrees of freedom are labeled by group elements of a finite and discrete group G . The ground states subspace of these quantum models are of particular interest since they possess topological properties. To showcase this, we choose a particular example of a QDM, known as Toric Code. This model is obtained when the gauge group is set to be $G = \mathbb{Z}_2$. The ground states of the Toric Code are sensitive to the underlying topology of the lattice amongst other properties. This comes as a result of the *long range entanglement* of the states. Moreover, in [33, 34] the *entanglement entropy* is recognized as a probe for long range entanglement (and topological order) in two dimensional quantum states. Specifically, they show the existence of a *universal* constant term that signals the presence of long range entanglement. This is known as *Topological Entanglement entropy* and is the main focus of the second part of this Chapter. In this sense, we end the discussion calculating the entanglement entropy for the Toric Code. The entanglement entropy exhibits an *area law* contribution together with the topological term, a known result [35, 36, 75, 76].

2.1 Quantum Double Models

To start our discussion about higher dimensional topological phases, let us begin with a brief description of the class of (2+1)D quantum models known as *Quantum Double Model* with group G ; These quantum models can be understood as the Hamiltonian formulation of a (2+1)D lattice gauge theories where the gauge group is G and were originally described in [82, 86] from a field theoretical point of view. The standard construction of the model [1] defines a many body interaction Hamiltonian over a two dimensional oriented lattice Λ_Σ (usually squared). The Hamiltonian is made of two types of localized projection operators that commute with each other, commonly referred to as *vertex* and *plaquette* operators. Every edge of the lattice is oriented and it joins two adjacent vertices. So the lattice consists on vertices, edges and plaquettes. Although both the lattice and the orientation of the edges can be arbitrary, we consider a square lattice with the orientation

shown in Fig.(2.1) for no other reason than simplicity. Moreover, a local Hilbert space \mathcal{H}_l is associated to each edge $l \in \Lambda$ spanned by basis elements $\{|g\rangle\}_{g \in G}$ labeled by the group elements $g \in G$. The dimension of this local vector space is $\dim \mathcal{H}_l = |G|$, the order of the group. Consequently, the full Hilbert space is naturally constructed as being the tensor product of all local Hilbert spaces \mathcal{H}_i , namely,

$$\mathcal{H} := \bigotimes_{i \in \Lambda} \mathcal{H}_i, \quad (2.1)$$

such that the orthonormal basis is written:

$$\{|g_1, g_2, \dots, g_N\rangle := |g_1\rangle \otimes |g_2\rangle \otimes \dots \otimes |g_N\rangle, g_i \in G\},$$

where N stands for the number of edges in Λ and the inner product in \mathcal{H} is the usual one.

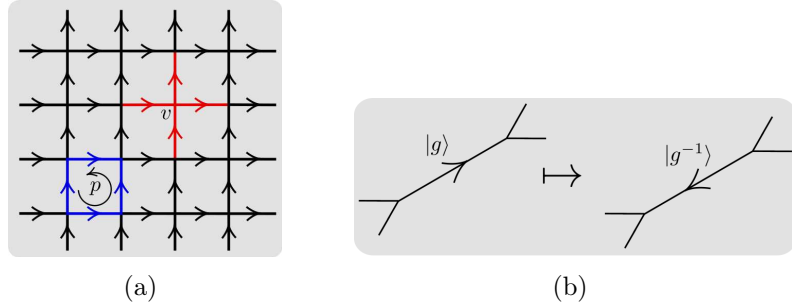


Figure 2.1: In (a), a square lattice with oriented edges is shown, where we highlight arbitrary vertex and plaquette, note that we fix the orientation of each plaquette of the lattice as being **counterclockwise**. Whereas, in (b) the process of inverting the orientation of an arbitrary link is exhibited.

Even though the lattice is oriented, meaning that both links and plaquettes have an orientation, the model itself is made orientation independent by assigning the following convention: Consider an arbitrary link l , associated to it there is a local basis element $|g\rangle$ ($g \in G$). A change in the orientation of the link l is related to the inversion of the group element that labels the basis state, as depicted in Fig. 2.1(b). Often and throughout this work we will use graphical representations of states. This is, let $|g_1, g_2, g_3, g_4, \dots, g_N\rangle$ be an arbitrary basis state where g_1, g_2, g_3 and g_4 are the link configurations around a particular vertex v , we will represent such state as:

$$|g_1, g_2, g_3, g_4, \dots, g_N\rangle \equiv \left| \begin{array}{c} \uparrow g_4 \\ \xrightarrow{g_1} \quad \xrightarrow{g_3} \\ \uparrow g_2 \end{array} \right\rangle.$$

The dynamics of the model is ruled by a Hamiltonian operator that consists on localized gauge transformations and localized holonomy measurements, as in a Lattice Gauge Theory. These localized operations are enhanced on quantum states by two operators, which we describe next. For any vertex v and plaquette p of the lattice \mathcal{L} we define a set of local operators A_v^g and B_p that act trivially on all edges in \mathcal{L} except for those *contained* in v and p respectively. This notion will be precisely stated in the following lines. So, consider an arbitrary vertex $v \in \mathcal{L}$, there are four (in a square lattice) local Hilbert spaces \mathcal{H}_i ,

$i \in v$ corresponding to the four adjacent edges, the *elementary gauge transformation*, A_v^g can be defined via its action on arbitrary basis states, as follows:

$$A_v^g \left| \begin{array}{c} \uparrow d \\ \xrightarrow{a} \quad \xrightarrow{c} \\ \uparrow b \end{array} \right\rangle = \left| \begin{array}{c} \uparrow d-g \\ \xrightarrow{g+a} \quad \xrightarrow{c-g} \\ \uparrow g+b \end{array} \right\rangle, \quad (2.2)$$

where we write the group operation in G additively. In the above expression it is understood that the operator acts trivially on the rest of the Hilbert space. This is, it acts non trivially on the four edges around the vertex v as shown in Fig.(2.2). Equivalently we say this operator acts on the subspace $\mathcal{H}_v \subset \mathcal{H}$ consisting on the four edges around the vertex in question. This operator is interpreted as implementing the local gauge symmetry labeled by group elements $g \in G$. The operator that is actually part of the Hamiltonian operator consists on a weighted sum over all gauge parameters, this is:

$$A_v := \frac{1}{|G|} \sum_{g \in G} A_v^g. \quad (2.3)$$

The other kind of local operators are in charge of making measurements, they consist on diagonal operators that measure the *holonomy* around the smallest closed loop, namely, a plaquette, defined by:

$$B_p \left| \begin{array}{ccc} & c & \\ d \uparrow & & \uparrow b \\ & a & \end{array} \right\rangle = \delta(a + b - c - d, 0) \left| \begin{array}{ccc} & c & \\ d \uparrow & & \uparrow b \\ & a & \end{array} \right\rangle. \quad (2.4)$$

where 0 is the identity element of G . In the above definition, j_1, \dots, j_4 are the edges at the boundary of p listed following a counterclockwise order.

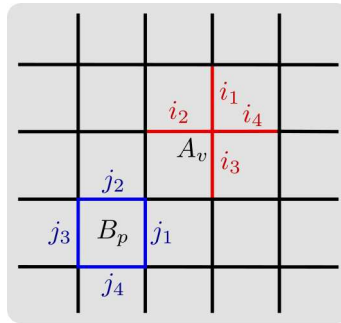


Figure 2.2: The local Hilbert spaces over which the A_v and B_p operators act are shown on the lattice in red and blue respectively.

Finally, we can write the Hamiltonian of the *Quantum Double Model*. Let Λ be a 2-dimensional oriented lattice and \mathcal{H} the Hilbert space of Eq.(2.1). We define $H_{QD} : \mathcal{H} \rightarrow \mathcal{H}$ as:

$$H_{QDM} := - \sum_{v \in \mathcal{L}} A_v - \sum_{p \in \mathcal{L}} B_p. \quad (2.5)$$

In the above definition, the A_v operators perform gauge transformations and thus project out states that are not invariant under the action of A_v^g for all elements $g \in G$ (and for each vertex $v \in \Lambda$). On the other hand, the operator B_p projects out the states with non-trivial holonomy around the plaquette $p \in \Lambda$. Both types of operators in H_{QD} are projectors and they all commute with each other. We refer the reader to [1, 84, 87] for details about the construction of this model.

2.1.1 Ground States

Let us now discuss, very briefly, about the topological nature of the QDMs. This analysis can be performed in a detailed fashion since both vertex and plaquette operators are mutually commuting projectors, namely

$$\begin{aligned} A_v^2 &= A_v, & B_p^2 &= B_p, \\ [A_v, B_p] &= 0, \quad \forall v, p \in \Lambda, \end{aligned} \quad (2.6)$$

which implies there is a simultaneous basis of eigenstates for all A_v and B_p . We are interested the subspace $\mathcal{H}_0 \subset \mathcal{H}$ that contains all states $|\psi\rangle$ with the lowest energy eigenvalue. In fact, it is easy to realize that the ground state subspace is defined by:

$$\mathcal{H}_0 := \{|\psi\rangle \in \mathcal{H} \mid A_v |\psi\rangle = |\psi\rangle, B_p |\psi\rangle = |\psi\rangle\}, \quad (2.7)$$

for all $v, p \in \Lambda$. There is a canonical way of constructing a basis for this subspace which we outline in the following lines. First of all we note that such subspace is not empty, to see this, consider the trivial *seed* state:

$$|\psi^0\rangle := \bigotimes_{l \in \Lambda} |0\rangle_l, \quad (2.8)$$

where e is the identity element of the group G . Notice that this state fulfills the ground state condition of Eq. (2.7) only for the plaquette operators, as the vertex operators perform gauge transformations by group multiplication. Therefore, we can construct an actual ground state if we perform all possible gauge transformations to the **seed** state of Eq. (2.8), this is, consider the following state:

$$|\psi_G^0\rangle := \prod_{v \in \Lambda} A_v |\psi^e\rangle,$$

it is straightforward to see that the above state is indeed a ground state of the H_{QD} from the following observation:

$$A_{v'} \prod_{v \in \Lambda} A_v = A_{v'} A_{v'} \prod_{v \neq v'} A_v = A_{v'} \prod_{v \neq v'} A_v = \prod_{v \in \Lambda} A_v,$$

a similar expression also holds for the product of plaquette operators B_p . This motivates the definition of an operator that projects basis states of \mathcal{H} to the ground state subspace \mathcal{H}_0 .

Definition 2.1.1 (Ground State Projector). Consider the operator $\Pi_0 : \mathcal{H} \rightarrow \mathcal{H}_0$ defined by:

$$\Pi_0 = \prod_{v \in \Lambda} A_v \prod_{p \in \Lambda} B_p, \quad (2.9)$$

where A_v and B_p are those of Equations (2.3) and (2.4), respectively.

To understand the ground state projector let us consider its action on an arbitrary basis state $|\psi\rangle \in \mathcal{H}$, this is:

1. The first part of the projector, namely, $\prod_{v \in \Lambda} A_v$ projects into gauge equivalent states, to see this it is enough to look at the notion of gauge equivalence. Consider two basis states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$, we say they are gauge equivalent if there is some gauge transformation A_v^g with $g \in G$ such that $A_v^g |\psi\rangle = |\phi\rangle$. Now, from Eq.(2.3) and Definition 2.1.1 it follows that:

$$\prod_{v \in \Lambda} A_v = \frac{1}{|G|^{|\Lambda^0|}} \prod_{v \in \Lambda} \sum_{g \in G} A_v^g,$$

as a consequence, two states $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ are gauge equivalent if $\prod_{v \in \Lambda} A_v |\psi\rangle = \prod_{v \in \Lambda} A_v |\phi\rangle$. Furthermore, from the above expression it is clear that $\prod_{v \in \Lambda} A_v$ maps a state $|\psi\rangle \in \mathcal{H}$ into a superposition of gauge equivalent states.

2. The part of the ground state projector that consists on a product of plaquette operators, namely, $\prod_{p \in \Lambda} B_p$ projects into states with trivial holonomy. This can be seen from the very definition of B_p in Eq. (2.4). Therefore, any state $|\psi\rangle$ is invariant under the action of $\prod_{p \in \Lambda} B_p$ only if it is invariant under the action of each B_p for all $p \in \Lambda$. This condition is exactly the same condition the state $|\psi\rangle$ must fulfill in order to be a ground state of the Hamiltonian in Equation (2.5).

From the above discussion, the action of Π_0 on an arbitrary basis state $|\psi\rangle \in \mathcal{H}$ is given by:

$$\Pi_0 |\psi\rangle = \begin{cases} |\psi\rangle, & \text{if } |\psi\rangle \in \mathcal{H}_0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, a basis for the ground state subspace is canonically obtained by the action of Π_0 on basis states of \mathcal{H} . Moreover, the ground state projector is a diagonal operator with only 1 and 0 as entries. Thus, the trace of such operator gives, precisely, the ground state degeneracy of the model, namely,

$$GSD = \text{tr}(\Pi_0).$$

In the next section we exhibit the most common example of a QDM. It corresponds of taking the gauge group to be $G = \mathbb{Z}_2$ and it already showcases the topological features of such models.

2.2 Example: $G = \mathbb{Z}_2$ (Toric Code)

In this section we describe an special case of the class of models known as QDMs. The model was originally proposed by Kitaev in [1] as the simplest topological quantum error correction code and usually called *Toric Code* (TC). This example is obtained by choosing the gauge group G to be the cyclic group of order 2 usually denoted as $\mathbb{Z}_2 = \{0, 1 : 1+1 = 0\}$, e being the identity element of the group. Canonically, the group algebra $\mathbb{C}(\mathbb{Z}_2)$ is a 2-dimensional vector space over the complex numbers spanned by the basis vectors $\{|0\rangle, |1\rangle\}$. In this case, the vertex operator A_v , is given by:

$$A_v = \frac{1}{2} (\mathbb{1}_{i_1} \otimes \mathbb{1}_{i_2} \otimes \mathbb{1}_{i_3} \otimes \mathbb{1}_{i_4} + \sigma_{i_1}^x \otimes \sigma_{i_2}^x \otimes \sigma_{i_3}^x \otimes \sigma_{i_4}^x). \quad (2.10)$$

On the other hand, the operator in charge of implementing the local flatness condition is given by:

$$B_p = \frac{1}{2} (\mathbb{1}_{j_1} \otimes \mathbb{1}_{j_2} \otimes \mathbb{1}_{j_3} \otimes \mathbb{1}_{j_4} + \sigma_{j_1}^z \otimes \sigma_{j_2}^z \otimes \sigma_{j_3}^z \otimes \sigma_{j_4}^z). \quad (2.11)$$

Then, the Hamiltonian of the \mathbb{Z}_2 Quantum Double model (Toric Code) is given by:

$$H_{TC} := - \sum_{v \in \mathcal{L}} A_v - \sum_{p \in \mathcal{L}} B_p, \quad (2.12)$$

where the plaquette and vertex operators are those obtained in Eqs.(2.11) and (2.10). The states of this model are elements of the Hilbert space \mathcal{H} which in turn is a tensor product of local Hilbert spaces $\mathcal{H}_i = \mathbb{C}(\mathbb{Z}_2)$ corresponding to each edge of the lattice \mathcal{L} . Clearly any state $|\psi_0\rangle \in \mathcal{H}$ such that

$$A_v |\psi_0\rangle = |\psi_0\rangle = B_p |\psi_0\rangle, \quad (2.13)$$

for any vertex v and plaquette p on the lattice \mathcal{L} , will be a ground state of the model. Consider the following state:

$$|\psi_0\rangle = \prod_{v \in \mathcal{L}} A_v \bigotimes_{i \in \mathcal{L}} |0\rangle_i, \quad (2.14)$$

where the product runs over all vertices in the lattice and by the tensor product we mean that each edge l of the lattice \mathcal{L} carries a $|e\rangle$ state. The state $|\psi_0\rangle$ is indeed a ground state of the Hamiltonian (2.12), i.e.

$$A_v |\psi_0\rangle = B_p |\psi_0\rangle = |\psi_0\rangle, \quad \forall v, p \in \mathcal{L}, \quad (2.15)$$

this can be shown by using the commutation relations between vertex and plaquette operators $[A_v, A_{v'}] = 0 = [A_v, B_p]$. This state is interpreted as a linear combination of **Loop** states, where the loops are a graphical way to represent the action of gauge transformations. To see this, consider the non-local operators:

$$Z(\gamma) := \bigotimes_{l \in \gamma} \sigma_l^z, \quad (2.16)$$

$$X(\gamma^*) := \bigotimes_{l \in \gamma^*} \sigma_l^x, \quad (2.17)$$

where γ and γ^* are paths in the direct and dual lattice, respectively. Notice that any product of vertex (plaquette) operators will act trivially on $|\psi_0\rangle$, equivalently, any operator $Z(\gamma)$ or $X(\gamma^*)$ where γ and γ^* define contractible loops on the direct and dual lattice, respectively, will act on $|\psi_0\rangle$ in a trivial way. This ground state is interpreted as being a **Loop Gas**, where the loops are the result of the action of vertex operators on the initial configuration $\bigotimes_{l \in \partial p} |e\rangle_l$. Each term on the product of eq.(2.14) will produce a combination of loops defined on the dual lattice and the ground state is a linear combination of all resulting states, some of them are shown in Fig.(2.3). The red closed loops on the graphical

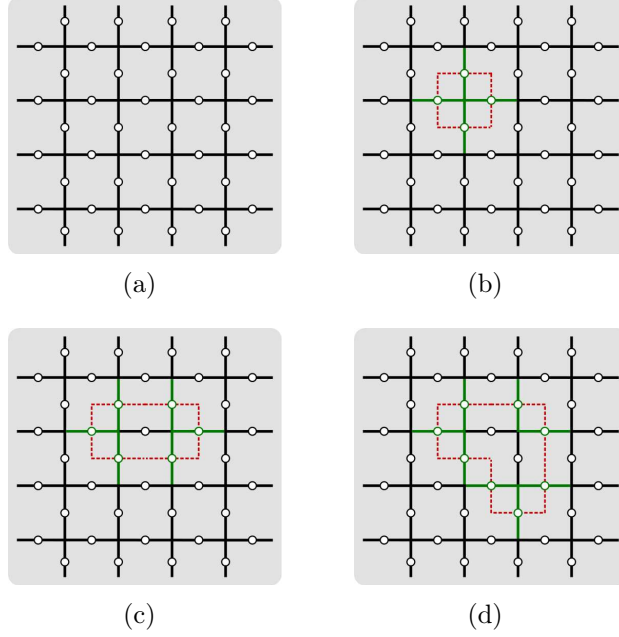


Figure 2.3: Some illustrative constituents of the ground state $|\psi_0\rangle$ are shown, where (a) corresponds to the first term in the expansion of Eq.(2.14), (b) corresponds to a term in this expansion for which there is a single A_v acting on the $|e\rangle$ states around v , in (c) and (d) we show terms that include the action of several vertex operators.

representation stand for the action of products of A_v operators on the state with all edges holding $|e\rangle$, it is in this sense that the ground state of the Toric Code can be interpreted as a **Loop Gas**, containing all possible contractible loops that could be defined on the lattice. From now on we consider the lattice to be embedded on the surface of a Torus. We know the torus has two non-contractible loops, see Fig.(2.4). Thus, the operators $X(\gamma_1^*)$, $X(\gamma_2^*)$, $Z(\gamma_1)$ and $Z(\gamma_2)$ can be defined and will not be made of products of vertex or plaquette operators, correspondingly. The action of the $X(\gamma_1^*)$ and $X(\gamma_2^*)$ operators is to interchange between different ground states since they commute both with the vertex and plaquette operators [1, 88, 89].

Consider the operator $X(\gamma_1^*)$ where the path γ_1^* winds the Torus along a horizontal non-contractible loop on the dual lattice, the action of this operator on the ground state

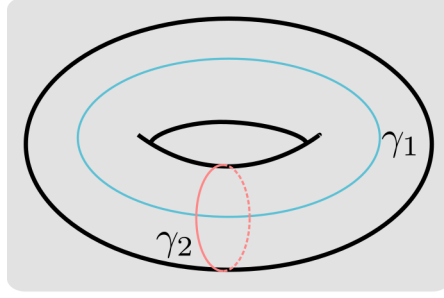


Figure 2.4: The non-contractible paths γ_1 and γ_2 are shown. The paths γ_1^* and γ_2^* are analogously drawn in the dual lattice.

$|\psi_0\rangle$ is given by:

$$\begin{aligned} X(\gamma_1^*) |\psi_0\rangle &= \prod_{v \in \Lambda^0} A_v X(\gamma_1^*) \bigotimes_{l \in \mathcal{L}} |0\rangle_l \\ &= \prod_{v \in \mathcal{L}} A_v \bigotimes_{l \notin \gamma_1^*} |0\rangle_l \bigotimes_{l' \in \gamma_1^*} |1\rangle_{l'} = |\psi_1\rangle, \end{aligned} \quad (2.18)$$

where in the first line we used the commutation relation $[X(\gamma_1^*), A_v] = 0$, and the action of the winding operator $X(\gamma_1^*)$ on the basis states $|e\rangle$ along the non-contractible loop γ_1^* consists on changing them into $|a\rangle$. Clearly this is still a ground state under the action of any vertex operator A_v , and it is straightforward to show that

$$B_p |\psi_1\rangle = |\psi_1\rangle, \quad (2.19)$$

by noticing that the plaquette operators acting on edges being crossed by γ_1^* will act trivially, since loop γ_1^* will necessarily cross any plaquette through two of its edges, therefore the configuration of the degrees of freedom on the plaquettes will be vortex free. Thus, we have shown that this new state is a ground state of the Toric Code Hamiltonian. Again it can be interpreted as a **Loop Gas**, some of its constituents are depicted in Fig.(2.5). Notice that the particular path defined by C'_1 along the dual lattice is not relevant as long as it winds the Torus through a non-contractible one, since all possible deformations of the path are already contained in the Loop Gas. Since this dual path cannot be written as a product of vertex operators, the ground state $|\psi_1\rangle$ cannot be written in terms of $|\psi_0\rangle$

Likewise, acting with $X(\gamma_2^*)$ on $|\psi_0\rangle$ we can create another ground state, namely:

$$X(\gamma_2^*) |\psi_0\rangle = |\psi_2\rangle. \quad (2.20)$$

Also by acting with $X(\gamma_1^*)$ and $X(\gamma_2^*)$ simultaneously on $|\psi_0\rangle$ we create another ground state, i.e.:

$$X(\gamma_1^*) X(\gamma_2^*) |\psi_0\rangle = |\psi_{1,2}\rangle. \quad (2.21)$$

Thus, $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_{1,2}\rangle$ are the four ground states of the Toric Code, and their existence is guaranteed as long as the lattice is embedded on a Torus. Note that if we allow the lattice to be embedded on a more general surface with genus g more ground states can be constructed, depending on the number of homotopically inequivalent non-contractible loops that can be defined on such surface. Hence the dependence of the ground state on the topological properties of the surface the model is defined in.

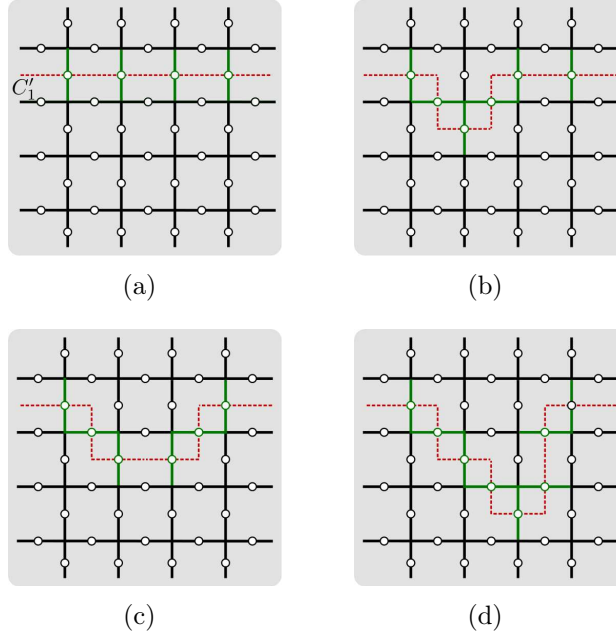


Figure 2.5: The path γ_1^* winds the torus along a non-contractible loop, the operator $X(\gamma_1^*)$ defined on this path transforms the degrees of freedom that lie on γ_1^* , the state $|\psi_1\rangle$ is composed of all such transformations that can be gotten by the action of vertex operators in \mathcal{L} .

2.3 Topological Entanglement Entropy

Entanglement arises amongst the fundamental features of quantum mechanics, it essentially refers to the nature of quantum states in systems that are composed of several subsystems. In particular, a many body quantum state that cannot be decomposed as a product of its subsystems' local states is said to be *entangled*. Naturally, we can ask whether there is a measurement for how much entangled a quantum state is, or equivalently, how much more entangled is quantum state compared to another. The concept of *entanglement entropy* appears as a way to answer this question, and the idea behind how this quantity reveals information about the entanglement of a quantum state is essentially the same as its classical counterpart, the *Shannon entropy*. The relation between these two is better understood in the context of information theory and we refer the reader to Nielsen and Chuang's book [90] for a detailed explanation. Consider, for instance, a source emitting information, it could be some signal detector emitting a bunch of 0's and 1's. Shannon asked: *what is the minimum physical resources needed to store the information emitted by the source?*; Remarkably, the answer to this question is precisely *Shannon entropy*, this is known as *Shannon's noiseless coding theorem* [91].

Similarly, *entanglement entropy* is a measure of how quantum information is stored in a quantum state and to define it we need to recall some basic notions from quantum mechanics. Let ρ be the density operator representing a given quantum state, its *von Neumann entropy* is defined as:

$$S(\rho) := -\text{Tr}(\rho \log \rho). \quad (2.22)$$

One of the practical uses of the above quantity is as a tool for describing the subsystems of a composite quantum system, such as the ones we treat in this work. The description of

subsystems is provided by the *reduced density matrix*. Suppose we have a physical system split in two parts A and B , more precisely, we are considering a quantum system described by the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_{A,B} \subset \mathcal{H}$ are subspaces corresponding to regions A and B . Then, the reduced density matrix of subsystem \mathcal{H}_A is given by:

$$\rho_A = \text{Tr}_B \rho, \quad (2.23)$$

where the operation $\text{Tr}_B : \mathcal{H} \rightarrow \mathcal{H}_A$ is known as *partial trace* over B . The *entanglement entropy* is nothing but the *von Neumann entropy* of the reduced density matrix,

$$S_A = -\text{Tr}(\rho_A \log \rho_A), \quad (2.24)$$

where the trace is now taken over subsystem A . To gain a better understanding about this quantity we will calculate it for two very simple quantum systems.

Example 2.3.1 (2 qubit system). Consider a quantum system composed of two qubits with a bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{B}$ such that each $\mathcal{H}_{A,B} = \text{span}(|0\rangle, |1\rangle)$. Equivalently,

$$\mathcal{H} = \text{span}(|00\rangle, |01\rangle, |10\rangle, |11\rangle).$$

Consider, also, the following pure state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

its density operator is just given by $\rho = |\psi\rangle\langle\psi|$ and the *reduced density matrix* is:

$$\rho_A = \text{Tr}_B \rho = \frac{1}{2}\mathbb{1}_{2 \times 2},$$

where $\mathbb{1}_{2 \times 2}$ is the identity operator. Since the reduced density matrix is proportional to the identity operator, the state is said to be *maximally mixed*, and the initial state $|\psi\rangle$ is *maximally entangled*. The calculation of the entanglement entropy now follows straightforwardly:

$$S_A = -\text{Tr}(\rho_A \log \rho_A), \quad (2.25)$$

$$= -2 \times \frac{1}{2} \log(1/2), \quad (2.26)$$

$$= \log(2). \quad (2.27)$$

usually the log is taken in base 2, so we get: $S_A = 1$. This result illustrates a useful way to understand *entanglement entropy*, since it essentially counts the number of entangled bits between regions A and B . If we had considered regions A and B to be composed of k qubits instead of 1, then the entanglement entropy of a maximally entangled state would have been given by: $S_A = k \log 2$. Moreover, for this maximally entangled state it is clear that S_A grows with the number of constituents in A , in other words,

$$S_A \sim \text{Volume}(A),$$

most states in the Hilbert space \mathcal{H} have this scaling for the entanglement entropy. However, ground states of local Hamiltonians often have entanglement entropies that obey special scaling laws, different from the Volume law. One interesting case of entropy scaling is called *Area Law*, where the entanglement entropy obeys:

$$S_A \sim \partial(A).$$

Remarkably, entanglement entropy arises as an interesting probe for the presence of topological order in many-body quantum states [32, 33, 34], in other words, entanglement entropy is able to detect states with no local order parameters. Essentially, topological order in quantum state is detected by the presence of a correction term in the entanglement entropy. To illustrate this, we outline the calculation of S_A for the *Toric Code* model of §2.2.

Example 2.3.2 (Toric Code). To showcase the usefulness of entanglement entropy as a source of information about the entanglement properties of topologically ordered states, let us outline the calculation of S_A for the ground states of the Toric Code, this quantity was first calculated in [35, 76], where they show how a constant term in the entanglement entropy indicates the presence of topological order in the state, this correction term is known as *topological entanglement entropy*. The calculation we outline here is, in spirit, the same as the one performed in chapter 5 to obtain the main result of this work. Hence, this section is also intended as a warm up for chapter 5.

Consider, then, the Toric Code model of §2.2 defined over a discretization of a Torus which we take to be a $k \times k$ square lattice. In this setting, there are $N_v = k^2$ vertices, $N_l = 2k^2$ links and $N_p = k^2$ plaquettes. The Hilbert space is just

$$\mathcal{H} = \bigotimes_l \mathcal{H}_l,$$

where each $\mathcal{H}_l = \text{span}(|0\rangle, |1\rangle)$. The Hamiltonian of the model is the one in Eq.(2.12) and we do not write it here. The starting point of the entanglement entropy calculation is the *density matrix* operator, ρ , which in this case we take to be proportional to the ground state projector of Definition 2.1.1 more precisely,

$$\rho := \frac{1}{GSD} \Pi_0 = \frac{1}{GSD} \prod_v A_v \prod_p B_p, \quad (2.28)$$

where A_v and B_p are the vertex and plaquette operators of Eqns.(2.10) and (2.11), respectively. Now, we consider a bipartition of the total Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ induced by the geometric partition shown in Fig. 2.6.

The next step is to find the *reduced density matrix* ρ_A , by taking a partial trace over B . To do so, we will introduce a very convenient way of expressing the ground state projector as a sum of group elements. Notice, by the form of A_v and B_p (see §2.2), that expanding the products on vertices and plaquettes in Π_0 results in a sum of operators,

$$\Pi_0 = \prod_v A_v \prod_p B_p = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g,$$

where each $g \in \mathcal{G}$ is an operator $g : \mathcal{H} \rightarrow \mathcal{H}$ of the form:

$$g = g_1 \otimes g_2 \otimes g_3 \otimes \cdots \otimes g_{2k^2},$$

and each local operator $g_l : \mathcal{H}_l \rightarrow \mathcal{H}_l$ is generated by $\{1, \sigma^x, \sigma^z\}$; Recall that σ^x and σ^z are both traceless operator while $\text{Tr}(1) = 2$, this fact will be important later. The order of the group $|\mathcal{G}|$, essentially counts the number of independent elements g , which in the

case of the Toric Code can be calculated very straightforwardly, to do that we need to set the dimensions of the lattice and its bipartition as shown in Fig. 2.6. This is, the lattice (a Torus) is of size $k \times k$ and the A region is of size $R \times R$. Since the model consider degrees of freedom on the links of the lattice, the dimension of the Hilbert space follows immediately,

$$\dim(\mathcal{H}) = 2^{N_l} = 2^{2k^2}.$$

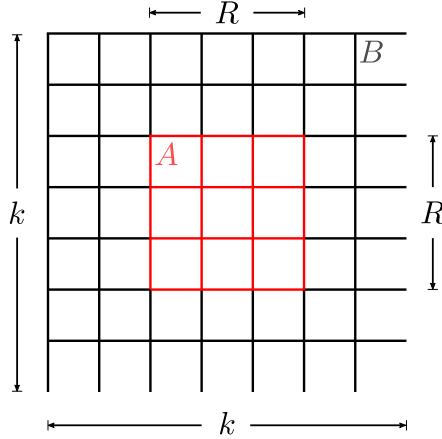


Figure 2.6: The shape and dimensions of the lattice and its bipartition into A and B are shown.

In order to calculate the order of \mathcal{G} we need to count the number of inequivalent elements $g = g_1 \otimes g_2 \otimes g_3 \otimes \cdots \otimes g_{2k^2}$. The operators that make the ground state projector Π_0 are:

$$A_v = \frac{1}{2} (A_v^0 + A_v^1),$$

$$B_p = \frac{1}{2} (B_p^0 + B_p^1),$$

the A_v^0 and B_p^0 operators are nothing but the identity operator and such terms do not generate inequivalent elements $g \in \mathcal{G}$ other than the trivial operator. Non trivial elements in \mathcal{G} are thus obtained from products involving A_v^1 and B_p^1 (see §2.2 for the exact form of these operators), moreover, it is not difficult to note that:

$$\prod_v A_v^1 = \mathbb{1}, \text{ and } \prod_p B_p^1 = \mathbb{1},$$

this is enough to calculate the number of independent elements $g \in \mathcal{G}$, from the above discussion there are 2^{N_v-1} non-equivalent operators coming from products of A_v and 2^{N_p-1} from the products of B_p . Hence, we have:

$$|\mathcal{G}| = 2^{2(k^2-1)}. \quad (2.29)$$

The density matrix is then written as:

$$\rho = \frac{1}{GSD} \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} g. \quad (2.30)$$

Note that the product $GSD |\mathcal{G}| = 2^2 2^{2(k^2-1)} = d \dim(\mathcal{H})$. Any $g \in \mathcal{G}$ can be written as $g = g_A g_B$ where $g_A = g_A \otimes \mathbb{1}_B$ and $g_B = \mathbb{1}_A \otimes g_B$ are operators acting non-trivially only on \mathcal{H}_A and \mathcal{H}_B respectively. The reduced density matrix is obtained by taking the partial trace of ρ , namely $\rho_A = \text{Tr}_B(\rho)$. Write $\{|i\rangle_B\}$, $i = \{1, 2, \dots, |\mathcal{H}_B|\}$ for a basis of \mathcal{H}_B . By writing $g = g_A g_B$ the reduced density matrix is written as:

$$\begin{aligned} \rho_A &= \frac{1}{(\dim \mathcal{H})} \sum_i \sum_g \langle i |_B g_A g_B |i\rangle_B, \\ &= \frac{1}{\dim(\mathcal{H})} \left(\sum_{g_A} g_A \right) \text{Tr}_B(\mathbb{1}_B), \end{aligned}$$

where the last line is obtained by observing that the only terms that *survive* the partial trace are $g_A = g_A \otimes \mathbb{1}_B$ since $\text{Tr}(\sigma^{x,z}) = 0$. The elements $g_A \in \mathcal{G}_A$ are generated by products of $A_v^{0,1}$ and $B_p^{0,1}$ that have support exclusively on \mathcal{H}_A . These operators essentially comprise vertex operators for the internal vertices only, and plaquette operators for all plaquettes in A . This gives us the order of the group \mathcal{G}_A by noting that there are $(R-1)^2$ internal vertices and R^2 plaquettes in region A , this is:

$$|\mathcal{G}_A| = 2^{R^2 + (R-1)^2}.$$

The trace of the identity operator is: $\text{Tr}(\mathbb{1}_B) = \dim(\mathcal{H}_B) = 2^{N_{l,B}}$, where $N_{l,B}$ is the number of links in region B . So, in the end we have for the reduced density matrix:

$$\begin{aligned} \rho_A &= \frac{\dim(\mathcal{H}_B)}{\dim(\mathcal{H})} \sum_{g \in \mathcal{G}_A} g_A, \\ &= \frac{1}{\dim(\mathcal{H}_A)} \sum_{g_A \in \mathcal{G}_A} g_A. \end{aligned}$$

The entanglement entropy $S_A = -\text{Tr}(\rho_A \log \rho_A)$, so we actually need to find $\log \rho_A$ first, this is done using series expansion so let us calculate the square of the reduced density matrix:

$$\begin{aligned} \rho_A^2 &= \left(\frac{1}{\dim(\mathcal{H}_A)} \right)^2 \left(\sum_{g_A \in \mathcal{G}_A} g_A \right) \left(\sum_{h_A \in \mathcal{G}_A} h_A \right) \\ &= \frac{|\mathcal{G}_A|}{\dim(\mathcal{H}_A)} \left(\frac{1}{\dim(\mathcal{H}_A)} \sum_{g_A} g_A \right), \end{aligned}$$

where in the last line we rearranged the sums into one and factored the equivalent elements. Hence, we can write $\rho_A^2 = \lambda \rho_A$. The constant factor λ contains all the essential information for the entanglement entropy, as we will see. It is not hard to realize that $\log \rho_A = \frac{\rho_A}{\lambda} \log(\lambda)$, which in turn gives for the entanglement entropy:

$$\begin{aligned} S_A &= -\text{Tr}(\rho_A \log \rho_A) \\ &= \log(1/\lambda), \end{aligned}$$

where we have used $\text{Tr}(\rho_A) = 1$. So, to find the entanglement entropy of the ground states of the Toric Code we only need to find λ , yielding for the entanglement entropy:

$$\begin{aligned} S_A &= \log \left(\frac{\dim(\mathcal{H}_A)}{|\mathcal{G}_A|} \right) \\ &= \log \left(\frac{2^{2R(R+1)}}{2^{R^2+(R-1)^2}} \right) \\ &= \log(2^{4R-1}). \end{aligned}$$

Taking the logarithm we obtain:

$$S_A = 4R - 1, \tag{2.31}$$

the first term exhibits the area law since $\partial(A) = 4R$ and the constant term (-1) is known as *topological entanglement entropy*.

Chapter 3

Quantum Models from Abelian Higher Gauge Theories

In Chapter 2 we exhibited a known class of models for topological phases, i.e., the Quantum Double Models with finite group G . In this chapter we introduce the first generalizations of the QDMs in the Abelian case, by example. We restrict for Abelian groups only in order to be able to use the mathematical apparatus of Homological Algebra in the next chapter. As a warm up, we start with the Abelian version of the QDMs casted as a 1-gauge theory in 2D, we do this to introduce the relevant concepts that are going to be generalized in what follows. Next, we introduce two natural generalizations of the QDMs in 2D, which we call 0,1-gauge and 1,2-gauge theories for reasons that will be clear in due time. We also exhibit a 3D example and by doing so we intend to show how the generalization is carried over higher dimensions. To end the chapter we write the most general case of an Abelian higher gauge theory, the central object of study in Chapter 4.

All models are described in the usual way, this is, as many-body quantum systems defined on a lattice. The Hilbert space \mathcal{H} is a tensor product of local spaces attached to the elements of a lattice such as vertices, links, plaquettes, etc.. The dynamics of all models is determined by a Hamiltonian operator $H : \mathcal{H} \rightarrow \mathcal{H}$ made of commuting local projectors, as we shall see.

With the intuition gained in this chapter, we dedicate Chapter 4 to showcase the formalism that allows us to treat all models of this class in a unified way for all dimensions, as defined in [3]. This is made possible by employing a few constructions coming from homology theory. At this point, we don't need to be concerned with all the details but we will point out some of the chain complexes that will be of essential importance for the construction of Chapter 4.

3.1 1-Gauge Theory in 2D

Let us start with the simplest example for $d = 2$, namely the Quantum Double Model based on a gauge group G [1, 92, 93]. The model has been defined in detail in §2.1 so we do not exhibit all the operators and relations between them in this section. There is nothing new here, of course, but it will help us fix notation over a familiar model.

We start with an oriented lattice that for simplicity we will think of as a squared lattice, representing the discretization of a surface Σ . Denote by K_0, K_1 and K_2 the set of vertices,

3. Quantum Models from Abelian Higher Gauge Theories

links, and faces of the lattice respectively. Contrary to the other examples -to come- we place quantum degrees of freedom on the links $l \in K_1$ only. In other words, for every link $l \in K_1$ there is a local Hilbert space \mathcal{H}_l generated by basis elements $\{|g\rangle\}, g \in G$. The inner product between any two vectors $|k\rangle, |m\rangle \in \mathcal{H}_l$ given by $\langle k|m\rangle = \delta(k, m)$. We consider all these G -spins to be independent, so the total Hilbert space is spanned by the tensor product of local ones,

$$\mathcal{H} = \bigotimes_{l \in K_1} \mathcal{H}_l. \quad (3.1)$$

The dynamics of the model is determined by the Hamiltonian operator which consists on two classes of projection operators: labeled by a vertex $v \in K_0$, the gauge transformation A_v acts on the links connected to v ; and labeled by a plaquette $p \in K_2$, the holonomy operator B_p acts on the links around plaquette p .

$$H_1 = - \sum_{v \in K_0} A_v - \sum_{p \in K_2} B_p, \quad (3.2)$$

All the operators in the two classes commute between each other (c.f. Appendix ??). The holonomy operator realizes the flatness condition, giving eigenvalue one only when the product of the degrees of freedom at links equals the identity of G , this is,

$$B_p \left| \begin{array}{c} \xrightarrow{c} \\ \uparrow \quad \uparrow \\ \xleftarrow{a} \end{array} \right\rangle = \delta(a + b - c - d, 0) \left| \begin{array}{c} \xrightarrow{c} \\ \uparrow \quad \uparrow \\ \xleftarrow{a} \end{array} \right\rangle. \quad (3.3)$$

On the other hand, the vertex operator performs all possible symmetries of the group; it consists of a normalized sum of the elementary transformations, A_v^g , namely,

$$A_v = \frac{1}{|G|} \sum_{g \in G} A_v^g, \text{ where } A_v^g \left| \begin{array}{c} \uparrow d \\ \xrightarrow{a} \quad \xrightarrow{c} \\ \uparrow b \end{array} \right\rangle = \left| \begin{array}{c} \uparrow gd \\ \xrightarrow{ag^{-1}} \quad \xrightarrow{gc} \\ \uparrow bg^{-1} \end{array} \right\rangle. \quad (3.4)$$

Section 2.1.1 is dedicated to the ground states of this model. There, we show that they are realized by simultaneously minimizing every operator in the Hamiltonian. Its degeneracy is related to the invariant of the discretized initial surface, its fundamental group $\pi_1(\Sigma)$.

We now pay attention on the algebraic data we need to cast this model into the general formalism of Chapter 4: a pair of chain complexes.

The first chain complex comes as a description of the lattice and its boundary relations. In this sense, C_n is the Abelian chain group freely generated by the finite sets K_n and the usual boundary maps between them. For a quick review on chain complexes we refer the reader to Appendix A.2. The composition of subsequent maps is null, $\partial_{n+1} \circ \partial_n = 0$. Hence in 2 dimensions, we have the following diagram,

$$0 \hookrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0. \quad (3.5)$$

The second chain complex describes the *abelian higher gauge* group we are dealing with.

$$0 \hookrightarrow G_2 \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \rightarrow 0. \quad (3.6)$$

where G_k are abelian groups and ∂_k^G are group homomorphisms such that $\partial_{k+1}^G \circ \partial_k^G = 0$. For the case of a 1-gauge theory there is a single gauge group G and this chain complex is

almost trivial, namely $G_2 = 0$, $G_1 = G$ and $G_0 = 0$, where the trivial groups are denoted by zeros:

$$0 \hookrightarrow 0 \rightarrow G \rightarrow 0 \rightarrow 0. \quad (3.7)$$

where we don't need to specify the obvious maps.

At this point, we already exhibited what we consider as being the most important ingredients in a lattice gauge theory, namely, the classical gauge configurations that span \mathcal{H} , the local gauge transformations, implemented by A_v , in charge of quantum entanglement, and the B_p operator that implements the local flatness condition by measuring the holonomy along an elementary plaquette and checking if is equal to the identity element, $e \in G$. We make emphasis on these three concepts as they are the ones to be generalized in order to construct what we call *Abelian Higher Gauge Theory* in Chapter 4. More precisely, a gauge configuration can be thought of as a map that assigns a group element to each edge $l \in K_1$, an elementary gauge transformation is labeled by a group element in G and is localized at (the co-boundary of) a vertex $v \in K_0$, whereas the holonomy measurement is carried along the boundary of an elementary plaquette $p \in K_2$.

3.1.1 Example: Toric Code ($G = \mathbb{Z}_2$)

We already know that the simplest example of a QDM is the so called Toric Code^[1], which we analyze in more detail in §2.2. Let us consider the discretization of a Torus (T^2) and the group $G = \mathbb{Z}_2$, this results on Eq.(3.2) being the Hamiltonian of the Toric Code. As we have seen in §??, the holonomy operator has eigenvalue one whenever there is an even number of red lines along the plaquette p , while the vertex operator is a normalized sum of the original state and one with the opposite configuration.

$$B_p \left| \begin{array}{c} \text{---} \rightarrow \text{---} \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{---} \leftarrow \text{---} \leftarrow \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \right\rangle = \left| \begin{array}{c} \text{---} \rightarrow \text{---} \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \\ \text{---} \leftarrow \text{---} \leftarrow \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \right\rangle, \quad (3.8)$$

$$A_v \left| \begin{array}{c} \rightarrow \quad \rightarrow \quad \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \right\rangle = \frac{1}{2} \left(\left| \begin{array}{c} \rightarrow \quad \rightarrow \quad \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \right\rangle + \left| \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \uparrow \quad \uparrow \quad \uparrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \uparrow \quad \uparrow \quad \uparrow \end{array} \right\rangle \right). \quad (3.9)$$

The ground state (see §2.2) is sometimes called *loop gas*, because it is composed by all possible closed red lines. The lines which are non contractible to a point, define different ground state sectors. Having two non-contractible loops on a torus results on a four-fold degeneracy.

3.2 0,1-Gauge Theory in 2D

Now we are ready to present a first generalization in the sense of a higher gauge theory. The model we construct in this section can be understood as being built on the top of the previous one (§3.1). In addition to the states localized on the links $l \in K_1$ and labeled by G_1 , we also place quantum degrees of freedom at the vertices $v \in K_2$ that are labeled by another Abelian group G_0 . This class of models corresponds to the Abelian version of the models constructed in [87, 94]. It is the lattice version of a scalar field coupled to a gauge field. The interaction between the two kind of degrees of freedom is driven by a group homomorphism $\partial_1^G : G_1 \rightarrow G_0$.

3. Quantum Models from Abelian Higher Gauge Theories

Let K_0, K_1 and K_2 be as in section 3.1. For each vertex $v \in K_0$ we have a Hilbert space H_v with basis $\{|h\rangle\}, h \in G_0$. For each link $l \in K_1$ we have a Hilbert space H_l with basis $\{|g\rangle\}, g \in G_1$. For this reason, we call this kind of model 0,1-gauge theories to indicate that there are quantum states associated with both sets K_0 (vertices) and K_1 (links). Ordinary gauge theories, as in the previous example, have degrees of freedom localized at elements of K_1 and are called 1-gauge theories.

The degrees of freedom are independent hence the total Hilbert space is just the tensor product over all vertices and links of the lattice, namely

$$\mathcal{H} = \bigotimes_{v \in K_0} \mathcal{H}_v \bigotimes_{l \in K_1} \mathcal{H}_l \quad (3.10)$$

The Hamiltonian operator resembles its 1-gauge analog given by Eq.(3.2) in the sense that it is made of mutually commuting projectors, that act locally. The novelty in the present case comes as a new set of operators, B_l , associated with the links $l \in K_1$ of the lattice. It is a diagonal operator similar in nature to B_p . Recall that B_p , one for each plaquette $p \in K_2$, measures the holonomy of the gauge configurations at the boundary of plaquette p . In the language of higher gauge theories, it is said to measure the 1-holonomy. The higher dimensional counterparts are called 2-holonomies, 3-holonomies and so on. What we have here is actually a lower dimensional counterpart. Thus, we say that B_l measures the 0-holonomy of the link l , this will be clear by looking at the precise definition of the operator in what follows.

Hence, the Hamiltonian operator consists on a sum of gauge transformations and operators that measure the two types of holonomy of the theory:

$$H_{0,1} = - \sum_{v \in K_0} A_v - \sum_{l \in K_1} B_l - \sum_{p \in K_2} B_p. \quad (3.11)$$

The 1-holonomy operator is identical to its 1-gauge theory analogue of Eq. (3.3). However, the 0-holonomy operator, B_l , compares the gauge fields of adjacent vertices with the map ∂_1 applied to the degree of freedom sitting in the link between them. More precisely,

$$B_l \left| \begin{array}{c} \textcircled{x} \xrightarrow{g} \textcircled{y} \end{array} \right\rangle = \delta(x - y, \partial(g)) \left| \begin{array}{c} \textcircled{x} \xrightarrow{g} \textcircled{y} \end{array} \right\rangle \quad (3.12)$$

Another difference from the models in §3.1 can be found in the action of the local gauge symmetry which is translated as the definition of the vertex operator, since it acts on the vertex as well as on the links around it,

$$A_v = \frac{1}{|G_1|} \sum_{g \in G_1} A_v^g, \text{ where } A_v^g \left| \begin{array}{c} \uparrow d \\ \textcircled{x} \xrightarrow{a} \textcircled{c} \\ \downarrow b \end{array} \right\rangle = \left| \begin{array}{c} \uparrow gd \\ \textcircled{x} \xrightarrow{ag^{-1}} \textcircled{c} \\ \downarrow bg^{-1} \end{array} \right\rangle. \quad (3.13)$$

With respect to the data needed to cast this model into the formalism of Chapter 4, namely the pair of chain complexes. The first one, describing the 2D lattice, has not changed.

$$0 \hookrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0. \quad (3.14)$$

The difference comes into the chain complex of Abelian groups, the higher gauge group. Since we now have two Abelian groups G_0, G_1 and a group morphism $\partial_1^G : G_1 \rightarrow G_0$.

This fixes the second chain complex to be:

$$0 \hookrightarrow G_2 \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \rightarrow 0, \quad (3.15)$$

where $G_2 = 0$ is the trivial group and ∂_2^G is the obvious map. In other words,

$$0 \hookrightarrow 0 \rightarrow G_1 \xrightarrow{\partial_1^G} G_0 \rightarrow 0, \quad (3.16)$$

The chain complexes (3.25) and (3.16) are what we need to recover this model from the formalism of Chap. 4. The classical gauge configurations in this case are defined by configurations in both vertices $v \in K_0$ and links $l \in K_1$. For this reason they can be thought of as maps $f = \{f_0, f_1\}$, with $f_i : K_i \rightarrow G_i$ for $i = 0, 1$ assigning G_0 -spins to the vertices and G_1 -spins to the links. Essentially, classical gauge configurations can be seen as maps:

$$\begin{array}{ccccccc} 0 & \hookrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \rightarrow 0 \\ & & & & \downarrow f_1 & & \downarrow f_0 \\ 0 & \hookrightarrow & 0 & \longrightarrow & G_1 & \longrightarrow & G_0 \rightarrow 0 \end{array}$$

Figure 3.1: Classical gauge configurations of a 0,1-gauge theory seen as maps $f = \{f_0, f_1\}$.

Similar to the case of a 1-gauge theory (§3.1); elementary gauge transformations are labeled by group elements $g \in G_1$ and located at vertices of the lattice $v \in K_0$. The difference being that in the present case the local symmetry acts on the vertex v itself along with all links around it, we will see how this notion gets codified naturally using the language of Chapter 4. Regarding the local flatness conditions; the 0-holonomy operator B_l favors flat configurations along links and taking into account the G_0 -spins at the vertices, whereas the 1-holonomy operator favors flat configurations along plaquettes just as in the 1-gauge case of §3.1. In the next section (3.3) we will see how the 1-holonomy measurement along plaquettes gets modified by the presence of G_2 -spins at plaquettes of the lattice.

For now, let us exhibit an example of a 0,1-gauge theory in detail. To do this, we choose the two gauge groups G_0 and G_1 together with the homomorphism between them ∂_1^G .

3.2.1 Example: $G_0 = \mathbb{Z}_2$, $G_1 = \mathbb{Z}_4$

We take the (square) lattice as coming from a discretization of a torus (T^2) with a $\mathbb{Z}_2 = \{1, -1\}$ degree of freedom associated with every vertex, and a $\mathbb{Z}_4 = \{1, i, -1, -i\}$ at every link. This is, the local Hilbert spaces are $\mathcal{H}_v = \mathbb{C}[\mathbb{Z}_2]$ and $\mathcal{H}_l = \mathbb{C}[\mathbb{Z}_4]$ at vertices and links respectively. The homomorphism in charge of the interaction between the gauge fields is defined by $\partial : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$, $i \mapsto -1$.

Just as in the Toric Code case (see §2.2) it is convenient to have a graphic representation of basis states: a link holding a $|-1\rangle_l$ state is represented by a red line, likewise we draw an oriented blue line for $|\pm i\rangle_l$ and nothing for $|+1\rangle_l$, as shown in Fig. 3.2. Yet for the vertices, we can draw a dual red surface for the state $|-1\rangle_v$ and nothing for vertices holding $|1\rangle_v$.

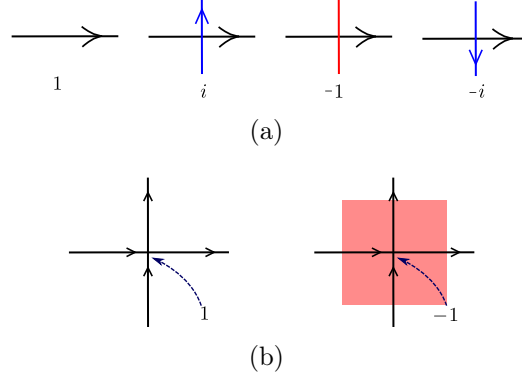


Figure 3.2: (a) The graphical representation of the \mathbb{Z}_4 -spin configurations at links is shown. From the left, we represent: $|1\rangle_l$, $|i\rangle_l$, $|-1\rangle_l$, $|-i\rangle_l$.
 (b) For the vertex \mathbb{Z}_2 -spins we use a red surface or the lack of it to represent $|-1\rangle_v$ and $|1\rangle_v$ respectively.

Operators and Hamiltonian

The Hamiltonian is that of Eq. (3.11) and we describe each kind of operator in detail. Let us begin with the vertex operator in charge of gauge transformations, defined in Eq.(3.13) it is given by:

$$A_v = \frac{1}{4} (A_v^1 + A_v^i + A_v^{-1} + A_v^{-i}), \quad (3.17)$$

and each term is written in terms of known matrices:

$$\begin{aligned} A_v^1 &= \mathbb{1}_v \otimes \mathbb{1}_{l_1} \otimes \mathbb{1}_{l_2} \otimes \mathbb{1}_{l_3} \otimes \mathbb{1}_{l_4}, & A_v^i &= \sigma_v^x \otimes X_{l_1}^3 \otimes X_{l_2}^3 \otimes X_{l_3} \otimes X_{l_4}, \\ A_v^{-1} &= \mathbb{1}_v \otimes X_{l_1}^2 \otimes X_{l_2}^2 \otimes X_{l_3}^2 \otimes X_{l_4}^2, & A_v^{-i} &= \sigma_v^x \otimes X_{l_1} \otimes X_{l_2} \otimes X_{l_3}^3 \otimes X_{l_4}^3, \end{aligned}$$

where σ^x is the usual Pauli shift operator and:

$$X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (3.18)$$

is the \mathbb{Z}_4 shift operator.

Let us now write the two holonomy measurement operators. The 0-holonomy operator for this example is given by:

$$B_l = \frac{1}{2} (\mathbb{1}_{v_1} \otimes \mathbb{1}_l \otimes \mathbb{1}_{v_2} + \sigma_{v_1}^z \otimes Z_l^2 \otimes \sigma_{v_2}^z),$$

where $v_1, v_2 \in K_0$ are the two vertices connected by the link $l \in K_1$; σ^z is the usual clock Pauli operator and:

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

is the \mathbb{Z}_4 clock operator. While the 1-holonomy of the theory is measured by the plaquette operators, B_p , which in this case is written as:

$$B_p = \frac{1}{4}(\mathbb{1}_{l_1} \otimes \mathbb{1}_{l_2} \otimes \mathbb{1}_{l_3} \otimes \mathbb{1}_{l_4} + Z_{l_1} \otimes Z_{l_2} \otimes Z_{l_3}^3 \otimes Z_{l_4}^3 + \\ + Z_{l_1}^2 \otimes Z_{l_2}^2 \otimes Z_{l_3}^2 \otimes Z_{l_4}^2 + Z_{l_1}^3 \otimes Z_{l_2}^3 \otimes Z_{l_3} \otimes Z_{l_4}),$$

where the links $l_1, l_2, l_3, l_4 \in \partial p$. Using the above expressions it is straightforward to check the commutativity of all operators in the Hamiltonian.

Ground States

Of particular interest are the ground states of this model, in particular we want to find out whether they carry some topological information or not. Resembling the analysis we did in §2.2, it is clear that the ground subspace, \mathcal{H}_0 , consists on all states satisfying the gauge equivalence and flatness conditions simultaneously, namely

$$\mathcal{H}_0 := \{|\psi\rangle \in \mathcal{H} \mid A_v |\psi\rangle = |\psi\rangle, B_l |\psi\rangle = |\psi\rangle \text{ and } B_p |\psi\rangle = |\psi\rangle = |\psi\rangle\},$$

for all $v \in K_0$, $l \in K_1$ and $p \in K_2$. To understand the nature of the ground states of this model, let us look at the ground state projector, very similar to the one of Definition 2.1.1 consists on the product of all operators in the Hamiltonian, this is,

$$\Pi_0 := \prod_{v \in K_0} A_v \prod_{l \in K_1} B_l \prod_{p \in K_2} B_p. \quad (3.19)$$

The action of this operator on an arbitrary state $|\psi\rangle \in \mathcal{H}$ consists, essentially on two parts:

1. Resembling the Toric Code case (see §2.2), the product of plaquette operators, $\prod B_p$, projects to states with trivial 1-holonomy, this is, satisfying $B_p |\psi\rangle = |\psi\rangle$. It is not difficult to note that such states will consist on configuration of loops, being them red or (oriented) blue. Thus, it projects into states in \mathcal{H} that can be pictured as a loop gas where the loops can be red or blue (oriented).

Likewise, the product of link operators, $\prod B_l$, projects into states with trivial 0-holonomy, something that was not there in the Toric Code. A little bit of inspection reveals that the trivial 0-holonomy configurations are such as the ones in Fig. 3.3. So, combining these two projections, we see that the configurations that satisfy both 0- and 1-flatness consist on closed red loops and blue (oriented) loops that act as domain walls for the red surfaces. Such configurations have trivial 0- and 1-holonomies but they are still not ground states of the Hamiltonian.

2. The first two products on the ground state projector Π_0 project into states in \mathcal{H} with trivial holonomies. To construct a ground state, we still have to apply the product over all vertex operators $\prod_v A_v$ which projects the state $|\psi\rangle$ into a superposition of states gauge equivalent to $|\psi\rangle$.

From the above discussion, the ground states are gauge equivalence classes of states with trivial 0- and 1-holonomy. We show one component of such a state in Fig. 3.4.

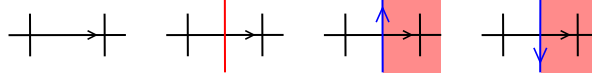


Figure 3.3: Typical configurations with trivial 0-holonomy are exhibited. Observe that the presence of a red line, $(|-1\rangle_l)$ has no effect on the vertex degrees of freedom; Yet a blue line, forces one of the vertex spin to be flipped, represented by the red surface.

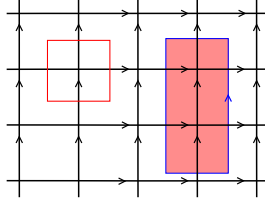


Figure 3.4: A component of a ground state showing the allowed configurations of both G_0 and G_1 spins in the 0,1-gauge model with groups $G_0 = \mathbb{Z}_2$, $G_1 = \mathbb{Z}_4$ of 3.2.1. Such configurations include red loops and oriented blue loops as domain walls for red surfaces.

This model has the same degeneracy of the Toric Code [3]. To see this, notice that the state with neither red loops nor blue domain walls has trivial 0- and 1-holonomy. Thus, its gauge equivalence class constitutes a, rather obvious, ground state. Which we call the *trivial state* and write as:

$$|\psi_0\rangle = \prod_{v \in K_0} A_v \bigotimes_{v \in K_0} |1\rangle_v \bigotimes_{l \in K_1} |1\rangle_l$$

In the torus, there are three additional ground states that correspond to red loops winding the torus along its non-contractible paths. They can be written in terms of the shift operator X^2 of \mathbb{Z}_4 and the trivial state $|\psi_0\rangle$, where:

$$X^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since its action on a the trivial link state: $X^2 |1\rangle_l = |-1\rangle_l$, or equivalently, to create a red curve crossing the link. For convinience let us define the following operator, for any (dual) path γ :

$$X^2(\gamma) = \bigotimes_{l \in \gamma} X_l^2,$$

that consists on a composition of X^2 shift operators acting on all links crossed by the path γ . Considering, then, the two non-contractible (dual) paths γ_1 and γ_2 shown in Fig. 2.4 the four ground states of the model can be written as:

$$|\psi_{i,j}\rangle = (X^2(\gamma_1))^i (X^2(\gamma_2))^j |\psi_0\rangle,$$

for $i, j = \{0, 1\}$. Leaving for the ground state degeneracy:

$$GSD = 4. \tag{3.20}$$

This result will be assured in the next chapter where we systematically calculate this number for all Abelian higher gauge theory.

3.3 1,2-Gauge Theory in 2D

The model we exhibit in this section can be thought of as yet another generalization of an ordinary gauge theory. To begin with, we have states associated with the plaquettes that are labeled by an Abelian group G_2 in addition to the states localized on the links and labeled by G_1 . This class of models corresponds to the Abelian version of the models constructed in [52], where 2-groups were considered. We are not going to discuss all the algebraic details regarding 2-groups and rather refer to [52, 53, 67] for details. Instead, we are working with a particular case of Abelian 2-groups where the only data we need is expressed by the group homomorphism $\partial_2^G : G_2 \rightarrow G_1$.

Just as in the two previous examples (§3.1 and §3.2) we consider a 2D lattice that comes from a discretization of a manifold, where K_0, K_1 and K_2 stand for the set of vertices, links and plaquettes respectively. To each link $l \in K_1$ we associate a Hilbert space \mathcal{H}_l with basis $\{|g\rangle, g \in G_1\}$. Likewise, for each plaquette $p \in K_2$ we have a Hilbert space \mathcal{H}_p with basis $\{|\alpha\rangle, \alpha \in G_2\}$. We call this kind of model 1,2-gauge theories to indicate that there are quantum states associated with both sets K_1 (links) and K_2 (plaquettes). This makes the total Hilbert space the tensor product over all the local Hilbert spaces.

$$\mathcal{H} = \bigotimes_{l \in K_1} \mathcal{H}_l \bigotimes_{p \in K_2} \mathcal{H}_p \quad (3.21)$$

The Hamiltonian operator is composed by three types of mutually commuting operators, namely, the 1-gauge transformation, A_v located at vertices and involving the G_1 symmetries for the links; the 2-gauge transformation, A_l , is located at links and implements the G_2 symmetry of the theory, finally we have the 1-holonomy operator B_p that is nothing but a slight modification of its 1-gauge analogue of Eq.(3.3).

$$H_{1,2} = - \sum_{v \in K_0} A_v - \sum_{l \in K_1} A_l - \sum_{p \in K_2} B_p \quad (3.22)$$

The vertex operator A_v is exactly the same as in a pure 1-gauge theory defined in Eq. (3.4), with the substitution $G = G_1$, so we do not write it here. The new class of gauge transformations, A_l , consists on the averaged sum of more elementary transformations applied over the plaquettes adjacent to the link in question. The elementary gauge transformation, A_l^β , acts on the adjacent plaquettes by $\beta \in G_2$ and on the link by $\partial_2^G(\beta)$, so $g' = g\partial_2^G(\beta)$, more precisely,

$$A_l = \frac{1}{|G_2|} \sum_{\beta \in G_2} A_l^\beta, \quad A_l^\beta \left| \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \alpha_1 \uparrow g \alpha_2 \\ \xrightarrow{\quad} \xrightarrow{\quad} \end{array} \right\rangle = \left| \begin{array}{c} \xrightarrow{\quad} \xrightarrow{\quad} \\ \alpha_1 \beta \uparrow g' \beta^{-1} \alpha_2 \\ \xrightarrow{\quad} \xrightarrow{\quad} \end{array} \right\rangle. \quad (3.23)$$

The 1-holonomy of the theory is measured at plaquettes $p \in K_2$, as in any other gauge theory we treat in this work. However, the inclusion of the G_2 degrees of freedom at plaquettes modifies the notion of this 1-holonomy. As a result, the 1-holonomy in a 1,2-gauge theory is measured by comparing the product of the link degrees of freedom along the plaquette with the result of applying the ∂_2^G map over the degree of freedom at the plaquette itself, this is,

$$B_p \left| \begin{array}{c} \xrightarrow{c} \xrightarrow{\quad} \\ d \uparrow \alpha \uparrow b \\ \xrightarrow{\quad} \xrightarrow{a} \end{array} \right\rangle = \delta(a + b - c - d, \partial_2^G(\alpha)) \left| \begin{array}{c} \xrightarrow{c} \xrightarrow{\quad} \\ d \uparrow \alpha \uparrow b \\ \xrightarrow{\quad} \xrightarrow{a} \end{array} \right\rangle. \quad (3.24)$$

3. Quantum Models from Abelian Higher Gauge Theories

For simplicity of notation, we will denote $\partial_2^G(\alpha)$ simple by $\partial\alpha$ whenever there is no danger of ambiguity.

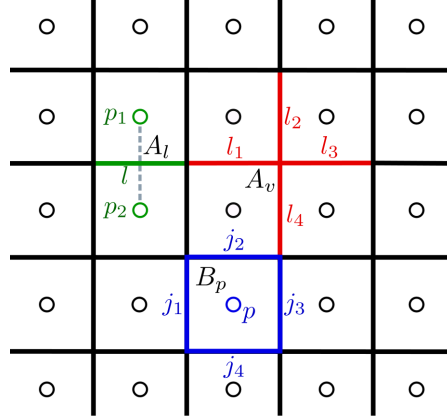


Figure 3.5: The non trivial support of the vertex, link and plaquette operators is shown.

To summarize the contents of this theory let us say a few words about the three concepts that give rise to the gauge theory. In a 1,2-gauge theory, the classical configurations (that label basis states of \mathcal{H}) are nothing but assignments of G_1 and G_2 labels to the links $l \in K_1$ and $p \in K_2$ respectively. We can be a bit more precise by considering the data needed to cast this model into the formalism of Chapter 4. The first chain complex, the geometric one, is common to all 2D examples and given as:

$$0 \hookrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0. \quad (3.25)$$

The algebraic content that plays the role of the higher gauge group is also a chain complex of the form

$$0 \hookrightarrow G_2 \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \rightarrow 0, \quad (3.26)$$

where $G_0 = 0$ and ∂_1^G the trivial map. In other words,

$$0 \hookrightarrow G_2 \xrightarrow{\partial_2^G} G_1 \rightarrow 0 \rightarrow 0. \quad (3.27)$$

In this terms, a classical gauge configuration is seen as being determined by a collection of maps $f = \{f_1, f_2\}$, where $f_i : K_i \rightarrow G_i$, $i = 0, 1$, this is,

$$\begin{array}{ccccccc} 0 & \hookrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \rightarrow 0 \\ & & \downarrow f_2 & & \downarrow f_1 & & \\ 0 & \hookrightarrow & G_2 & \longrightarrow & G_1 & \longrightarrow & 0 \rightarrow 0 \end{array}$$

Figure 3.6: The classical configurations of a 1,2-gauge theory are determined by such maps.

Regarding gauge transformations, this model presents two different kinds of them the first being the usual 1-gauge transformation of a pure 1-gauge theory. We would like to point out that the presence of G_2 degrees of freedom does not alter the nature of how the 1-gauge symmetry is implemented in the theory by the vertex operator A_v , leaving this operator unchanged from that of a pure 1-gauge theory. However, the G_2 -spins

at plaquettes make a 1,2-gauge theory to have a second type of gauge transformations, which we call 2-gauge transformations since they enhance symmetry operations labeled by the second group G_2 . We will see in Chapter 4 how all these different notions of gauge transformations are treated on equal footing and in a compact way using some elements of Homological Algebra.

Finally, the concept of holonomy is essential for a gauge theory. Different from the previous case (0,1-gauge) of §3.2, an Abelian 1,2-gauge theory in 2 dimensions has one type of holonomy only, namely the 1-holonomy with a subtle modification. Since the 1-holonomy of a plaquette $p \in K_2$ also considers the degree of freedom sitting at the plaquette itself.

In what follows we exhibit the main features of a 2 dimensional 1,2-gauge theory by an example. We will see how the ground state of such example contains topological information different from what we have seen for both the pure 1-gauge theory and the 1,2-gauge theory.

3.3.1 Example: $G_1 = \mathbb{Z}_2$, $G_2 = \mathbb{Z}_4$

In this example, we take the lattice as coming from a discretized sphere S^2 . The groups that label degrees of freedom are chosen to be $G_1 = \mathbb{Z}_2 = \{\pm 1\}$ and $G_2 = \mathbb{Z}_4 = \{\pm 1, \pm i\}$. Moreover, the homomorphism that relates both gauge fields gets completely defined by $\partial(i) := -1$. As usual, we use a graphical description for the basis states by assigning a transverse red line for every link holding a $|-1\rangle_l$ state; Similarly, for the plaquette degrees of freedom, we can assign a (\pm) blue dot to any plaquette that holds a $|\pm i\rangle_p$ state, and a red dot whenever it holds a $|-1\rangle_p$.

The Hamiltonian is that of Eq. (3.11), of course. The operators that make the Hamiltonian can be written in terms of shift and clock matrices, as follows. The 1-gauge symmetry implemented by the vertex operator is:

$$A_v = \frac{1}{2} (\mathbb{1}_{l_1} \otimes \mathbb{1}_{l_2} \otimes \mathbb{1}_{l_3} \otimes \mathbb{1}_{l_4} + \sigma_{l_1}^x \otimes \sigma_{l_2}^x \otimes \sigma_{l_3}^x \otimes \sigma_{l_4}^x),$$

where σ^x is the shift Pauli matrix and X the shift operator of \mathbb{Z}_4 as in Eq. (3.18). Note that the above expression is identical to the one in Eq.(3.9). The 2-gauge transformation defined in Eq. (3.23) is localized at links $l \in K_1$ performing the transformation at the link and the two adjacent plaquettes, which for this particular example can be written as:

$$A_l = \frac{1}{4} (\mathbb{1}_{p_1} \otimes \mathbb{1}_l \otimes \mathbb{1}_{p_2} + X_{p_1} \otimes \sigma_l^x \otimes X_{p_2}^3 + X_{p_1}^2 \otimes \mathbb{1}_l \otimes X_{p_2}^2 + X_{p_1}^3 \otimes \sigma_l^x \otimes X_{p_2}),$$

where $p_1, p_2 \in K_2$ are the two plaquettes adjacent to link $l \in K_1$. We now turn into the only holonomy operator of the theory, the 1-holonomy is measured at plaquettes $p \in K_2$, it takes values in G_1 and is written as:

$$B_p = \frac{1}{2} (\mathbb{1}_p \otimes \mathbb{1}_{l_1} \otimes \mathbb{1}_{l_2} \otimes \mathbb{1}_{l_3} \otimes \mathbb{1}_{l_4} + Z_p^2 \otimes \sigma_{l_1}^z \otimes \sigma_{l_2}^z \otimes \sigma_{l_3}^z \otimes \sigma_{l_4}^z),$$

where the links $l_1, l_2, l_3, l_4 \in \partial p \subset K_1$. Again, the commutation relations of these operators can be readily checked using the above expressions.

Ground States

It should be clear by analogy with the previous examples, that the ground state subspace of the model is defined by states that are invariant under the action of each one of the operators that constitute the Hamiltonian, in other words,

$$\mathcal{H}_0 := \{|\psi\rangle \in \mathcal{H} \mid A_v |\psi\rangle = |\psi\rangle, A_l |\psi\rangle = |\psi\rangle, B_p |\psi\rangle = |\psi\rangle\},$$

for all $v \in K_0$, $l \in K_1$ and $p \in K_2$. These states can be obtained by means of the ground state projector, $\Pi_0 : \mathcal{H} \rightarrow \mathcal{H}_0$, which in this case is given by:

$$\Pi_0 := \prod_{v \in K_0} A_v \prod_{l \in K_1} A_l \prod_{p \in K_2} B_p. \quad (3.28)$$

The action of Π_0 on an arbitrary state $|\psi\rangle \in \mathcal{H}$ is best understood in two parts. The products in Eq. (3.28) that involve 1-holonomy operators B_l projects into what we call *flat states*, states invariant under the action of plaquette operators. This flatness condition favors configurations having closed red loops or red lines with blue dot endings and plaquettes with red dots as depicted in Fig. 3.7. So, applying all inequivalent gauge transformations over a flat state is that we obtain the actual ground state of the model. This gets done by the products in Π_0 that involve both gauge transformations A_v and A_l . In other words, a state $|GS\rangle \in \mathcal{H}$ is a ground state of the model if and only if, $\Pi_0 |GS\rangle = |GS\rangle$.

In this particular case, the ground state is two fold degenerate; Both vacua consist on gauge invariant states whose only difference is the basis state that is chosen as a *seed*. One one hand, we have a seed state that consists on $|1\rangle_p$ at every plaquette and $|1\rangle_l$ at every link, the trivial state. The other seed state has a single (arbitrary) plaquette degree of freedom in the $|-1\rangle_p$ state, more precisely,

$$|GS_1\rangle = \prod_v A_v \prod_l A_l \bigotimes_{l \in K_1} |1\rangle_l \bigotimes_{p \in K_2} |1\rangle_p, \quad (3.29)$$

$$|GS_{-1}\rangle = \prod_v A_v \prod_l A_l \bigotimes_{l \in K_1} |1\rangle_l \bigotimes_{p \neq p'} |1\rangle_p \otimes |-1\rangle_{p'}. \quad (3.30)$$

Leaving for the ground state degeneracy:

$$GSD = 2.$$

The two ground states are identified from each other by measuring the global 2-holonomy. In this case, the operator that performs such measurement is given by:

$$h_2 := \prod_p Z_p, \quad \forall p \in \mathcal{L},$$

clearly,

$$\begin{aligned} h_2 |G_1\rangle &= |G_1\rangle, \\ h_2 |G_2\rangle &= -|G_2\rangle, \end{aligned}$$

Notice that the ground states can be stabilized (protected) by the bulk when the model is defined on $S^2 \times I$, where I is the interval, since in this case no local perturbation can lift the degeneracy.

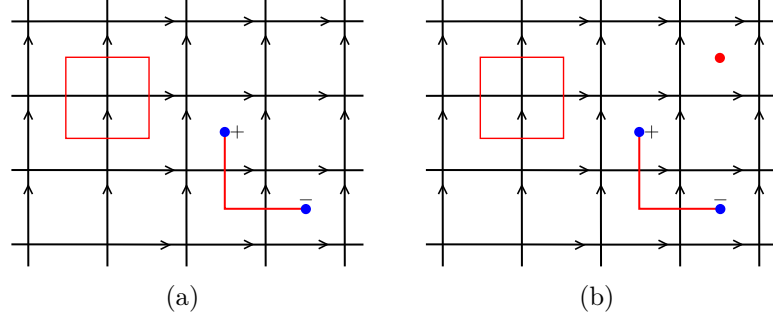


Figure 3.7: (a) A representative configuration of the ground state, $|GS_1\rangle$, for the 1,2-gauge model with groups $G_1 = \mathbb{Z}_2$, $G_2 = \mathbb{Z}_4$, as in example 3.3.1. (b) A representative of $|GS_{-1}\rangle$

3.4 1,2,3-Gauge Theory in 3D

Our final example will be a 3D model, which we hope will help understanding how the generalization works by going to higher dimensions. Let K_0, K_1, K_2, K_3 be the sets of vertices, links, plaquettes, and cubes of a cubic lattice. A 1, 2, 3-gauge configuration consists of assigning G_1, G_2, G_3 -spins respectively to the links, plaquettes, and cubes of the lattice. In other words, we have local Hilbert spaces $\{\mathcal{H}_l, l \in K_1\}$, $\{\mathcal{H}_p, p \in K_2\}$ and $\{\mathcal{H}_c, c \in K_3\}$ with basis $\{|g\rangle, g \in G_1\}$, $\{|\alpha\rangle, \alpha \in G_2\}$ and $\{|\xi\rangle, \xi \in G_3\}$, respectively.

Again all the degrees of freedom are independent, so we consider the tensor product of the local Hilbert spaces as the total Hilbert space.

$$\mathcal{H} := \bigotimes_{l \in K_1} \mathcal{H}_l \bigotimes_{p \in K_2} \mathcal{H}_p \bigotimes_{c \in K_3} \mathcal{H}_c. \quad (3.31)$$

The Hamiltonian is composed by a sum of commuting projectors as follows

$$H_{1,2,3} = - \sum_{v \in K_0} A_v - \sum_{l \in K_1} A_l - \sum_{p \in K_2} A_p - \sum_{p \in K_2} B_p - \sum_{c \in K_3} B_c. \quad (3.32)$$

The definition of most of the operators in $H_{1,2,3}$ is analogue to the previous examples. Let us discuss them one by one.

The 1-gauge transformation is performed by A_v , labeled by vertices $v \in K_0$ it is defined as:

$$A_v = \frac{1}{|G_1|} \sum_{g \in G_1} A_v^g, \text{ where: } A_v^g \left| \begin{array}{c} \epsilon \\ a \end{array} \begin{array}{c} d \\ c \\ b \\ f \end{array} \right\rangle = \left| \begin{array}{c} \epsilon g^{-1} \\ ag^{-1} \end{array} \begin{array}{c} gd \\ gc \\ bg^{-1} \\ gf \end{array} \right\rangle, \quad (3.33)$$

this operator establishes the local 1-gauge symmetry of the theory and acts on 1-gauge degrees of freedom only.

On the other hand, the 2-gauge transformation, implemented by A_l is a resemblant of the one in Eq.(3.23), it acts on the link $l \in K_1$ and all plaquettes $p \in K_2$ adjacent to the link. Its action is an averaged sum of more elementary 2-gauge transformations labeled by elements in G_2 , namely

$$A_l = \frac{1}{|G_2|} \sum_{\beta \in G_2} A_l^\beta, \quad (3.34)$$

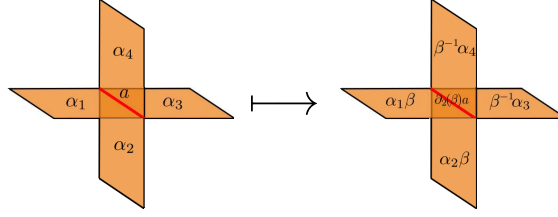


Figure 3.8: The action of A_l on an arbitrary basis state is shown, involving the link in question $l \in K_1$ and the four adjacent plaquettes.

to see this action, let $|a, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots\rangle$ be an arbitrary basis state in \mathcal{H} , whose configuration is depicted in Fig.3.8(a); Then, the action of the elementary 2-gauge transformation A_l^β on this state is given by:

$$A_l^\beta |a, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots\rangle = |\partial_2(\beta)a, \alpha_1\beta, \alpha_2\beta, \beta^{-1}\alpha_3, \beta^{-1}\alpha_4, \dots\rangle,$$

which is depicted in Fig.3.8(b).

Analogously, the local 3-gauge transformation A_p^ζ , with $\zeta \in G_3$ transforms by ζ the degrees of freedom at cubes next to p and acts at p by $\partial_3^G \zeta$, this is,

$$A_p = \frac{1}{|G_3|} \sum_{\zeta \in G_3} A_p^\zeta, \text{ where: } A_p^\zeta \left| \begin{array}{c} \text{cube with faces } \epsilon_1, \epsilon_2, \alpha \end{array} \right\rangle = \left| \begin{array}{c} \text{cube with faces } \zeta\epsilon_1, \zeta\epsilon_2, \alpha\zeta \end{array} \right\rangle. \quad (3.35)$$

Regarding the two operators that enhance the flatness conditions of the theory, the 1-holonomy operator B_p is equivalent to its 1,2-gauge analogue of Eq. (3.24), while the 2-holonomy operator collects the values at faces of the cube with the result of applying ∂_3^G to the degree of freedom at the cube, in other words,

$$B_c \left| \begin{array}{c} \text{cube with faces } \alpha_1, \alpha_2, \alpha_3 \end{array} \right\rangle = \delta\left(\sum_j (-1)^{o_j} \alpha_j, \partial_3^G \xi\right) \left| \begin{array}{c} \text{cube with faces } \alpha_1, \alpha_2, \alpha_3 \end{array} \right\rangle, \quad (3.36)$$

where $o_j = \{0, 1\}$ takes care of the orientation dependent sum, giving a negative sign to the group element that is sitting in a face with orientation opposite to the cube orientation. To encode the essential information about the gauge theory we, once again, use a pair of chain complexes. The Geometrical chain complex is, essentially, an extended version of the chain complex of diagram (3.5), with the usual boundary maps.

$$0 \hookrightarrow C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0. \quad (3.37)$$

The higher gauge symmetry is encoded in the second chain complex, where we now have three Abelian groups G_1 , G_2 and G_3 and two group homomorphisms, $\partial_3^G : G_3 \rightarrow G_2$, $\partial_2^G : G_2 \rightarrow G_1$, this is

$$0 \hookrightarrow G_3 \xrightarrow{\partial_3^G} G_2 \xrightarrow{\partial_2^G} G_1 \rightarrow 0. \quad (3.38)$$

Extending the interpretation we made in the previous sections, 1,2,3-gauge configurations are determined by collections of maps $f = \{f_1, f_2, f_3\}$, where $f_i : C_i \rightarrow G_i$, for $i = \{1, 2, 3\}$ as pictured in Fig. 3.9. We now showcase some details of a 1,2,3-gauge theory by looking at an specific example

$$\begin{array}{ccccccc}
 0 & \hookrightarrow & C_3 & \longrightarrow & C_2 & \longrightarrow & C_1 \longrightarrow C_0 \rightarrow 0 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 \\
 0 & \hookrightarrow & G_3 & \longrightarrow & G_2 & \longrightarrow & G_1 \rightarrow 0
 \end{array}$$

Figure 3.9: The classical configurations of a 1,2,3-gauge theory are determined by such maps.

3.4.1 Example: $G_1 = G_2 = G_3 = \mathbb{Z}_4$

Let us consider the lattice as coming from a discretization of a solid ball, S^3 , with groups $G_1 = G_2 = G_3 = \mathbb{Z}_4 = \{\pm 1, \pm i\}$, while the homomorphisms between them are defined by: $\partial'_3(i) = \partial'_2(i) = -1$.

To understand the dynamics of the model we can adopt a convenient graphic representation of states, depicted in Fig. 3.11 and consisting of:

- a (\pm) blue dot at a cube whenever it holds a $|\pm i\rangle_c$ state, and a red dot if the volume holds a $|-1\rangle_c$,
- a red line through a plaquette when it holds a $|-1\rangle_p$ state, and an oriented blue line for the $|\pm i\rangle_p$ states,
- for the links we picture a red surface orthogonal to it for $|-1\rangle_l$ state, and oriented blue surfaces for $|\pm i\rangle_l$ states.

The Hamiltonian is the one of Eq. (3.32) and we write each operator using the shift and clock matrices of \mathbb{Z}_4 . The 1-gauge transformation is the usual one and it is given by:

$$\begin{aligned}
 A_v = \frac{1}{6} & (\mathbb{1}_{l_1} \otimes \mathbb{1}_{l_2} \otimes \mathbb{1}_{l_3} \otimes \mathbb{1}_{l_4} \otimes \mathbb{1}_{l_5} \otimes \mathbb{1}_{l_6} + X_{l_1} \otimes X_{l_2} \otimes X_{l_3} \otimes X_{l_4}^3 \otimes X_{l_5}^3 \otimes X_{l_6}^3 + \\
 & + X_{l_1}^2 \otimes X_{l_2}^2 \otimes X_{l_3}^2 \otimes X_{l_4}^2 \otimes X_{l_5}^2 \otimes X_{l_6}^2 + X_{l_1}^3 \otimes X_{l_2}^3 \otimes X_{l_3}^3 \otimes X_{l_4} \otimes X_{l_5} \otimes X_{l_6}),
 \end{aligned}$$

where $\{l_i\}_{i=1}^6$ are labels for the 6 links that meet at vertex v . The 2-gauge transformations are performed by the A_l operators labeled by the links of the lattice since their non trivial support consists on the link l and the four faces that meet at l (Fig.(3.10)) in the following way:

$$\begin{aligned}
 A_l = \frac{1}{4} & (\mathbb{1}_l \otimes \mathbb{1}_{p_1} \otimes \mathbb{1}_{p_2} \otimes \mathbb{1}_{p_3} \otimes \mathbb{1}_{p_4} + X_l^2 \otimes X_{p_1} \otimes X_{p_2} \otimes X_{p_3}^3 \otimes X_{p_4}^3 + \\
 & + \mathbb{1}_l \otimes X_{p_1}^2 \otimes X_{p_2}^2 \otimes X_{p_3}^2 \otimes X_{p_4}^2 + X_l^2 \otimes X_{p_1}^3 \otimes X_{p_2}^3 \otimes X_{p_3} \otimes X_{p_4})
 \end{aligned}$$

Finally, the 3-gauge transformations are performed by the A_p operators, they are labeled by the plaquettes of the lattice as their non trivial action support consists on a face p and the two volume degrees of freedom adjacent to p (see fig.(3.10)), i.e., the operator is given by:

$$A_p = \frac{1}{4} (\mathbb{1}_p \otimes \mathbb{1}_{c_1} \otimes \mathbb{1}_{c_2} + X_{c_1} \otimes X_p^2 \otimes X_{c_2}^3 + X_{c_1}^2 \otimes \mathbb{1}_p \otimes X_{c_2}^2 + X_{c_1}^3 \otimes X_p^2 \otimes X_{c_2})$$

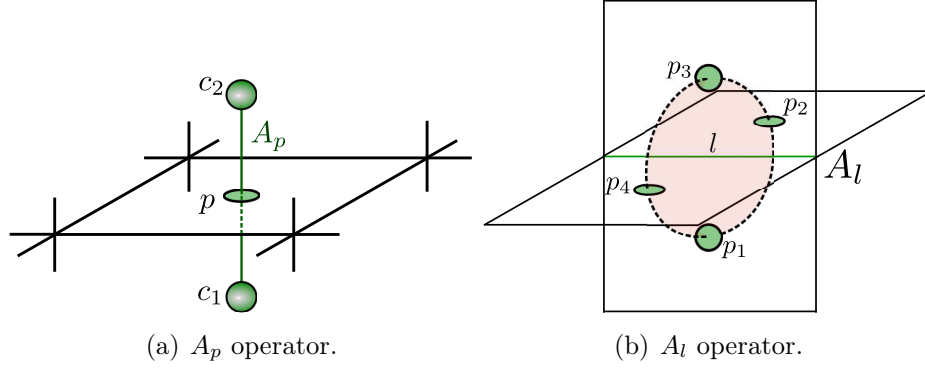


Figure 3.10: The non trivial support of the 3-gauge transformation operator A_p and the 2-gauge transformation operator A_l are shown. We represent the volume degrees of freedom as dual vertices.

Ground States

As usual for the gauge theories we study in this work, the ground states subspace is composed by all states $|\psi\rangle \in \mathcal{H}$ satisfying $\Pi_0 |\psi\rangle = |\psi\rangle$, with Π_0 the ground state projector given by:

$$\Pi_0 := \prod_{v \in K_0} A_v \prod_{l \in K_1} A_l \prod_{p \in K_2} A_p \prod_{p \in K_2} B_p \prod_{c \in K_3} B_c.$$

The flat configurations of the theory (seed states) are those that are invariant under B_p and B_c for all plaquettes $p \in K_2$ and cubes $c \in K_3$ of the lattice. Such configurations consist, in general, of blue loops, red loops or open red lines ending at oriented blue points, conditions that are enforced by B_c . The 1-holonomy operator, B_p , implies that every blue loop encloses a red surface.

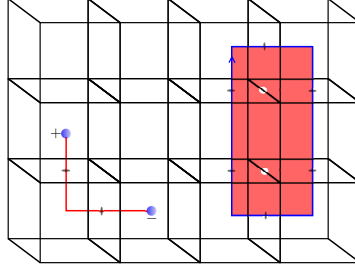


Figure 3.11: Example of a flat configuration for the 1,2,3-gauge model with groups $G_1 = G_2 = G_3 = \mathbb{Z}_4$, in 3.4.1.

Note that a configuration with an arbitrary cube holding $|-1\rangle_c$ satisfies all flatness conditions, this makes the ground state subspace two fold degenerate; one ground state comes from summing over all states gauge equivalent to the trivial state, for which every degree of freedom is at the identity element $1 \in G_n$. The other ground state considers a state with one single arbitrary cube holding a $|-1\rangle_c$, more precisely

$$|G_1\rangle = \prod_v A_v \prod_l A_l \prod_p A_p \bigotimes_l |1\rangle_l \bigotimes_p |1\rangle_p \bigotimes_c |1\rangle_c, \quad (3.39)$$

$$|G_2\rangle = \prod_v A_v \prod_l A_l \prod_p A_p \bigotimes_l |1\rangle_l \bigotimes_p |1\rangle_p \bigotimes_{c \neq c'} |1\rangle_c \otimes |-1\rangle_{c'}. \quad (3.40)$$

Clearly the seed states are not gauge equivalent since there is no 3-gauge transformation that can map between the two seed states. In this case the ground states are distinguished from each other by the global 3-holonomy operator defined as:

$$h_3 := \prod_c Z_c, \tag{3.41}$$

it can be checked by a straightforward computation that:

$$\begin{aligned} h_3 |G_1\rangle &= |G_1\rangle, \\ h_3 |G_2\rangle &= -|G_2\rangle. \end{aligned}$$

Leaving for the ground state degeneracy:

$$GSD = 2.$$

The above is not the only example of a gauge theory in 3 dimensions since one could choose to place degrees of freedom on vertices as well, thus defining a 0,1,2,3-gauge theory, or we could decide to give up the degrees of freedom at cubes and obtain a 0,1,2-gauge theory, and so on. The detailed exploration of such models goes beyond the scope of this work, thus we do not include them here. We would like to say, however, that there are many things to be done yet. In particular, we hope these models help to deepen the understanding of topological phases in 3 dimensions, by studying the excited states for instance; The complete classification of (3+1)D topological phases remains being an open problem [52, 53, 95, 96]. Moreover, there is a growing interest on the relation between higher gauge theories and topological phases [54, 97, 98, 99, 100]; In what follows we focus on the Abelian version of higher gauge theories on the lattice, to do that we first introduce them as a natural higher dimensional extension of the models of §3.1, §3.2, §3.3 and §3.4. The more general framework that is going to be exhibited in the next chapter also allows for the straightforward calculation of the entanglement entropy, in the more general case, as we will show in Chapter 5.

Chapter 4

Abelian Higher Gauge Theories

In Chapter 3 we have exhibited several examples of Abelian higher gauge theories in 2 and 3 spatial dimensions with the intent of providing the reader with a detailed account of the kind of models we will treat in this work, more importantly we expect Chap. 3 to have helped the reader to develop some intuition about the models and their features. The most general case of an Abelian higher gauge theory is going to be treated in this Chapter. Remarkably the study of the ground states of these models gets extremely simplified when describing using tools borrowed from Homological Algebra. The use of such language has already been shown useful for the study of topological models [101, 102, 103] and this chapter is dedicated to show how this holds for the class of models we call Abelian higher gauge theories [3].

As it was stressed out in the previous chapter, by *higher gauge theory* we mean a theory that allows degrees of freedom associated to higher (and lower) dimensional objects. This is not enough to define a gauge theory though, an appropriate generalized notion of gauge equivalence between states as well as a generalized notion of holonomy must also be given. For instance, we have seen in §3.3 that a 1,2-gauge theory consists on gauge fields living in both links $x \in K_1$ and faces $y \in K_2$ of the complex K . More than that, the theory has two types of gauge transformations corresponding to each gauge group as well as a generalized notion of 1-holonomy.

We have also indicated in the previous Chapter that the models are parametrized by two chain complexes. The first one is geometrical in nature and accounts for the structure of the lattice. As for the second, it is a chain complex of finite Abelian groups encoding the higher gauge group of the model. There will be one model (Hilbert space and Hamiltonian) for any such choice of chain complexes. These two chain complexes actually form the co-chain complex $(\text{hom}(C, G)^p, \delta^p)$ (see Appendix 6). In this chapter we show the last complex is enough to describe an Abelian higher gauge theory in full generality. For this reason, we begin the chapter by describing the model in Chapter 3 in terms of Homology: Starting from the generalized notion of gauge configurations, we define the Hilbert space, \mathcal{H} of the model. The Hamiltonian of the model is written in homological terms to show how the ground state subspace $\mathcal{H}_0 \subset \mathcal{H}$ gets naturally labeled by elements in a special cohomology group, this is the main result of [3]. We end the Chapter with a couple of illustrative examples showing how the GSD of such models is sensitive to the topological features of the discretized space where they are defined.

4.1 General Description of the Model

The beginning point of this construction resides in the notion of gauge configurations and consequently the definition of the Hilbert space \mathcal{H} . In Chapter 3 we have seen that in the case of 1-gauge theories (cf. §3.1) this notion consists on an assignment of labels $g \in G$ to 1-dimensional simplices $x \in K_1$. This is, two mathematical structures were used to define the model: a geometrical structure in the form of a 2-dimensional simplicial chain complex $C(K)$ and a higher gauge group G that was casted in the form of the chain complex of Eq.(3.6). Furthermore, we have also shown how more general theories can be constructed, such as the 1,2-gauge theory of §3.3 where we allow for a second group G_2 to label the 2-simplices $p \in K_2$ of the lattice. A simplicial decomposition is a natural choice for lattices of any dimension. Although the formalism can accommodate for any finite cell decomposition we will assume that the lattice K is made of simplices:

$$K = K_0 \cup K_1 \cup \dots \cup K_d,$$

where K_n is the (finite) set of n -dimensional simplices. We would like to point out that there are no further assumptions on K , this makes the formalism very flexible. For instance, K may have a boundary and may not have a uniform dimension.

It is a standard procedure [106] to associate to K a chain complex

$$C_d \xrightarrow{\partial_d^C} C_{d-1} \xrightarrow{\partial_{d-1}^C} \dots \xrightarrow{\partial_2^C} C_1 \xrightarrow{\partial_1^C} C_0 \quad (4.1)$$

that we will denote by $(C(K), \partial^C)$. We recall that C_n is the Abelian group freely generated by K_n . In other words, if we write the group operation as an addition operation, $c \in C_n$ is given by a formal linear combination

$$c = \sum_{x \in K_n} n(x)x \quad (4.2)$$

with $n(x) \in \mathbb{N}$. The homomorphisms $\partial_n^C : C_n \rightarrow C_{n-1}$ are the usual boundary maps. On the other hand, we need a mathematical structure that plays the role of a gauge group. In the previous chapter we saw that the model considers generalized spins placed at every simplex of the geometrical complex. Such spins are labeled by groups G_n . In this sense, we consider the abstract chain complex of Abelian groups $(G_\bullet, \partial_\bullet^G)$ as the algebraic content of the model and the one in charge of the higher gauge symmetry, the *higher gauge group*. Schematically we have,

$$0 \hookrightarrow G_d \xrightarrow{\partial_d^G} G_{d-1} \xrightarrow{\partial_{d-1}^G} \dots \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \rightarrow 0, \quad (4.3)$$

where all groups G_n are finite Abelian groups and $\partial_n^G : G_n \rightarrow G_{n-1}$ are group homomorphisms, such that $\partial_n \circ \partial_{n+1} = 0$. Note that the ∂ in (4.1) and the one in eq.(4.3) stand for different group morphisms.

4.1.1 Gauge Configurations and Hilbert Space

The notion of gauge configurations, as we have repeatedly seen in Chapter 3, consists on the assignment of G_n -labels to the n -simplices, $x \in K_n$ for all $n = \{0, 1, \dots, d\}$ of

the d -dimensional geometric chain complex $C(K)$. We now introduce a way to describe such assignments as determined by collections of maps $f = \{f_n\}$ for $n = 0, 1, \dots, d$ where each map $f_n : C_n \rightarrow G_n$ is defined by its action on the generators of C_n , $x \in K_n$. More precisely, we define a gauge configuration f to be an assignment of a group element $g \in G_n$ for each element $x \in K_n$. In other words, a sequence $f = \{f_n\}_{n=0,1,2,\dots,d}$ of functions

$$f_n : K_n \rightarrow G_n, \quad (4.4)$$

$$x \mapsto f_n(x). \quad (4.5)$$

Strictly speaking, we should call f a higher-gauge configuration. Only in the case when all groups except G_1 are trivial f is a proper gauge configuration as can be seen from the examples of last section. For simplicity, we will keep using "gauge configuration" (and gauge transformation) to mean a generic f .

Each map f_n in (4.4) defines a unique group homomorphism $f_n : C_n \rightarrow G_n$, as f_n is extended by linearity. Let $c \in C_n$ as in (4.2), then

$$f_n(c) = \sum_{x \in K_n} n(x) f_n(x).$$

The set $\text{Hom}(C_n, G_n)$ of homomorphisms is also an Abelian group if we set

$$(f_n + \tilde{f}_n)(x) = f_n(x) + \tilde{f}_n(x), \quad f_n, \tilde{f}_n \in \text{Hom}(C_n, G_n).$$

It is useful to collect all such Abelian groups in a single direct sum. This simple fact allows us to view a gauge configuration f as an element of the direct sum

$$\text{hom}(C, G)^0 := \bigoplus_{n=0}^d \text{Hom}(C_n, G_n) \quad (4.6)$$

of Abelian groups. That can be represented by a map between chain complexes as depicted by the diagram in Fig. 4.1.

Recall from Chapter 3 that the Hilbert space of the models consisted of a tensor product of local spaces such that: For each n -simplex $x \in K_n$ there is a local Hilbert space \mathcal{H}_x whose basis $\{|g\rangle\}$ is labeled by group elements $g \in G_n$, for all $n = \{0, 1, \dots, d\}$. In other words, the Hilbert space is given by

$$\mathcal{H} := \bigotimes_n \bigotimes_{x \in K_n} \mathcal{H}_x, \quad (4.7)$$

where, for $x \in K_n$, the local Hilbert space \mathcal{H}_x has its basis $\{|g\rangle\}$ labeled by group elements $g \in G_n$ with the usual inner product: $\langle f|g\rangle = \delta(f, g)$.

We will now see how the notion of higher-gauge configurations comes in handy for writing a very convenient basis for the Hilbert space \mathcal{H} . To begin, note that for each $n = \{0, 1, \dots, d\}$ the collection of all maps $f_n : C_n \rightarrow G_n$ forms an additive Abelian group $\text{Hom}(C_n, G_n)$ which consists of all group morphisms from the n -chain group C_n into the gauge group G_n . The group operation is inherited from that of G_n , to see this consider $f_n, f'_n \in \text{Hom}(C_n, G_n)$, their sum which we denote as $(f + f')_n$ is also an element of $\text{Hom}(C_n, G_n)$ defined by:

$$(f + f')_n(x) = f_n(x) + f'_n(x) \in G_n,$$

this is true for each $n \in \{0, 1, \dots, d\}$. Each $\text{Hom}(C_n, G_n)$ group encodes the different ways G_n -labels can be assigned to all n -simplices, $x \in K_n$ in order to label the local basis states of \mathcal{H}_x .

The total number of such assignments can be obtained by observing that the n -chain group, C_n is *finitely generated*, which means that $|K_n|$ is finite, for each $n \in \{0, 1, \dots, d\}$. Additionally, every Abelian group on the chain complex of Eq. (4.3) is finite, giving precisely $|G_n|$ ways to color a single n -simplex $x \in K_n$. Thus the total number of possible assignments of G_n -labels to the n -simplices is simply given by $|G_n|^{|K_n|} = |\text{Hom}(C_n, G_n)|$. More importantly, this number coincides with the dimension of the n -th level Hilbert space $\mathcal{H}_n := \bigotimes_{x \in K_n} \mathcal{H}_x$, the tensor product over of all local spaces \mathcal{H}_x for all $x \in K_n$ and a single n . Consequently a basis for \mathcal{H}_n can be given in terms of the elements of $\text{Hom}(C_n, G_n)$ for each n .

Collecting the groups $\text{Hom}(C_n, G_n)$ for all $n = \{0, 1, \dots, d\}$ we construct a new additive Abelian group:

$$\text{hom}(C, G)^0 := \bigoplus_n \text{Hom}(C_n, G_n), \quad (4.8)$$

an element $f \in \text{hom}(C, G)^0$ is a set of maps $\{f_n\}_{n=0}^d$ where each $f_n \in \text{Hom}(C_n, G_n)$ and the group operation is inherited from that of $\text{Hom}(C_n, G_n)$ for each n , we refer the reader to Appendix 6 for a detailed discussion on this group. Then, for a given $f \in \text{hom}(C, G)^0$ the state,

$$|f\rangle := \bigotimes_{n=0}^d \bigotimes_{x \in K_n} |f_n(x)\rangle,$$

where $f_n(x) \in G_n$, is a basis element of the full Hilbert space \mathcal{H} . In other words, the state $|f\rangle \in \mathcal{H}$ is determined by an element $f \in \text{hom}(C, G)^0$ which in turn consist of a collection of morphisms $\{f_n\}$, $\forall n \in \{0, 1, 2, \dots, d\}$ as depicted in Fig.(4.1). Each group element $f \in \text{hom}(C, G)^0$ defines a *classical configuration* of the generalized gauge field, the *higher gauge field* and a basis of \mathcal{H} is obtained by collecting all such elements, $\{|f\rangle\}_{f \in \text{hom}(C, G)^0}$. Any general state $|\psi\rangle \in \mathcal{H}$ can be written as a linear combination:

$$|\psi\rangle = \sum_f \psi(f) |f\rangle \quad \text{with} \quad f \in \text{hom}(C, G)^0,$$

and it corresponds to quantum field configurations living on the space associated to the geometrical chain complex $C(K)$. The order of the classical gauge configurations group is the dimension of the total Hilbert space, namely

$$\dim(\mathcal{H}) = |\text{hom}(C, G)^0| = \prod_n |G_n|^{|K_n|} < \infty. \quad (4.9)$$

Amongst all classical gauge configurations the one defined by the identity element, $0 \in \text{hom}(C, G)^0$, determines what we call *trivial state* and it is given by the product state:

$$|0\rangle = \bigotimes_p \bigotimes_{x \in K_n} |0\rangle_x,$$

where 0 is the identity element of G_n , associated to the n -dimensional simplex $x \in K_n$. As we will see in the following sections, this state will serve as a *seed* state that allows the analysis of the model's topological features from the properties of the operators of the theory only.

$$\begin{array}{ccccccc}
 0 \hookrightarrow C_d & \longrightarrow & C_{d-1} & \longrightarrow & \dots & \longrightarrow & C_1 \longrightarrow C_0 \rightarrow 0 \\
 & \downarrow f_d & \downarrow f_{d-1} & & \dots & & \downarrow f_1 & \downarrow f_0 \\
 0 \hookrightarrow G_d & \longrightarrow & G_{d-1} & \longrightarrow & \dots & \longrightarrow & G_1 \longrightarrow G_0 \rightarrow 0
 \end{array}$$

Figure 4.1: The higher gauge configurations as maps belonging to $\text{hom}(C, G)^0$ are shown.

4.1.2 Higher Gauge Transformations and Holonomies

The two other notions that are essential for a higher gauge theory are: *gauge transformations* and *holonomies*, the former implements the higher gauge symmetry of the theory on actual quantum states whereas the latter imposes the local flatness conditions. The local notions of higher gauge symmetry and flatness are model-dependent as we have shown in chapter 3; However, the language we develop in this chapter will allow us to treat these notions in a unified way; We now proceed to discuss the mathematical structures that allow for this generalization. In the previous section we saw how a convenient basis for the global Hilbert space can be given in terms of the elements of an Abelian group, $f \in \text{hom}(C, G)^0$. Besides $\text{hom}(C, G)^0$ more general groups can be constructed by considering morphisms $g_n : C_n \rightarrow G_{n-p}$ also called p -maps. For a given n , the collection of all p -maps defines the Abelian group $\text{Hom}(C_n, G_{n-p})$. Then, the set $\{g_n\}$ for $n \in \{0, 1, \dots, d\}$ is the following additive Abelian group:

$$\text{hom}(C, G)^p := \bigoplus_{n=0}^d \text{Hom}(C_n, G_{n-p}), \quad (4.10)$$

constituted by p skewed maps such as the ones shown in Fig. 4.2 for arbitrary elements $f \in \text{hom}(C, G)^0$ and $g \in \text{hom}(C, G)^1$. Moreover, there is a co-boundary operator $\delta^p : \text{hom}(C, G)^p \rightarrow \text{hom}(C, G)^{p+1}$ given by:

$$(\delta^p h)_n = h_{n-1} \circ \partial_n^C - (-1)^p \partial_{n-p}^G \circ h_n, \quad (4.11)$$

where $h \in \text{hom}(C, G)^p$. Observe that $\delta^p h \in \text{hom}(C, G)^{p+1}$ is actually a collection of maps $\{(\delta^p h)_n\}$ for $n = \{0, 1, \dots, d\}$ and each of these maps $(\delta^p h)_n : C_n \rightarrow G_{n-(p+1)}$. It is straightforward to show that $\delta^{p+1} \delta^p = 0$ which makes the sequence $(\text{hom}(C, G)^\bullet, \delta^\bullet)$ a co-chain complex (cf. Appendix 6), of particular interest is the following part of the sequence:

$$\text{hom}(C, G)^{-1} \xrightarrow{\delta^{-1}} \text{hom}(C, G)^0 \xrightarrow{\delta^0} \text{hom}(C, G)^1, \quad (4.12)$$

since, as we will now see, it is enough to encode all the information of a higher gauge theory. We refer the reader to [3, 81] for a detailed account.

Let us now delve into the discussion about the way the sequence in Eq. (4.12) is enough to describe the generalized notions of gauge symmetry and holonomy. For instance, in §3.1 we saw that the elementary gauge transformations of a 1-gauge theory are implemented on quantum states by means of the vertex operator, A_v^g , located at a vertex $v \in K_0$ and labeled by a group element $g \in G_1$, see Eq. (3.4). Thus, in a 1-gauge theory the local gauge symmetry is implemented around vertices $v \in K_0$ and its action is labeled by group elements $g \in G_1$. This suggests that the elementary gauge transformations of a 1-gauge theory can be parametrized by assignments of G_1 -labels to vertices $v \in K_1$ of the

$$\begin{array}{ccccccc}
 0 \hookrightarrow C_d & \longrightarrow & C_{d-1} & \longrightarrow & \dots & \longrightarrow & C_1 \longrightarrow C_0 \rightarrow 0 \\
 \downarrow f_d & \searrow g_d & \downarrow f_{d-1} & \searrow g_{d-1} & & \searrow g_2 & \downarrow f_1 \searrow g_1 \downarrow f_0 \\
 0 \hookrightarrow G_d & \longrightarrow & G_{d-1} & \longrightarrow & \dots & \longrightarrow & G_1 \longrightarrow G_0 \rightarrow 0
 \end{array}$$

Figure 4.2: Set of functions $f \in \text{hom}(C, G)^0$ and $g \in \text{hom}(C, G)^1$, we show the actual functions $\{g_n\}$ in green and $\{f_n\}$ in black.

lattice. Hence, all elementary gauge transformations of a 1-gauge theory are parametrized by elements of the group $\text{Hom}(C_0, G_1)$, since an element $t \in \text{Hom}(C_0, G_1)$ associates 0-simplices (vertices) $v \in K_0$ to group elements $t(v) \in G_1$.

Take now the example of a 1,2-gauge theory (§3.3), two kinds of gauge transformations implemented the higher gauge symmetry in this case. The first kind, which we call 1-gauge transformations are nothing but the usual vertex operators A_v^g with $v \in K_0$ and $g \in G_1$, and again the group $\text{Hom}(C_0, G_1)$ parametrizes all such elementary transformations. Similarly, the 2-gauge elementary transformation is locally enhanced by the link operator A_l^β of Eq.(3.23), located at links $l \in K_1$ and labeled by group elements $\beta \in G_2$; this in turn suggests that all elementary 2-gauge transformations are parametrized by elements of the group $\text{Hom}(C_1, G_2)$, corresponding to all possible assignments of G_2 -labels to 1-simplices $l \in K_1$. The parametrization of elementary gauge transformations by elements of $\text{Hom}(C_n, G_{n+1})$ will allow us to define the more general notion of a higher gauge transformation. In the most general case, where we could have all kinds of gauge transformations, the elementary gauge transformations are all parametrized (at once) by elements of the following additive Abelian group:

$$\text{hom}(C, G)^{-1} := \bigoplus_n \text{Hom}(C_n, G_{n+1}),$$

which consists of maps from 0-simplices to elements of the group G_1 , 1-simplices to G_2 , 2-simplices to elements of G_3 and so on. The actual implementation of the higher gauge symmetry on quantum states of $|\psi\rangle \in \mathcal{H}$ is a task for the elementary operators that are going to be defined in the next section. The group $\text{hom}(C, G)^{-1}$ parametrizes elementary gauge transformations in a global and general way.

The last ingredient needed to construct a higher gauge theory has to do with idea of holonomy. For instance, in a 1-gauge theory, the notion of holonomy corresponds to collecting G_1 -values along plaquettes $p \in K_2$ of the lattice, as we saw in §3.1. This is, holonomy measurements associate G_1 -labels to plaquettes $p \in K_2$ and the collection of all such assignments for all $p \in K_2$ is parametrized by the set of all morphisms from the 2-chains C_2 to the group G_1 , or $\text{Hom}(C_2, G_1)$. To see how this generalizes consider the example of a 0,1-gauge theory (§3.2) where we had two different notions of holonomy, namely, the 0-holonomy which is measured by the link operator B_l and the usual 1-holonomy measured by the plaquette operators, B_p . Thus, in a 0,1-gauge theory, the 0-holonomy values can be parametrized by maps $m_1 : C_1 \rightarrow G_0$ that send 1-simplices (links) $l \in K_1$ to group elements $m_1(x) \in G_0$; In other words, they are parametrized by the group $\text{Hom}(C_1, G_0)$; Likewise the 1-holonomy values can be thought of as associating G_1 -labels to the plaquettes $p \in K_2$; They are parametrized by elements of the group $\text{Hom}(C_2, G_1)$. So, as the reader may now guess, in the most general case all holonomy

values of a higher gauge theory in d -dimensions are parametrized by the group:

$$\mathrm{hom}(C, G)^1 := \bigoplus_{n=0}^d \mathrm{Hom}(C_n, G_{n-1}). \quad (4.13)$$

This additive Abelian group parametrizes all generalized holonomy values. However, the measurement of such values is performed through the representations of $\mathrm{hom}(C, G)^1$; Since this group is Abelian, all its irreducible representations are one-dimensional and they form a group by themselves, which we call the *dual group* and denote

$$\mathrm{hom}(C, G)_1 := \mathrm{Hom}(\mathrm{hom}(C, G)^1, U(1)) \simeq \bigoplus_{n=0}^d \mathrm{Hom}(C_n, \widehat{G}_{n-p}),$$

where \widehat{G}_{n-p} is the dual group of G_{n-p} . We refer the reader to Appendix 6 for a detailed discussion about dual groups. Thus, the operators that actually implement the holonomy measurements are defined using elements $m \in \mathrm{hom}(C, G)_1$ as we will see in the next section.

4.1.3 Elementary Operators

In the previous two sections we spend some time arguing to some level of detail about the sequence in Eq.(4.12),

$$\mathrm{hom}(C, G)^{-1} \xrightarrow{\delta^{-1}} \mathrm{hom}(C, G)^0 \xrightarrow{\delta^0} \mathrm{hom}(C, G)^1,$$

being enough to encode the necessary ingredients needed to construct an Abelian higher gauge theory in full generality. We started showing how the group $\mathrm{hom}(C, G)^0$ can be used to label the basis elements of the total Hilbert space \mathcal{H} (§ 4.1.1), elements of this group $f \in \mathrm{hom}(C, G)^0$ are called *classical gauge configurations* and are the starting point of our construction. Next, in §4.1.2, we discussed the notion of higher gauge symmetry and how it gets parametrized by the group $\mathrm{hom}(C, G)^{-1}$, an element of this group $t \in \mathrm{hom}(C, G)^{-1}$ will define a generalized gauge transformations that acts on basis states $|f\rangle$, $f \in \mathrm{hom}(C, G)^0$ via the coboundary operator δ^{-1} as we will precisely see in this section. Moreover, the group $\mathrm{hom}(C, G)^1$ parametrizes the generalized holonomy values of the theory and their measurement is performed by means of the coboundary operator δ^0 and the dual group $\mathrm{hom}(C, G)_1$ as we will see.

Now that the global Hilbert space has its basis labeled by the generalized notion of classical gauge configurations, we are ready to define the elementary operators of the theory. They come as the generalizations of the operators in Definition ?? called *shift* and *clock* operators. Consider then the group of classical gauge configurations $\mathrm{hom}(C, G)^0$ and its dual¹ group $\mathrm{hom}(C, G)_0$, consider also an arbitrary element $f \in \mathrm{hom}(C, G)^0$ and its corresponding basis state $|f\rangle \in \mathcal{H}$.

¹The group $\mathrm{hom}(C, G)^0$ is Abelian, therefore all its irreducible representations are of dimension one and they form a group by themselves, which we call the *dual group* and denote $\mathrm{hom}(C, G)_0$, see Appendix 6 for a detailed account.

Definition 4.1.1 (Shift and Clock Operators). Let $|f\rangle \in \mathcal{H}$ a basis state. Given $t \in \text{hom}(C, G)^0$ and $m \in \text{hom}(C, G)_0$, we define the operators $P_t, Q_m : \mathcal{H} \rightarrow \mathcal{H}$ by:

$$\begin{aligned} P_t |f\rangle &= |f + t\rangle, \\ Q_m |f\rangle &= \chi_m(f) |f\rangle, \end{aligned}$$

where $\chi_m(f)$ is the character of $f \in \text{hom}(C, G)^0$ in the representation $m \in \text{hom}(C, G)_0$ and it is given by

$$\chi_m(f) = \prod_{n=0}^d \prod_{x \in K_n} m(f(x)).$$

These operators enhance the most fundamental operations of the theory, this is, P_t performs a permutation between basis states that is parametrized by $t \in \text{hom}(C, G)^{-1}$ while Q_m measures the generalized holonomy of basis states in the representation $m \in \text{hom}(C, G)_1$ via its character. The algebraic relations these operators satisfy are obtained by a mere application on basis states so it is straightforward to show that

$$\begin{aligned} P_s P_t &= P_{s+t}, & Q_m Q_r &= Q_{m+r}, \\ Q_m P_t &= \chi_m(t) P_t Q_m, \end{aligned} \tag{4.14}$$

for all $s, t \in \text{hom}(C, G)^0$ and $m, r \in \text{hom}(C, G)_0$.

Note that the shift operator can be used P_t to write arbitrary basis states $|t\rangle \in \mathcal{H}$, with $t \in \text{hom}(C, G)^0$ as:

$$|t\rangle = P_t |0\rangle,$$

where $|0\rangle \in \mathcal{H}$ is the *trivial state* of Eq.(4.10) labeled by the identity element $0 \in \text{hom}(C, G)^0$. On top of these elementary operators we construct generalized notion of gauge transformations and of holonomy measurements of the theory. The usual construction shown in Chapter 3 utilizes these notions locally, this means, the operators that perform gauge transformations and holonomy measurements are local. However, the use of elements in $\text{hom}(C, G)^0$ to label basis states allows us to define a more general notion of gauge transformations and holonomy measurements.

Definition 4.1.2 (Generalized Gauge Transformation). Let $|f\rangle \in \mathcal{H}$ with $f \in \text{hom}(C, G)^0$. Let also $t \in \text{hom}(C, G)^{-1}$. The generalized gauge transformation $A_t : \mathcal{H} \rightarrow \mathcal{H}$ is defined by:

$$A_t |f\rangle = |f + \delta^{-1}(t)\rangle,$$

where $\delta^{-1} : \text{hom}(C, G)^{-1} \rightarrow \text{hom}(C, G)^0$.

Equivalently we could write, $A_t := P_{\delta^{-1}t}$. Given this general idea of gauge transformation, the notion of gauge equivalence between states naturally extends to the higher gauge case:

Definition 4.1.3 (Gauge Equivalence). Two states $|f\rangle, |g\rangle \in \mathcal{H}$, with $f, g \in \text{hom}(C, G)^0$, are said to be **gauge equivalent** if there exists a $t \in \text{hom}(C, G)^{-1}$ such that:

$$|f\rangle = A_t |g\rangle = |f + \delta^{-1}t\rangle,$$

or equivalently, $g = f + \delta^{-1}t$.

On the other hand, in §4.1.2 the group $\text{hom}(C, G)^1$ was shown to parametrize all generalized holonomy values; Now, the measurement of such values is carried over by the dual group $\text{hom}(C, G)_1$, or the group formed by *irreducible representations* of $\text{hom}(C, G)^1$.

Definition 4.1.4 (Generalized Holonomy Measurement). Let $|f\rangle \in \text{hom}(C, G)^0$ and $m \in \text{hom}(C, G)_1$. Define the operator $B_m : \mathcal{H} \rightarrow \mathcal{H}$ by:

$$B_m |f\rangle = \chi_m(\delta^0 f) |f\rangle,$$

Now, we appeal to the dual nature between homology and cohomology. The sequence of Eq.(4.12) has, naturally associated to it, a dual sequence:

$$\text{hom}(C, G)_{-1} \xleftarrow{\delta_0} \text{hom}(C, G)_0 \xleftarrow{\delta_1} \text{hom}(C, G)_1, \quad (4.15)$$

where the boundary operator δ_p is defined from δ^{p-1} , see A.4.9. Thus, the definition of the holonomy measurement operator can be given in terms of such boundary operators by noting that: $\chi_{\delta_1 m}(f) := \chi_m(\delta^0 f)$ for $f \in \text{hom}(C, G)^0$ and $m \in \text{hom}(C, G)_1$, or equivalently:

$$B_m := Q_{\delta_1 m},$$

where $Q_{\delta_1 m}$ is as in Definition 4.1.1 and δ_1 is the boundary operator of Definition A.4.9. So, the boundary operator δ_1 is the one in charge measuring the general notion of holonomy. As we have already seen in the examples (Chapter 3), holonomies not only measure the gauge fields along the boundary of an n -chain but also consider the gauge value of the n -chain itself. To see how this general notions of gauge transformations and holonomy measurements reduce to the familiar ones, in §4.3 we will show how to cast the examples of Chapter 3 utilizing the language we have just developed.

The operators in Definitions 4.1.2 and 4.1.4 can be easily shown to satisfy the following relations:

$$\begin{aligned} A_t A_{t'} &= A_{t+t'} = A_{t'} A_t ; B_m B_{m'} = B_{m+m'} = B_{m'} B_m \\ B_m A_t &= A_t B_m. \end{aligned} \quad (4.16)$$

For instance, the last identity comes from:

$$B_m A_t = Q_{\delta_1 m} P_{\delta^{-1} t} = \chi_{\delta_1 m}(\delta^{-1} t) P_{\delta^{-1} t} Q_{\delta_1 m} = \chi_m(\delta^0 \delta^{-1} t) P_{\delta^{-1} t} Q_{\delta_1 m} = A_t B_m,$$

where relation (4.14) was used; Also, the last step follows from $\delta^0 \delta^{-1} = 0$ and $\chi_m(0) = 1$ for all $m \in \text{hom}(C, G)_1$. The above operators are the ones that enhance the general notions of gauge transformations and holonomy measurements on quantum states, they are of global nature since the elements of $\text{hom}(C, G)^{-1}$ and $\text{hom}(C, G)_1$ are defined on the entire complex $C(K)$.

There is another set of operators that will become very important when analyzing the ground state subspace.

Definition 4.1.5 (Generalized Projection Operators). Given $s \in \text{hom}(C, G)_{-1}$ and $v \in \text{hom}(C, G)^1$, we define:

$$\mathcal{A}_s = \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t \chi_s(t) A_t, \quad (4.17)$$

$$\mathcal{B}_v = \frac{1}{|\text{hom}(C, G)_1|} \sum_m \chi_m(v) B_m, \quad (4.18)$$

where $t \in \text{hom}(C, G)^{-1}$ and $m \in \text{hom}(C, G)_1$.

The above operators form a complete set of mutually commuting projectors, by this we mean that they are projectors, they commute within themselves as well as with each other and their sum adds up to the identity operator. More precisely:

Proposition 4.1.6. *For all $s, s' \in \text{hom}(C, G)_{-1}$ and $v, v' \in \text{hom}(C, G)^1$, the following relations for the generalized projector operators of Definition 4.1.5 are satisfied:*

1. *Pairwise commutation:*

$$\mathcal{A}_s \mathcal{B}_v = \mathcal{B}_v \mathcal{A}_s.$$

2. *Orthogonality:*

$$\mathcal{A}_s \mathcal{A}_{s'} = \delta(s, s') \mathcal{A}_s, \quad \mathcal{B}_v \mathcal{B}_{v'} = \delta(v, v') \mathcal{B}_v,$$

where $\delta(\cdot, \cdot)$ is the Kronecker delta.

3. *Completeness:*

$$\sum_s \mathcal{A}_s = \mathbb{1}, \quad \sum_v \mathcal{B}_v = \mathbb{1},$$

where $\mathbb{1}$ is the identity operator.

Proof. 1. The pairwise commutation follows directly from $A_t B_m = B_m A_t$.

2. The orthogonality relations follows from the orthogonality of characters, for instance:

$$\begin{aligned} \mathcal{A}_s \mathcal{A}_{s'} &= \frac{1}{|\text{hom}(C, G)^{-1}|^2} \sum_t \chi_s(t) A_t \sum_p \chi_{s'}(p) A_p \\ &= \frac{1}{|\text{hom}(C, G)^{-1}|^2} \sum_{t,p} \chi_s(t) \chi_{s'}(p) A_{t+p}, \end{aligned}$$

let us perform the following change of variables: $q = t + p$, the above expression would be:

$$\begin{aligned} \mathcal{A}_s \mathcal{A}_{s'} &= \frac{1}{|\text{hom}(C, G)^{-1}|^2} \sum_{q-p} \sum_p \chi_s(q-p) \chi_{s'}(p) A_q \\ &= \frac{1}{|\text{hom}(C, G)^{-1}|^2} \sum_{q-p} \sum_p \chi_s(q) \overline{\chi_s(p)} \chi_{s'}(p) A_q, \end{aligned} \tag{4.19}$$

where $\overline{\chi_s(p)}$ stands for $\chi_s(-p)$, $-p$ being the inverse of p since the group operation is being written additively. Recalling one of Schur's orthogonality relations [104, 105], which in this case reads:

$$\sum_p \overline{\chi_s(p)} \chi_{s'}(p) = |\text{hom}(C, G)^{-1}| \delta(s, s'),$$

on Eq.(4.19) we get:

$$\begin{aligned}\mathcal{A}_s \mathcal{A}_{s'} &= \delta(s, s') \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_{q=p} \chi_s(q) A_q \\ &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_q \chi_s(q) A_q \\ &= \delta(s, s') \mathcal{A}_s,\end{aligned}$$

where in the second line we note that $\sum_{q=p} \equiv \sum_q$ since the sum is carried over the entire group $\text{hom}(C, G)^{-1}$. The identity involving the generalized projectors \mathcal{B}_v follows from a very similar calculation.

3. The last identity of Property 4.1.6 reads:

$$\begin{aligned}\sum_s \mathcal{A}_s &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_s \sum_t \chi_s(t) A_t \\ &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_s \sum_t \chi_s(t) \overline{\chi_s(0)} A_t,\end{aligned}$$

where in the last expression we introduced a trivial character, i.e., $\chi_s(0) = 1$ for $0 \in \text{hom}(C, G)_{-1}$ the trivial representation of the group $\text{hom}(C, G)^{-1}$. Thus, we get:

$$\begin{aligned}\sum_s \mathcal{A}_s &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_s \sum_t \chi_s(t) \overline{\chi_s(0)} A_t \\ &= \frac{|\text{hom}(C, G)^{-1}|}{|\text{hom}(C, G)^{-1}|} \sum_t \delta(t, 0) A_t \\ &= A_0 = \mathbb{1}.\end{aligned}$$

□

Remark 4.1.7. We highlight two particular cases of the generalized projectors of Definition 4.1.5:

1. The generalized projector $\mathcal{A}_0 := \mathcal{A}_{s|s=0}$, written as the sum:

$$\mathcal{A}_0 = \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t A_t, \quad t \in \text{hom}(C, G)^{-1}, \quad (4.20)$$

which projects any state $|f\rangle \in \mathcal{H}$ into an equal sum of gauge equivalent states and consequently, can be used to characterize two states $|f\rangle, |g\rangle \in \mathcal{H}$ as being gauge equivalent if $\mathcal{A}_0 |f\rangle = \mathcal{A}_0 |g\rangle$, which is evident from the generalized notion of gauge equivalence (Def. 4.1.3).

2. The generalized $\mathcal{B}_0 := \mathcal{B}_{v|v=0}$, written as the sum:

$$\mathcal{B}_0 = \frac{1}{|\text{hom}(C, G)_1|} \sum_m B_m, \quad m \in \text{hom}(C, G)_1, \quad (4.21)$$

which projects such states $|f\rangle \in \mathcal{H}$ that satisfy $f \in \ker(\delta^0)$. A relation which is easily obtained by using its definition and the characters orthogonality relations.

Finally, the algebraic relations between the operators of Definition 4.1.5 and the ones in Defns. 4.1.2 and 4.1.4 will show to be very useful in the next section when studying the ground state of the model. For this reason:

Proposition 4.1.8. *Let $f \in \text{hom}(C, G)^0$ and $m \in \text{hom}(C, G)_0$ be arbitrary elements. It holds:*

1. $\mathcal{B}_v P_f = P_f \mathcal{B}_{v+\delta^0 f}$, where $v \in \text{hom}(C, G)^1$.
2. $Q_m \mathcal{A}_s = \mathcal{A}_{s+\delta_0 m} Q_m$, where $s \in \text{hom}(C, G)_{-1}$.

Proof. 1. Expanding the left hand side:

$$\begin{aligned}
 \mathcal{B}_v P_f &= \frac{1}{|\text{hom}(C, G)_1|} \sum_m \chi_m(v) Q_{\delta_1 m} P_f \\
 &= \frac{1}{|\text{hom}(C, G)_1|} \sum_m \chi_m(v) \chi_{\delta_1 m}(f) P_f Q_{\delta_1 m} \\
 &= \frac{P_f}{|\text{hom}(C, G)_1|} \sum_m \chi_m(v) \chi_m(\delta^0 f) Q_{\delta_1 m} \\
 &= \frac{P_f}{|\text{hom}(C, G)_1|} \sum_m \chi_m(v + \delta^0 f) Q_{\delta_1 m} = P_f \mathcal{B}_{v+\delta^0 f}.
 \end{aligned}$$

2. The procedure is very similar to the previous one, hence we will skip some intermediate steps. Expanding the left hand side we get:

$$\begin{aligned}
 Q_m \mathcal{A}_s &= \frac{Q_m}{|\text{hom}(C, G)^{-1}|} \sum_t \chi_s(t) P_{\delta^{-1} t} \\
 &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t \chi_s(t) \chi_m(\delta^{-1} t) P_{\delta^{-1} t} Q_m \\
 &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t \chi_s(t) \chi_{\delta_0 m}(t) A_t Q_m \\
 &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t \chi_{s+\delta_0 m}(t) A_t Q_m \\
 &= \mathcal{A}_{s+\delta_0 m} Q_m
 \end{aligned}$$

□

In the next section we will exhibit a special set of maps in $\text{hom}(C, G)_{-1}$ and $\text{hom}(C, G)^1$ that will allow us to localize the generalized projectors of Definition 4.1.5. Making, thus, the connection with the projectors that make part of the Hamiltonians we have presented in chapter 3. The true power of this formalism will be seen when tackling the problem of characterizing the ground state subspace of the model in §4.2.

4.1.4 Local Decomposition and Hamiltonian of the Model

The previous section was devoted to the study of the elementary operators of the theory, we started defining the generalized notions of gauge transformations and holonomies and from there we constructed more complicated operators. In this section we use such operators to actually write the Hamiltonian of an *Abelian higher gauge theory*. In that sense we begin by exhibiting what we call as *local maps* that will lay down the connection between the projectors of Def. 4.1.5 and the operators that make up the models of chapter 3. To be able to write the Hamiltonian of the model we first need to have a notion of locality for the operators we have defined so far; This notion of locality is obtained by considering generating sets of $\text{hom}(C, G)^p$ and $\text{hom}(C, G)_p$, as we are about to see.

Definition 4.1.9 (Local Map 1). Let $x \in K_n$ and $g \in G_{n-p}$. The local map $e[g, n, x] \in \text{hom}(C, G)^p$ is given by:

$$e[g, n, x](y) = \begin{cases} g, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases}$$

where $y \in K_m$ for $0 \leq m \leq d$ and $0 \in G_{n-p}$ is the identity element.

Recalling that a map $f \in \text{hom}(C, G)^p$ consists on a collection of maps $\{f_n\}_{n=0}^d$, each one $f_n \in \text{Hom}(C_n, G_{n-p})$; f can be written in terms of the above local maps, as follows:

$$f = \sum_{n=0}^d \sum_{x \in K_n} e[f_n(x), n, x], \quad (4.22)$$

where $f_n(x) \in G_{n-p}$. In other words, the set of maps $\{e[g, n, x]\}$ is a generating set of the group $\text{hom}(C, G)^p$.

Similarly, a generating set for the group $\text{hom}(C, G)_p$ can be defined in terms of the following local maps:

Definition 4.1.10 (Local Map 2). Let $x \in K_n$, $r \in \hat{G}_{n-p}$ and $t \in \text{hom}(C, G)^p$ define the map $\hat{e}[r, n, x] \in \text{hom}(C, G)_p$ by:

$$\hat{e}[r, n, x](t) = \chi_r(t_n(x))$$

where $\chi_r(t_n(x))$ stands for the character of $t_n(x) \in G_{n-p}$ in the representation $r \in \hat{G}_{n-p}$.

To see that it is a generating set consider an arbitrary element $m \in \text{hom}(C, G)_p$, it can be written as:

$$m = \sum_{n=0}^d \sum_{x \in K_n} \hat{e}[m_n(x), n, x], \quad (4.23)$$

where each $m_n \in \text{Hom}(C_n, \hat{G}_{n-p})$, in other words: for all $x \in K_n$, $m_n(x) \in \hat{G}_{n-p}$ is an irreducible representation of the group G_{n-p} .

Note that the above *local maps* can be used to expand the generalized gauge transformations and holonomies (the operators of Def. 4.1.2 and 4.1.4) in terms of local operators. So, let us take arbitrary elements $t \in \text{hom}(C, G)^{-1}$ and $m \in \text{hom}(C, G)_1$, we can write:

$$A_t = A_{\sum_n \sum_x e[t_n(x), n, x]} = \prod_{n=0}^d \prod_{x \in K_n} A_{e[t_n(x), n, x]}, \quad (4.24)$$

$$B_m = B_{\sum_n \sum_x \hat{e}[m_n(x), n, x]} = \prod_{n=0}^d \prod_{x \in K_n} B_{\hat{e}[m_n(x), n, x]}, \quad (4.25)$$

where the r.h.s. of each expression follows from Eq.(4.16). The operators $A_{e[t_n(x), n, x]}$ and $B_{\hat{e}[m_n(x), n, x]}$ act locally (around $x \in K_n$) and to see this it is enough to act with them on arbitrary basis states, $|f\rangle \in \mathcal{H}$. We are one step away of writing the actual operators that go into the Hamiltonian operator. However, before we do that let us spend some time showing the usefulness of the local maps by proving the following identities:

Proposition 4.1.11. *Let $s \in \text{hom}(C, G)_{-1}$ and $v \in \text{hom}(C, G)^1$, it holds:*

$$\begin{aligned} 1. \quad & \sum_{t \in \text{hom}(C, G)^{-1}} \chi_s(t) A_t = \prod_{n=0}^d \prod_{x \in K_n} \sum_{t_n(x) \in G_{n+1}} \chi_s(e[t_n(x), n, x]) A_{e[t_n(x), n, x]}, \\ 2. \quad & \sum_{m \in \text{hom}(C, G)_1} \chi_m(v) B_m = \prod_{n=0}^d \prod_{x \in K_n} \sum_{m_n(x) \in \hat{G}_{n-1}} \chi_{\hat{e}[m_n(x), n, x]}(v) B_{\hat{e}[m_n(x), n, x]}. \end{aligned}$$

Proof. From Eq.(4.24) and realizing that summing over all $t \in \text{hom}(C, G)^{-1}$ is equivalent to sum over all possible values of $g_x := t(x)$, this is, all possible values each n -simplex $x \in K_n$ can be mapped to by $t \in \text{hom}(C, G)^{-1}$. We can write the l.h.s. of expression 1 as:

$$\begin{aligned} \sum_{t \in \text{hom}(C, G)^{-1}} \chi_s(t) A_t &= \sum_{g_x} \chi_s \left(\sum_n \sum_{x \in K_n} e[g_x, n, x] \right) A_{\sum_n \sum_x e[g_x, n, x]}, \\ &= \sum_{g_x} \prod_{n=0}^d \prod_{x \in K_n} \chi_s(e[g_x, n, x]) A_{e[g_x, n, x]}, \\ &= \prod_{n=0}^d \prod_{x \in K_n} \sum_{t_n(x) \in G_{n+1}} \chi_s(e[t_n(x), n, x]) A_{e[t_n(x), n, x]}. \end{aligned}$$

A very similar proof holds for the second identity. \square

Now, we are ready to define the local projectors that make part of the Hamiltonian operator of the theory, they come as the generalization of the operators defined in sections ??, 3.1, 3.2, 3.3 and 3.4.

Definition 4.1.12 (Local Projectors). Given $x \in K_n, g \in G_{n-1}, r \in \hat{G}_{n+1}$ define the local gauge projector and local holonomy projector as:

$$A_{n,x}^r := \mathcal{A}_{\hat{e}[r,n,x]} = \frac{1}{|G_{n+1}|} \sum_{g \in G_{n+1}} \chi_r(g) A_{e[g,n,x]} \quad \text{and}$$

$$B_{n,x}^g := \mathcal{B}_{e[g,n,x]} = \frac{1}{|G_{n-1}|} \sum_{r \in \hat{G}_{n-1}} \chi_r(g) B_{\hat{e}[r,n,x]}.$$

Furthermore, the generalized projectors of Definition 4.1.5 can be written in terms of the above local projectors, this is, a local decomposition.

Proposition 4.1.13. Let $s \in \text{hom}(C, G)_{-1}, m \in \text{hom}(C, G)^1$, then we can write:

$$\mathcal{A}_s := \prod_{n=0}^d \prod_{x \in K_n} A_{n,x}^{s_n(x)} \quad , \quad \mathcal{B}_m := \prod_{n=0}^d \prod_{x \in K_n} B_{n,x}^{m_n(x)} \quad , \quad (4.26)$$

where $s_n(x) \in \hat{G}_{n+1}$ and $m_n(x) \in G_{n-1}$

Proof. From Definition 4.1.5 we write:

$$\begin{aligned} \mathcal{A}_s &= \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t \chi_s(t) A_t, \\ &= \frac{1}{|\text{hom}(C, G)^{-1}|} \prod_{n=0}^d \prod_{x \in K_n} \sum_{t_n(x)} \chi_s(e[t_n(x), n, x]) A_{e[t_n(x), n, x]}, \end{aligned}$$

where $t \in \text{hom}(C, G)^{-1}$ and we have used Proposition 4.1.11 in the last line. Recall that $s \in \text{hom}(C, G)_{-1}$ and the local map $e[t_n(x), n, x] \in \text{hom}(C, G)^{-1}$, thus evaluating the character yields:

$$\begin{aligned} \chi_s(e[t_n(x), n, x]) &= \prod_{m=0}^d \prod_{y \in K_m} s(e[t_n(x), n, x](y)), \\ &= s(t_n(x)), \end{aligned}$$

where we used Definition 4.1.9 to evaluate the local map $e[t_n(x), n, x](y)$ in the first line; We then note that $s(t_n(x)) = s_n(t_n(x)) = \chi_{s_n(x)}(t_n(x))$, where $s_n(x) \in \hat{G}_{n+1}$. Also, recall that the order of the group:

$$|\text{hom}(C, G)^{-1}| = \prod_{n=0}^d |G_{n+1}|^{|K_n|}.$$

With all these considerations in hand we can write, for the generalized projection operator:

$$\begin{aligned} \mathcal{A}_s &= \prod_{n=0}^d \prod_{x \in K_n} \frac{1}{|G_{n+1}|} \sum_{t_n(x) \in G_{n+1}} \chi_{s_n(x)}(t_n(x)) A_{e[t_n(x), n, x]}, \\ &= \prod_{n=0}^d \prod_{x \in K_n} A_{n,x}^{s_n(x)}. \end{aligned}$$

A very similar proof holds for the second identity. □

Finally, we are ready to define the Hamiltonian of the model:

Definition 4.1.14 (Hamiltonian Operators). Define the operator $H : \mathcal{H} \rightarrow \mathcal{H}$ by:

$$H := - \sum_{n=0}^d \sum_{x \in K_n} A_{n,x}^0 - \sum_{n=0}^d \sum_{y \in K_n} B_{n,y}^0,$$

where $0 \in \widehat{G}_{n+1}$ in $A_{n,x}^0$ is the trivial representation, whereas $0 \in G_{n-1}$ in $B_{n,y}^0$ is the identity element of the corresponding group. From now on we will write $A_{n,x}^0 = A_{n,x}$ and $B_{n,y}^0 = B_{n,y}$ for the operators that go into the Hamiltonian.

This Hamiltonian enforces two kinds of constraints on the ground states. The first kind, related to $A_{n,x}$, implies ground states must be gauge invariant. The second kind, related $B_{n,x}$, projects to the trivial holonomy sector. As we shall see in the next section, this leads to the topological features of the model and, in particular, to a topological degeneracy for the ground state.

4.2 Ground States and GSD

In this section we focus on the properties of the ground state subspace $\mathcal{H}_0 \subset \mathcal{H}$ of the Hamiltonian in Definition 4.1.14, and its relation with the topology of the underlying $(\text{hom}(C, G)^p, \delta^p)$ cochain complex of Definition A.4.4. The result, which is the main feature of [3], is presented in the form of Theorem 4.2.6. In order to begin, let us make certain preliminary considerations about a suitable characterization of the ground state subspace $\mathcal{H}_0 \subset \mathcal{H}$. In fact, it is clear from Definition 4.1.14 that the ground state subspace \mathcal{H}_0 is defined by:

$$\mathcal{H}_0 := \{ |\Psi\rangle \in \mathcal{H} \mid A_{n,x}^0 |\Psi\rangle = |\Psi\rangle \text{ and } B_{n,x}^0 |\Psi\rangle = |\Psi\rangle \}, \quad (4.27)$$

for all $x \in K_n$, $n = 0, 1, \dots, d$. Furthermore, it can be easily shown that \mathcal{H}_0 is non empty, therefore, frustration free. Nonetheless, as we stressed in the previous section, we can also study the ground state space from an operator perspective only. Moving towards that direction, we define:

Definition 4.2.1 (Ground State Projector). Define the operator $\Pi_0 : \mathcal{H} \rightarrow \mathcal{H}_0$ by:

$$\Pi_0 := \mathcal{A}_{s=0} \mathcal{B}_{v=0},$$

where $s = 0 \in \text{hom}(C, G)_{-1}$ is the trivial representation and $v = 0 \in \text{hom}(C, G)^1$ is the identity element.

It is clear that Π_0 is a projector operator as the name suggests, this follows easily from Property 4.1.6 (ii). The fact that Π_0 projects into the ground state space \mathcal{H}_0 follows from the additive form of the Hamiltonian 4.1.14. This motivates the following:

Proposition 4.2.2 (Ground states). *A state $|\Psi\rangle \in \mathcal{H}$ is a ground state if and only if satisfies*

$$\Pi_0 |\Psi\rangle = |\Psi\rangle \quad ,$$

where Π_0 is the ground state projector defined in 4.2.1.

The latter is the characterization we were looking for and represents the ground states as the eigenvectors of the Π_0 operator with eigenvalue one. However, to get a more physical intuition, it is enough to consider the action Π_0 on an arbitrary basis state $|f\rangle \in \mathcal{H}$:

1. \mathcal{A}_0 projects to gauge equivalent states. This can be easily seen by considering the notion of gauge equivalence in this formalism. Consider $f, g \in \text{hom}(C, G)^0$ and the associated basis states $|f\rangle, |g\rangle \in \mathcal{H}$. These states are said to be *gauge equivalent* if there exists a $t \in \text{hom}(C, G)^{-1}$ such that $|f\rangle = A_t |g\rangle = |g + \delta^{-1}t\rangle$. Now, recall from Definition 4.1.5 and Remark 4.1.7, that the generalized projector \mathcal{A}_0 is written as a sum over all elements $t \in \text{hom}(C, G)^{-1}$:

$$\mathcal{A}_0 = \frac{1}{|\text{hom}(C, G)^{-1}|} \sum_t A_t. \quad (4.28)$$

Consequently, two states $|f\rangle, |g\rangle \in \mathcal{H}$ are gauge equivalent if $\mathcal{A}_0 |f\rangle = \mathcal{A}_0 |g\rangle$. Moreover, \mathcal{A}_0 maps a state $|f\rangle \in \mathcal{H}$ into an equal weight superposition of all gauge equivalent states which is evident from Eq.(4.28).

2. \mathcal{B}_0 projects to configurations with trivial generalized holonomy. We can see this from the local decomposition of 4.26, in particular, \mathcal{B}_0 operator can be written in the form:

$$\mathcal{B}_0 = \prod_n \prod_{x \in K_n} B_{n,x}^0,$$

where $B_{n,x}^0$ is as in Definition 4.1.12. Therefore, any state $|g\rangle$ is invariant under the action of \mathcal{B}_0 only if it is invariant under the action of all $B_{n,x}^0$ for all $x \in K_n$ and all $n = 0, 1 \dots, d$. This condition is exactly the same condition the state $|g\rangle$ must fulfill in order to be a ground state of the Hamiltonian in Definition 4.1.14.

From the above discussion, a gauge invariant state can be constructed out of any arbitrary state $|\Psi\rangle \in \mathcal{H}$ by the action of the \mathcal{A}_0 projector, this is, consider the state $|\Psi_G\rangle \in \mathcal{H}$ given by:

$$\begin{aligned} |\Psi_G\rangle &:= \mathcal{A}_0 |\Psi\rangle, \\ &= \sum_f \Psi(f) \mathcal{A}_0 |f\rangle, \\ &= \sum_f \Psi(f) |f_G\rangle \end{aligned} \quad (4.29)$$

where $f \in \text{hom}(C, G)^0$ labels the basis states of \mathcal{H} and the state $|f_G\rangle := \mathcal{A}_0 |f\rangle$, is the gauge invariant state related to the equivalence class of $f \in \text{hom}(C, G)^0$ under gauge transformations. From here is immediate that:

Proposition 4.2.3. *The ground state subspace $\mathcal{H}_0 \subset \mathcal{H}$ is not empty.*

Proof. We prove this by construction. Applying the operator Π_0 over the state

$$|0_G\rangle := \mathcal{A}_0 |0\rangle \quad \text{gives} \quad \Pi_0 |0_G\rangle = |0_G\rangle, \quad (4.30)$$

which, by Proposition 4.2.2, proves that $|0_G\rangle \in \mathcal{H}_0$. Moreover, this state is always non zero and hence $\mathcal{H}_0 \neq \emptyset$, since \mathcal{H}_0 always has at least one element. \square

Notice that the state $|0_G\rangle$, by means of Eq. (4.29), can be understood as a superposition of all basis states that are *gauge equivalent* to the trivial state $|0\rangle$. This state will have an important role in the following section.

4.2.1 Ground State Degeneracy

So far, we have shown that \mathcal{H}_0 is non empty. However, we can show that if this subspace is degenerate, its degeneracy is topological in a sense that will be clear below. To begin, we will show that the states in \mathcal{H}_0 are labeled by the subgroup $\ker(\delta^0) \subset \text{hom}(C, G)^0$, the kernel of the coboundary map δ^0 , more precisely:

Proposition 4.2.4. *Let $f \in \text{hom}(C, G)^0$. The gauge equivalent state $|f_G\rangle := \mathcal{A}_0 |f\rangle$, belongs to the ground state subspace \mathcal{H}_0 , if and only if, $f \in \ker(\delta^0)$.*

Proof. We prove this proposition in two steps:

1. Let us take an arbitrary state $|\Psi\rangle \in \mathcal{H}$. From Proposition 4.2.2, the state $|\Psi\rangle \in \mathcal{H}_0$ if and only if $\Pi_0 |\Psi\rangle = |\Psi\rangle$, which in turn implies that $|\Psi\rangle$ must be gauge invariant. Moreover, from Eq.(4.29) any gauge invariant state is a linear combination of $|f_G\rangle := \mathcal{A}_0 |f\rangle$, for $f \in \text{hom}(C, G)^0$; Hence it is enough to analyze $|f_G\rangle$.
2. We then take our prototype state to be $|f_G\rangle := \mathcal{A}_0 |f\rangle$, for any arbitrary basis state $|f\rangle \in \mathcal{H}$. Let us start assuming that $|f_G\rangle \in \mathcal{H}_0$, this means:

$$\Pi_0 |f_G\rangle = |f_G\rangle, \quad (4.31)$$

using the expansion of Eq.(4.21), we obtain:

$$\begin{aligned} \Pi_0 |f_G\rangle &= \mathcal{B}_0 P_f |0_G\rangle, \\ &= \frac{1}{|\text{hom}(C, G)_1|} \sum_m B_m P_f |0_G\rangle, \\ &= \frac{1}{|\text{hom}(C, G)_1|} \sum_m \chi_m(\delta^0 f) P_f |0_G\rangle, \\ &= \left(\frac{1}{|\text{hom}(C, G)_1|} \sum_m \chi_m(\delta^0 f) \right) |f_G\rangle, \end{aligned}$$

where we have used that $|f_G\rangle = P_f |0_G\rangle$ and $B_m P_f = \chi_m(\delta^0 f) P_f B_m$; Clearly, in order to satisfy Eq.(4.31) necessarily:

$$\frac{1}{|\text{hom}(C, G)_1|} \sum_m \chi_m(\delta^0 f) = 1,$$

which is only true if $\delta^0 f = 0 \in \text{hom}(C, G)^0$, or equivalently, if $f \in \ker(\delta^0)$. Similarly, one just needs to reverse the argument to show that given an $f \in \ker(\delta^0)$, the state $|f_G\rangle$ is a ground state of the model, in other words, $|f_G\rangle \in \mathcal{H}_0$.

□

We can understand the above proposition as completely characterizing the ground state subspace, since all configurations are exhausted by the group $f \in \text{hom}(C, G)^0$. In other words, any state $|f_G\rangle$ with $f \in \ker(\delta^0)$ is a ground state of the model. However, to determine the ground state degeneracy of the model we also need to characterize the basis states of the ground state subspace, which we do below.

Proposition 4.2.5. *The states $\{|f_G\rangle \mid f \in \ker(\delta^0)\}$ form a basis of the ground state subspace and are in one-to-one correspondence with elements of $\mathcal{H}^0(C; G) := \ker(\delta^0) / \text{im}(\delta^{-1})$, the 0-th cohomology group of the co-chain complex $(\text{hom}(C, G)^\bullet, \delta^\bullet)$, see Def. A.4.4 in Appendix 6.*

Proof. It is clear from Proposition 4.2.4 that $|f_G\rangle \in \mathcal{H}_0$ if, and only if, $f \in \ker(\delta^0)$. Also, by gauge equivalence, $|f_G\rangle = |g_G\rangle$ if, and only if, $f - g \in \text{im}(\delta^{-1})$. Therefore, the equivalence class $[f] \in \mathcal{H}^0(C; G)$ with representative $|f_G\rangle$ is well defined; Furthermore, we have a bijection between $\{|f_G\rangle \mid f \in \ker(\delta^0)\}$ and $\mathcal{H}^0(C; G)$. Projecting the basis $\{|f\rangle\}$ of \mathcal{H} into a basis of \mathcal{H}_0 using the ground state projector Π_0 leads to $\{\Pi_0 |f\rangle \mid \Pi_0 |f\rangle \neq 0\} = \{|f_G\rangle \mid \mathcal{B}_0 |f_G\rangle \neq 0\} = \{|f_G\rangle \mid f \in \ker(\delta^0)\}$, and the result follows. \square

Furthermore, proposition 4.2.5 also implies that, for each cohomology class $[f] \in \mathcal{H}^0(C; G)$, there is a well defined operator

$$P_{[f]} := \prod_{g \sim f} P_g = P_f \mathcal{A}_0, \quad (4.32)$$

that creates the ground states from the state $|0\rangle$, this last observation allows us to state the main result of [3]: Given a finite simplicial complex K , its associated chain complex $(C(K), \partial^C)$, a graded group $\{G_n\}$, its associated chain complex of finite abelian groups (G, ∂^G) and \mathcal{H} the Hilbert space of Def. 4.7 with Hamiltonian $H : \mathcal{H} \rightarrow \mathcal{H}$ as defined in 4.1.14. The following theorem follows:

Theorem 4.2.6 (Dimension of the ground state subspace). *The dimension of the ground state subspace \mathcal{H}_0 is given by:*

$$\text{GSD} = |\mathcal{H}^0(C; G)| \cong \prod_i |H^i(C, H_i(G))|$$

Proof. The proof follows immediately from Propositions 4.2.4 and 4.2.5, as well as the Theorem A.4.5 found in the Appendix 6. \square

In physical terms, Theorem 4.2.6 underscores a very useful way to understand the GSD; this is, there is a contribution coming from each individual usual cohomology group $H^i(C, H_i(G))$. Moreover, intricate relations between *geometrical quantities* (related to the C complex) and *gauge quantities* (related to the G complex) can be present. Let us exemplify the latter by restricting C as coming from a closed triangulable manifold. This allows us to use the universal coefficient theorem (See [106] for a general reference), such that we can decompose each contribution as:

$$H^i(C, H_i(G)) = \text{Hom}(H_i(C), H_i(G)) \oplus \text{Ext}^1(H_{i-1}(C), H_i(G)), \quad (4.33)$$

for all $0 \leq i \leq n$. Here is explicit that, if the G complex is not changed, any two homological triangulable manifolds ($X_1 \cong X_2 \Rightarrow C(X_1) = C(X_2)$) will give the same

GSD. On the other hand, the appearance of two different homology groups ($H_i(C)$ and $H_{i-1}(C)$) in the decomposition (4.33) makes the physical interpretation of the terms somewhat cumbersome since it calls for a case by case study. Reason for which we write the examples of Chapter 3 in the next section.

4.3 Examples

We started the discussion on higher gauge theories in Chapter 3 by introducing them through examples. In this section we review these examples using the language we developed in §4.1; Furthermore, we exhibit how Theorem 4.2.6 readily applies to this examples to find the *GSD* which is in agreement with Chapter 3.

4.3.1 1-Gauge Theory

For simplicity we stick to a 2-dimensional simplicial complex $C(K)$ although the formalism accounts for any finite dimensional complex to play the role of the lattice. In the case of a *1-gauge* theory the classical gauge configuration group is simply given by:

$$\text{hom}(C, G)^0 \equiv \text{Hom}(C_1, G_1), \quad (4.34)$$

as a 1-gauge theory has G_1 spins placed at 1-simplices (links) $l \in K_1$. An element of this group $f \in \text{hom}(C, G)^0$ is just a single map $f_1 : C_1 \rightarrow G_1$ that assigns G_1 -spins to the links of the lattice, $l \in K_1$ (Fig. 4.3). Thus, for every link there is a local Hilbert space \mathcal{H}_l spanned by basis elements $|f_1(l)\rangle$. The total Hilbert space is just a tensor product over all local Hilbert spaces:

$$\mathcal{H} = \bigotimes_{l \in K_1} \mathcal{H}_l,$$

with dimension $\dim(\mathcal{H}) = |G_1|^{|K_1|} = |\text{Hom}(C_1, G_1)|$.

The higher gauge symmetry is parametrized by $\text{hom}(C, G)^{-1} \equiv \text{Hom}(C_0, G_1)$; an element of this group is a map $t_0 : C_0 \rightarrow G_1$ that assigns G_1 labels to the vertices of the complex, $v \in K_0$ (Fig. 4.3). So, given $t \in \text{hom}(C, G)^{-1}$ and a classical gauge configuration $f \in \text{hom}(C, G)^0$, from Definition 4.1.2 we have:

$$\begin{aligned} A_t |f\rangle &= |f + \delta^1(t)\rangle, \\ &= |f + t \circ \partial_1^C\rangle, \\ &= |f(x_1) + t \circ \partial_1^C(x_1)\rangle \otimes \dots \otimes |f(x_{|K_1|}) + t \circ \partial_1^C(x_{|K_1|})\rangle, \end{aligned}$$

where in the last line we expanded the actual state to show the action of a gauge transformation on each local basis state.

The notion of holonomy in a 1-gauge theory reduces to the usual 1-holonomy that is measured along the boundary of 2-chains, see §2.1 and §3.1. More precisely, the holonomy values of the theory are parametrized by elements of the group $\text{hom}(C, G)^1 \equiv \text{Hom}(C_2, G_1)$ that associates G_1 -values to 2-simplices (plaquettes), $p \in K_2$. Hence, the 1-holonomy is measured by means of the dual group $\text{hom}(C, G)_1 \equiv \text{Hom}(C_2, \widehat{G}_1)$, such that for $m \in$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 \longrightarrow 0 \\
 & & & \searrow m_2 & \downarrow f_1 & \swarrow t_0 & \\
 & & 0 & \longrightarrow & G_1 & \longrightarrow & 0
 \end{array}$$

Figure 4.3: The chain complexes and maps for the Abelian QDM of §3.1 are shown.

$\text{hom}(C, G)_1$ consisting on a single map $m_2 : C_2 \rightarrow G_1$ (Fig. 4.3) and a classical gauge configuration $f \in \text{hom}(C, G)^0$ we have:

$$\begin{aligned}
 B_m |f\rangle &= \chi_m(\delta^0 f) |f\rangle, \\
 &= \prod_{p \in K_2} m_2(f_1 \circ \partial_2^C(p)) |f\rangle,
 \end{aligned}$$

where we have used the fact that the only non-trivial component of $\delta^0 f$ is:

$$(\delta^0 f)_2 = f_1 \circ \partial_2^C \in \text{Hom}(C_2, G_1). \quad (4.35)$$

From Def. 4.1.14, the Hamiltonian of an Abelian 1-gauge theory reduces to:

$$H = - \sum_{x \in K_0} A_{0,x}^0 - \sum_{y \in K_2} B_{2,y}^0, \quad (4.36)$$

$$= - \sum_{x \in K_0} \left(\frac{1}{|G_1|} \sum_{g \in G_1} A_{e[g,0,x]} \right) - \sum_{x \in K_0} \left(\frac{1}{|G_1|} \sum_{r \in \widehat{G}_1} B_{\hat{e}[r,2,y]} \right), \quad (4.37)$$

where in the last line we used Definition 4.1.12 to show that it coincides with the ones in Eq. (3.2) and Def. ???. To see how the formula for the GSD of Theorem 4.2.6 gives the correct answer let us look at a particular example of a 1-gauge theory in 2 spatial dimensions. We saw in §3.1.1 that the Toric Code model is recovered when the gauge group $G_1 = \mathbb{Z}_2$; The geometrical chain complex $C(K)$ comes from a discretization of the Torus (T^2). From Theorem 4.2.6, the ground state degeneracy of the model is given by:

$$GSD = |\mathcal{H}^0(C; G)| = |H^1(C, H_1(G))| = |\text{Hom}(H_1(C), H_1(G))| = 2^2,$$

where we have used the universal coefficient theorem of Eq.(4.33) to expand the terms in $\mathcal{H}^0(C; G)$; Also $H_1(C) = H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_1(G) = \mathbb{Z}_2$, from which the above result follows.

4.3.2 0,1-Gauge Theory

Let us now look at the example we introduced in §3.2, where we allow the vertices $v \in K_0$ to hold G_0 -degrees of freedom along with the usual G_1 -spins at links $l \in K_1$ of the lattice. We consider a 2-dimensional torus for the chain complex $C(K) \simeq T^2$. The classical gauge configurations are elements of the group:

$$\text{hom}(C, G)^0 \equiv \text{Hom}(C_0, G_0) \oplus \text{Hom}(C_1, G_1), \quad (4.38)$$

in other words, an element $f \in \text{hom}(C, G)^0$ is made of a pair of maps $f = \{f_0, f_1\}$ where $f_0 : C_0 \rightarrow G_0$ labels the vertices with G_0 -spins and $f_1 : C_1 \rightarrow G_1$ assigns G_1 -labels to the links of the lattice. These maps are used to construct a basis $\{|f\rangle\}_{f \in \text{hom}(C, G)^0}$ for the global Hilbert space \mathcal{H} , which in this case is given by:

$$\mathcal{H} = \bigotimes_{v \in K_0} \mathcal{H}_v \bigotimes_{l \in K_1} \mathcal{H}_l,$$

with dimension $\dim(\mathcal{H}) = |\text{Hom}(C_0, G_0)| |\text{Hom}(C_1, G_1)| = |G_0|^{|K_0|} |G_1|^{|K_1|}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 \longrightarrow 0 \\ & & & \searrow m_2 & \downarrow f_1 & \swarrow m_1 & \downarrow f_0 \\ & & 0 & \longrightarrow & G_1 & \xrightarrow{\partial_1^G} & G_0 \longrightarrow 0 \end{array}$$

Figure 4.4: The relevant maps for a 0,1-gauge model are shown. Where $f = \{f_0, f_1\} \in \text{hom}(C, G)^0$ are the classical gauge configurations, the maps $t = \{t_0\} \in \text{hom}(C, G)^{-1}$ parametrize the higher gauge symmetry and $m = \{m_1, m_2\} \in \text{hom}(C, G)^1$ encode the 0- and 1-holonomy values of the theory.

The higher gauge symmetry of the model is parametrized by elements of the group $\text{hom}(C, G)^{-1} \equiv \text{Hom}(C_0, G_1)$ that assigns G_1 -values to the vertices of the lattice. An element $t \in \text{hom}(C, G)^{-1}$ is a map $t_0 : C_0 \rightarrow G_1$ (see Fig. 4.4); To see how the symmetry is implemented on actual quantum states we need to understand the co-boundary operator $\delta^{-1} : \text{hom}(C, G)^{-1} \rightarrow \text{hom}(C, G)^0$ that maps gauge symmetries into gauge configurations. So, given a gauge symmetry $t \in \text{hom}(C, G)^{-1}$, the resulting map $\delta^{-1}t \in \text{hom}(C, G)^0$ has two non-trivial components, namely,

$$\begin{aligned} (\delta^{-1}t)_0 &= \partial_1^G \circ t_0, \\ (\delta^{-1}t)_1 &= t_0 \circ \partial_1^C, \end{aligned}$$

where $\partial_1^G : G_1 \rightarrow G_0$ and $\partial_1^C : C_1 \rightarrow C_0$; Notice also that $(\delta^{-1}t)_0 \in \text{Hom}(C_0, G_0)$ and $(\delta^{-1}t)_1 \in \text{Hom}(C_1, G_1)$, the former transforms the G_0 -spins at vertices whereas the latter transforms the G_1 -spins at links, as we shall see. Consider a gauge configuration $f \in \text{hom}(C, G)^0$, a vertex $x \in K_0$ and a link $y \in K_1$, both arbitrary. The gauge transformation:

$$A_t |f\rangle = \cdots \otimes |f_0(x) + \partial_1^G \circ t_0(x)\rangle \otimes |f_1(y) + t_0 \circ \partial_1^C(y)\rangle \cdots,$$

observe how all maps act consistently on the corresponding vertex and link degrees of freedom.

On the other hand, the group $\text{hom}(C, G)^1$ parametrizes the generalized holonomy values of the theory, an element of such group consists on a pair of maps $m = \{m_1, m_2\}$ where $m_1 : C_1 \rightarrow G_0$ parametrizes the 0-holonomy associated to 1-chains (links) and $m_2 : C_2 \rightarrow G_1$ encodes the 1-holonomy values associated to 2-chains (plaquettes). The actual measurement of the holonomy values is obtained through representations of this group. Such that, for $f \in \text{hom}(C, G)^0$:

$$B_m |f\rangle = \chi_m(\delta^0 f) |f\rangle,$$

where the non-trivial components of $\delta^0 f$ are:

$$\begin{aligned} (\delta^0 f)_1 &= f_0 \circ \partial_1^C - \partial_1^G \circ f_1 \in \text{Hom}(C_1, G_0), \\ (\delta^0 f)_2 &= f_1 \circ \partial_2^C \in \text{Hom}(C_2, G_1). \end{aligned}$$

Observe that the second term, $(\delta^0 f)_2$ is identical to that of a 1-gauge theory, see Eq.(4.35); The novelty brought by the inclusion of G_0 degrees of freedom at vertices is the first term, $(\delta^0 f)_1$, that encodes the 0-holonomy values of the theory. Consequently, the action of the measurement operator is given by:

$$B_m |f\rangle = \prod_{x \in K_1} m_1(f_0 \circ \partial_1^C(l)) m_1(\partial_1^G \circ f_1(l))^{-1} \prod_{y \in K_2} m_2(f_1 \circ \partial_2^C(p)) |f\rangle,$$

where the product over links, $x \in K_1$ measures the generalized values of 0-holonomies while the product over plaquettes $y \in K_2$ does it for the 1-holonomies.

The Hamiltonian operator of the theory is readily obtained from Definition 4.1.14 and in this case reads:

$$H = - \sum_{x \in K_0} A_{0,x} - \sum_{y \in K_1} B_{1,y} - \sum_{z \in K_2} B_{2,z}. \quad (4.39)$$

The three type of operators in the above Hamiltonian are in charge of implementing the higher gauge symmetry and to impose the flat-holonomy constraints on quantum states. In particular, for $x \in K_0$, the vertex operator is given by:

$$A_{0,x} = \frac{1}{|G_1|} \sum_{g \in G_1} A_{e[g,0,x]},$$

and implements what we call as *1-gauge* symmetry, since the symmetry transformations are labeled by elements of G_1 , it is precisely the operator in Eqns. (3.4) and (2.3). The next kind of operator in Eq.(4.39) has its action localized at links, so for $y \in K_1$ we have:

$$B_{1,y} = \frac{1}{|G_0|} \sum_{r \in \hat{G}_1} B_{\hat{e}[r,1,y]},$$

in charge of implementing the *0-holonomy*¹ flatness constraint, it is the operator in Eq. (3.12). Finally, the second type of holonomy constraint is implemented by the plaquette operators, so for $z \in K_2$:

$$B_{2,z} = \frac{1}{|G_1|} \sum_{r \in \hat{G}_1} B_{\hat{e}[r,2,z]},$$

in charge of imposing the *1-holonomy* flatness constraint on quantum states and it corresponds to the operators in Eqns. (??) and (3.3).

Now, let us showcase the appliance of Theorem 4.2.6 to this model. In this sense, let us consider the example of §3.2.1, where we set $G_0 = \mathbb{Z}_2$ and $G_1 = \mathbb{Z}_4$ and a lattice that comes from the discretization of a torus T^2 . Then, the ground state degeneracy of the model is given by:

$$GSD = |\mathcal{H}^0(C; G)| = |H^1(, H_1(G))| = |\text{Hom}(H_1(T^2), H_1(G))|,$$

¹We call this quantity as *0-holonomy* since it takes values in G_0 .

this number is essentially counting the number of ways we can color the generators of the Homology group $H_1(T^2)$ with elements of $H_1(G)$. For this particular 0,1-gauge theory, we have:

$$H_1(T^2) \simeq \mathbb{Z} \oplus \mathbb{Z},$$

the first homology group of a discrete Torus. While the first homology group of the algebraic sequence $G_1 \xrightarrow{\partial_1^G} G_0$ reads:

$$H_1(G) = \frac{\ker \partial_1^G}{\text{Im} \partial_2^G} \simeq \frac{\mathbb{Z}_2}{\{ \}} \simeq \mathbb{Z}_2,$$

leaving for the dimension of the ground state subspace:

$$GSD = 2^2 = 4,$$

which coincides with the result of Eq.(3.20).

4.3.3 1,2-Gauge Theory

We now move on to the next higher gauge theory we exhibited in Chapter 3. We stick to a 2-dimensional lattice $C(K)$ but the construction generalizes straightforwardly to any dimensions, as we will show by the end of the section. Recall that in a 1,2-gauge theory we allow for the existence of degrees of freedom at links and plaquettes of the lattice. In other words, we place G_1 -spins at links $l \in K_1$ and G_2 -spins at plaquettes $p \in K_2$. Naturally the total Hilbert space is

$$\mathcal{H} = \bigotimes_{l \in K_1} \mathcal{H}_l \bigotimes_{p \in K_2} \mathcal{H}_p.$$

Again, the group of classical gauge configurations $\text{hom}(C, G)^0$ is used to label a global basis for \mathcal{H} denoted $\{|f\rangle\}_{f \in \text{hom}(C, G)^0}$. An element of this group, i.e., a classical gauge configuration is made of a pair of maps $f = \{f_1, f_2\}$ where $f_1 \in \text{Hom}(C_1, G_1)$ and $f_2 \in \text{Hom}(C_2, G_2)$, this is,

$$\text{hom}(C, G)^0 \equiv \text{Hom}(C_1, G_1) \oplus \text{Hom}(C_2, G_2), \quad (4.40)$$

we encourage the reader to compare the structure of this group with those of Eqns.(4.34) and (4.38). The dimension of the Hilbert space is :

$$\dim(\mathcal{H}) = |\text{Hom}(C_1, G_1)| |\text{Hom}(C_2, G_2)| = |G_1|^{K_1} |G_2|^{K_2}.$$

The higher gauge symmetry of the model is parametrized by the group $\text{hom}(C, G)^{-1}$ which in this case has two non-trivial components, namely,

$$\text{hom}(C, G)^{-1} \equiv \text{Hom}(C_0, G_1) \oplus \text{Hom}(C_1, G_2),$$

this is, an element $t \in \text{hom}(C, G)^{-1}$ actually consists of a pair $t = \{t_0, t_1\}$ where $t_0 : C_0 \rightarrow G_1$ and $t_1 : C_1 \rightarrow G_2$, see Fig. 4.5. The structure of $\text{hom}(C, G)^{-1}$ is already telling us that there are two types of gauge transformations, fact we already know from §3.3. To see

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 \longrightarrow 0 \\
 & & \downarrow f_2 & \swarrow m_2 & \downarrow f_1 & \searrow t_0 & \\
 0 & \longrightarrow & G_2 & \xrightarrow{\partial_2^G} & G_1 & \longrightarrow & 0
 \end{array}$$

Figure 4.5: The set of relevant maps for 1,2-gauge theory are shown. Classical gauge configurations $f = \{f_1, f_2\}$ and higher gauge transformations $t = \{t_0, t_1\}$ com in pairs of maps, whereas the generalized holonomy values consist of a single map $m = \{m_2\}$.

how the higher gauge symmetry is implemented on quantum states consider a classical gauge configuration $f \in \text{hom}(C, G)^0$ and a higher gauge symmetry $t \in \text{hom}(C, G)^{-1}$, from Definition 4.1.2 we have:

$$A_t |f\rangle = |f + \delta^{-1}t\rangle,$$

so, we need to find the non-trivial components of $\delta^{-1}t$ which in this case turn out to be:

$$\begin{aligned}
 (\delta^{-1}t)_1 &= t_0 \circ \partial_1^C - \partial_2^G \circ t_1, \\
 (\delta^{-1}t)_2 &= t_1 \circ \partial_2^C,
 \end{aligned}$$

such that for arbitrary link $x \in K_1$ and plaquette $y \in K_2$, we have:

$$A_t |f\rangle = \cdots \otimes |f_1(x) + t_0 \circ \partial_1^C(x) + \partial_2^G \circ t_1(x)\rangle \otimes |f_2(y) + t_1(y)\rangle \cdots$$

The generalized holonomy values of the theory are measured via representations in $\text{hom}(C, G)_1$ which in this case takes the following form:

$$\text{hom}(C, G)_1 \equiv \text{Hom}(C_2, \widehat{G}_1), \quad (4.41)$$

this is, the only notion of holonomy in a (2-dimensional) 1,2-gauge theory is the 1-holonomy associated to 2-chains (plaquettes) in C_2 . A representation $m \in \text{hom}(C, G)_1$ is a map $m_2 : C_2 \rightarrow \widehat{G}_1$, such that:

$$B_m |f\rangle = \chi_m(\delta^0 f) |f\rangle,$$

where the map $\delta^0 f \in \text{hom}(C, G)^1$ has one non-trivial component, namely,

$$(\delta^0 f)_2 = f_1 \circ \partial_2^C - \partial_2^G \circ f_2, \quad (4.42)$$

thus yielding:

$$B_m |f\rangle = \prod_{x \in K_2} m_2(f_1 \circ \partial_2^C(x)) m_2(\partial_2^G \circ f_2(x))^{-1} |f\rangle.$$

where the local measurements are being carried over plaquettes $x \in K_2$. The Hamiltonian is obtained from Def. 4.1.14, giving:

$$H = - \sum_{x \in K_0} A_{0,x} - \sum_{y \in K_1} A_{1,y} - \sum_{z \in K_2} B_{2,z}. \quad (4.43)$$

The above operator is exactly the same as the one in Eq.(3.22). The 1-gauge symmetry is implemented by vertex operators $A_{0,x}$ just as in a usual 1-gauge theory (cf. §3.1 and §4.3.1). There is a new kind of gauge transformation, implemented on quantum states through the link operator, for $y \in K_1$

$$A_{1,y} = \frac{1}{|G_2|} \sum_{\alpha \in G_2} A_{e[\alpha,1,y]},$$

which we call *2-gauge* transformation since the symmetry transformations are labeled by elements $\alpha \in G_2$. Note that this operator is exactly the same as the one in Eq.(3.23).

The last operator in Eq.(4.43) is in charge of enhancing the 1-holonomy flatness condition. From Eq.(3.24) in §3.3 we know that the notion of 1-holonomy of a 1,2-gauge theory has a slight modification from that of a pure 1-gauge theory, this can be seen comparing Eq.(4.42) and Eq.(4.35). The local projection is carried over by the plaquette operator,

$$B_{2,z} = \frac{1}{|G_1|} \sum_{r \in \hat{G}_1} B_{\hat{e}[r,2,y]},$$

where $z \in K_2$ is an arbitrary plaquette.

Let us, now, consider the example of §3.3.1, this is, a 1,2-gauge theory defined on a discretization of a sphere $C(K) \simeq S^2$; Where the gauge groups are $G_1 = \mathbb{Z}_2 = \{1, -1\}$ and $G_2 = \mathbb{Z}_4 = \{1, i, -1, -i\}$ and the homomorphism between them $\partial_2^G : G_2 \rightarrow G_1$ is defined by $\partial_2^G(i) = -1$. Theorem 4.2.6 gives the ground state degeneracy of the model as:

$$GSD = |\mathcal{H}^0(C; G)| = |H^2(C, H_2(G))| = |\text{Hom}(H_2(C), H_2(G))|,$$

now

$$H_2(G) = \frac{\ker \partial_2}{\text{Im } \partial_3} = \frac{\mathbb{Z}_2}{\{\}} \simeq \mathbb{Z}_2.$$

On the other hand, $H_2(S^2) = \mathbb{Z}$, therefore:

$$GSD = |\text{Hom}(\mathbb{Z}, \mathbb{Z}_2)| = 2,$$

where the last equality holds since there are only two morphisms from the free Abelian group \mathbb{Z} to \mathbb{Z}_2 , namely the one that sends every element of \mathbb{Z} into the trivial element $0 \in \mathbb{Z}_2$ and the one that sends the generator of \mathbb{Z} into the generator of \mathbb{Z}_2 . Observe that this accounts for the ground states shown in §3.3.1.

We have seen that, in 2 spatial dimensions, the formalism of §4.1 accounts for the models we presented in Chapter 3. To showcase the power of this formalism, let us see how the algebraic structures are naturally modified by considering a 3-dimensional 1,2-gauge theory. It is not difficult to note that the main difference between a 2-dim 1,2-gauge theory and a 3-dim one comes in the generalized notion of Holonomies. To see this, it is enough to look at the groups that parametrize gauge configurations, gauge transformations and holonomy values, namely,

$$\begin{aligned} \text{hom}(C, G)^0 &\equiv \text{Hom}(C_1, G_1) \oplus \text{Hom}(C_2, G_2), \\ \text{hom}(C, G)^{-1} &\equiv \text{Hom}(C_0, G_1) \oplus \text{Hom}(C_1, G_2), \\ \text{hom}(C, G)^1 &\equiv \text{Hom}(C_2, G_1) \oplus \text{Hom}(C_3, G_2), \end{aligned}$$

it is enough to compare the last expression with the one in Eq.(4.41) to see that a 1,2-gauge theory in 3 spatial dimensions has a new notion of holonomy whose values are parametrized by elements in $\text{Hom}(C_3, G_2)$. Let us pay attention to this group, its elements associate G_2 -values to 3-chains (volumes) of the lattice, which we call *2-holonomies*. As a consequence, a new operator kind of operator appears in the Hamiltonian of the theory that enforces the flatness condition over these 2-holonomies. The details of how this operator arises are similar to the example in the next section.

4.3.4 1,2,3-Gauge Theory

To end the detailed exposure of Abelian higher gauge theories and the formalism we use to describe them in full generality we exhibit a 1,2,3-gauge theory in 3 spatial dimensions. This theory was introduced in §3.4 where we showed all operators and the ground states for a particular example. In this section, we re-write the model using the formalism we developed in this chapter. Let us begin with the three groups in the sequence of Eq.(4.12), namely:

- Classical gauge configurations are parametrized by elements in:

$$\text{hom}(C, G)^0 \equiv \text{Hom}(C_1, G_1) \oplus \text{Hom}(C_2, G_2) \oplus \text{Hom}(C_3, G_3),$$

in other words, we have G_1 -spins at links $l \in K_1$, G_2 -spins at plaquettes $p \in K_2$ and G_3 -spins at cubes $c \in K_3$.

- The higher gauge symmetry is labeled by the group:

$$\text{hom}(C, G)^{-1} \equiv \text{Hom}(C_0, G_1) \oplus \text{Hom}(C_1, G_2) \oplus \text{Hom}(C_2, G_3),$$

the structure of this group already tells us that there are three different notions of gauge transformations, the *1-*, *2-* and *3-gauge* transformations.

- Finally, the generalized holonomies of the theory take values in the group:

$$\text{hom}(C, G)^1 \equiv \text{Hom}(C_2, G_1) \oplus \text{Hom}(C_3, G_2),$$

the first term accounts for the *1-holonomies* of the theory, whereas the second term $\text{Hom}(C_3, G_2)$ does it for the *2-holonomies*.

Elements of the above groups are shown in Fig. 4.6.

The Hamiltonian of the model is given by:

$$H = - \sum_{x \in K_0} A_{0,x} - \sum_{y \in K_1} A_{1,y} - \sum_{z \in K_2} A_{2,z} - \sum_{z' \in K_2} B_{2,z'} - \sum_{w \in K_3} B_{3,w}, \quad (4.44)$$

and it corresponds to the Hamiltonian of Eq. (3.32). Consider the 1,2,3-gauge theory defined on a discretization of S^3 . Moreover, let $G_1 = G_2 = G_3 = \mathbb{Z}_4$, or:

$$1 \xrightarrow{\partial_4} G_3 \xrightarrow{\partial_3} G_2 \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} 1, \quad (4.45)$$

where ∂_1 and ∂_4 are trivial and ∂_2, ∂_3 are both defined by

$$\partial_j(i) = -1, \quad j = 2, 3,$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C_3 & \xrightarrow{\partial_3^C} & C_2 & \xrightarrow{\partial_2^C} & C_1 & \xrightarrow{\partial_1^C} & C_0 & \longrightarrow & 0 \\
 & & \downarrow f_3 & \swarrow m_3 & \downarrow t_2 f_2 & \swarrow m_2 & \downarrow t_1 & \swarrow f_1 & \downarrow t_0 & & \\
 0 & \longrightarrow & G_3 & \xrightarrow{\partial_3^G} & G_2 & \xrightarrow{\partial_2^G} & G_1 & \longrightarrow & 0
 \end{array}$$

Figure 4.6: The set of relevant maps for a 1,2,3-gauge theory in 3 spatial dimensions are shown. Classical gauge configurations $f = \{f_1, f_2, f_3\}$ are shown in solid. Higher gauge transformations $t = \{t_0, t_1, t_2\}$ in dotted, whereas the generalized holonomy values $m = \{m_2, m_1\}$ are shown as dashed lines.

where i is the generator of $\mathbb{Z}_4 = \{1, i, -1, -i\}$. We have seen, in §3.4 that this model has two ground states when defined in a discretization of the 3-sphere S^3 . This ground state degeneracy of the model coincides with the result of Theorem 4.2.6, namely,

$$GSD = |\mathcal{H}^0(C; G)| = |\text{Hom}(H_3(S^3), H_3(G))| = 2,$$

where the last equation comes from observing that the only non trivial homology group of S^3 is the third one. In particular $H_3(S^3) = \mathbb{Z}$ and the third homology group of the abstract chain complex of eq.(4.45) is given by:

$$H_3(G) = \frac{\ker \partial_3}{\text{Im } \partial_4} = \frac{\mathbb{Z}_2}{\{ \}} \simeq \mathbb{Z}_2.$$

The presence of topological order of all degrees can be reached by cleverly choosing the groups and the maps between them. For instance, consider the following abstract chain complex:

$$1 \xrightarrow{\partial_4} \mathbb{Z}_4 \xrightarrow{\partial_3} \mathbb{Z}_8 \xrightarrow{\partial_2} \mathbb{Z}_4 \xrightarrow{\partial_1} 1,$$

where the boundary maps are given by:

$$\begin{aligned}
 \partial_3(i) &= \omega^4, \\
 \partial_2(\omega) &= -1,
 \end{aligned}$$

where i is the generator of \mathbb{Z}_4 and ω is the generator of \mathbb{Z}_8 . The non trivial homology groups of the complex chain are:

$$H_3(G) = \ker \partial_3 \simeq \mathbb{Z}_2, \quad H_2(G) = \frac{\ker \partial_2}{\text{Im } \partial_3} = \frac{\mathbb{Z}_4}{\mathbb{Z}_2} \simeq \mathbb{Z}_2, \quad \text{and} \quad H_1(G) = \frac{\mathbb{Z}_4}{\text{Im } \partial_2} \simeq \mathbb{Z}_2, \quad (4.46)$$

thus, if the model is defined on $S^2 \times S^1$, for example, it would have \mathbb{Z}_2 1,2 and 3-topological order.

Chapter 5

Entanglement Entropy in Abelian Higher Gauge Theories

The previous chapter was devoted to introduce the Homological description of Abelian Higher Gauge Theories; Furthermore, Theorem 4.2.6 showcases the power of such description by efficiently encoding all the topological information of the models in the cohomology group $\mathcal{H}^0(C; G)$. Remarkably, this formalism seems adequate to extract information about the Entanglement Entropy of such models, as we will precisely see. Hence, in this Chapter we explicitly calculate the entanglement entropy for the class of models presented in Chapter 3 using the formalism developed in Chapter 4 and [3]. We show that the obtention of the entanglement entropy follows from a calculation very similar to that of 2.3. To this intent, we begin the Chapter by precisely defining the bipartition of the $(C(K), \partial^C)$ chain complex (the lattice) into a subcomplex, $(C(K_A), \partial_A^C)$, and its complement. We then observe that a generalized gauge theory can be defined in the subcomplex $(C(K_A), \partial_A^C)$ which will be relevant for the interpretation of the entanglement entropy. We end the Chapter by showing how this calculation applies for specific examples.

5.1 Bipartition of the Geometrical Chain Complex

As it was shown in Chapter 4, the geometrical content of the model is encoded in a chain complex, which plays the role of the lattice; Again, we consider a simplicial chain complex, for convenience. Recall from §?? that the entanglement entropy calculation starts off by defining a bipartition of the lattice. So, in order to calculate the entanglement entropy for the class of models defined by the Hamiltonian in definition 4.1.14 we first need to be clear about the bipartition that is going to be considered. In this sense, the system is divided into two regions A and B , as in §2.3 and [66, 75, 76]. This essentially means that the total Hilbert space is split into $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where \mathcal{H}_A is the Hilbert space associated to region A , the region to which we will have access to. For convenience we want such region to be algebraically described by a subcomplex of $(C(K), \partial^C)$.

Recalling Eq. (4.1), the $(C(K), \partial^C)$ chain complex consists on a family of the finitely generated chain groups C_n made by formal sums of the generators $x \in K_n$. Consider then, $K = \bigcup_{n=0}^d K_n$ a d -dimensional simplicial complex that we split into two regions A and B in the following way: For each $0 \leq n \leq d$, we divide the set of n -simplices in the form $K_n = K_{n,A} \cup K_{n,B}$ where $K_{n,A}$ stands for the set of n -simplices in region A and $K_{n,B}$

the set of simplices in region B . We do this in a way such that $K_A = \bigcup_{n=0}^d K_{n,A}$ is a subcomplex of K , in the sense of Definition A.4.1 in App.(6).

Essentially, we are splitting the simplicial complex K into a subcomplex K_A and its complement, which in general is not a simplicial complex on its own. This is done on purpose since the chain complex structure of K_A will allow us to perform the calculation of the entanglement entropy in the most general case using the formalism of Chapter 4. In fact, a generalized gauge theory strictly defined in region A can be described by the homological algebra language and this ultimately allow us to have a physical interpretation of the entanglement entropy in Abelian higher gauge theories.

To this intent, let $C_{n,A}$ be the n -chain group generated by n -simplices, $x \in K_{n,A}$, and let also $\partial_{n,A}^C : C_{n,A} \rightarrow C_{n-1,A}$ be the canonical restriction of ∂_n^C into the subset $K_{n,A}$, this is, $\partial_{n,A}^C := \partial_n^C|_A$. It is not difficult to show that $(C(K_A), \partial_A^C)$ is, in fact, a chain complex:

$$0 \hookrightarrow C_{d,A} \xrightarrow{\partial_{d,A}^C} C_{d-1,A} \xrightarrow{\partial_{d-1,A}^C} \cdots \xrightarrow{\partial_{2,A}^C} C_{1,A} \xrightarrow{\partial_{1,A}^C} C_{0,A} \rightarrow 0, \quad (5.1)$$

since $\partial_{n,A}^C \circ \partial_{n+1,A}^C = 0$. Following the construction of [3] and Chapter 4, we also consider the higher gauge group as the usual abstract chain complex of Abelian groups of Eq.(4.3), namely

$$0 \hookrightarrow G_d \xrightarrow{\partial_d^G} G_{d-1} \xrightarrow{\partial_{d-1}^G} \cdots \xrightarrow{\partial_2^G} G_1 \xrightarrow{\partial_1^G} G_0 \rightarrow 0, \quad (5.2)$$

where all groups G_n are finite Abelian groups and $\partial_n^G : G_n \rightarrow G_{n-1}$ are group homomorphisms, such that $\partial_n^G \circ \partial_{n+1}^G = 0$. Note that the ∂ in Eq.(5.1) and the one in Eq.(5.2) represent different group morphisms.

With the two chain complexes of Eqns. (5.1) and (5.2) in hand, it is natural to define homomorphisms between them. In general, we can consider the group

$$\text{hom}(C_A, G)^p := \bigoplus_{n=0}^d \text{Hom}(C_{n,A}, G_{n-p}),$$

that consists on the collection of all group morphisms $f_{n,A} : C_{n,A} \rightarrow G_{n-p}$ whose support lies exclusively on K_A . Chapter 4 was devoted to discuss how the three essential notions of a generalized higher gauge theory are naturally parametrized by the groups corresponding to $p = -1, 0, 1$ and their representations.

Of particular interest is the group:

$$\text{hom}(C_A, G)^0 = \bigoplus_n \text{Hom}(C_{n,A}, G_n),$$

which is made up of collections of maps $f_A = \{f_{n,A}\}$ for $n = 0, 1, \dots, d$ that we understand as the classical gauge configurations restricted to region A :

$$\begin{aligned} f_{n,A} : C_{n,A} &\rightarrow G_n, \\ x &\mapsto f_{n,A}(x), \end{aligned}$$

where $x \in K_{n,A}$ and $f_{n,A}(x) \in G_n$. The group elements $f \in \text{hom}(C_A, G)^0$ are then used to label a basis $\{|f\rangle_A\}$ of the Hilbert space $\mathcal{H}_A \subset \mathcal{H}$, such that any arbitrary state $|\Psi\rangle_A \in \mathcal{H}_A$ is written as:

$$|\Psi\rangle_A = \sum_{f \in \text{hom}(C_A, G)^0} \Psi(f) |f\rangle_A.$$

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Similarly, consider the groups $\text{hom}(C_A, G)^{-1}$ and $\text{hom}(C_A, G)^1$, corresponding to parameters of the generalized gauge transformations and the holonomy values, respectively. We know from Chapter 4 that a higher gauge theory can be constructed by means of the following co-chain complex:

$$\text{hom}(C_A, G)^{-1} \xrightarrow{\delta_A^{-1}} \text{hom}(C_A, G)^0 \xrightarrow{\delta_A^0} \text{hom}(C_A, G)^1, \quad (5.3)$$

where the co-boundary map $\delta_A^p : \text{hom}(C_A, G)^p \rightarrow \text{hom}(C_A, G)^{p-1}$ is defined by,

$$(\delta_A^p f)_n := f_{n-1, A} \circ \partial_{n, A}^C - (-1)^p \partial_{n-p}^G \circ f_{n, A},$$

for any $f \in \text{hom}(C_A, G)^p$. This higher gauge theory is restricted to region A and the corresponding Hamiltonian operator can be written straightforwardly from Definition 4.1.14. However, for reasons that will be clear in §5.3, we want to further restrict this theory by allowing gauge transformations that act on the interior of K_A only; Thus discarding gauge transformations that act at the boundary $\partial(A)$. This can be easily done by considering a special subgroup of $\text{hom}(C_A, G)^{-1}$. To that intent, let us give a precise notion of the *interior* of A .

Definition 5.1.1 (Interior of A). Let $K_{n, \tilde{A}} = \{x \in K_{n, A} \mid x \cap \partial(A) = \emptyset\}$, this is, the set of n -simplices whose intresection with the boundary of A is null. Then, we define the *interior* of A as

$$\text{int}(A) = \tilde{A} := \bigcup_{n=0}^d K_{n, \tilde{A}}.$$

The notion of interior allows us to define a subgroup of $\text{hom}(C_A, G)^p$ whose support is contained on \tilde{A} only. More precisely, let us consider homomorphisms whose support lie on the interior of A (denoted \tilde{A}), namely $\text{Hom}(C_{n, \tilde{A}}, G_{n-p})$. They define the group:

$$\text{hom}(C_{\tilde{A}}, G)^p := \bigoplus_{n=0}^d \text{Hom}(C_{n, \tilde{A}}, G_{n-p}),$$

with elements $f \in \text{hom}(C_{\tilde{A}}, G)^p$, consisting of collection of maps $f = \{f_n\}$:

$$\begin{aligned} f_n : C_{n, \tilde{A}} &\rightarrow G_{n+p}, \\ x &\mapsto f_n(x), \end{aligned}$$

where $x \in K_{n, \tilde{A}}$ and $f_n(x) \in G_{n+p}$. It is straightforward to show that $\text{hom}(C_{\tilde{A}}, G)^p$ is a subgroup of $\text{hom}(C_A, G)^p$. Moreover, we can consider the restriction of the co-boundary operator δ^p into the interior of A , this is, $\delta_{\tilde{A}}^p := \delta^p|_{\tilde{A}}$, such that the following particular sequence:

$$\text{hom}(C_{\tilde{A}}, G)^{-1} \xrightarrow{\delta_{\tilde{A}}^{-1}} \text{hom}(C_{\tilde{A}}, G)^0 \xrightarrow{\delta_{\tilde{A}}^0} \text{hom}(C_{\tilde{A}}, G)^1, \quad (5.4)$$

is a co-chain complex, namely $\delta_{\tilde{A}}^0 \circ \delta_{\tilde{A}}^{-1} = 0$. This co-chain complex encodes an Abelian Higher Gauge Theory over K_A where the gauge transformations are restricted to act on

the interior of A only (which we denote as \tilde{A}). The Hamiltonian operator, $H_{\tilde{A}} : \mathcal{H}_A \rightarrow \mathcal{H}_A$, of such theory follows neatly from Definition 4.1.14:

$$H_{\tilde{A}} = \sum_{n=0}^d \sum_{x \in K_{n,\tilde{A}}} A_{n,x} - \sum_{n=0}^d \sum_{y \in K_{n,A}} B_{n,y}. \quad (5.5)$$

More importantly, theorem 4.2.6 provides the dimension of the ground state subspace (GSD) of this model, which reads:

$$GSD_{\tilde{A}} = \frac{|\ker \delta_A^0|}{|\operatorname{Im} \delta_{\tilde{A}}^{-1}|}. \quad (5.6)$$

In this case $GSD_{\tilde{A}}$ is essentially counting the number of flat configurations in A , $|\ker \delta_A^0|$, modulo internal gauge transformations $|\operatorname{Im} \delta_{\tilde{A}}^{-1}|$. In the next section we show how this restricted gauge theory, and in particular its ground state degeneracy ($GSD_{\tilde{A}}$) makes the obtention of the reduced density matrix of a higher gauge theory an easy task.

5.2 Reduced Density Matrix

Recall from §?? that the next step in the entanglement entropy calculation regards the density matrix of the model. In general, we are considering Abelian higher gauge theories whose dynamics are determined by the Hamiltonian operator of Definition 4.1.14, this is:

$$H := - \sum_{n=0}^d \sum_{x \in K_n} A_{n,x} - \sum_{n=0}^d \sum_{y \in K_n} B_{n,y},$$

where $A_{n,x}$ and $B_{n,y}$ are the local implementation of gauge transformations and flat generalized holonomy constraints (see Definition 4.1.12). We are interested in the entanglement properties of the ground state subspace \mathcal{H}_0 of the above Hamiltonian, for this reason:

Definition 5.2.1 (Density Matrix). Consider the above Hamiltonian operator, the ground state projector of Definition 4.2.1, $\Pi_0 : \mathcal{H} \rightarrow \mathcal{H}_0$. Let also GSD be the *ground state degeneracy* of Theorem 4.2.6. We define the density matrix of the model as:

$$\rho := \frac{\Pi_0}{\operatorname{tr}(\Pi_0)} = \frac{\Pi_0}{GSD}. \quad (5.7)$$

The density matrix can be written in a more convenient way by making use of the generalized projectors. Recall, from §4.2 that the ground state projector is written:

$$\Pi_0 = \mathcal{A}_0 \mathcal{B}_0,$$

the above expression is expanded in terms of the generalized gauge transformations and holonomy measurement operators (cf. remark 4.1.7), this is,

$$\begin{aligned} \Pi_0 &= \left(\frac{1}{|\operatorname{hom}(C, G)^{-1}|} \sum_{t' \in \operatorname{hom}(C, G)^{-1}} A_{t'} \right) \left(\frac{1}{|\operatorname{hom}(C, G)_1|} \sum_{m' \in \operatorname{hom}(C, G)_1} B_{m'} \right) \\ &= \left(\frac{1}{|\operatorname{Im}(\delta^{-1})|} \sum_{t \in \operatorname{hom}(C, G)^{-1} / \ker(\delta^{-1})} A_t \right) \left(\frac{1}{|\operatorname{Im}(\delta_1)|} \sum_{m \in \operatorname{hom}(C, G)_1 / \ker(\delta_1)} B_m \right), \end{aligned}$$

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where in the last line we factored $\text{hom}(C, G)^{-1}$ into the gauge transformations that act trivially on gauge configurations, parametrized by $\ker(\delta^{-1})$, and the image of the co-boundary map $\text{Im}(\delta^{-1})$ (see Appendix A.4). Similarly, in the second sum we factor $\text{hom}(C, G)_1$ into $\ker(\delta_1)$ and $\text{Im}(\delta_1)$. By doing this we ensure that the sums run over independent group elements only, in other words, we factor out redundancies. This yields for the density matrix:

$$\rho = \frac{1}{GSD} \frac{1}{|\text{Im}(\delta^{-1})||\text{Im}(\delta_1)|} \left(\sum_{t \in \text{hom}(C, G)^{-1}/\ker(\delta^{-1})} A_t \right) \left(\sum_{m \in \text{hom}(C, G)_1/\ker(\delta_1)} B_m \right). \quad (5.8)$$

Provided with the density matrix ρ we can now calculate the reduced density matrix by considering the bipartition of the geometric chain complex $(C_\bullet, \partial_\bullet^C)$ described in §5.1, which essentially splits the Hilbert space into two subspaces $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Definition 5.2.2 (Partial trace). The partial trace of a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is the linear map $\text{Tr}_B : \mathcal{H} \rightarrow \mathcal{H}_A$ determined by:

$$\text{Tr}_B(\mathcal{O}_A \otimes \mathcal{O}_B) = \mathcal{O}_A \text{Tr}(\mathcal{O}_B),$$

where $\mathcal{O}_A : \mathcal{H}_A \rightarrow \mathcal{H}_A$ and $\mathcal{O}_B : \mathcal{H}_B \rightarrow \mathcal{H}_B$ are operators acting on A and B respectively.

The partial trace is useful to obtain information about observables of a subsystem since $\langle \mathcal{O}_A \rangle = \text{Tr}(\mathcal{O}_A \rho_A)$, where ρ_A is defined next.

Definition 5.2.3 (Reduced density matrix). The *reduced density matrix* is obtained by taking a partial trace over the B region, interpreted as the region where we have no access to, this is:

$$\rho_A := \text{Tr}_B(\rho). \quad (5.9)$$

To obtain the reduced density matrix, we need to perform the partial trace of ρ over region B . In this sense, let us consider a basis $\{|f_{n,B}\rangle\}$ for \mathcal{H}_B , where $f_{n,B} \in \text{hom}(C_B, G)^0$ is the restriction of $\text{hom}(C, G)^0$ to B . For simplicity, let us denote this basis as $\{|b_i\rangle\}$, with $i = 1, 2, \dots, |\mathcal{H}_B|$. Then, the reduced density matrix is given by:

$$\rho_A = \frac{1}{GSD} \frac{1}{|\text{Im}(\delta^{-1})||\text{Im}(\delta_1)|} \sum_i \langle b_i | \left(\sum_{t,m} A_t B_m \right) | b_i \rangle,$$

where $t \in \frac{\text{hom}(C, G)^{-1}}{\ker(\delta^{-1})}$ label the non-trivial higher gauge transformations and $m \in \frac{\text{hom}(C, G)_1}{\ker(\delta_1)}$ label the non-trivial representations of the holonomy values. Now, both A_t and B_m are traceless operators unless they are equal to the identity operator, as we show in Property C.4.12 of Appendix A.4. Because of this, the only terms that survive the partial trace are those for which A_t and B_m act trivially in \mathcal{H}_B , yielding for the reduced density matrix:

$$\rho_A = \frac{1}{GSD} \frac{1}{|\text{Im}(\delta^{-1})||\text{Im}(\delta_1)|} \left(\sum_{p,q} A_p B_q \right) \text{Tr}_B(\mathbb{1}_B), \quad (5.10)$$

where the sums now run over: the independent internal gauge transformations, $p \in \frac{\text{hom}(C_{\tilde{A}}, G)^{-1}}{\ker(\delta_{\tilde{A}}^{-1})}$, and the non-trivial holonomy values in A , $q \in \frac{\text{hom}(C_A, G)_1}{\ker(\delta_1|_A)}$. Finally, by noting that $\text{Tr}_B(\mathbb{1}_B) = \dim(\mathcal{H}_B)$, we reach:

$$\rho_A = \frac{1}{GSD} \frac{\dim(\mathcal{H}_B)}{|\text{Im}(\delta^{-1})||\text{Im}(\delta_1)|} \left(\sum_{p,q} A_p B_q \right).$$

We can further simplify this expression. Applying the first isomorphism theorem [107] on the sequence of Eq. (4.12) it is easy to show that $\dim(\mathcal{H}) = |\text{hom}(C, G)^0| = |\ker(\delta^0)| |\text{Im}(\delta^0)|$, where $|\ker(\delta^0)|$ counts the classical configurations with trivial holonomy (flat basis states) while $|\text{Im}(\delta^0)|$ counts all possible non-trivial holonomy values. Moreover, in Appendix A.4 we show that $|\text{Im}(\delta^0)| = |\text{Im}(\delta_1)|$ which allows us to write the following identity:

$$GSD |\text{Im}(\delta^{-1})||\text{Im}(\delta_1)| = \dim(\mathcal{H}) = \dim(\mathcal{H}_A) \dim(\mathcal{H}_B),$$

replacing the above into the expression for the reduced density matrix we get:

$$\rho_A = \frac{1}{\dim(\mathcal{H}_A)} \left(\sum_{p,q} A_p B_q \right), \quad (5.11)$$

where, again, the sums are carried over, $p \in \frac{\text{hom}(C_{\tilde{A}}, G)^{-1}}{\ker(\delta_{\tilde{A}}^{-1})}$, the internal (non-trivial) higher gauge transformation and $q \in \frac{\text{hom}(C_A, G)_1}{\ker(\delta_1|_A)}$, the non trivial holonomy values in A . As we will see in the next section, the expression we got for the reduced density matrix will make the calculation of the entanglement entropy a rather straightforward calculation.

5.3 Entanglement Entropy

Having found the reduced density matrix, Eq. (5.11), we are able to calculate its Von Neumann entropy, also known as *entanglement entropy*:

$$S_A = -\text{Tr}(\rho_A \log(\rho_A)),$$

where the trace is taken over region A . To calculate this quantity we first need to evaluate the logarithm of the reduced density matrix, $\log(\rho_A)$; Usually this is done using series expansion so we first calculate the square of ρ_A , namely,

$$\rho_A^2 = \frac{1}{\dim(\mathcal{H}_A)^2} \left(\sum_{p,q} A_p B_q \right) \left(\sum_{p',q'} A_{p'} B_{q'} \right) = \frac{|\text{Im}(\delta_A^0)||\text{Im}(\delta_{\tilde{A}}^{-1})|}{\dim(\mathcal{H}_A)^2} \left(\sum_{p,q} A_p B_q \right),$$

where in the last equality the factors in the numerator come from rearranging the sums over $p' \in \frac{\text{hom}(C_{\tilde{A}}, G)^{-1}}{\ker(\delta_{\tilde{A}}^{-1})}$ and over $q' \in \frac{\text{hom}(C_A, G)_1}{\ker(\delta_1|_A)}$. This leaves for the square of the density matrix:

$$\rho_A^2 = \frac{|\text{Im}(\delta_A^0)||\text{Im}(\delta_{\tilde{A}}^{-1})|}{\dim(\mathcal{H}_A)} \rho_A = \lambda \rho_A. \quad (5.12)$$

From the above expression we know that the reduced density matrix ρ_A is actually representing a mixed state which is a consequence of the initial state (the ground state of the model) being indeed entangled; The reduced system is left in a mixed state since some information was lost in the process of taking the partial trace.

Now we can calculate the logarithm of ρ_A by series expansion, which yields:

$$\log(\rho_A) = \frac{\log(\lambda)}{\lambda} \rho_A.$$

Thus leaving for the *entanglement entropy*:

$$S_A = -\text{Tr}(\rho_A \log(\rho_A)) = -\text{Tr}(\rho_A \log(\lambda)) = \log(1/\lambda) \text{Tr}(\rho_A) = \log(1/\lambda), \quad (5.13)$$

where we have used $\text{Tr}(\rho_A) = 1$. Let us look at the λ factor more carefully, since it encodes the essential information about the entanglement entropy of the model. Essentially, we have:

$$\frac{1}{\lambda} = \frac{\dim(\mathcal{H}_A)}{|\text{Im}(\delta_A^0)| |\text{Im}(\delta_{\tilde{A}}^{-1})|} = \frac{|\ker(\delta_A^0)|}{|\text{Im}(\delta_{\tilde{A}}^{-1})|}, \quad (5.14)$$

where we use $\dim(\mathcal{H}_A) = |\text{hom}(C_A, G)^0| = |\ker(\delta_A^0)| |\text{Im}(\delta_A^0)|$. Equation (5.14) is already very interesting since it relates $1/\lambda$ to the GSD of a model restricted to \mathcal{H}_A and for which gauge transformations only act in the interior of A , which we denote \tilde{A} . By replacing this expression into Eq.(5.13) we are able to state our main result, that the entanglement entropy is given by:

$$S_A = \log \left(\frac{|\ker(\delta_A^0)|}{|\text{Im}(\delta_{\tilde{A}}^{-1})|} \right) = \log(\text{GSD}_{\tilde{A}}). \quad (5.15)$$

We want to highlight that the only requirement we asked for the bipartition is that the simplicial complex K is divided into a subcomplex K_A and its complement. Therefore, this result is very general, since it is valid for any Higher Gauge Theory of the type described in chapters 3, 4 and constructed in [3].

In the following section we exhibit the power of this result by calculating the entanglement entropy of several examples coming from Higher Gauge Theories, which include the familiar Quantum Double Models in their Abelian version and in particular the Toric Code.

5.4 Examples

In this section we showcase the generality of our result by calculating the entanglement entropy of the models shown in chapter 3. In simple terms, the expression we obtained for S_A requires us to find the number of basis states in \mathcal{H}_A with trivial holonomy, $|\ker \delta_A^0|$, that are not gauged away by transformations restricted to act on the interior of A given by $|\text{Im} \delta_{\tilde{A}}^{-1}|$. Such counting calls for a case by case study, as we shall see.

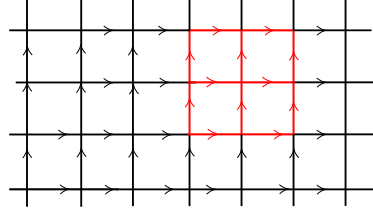


Figure 5.1: Region A of a 2D squared lattice is shown in red

5.4.1 (2D) 1-Gauge: \mathbb{Z}_2 Toric Code

We start with the model we introduced in §2.2, and in §3.1.1; whose entanglement entropy was already calculated in the literature [35, 108]. The model consists of plaquette operators, B_p , for the 1-holonomy, and vertex operators, A_v , for the gauge transformations. Considering the partition of the lattice shown in Fig. 5.1; Eq. 5.15 tells us that obtaining the entanglement entropy reduces to counting the number of flat configurations in \mathcal{H}_A modulo internal gauge transformations.

Following the graphical representation of states we introduced in §??, flat configurations in region A are described by red curves that cross plaquettes an even number of times. Thus, it is not difficult to realize that the different gauge equivalence classes are given by all possible configurations of red curves at the plaquettes on the boundary of A . So by counting the number of non equivalent configurations for all plaquettes in the boundary of A , we recover the area law together with the topological correction for the entanglement entropy of the Toric Code, this is:

$$S_A = \log_2 \left(\frac{|\ker(\delta_A^0)|}{|\text{Im}(\delta_A^{-1})|} \right) = \log_2 (GSD_{\bar{A}}) = \log_2 \left(\frac{2^{\partial A}}{2} \right) = \partial A - 1 \quad (5.16)$$

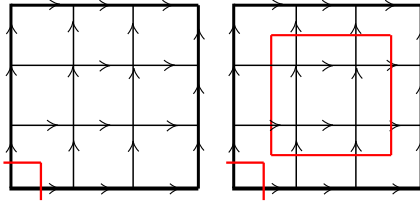


Figure 5.2: Region A: configurations for the corner plaquette, which are gauge equivalent in the calculation of the entanglement entropy S_A for the Toric Code.

The Toric Code with open boundary conditions was considered in [108] where they calculate $GSD_{\bar{A}}$ in detail and coincides with our result. Moreover, a calculation of entanglement entropy is also performed in [108] giving the exact same result we obtain here.

5.4.2 (2D) 0,1-Gauge: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2$

We now consider an specific example from the models exhibited in §3.2 where the expression for the operators and the Hamiltonian can be found. In particular, we saw in

Example 3.2.1 that when the gauge groups are set to be $G_1 = \mathbb{Z}_4$ and $G_0 = \mathbb{Z}_2$ the model exhibits topological degeneracy, very similar in nature to that of the Toric Code. This suggests the appearance of a corresponding topological term in the entanglement entropy, S_A , as we shall see.

Again, we take a region A of the two dimensional lattice, such as the one shown in Fig. 5.1. Calculating the entanglement entropy, as given by Eq. (5.15), requires us to count the number of flat configurations $|\ker \delta_A^0|$ divided by the number of non-trivial internal gauge transformations $|\text{Im} \delta_{\bar{A}}^{-1}|$. It is not difficult to realize that this counting is nothing more than identifying all non-equivalent flat configurations at the boundary of the region in question, ∂A . From the details of the model (see Example 3.2.1) note that for each plaquette, $p \in \partial A$, there are four (0,1)-flat configurations. Making $4^{\partial A}$ the total number of such configurations. Yet there are four configurations with internal loops which must be gauged away by internal transformations, as shown in Figure 5.3.

This leaves for the entanglement entropy of the 0,1-gauge theory:

$$S_A = \log_4 \left(\frac{|\ker(\delta_A^0)|}{|\text{Im}(\delta_{\bar{A}}^{-1})|} \right) = \log_4 (GSD_{\bar{A}}) = \log_4 \left(\frac{4^{\partial A}}{4} \right) = \partial A - 1. \quad (5.17)$$

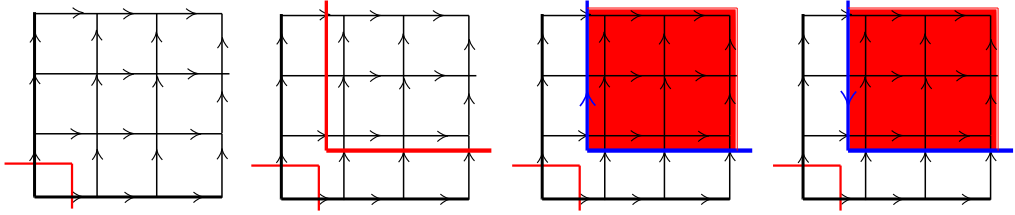


Figure 5.3: Zoom of region A in the down-left corner: four types of configurations gauge equivalent for the corner plaquette for the entanglement entropy calculation S_A in the 0,1-gauge model with groups $G_0 = \mathbb{Z}_2$, $G_1 = \mathbb{Z}_4$, discussed in 5.4.2.

5.4.3 (2D) 1,2-Gauge: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_2$

This time we consider the case of the 1,2-gauge theory (§3.3) shown in Example 3.3.1, where the gauge groups are $G_2 = \mathbb{Z}_4$ and $G_1 = \mathbb{Z}_2$. As already shown in Ex. 3.3.1 and [3], this model presents a 2-fold ground state degeneracy for any connected two dimensional lattice, the ground states of Eqns. (3.29) and (3.30). Although this degeneracy seems not to have any topological origin, it is related to the second cohomology group $H^2(C, H_2(G)) = \text{Hom}(H_2(C), H_2(G))$ as shown in [3]. The topological nature of the ground state degeneracy can be reassured by finding the entanglement entropy of the model.

Once more, we consider a special region of the lattice, A , such as the one in Fig. 5.1. Note, from Eq. (3.22), that the flatness condition of the model is enforced by the plaquette operators, B_p , of Eq. (3.24). Thus, the entanglement entropy calculation reduces to finding the number of non-equivalent flat configurations at the boundary of A . Observe that, for each $p \in \partial A$, there are $2^{\partial A}$ boundary flat configurations of the type of

Fig.(sth), likewise there are $2^{\partial A}$ boundary flat configurations of the kind shown in Fig(sth), giving $2^{2\partial A}$ boundary flat configurations. Nevertheless, there is redundancy coming from the internal 1-gauge transformations that relates pairs of boundary flat configurations. This leaves for the entanglement entropy:

$$S_A = \log_4 \left(\frac{|\ker(\delta_A^0)|}{|\text{Im}(\delta_A^{-1})|} \right) = \log_2 (GSD_{\hat{A}}) = \log_4 \left(\left| \frac{2^{2\partial A}}{2} \right| \right) = 2(\partial A) - 1. \quad (5.18)$$

We can see from the above expression the area law term $\sim \partial A$ and the topological term -1 .

5.4.4 (3D) 1,2,3-Gauge: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$

In the three dimensional case we consider region A as being a cube of R links per side. This model is analyzed in section 3.4 and 3.4.1. The holonomy operators involved are B_p , B_c , and Figure 3.11 exhibits the allowed configurations in the ground state subspace.

In region A , the different configurations for the flatness condition are equal to the number of boundary plaquettes in the cube. Every boundary plaquette can present four different ground state configurations: empty, with a red line exiting the cube, with a crossing blue curve oriented, as shown in figure 5.4.

On the other hand the gauge transformations in the interior of A can add a red loop or blue loops with two different orientations. So the configurations are the number of boundary plaquettes minus 1.

$$S_A = \log_4 \left(\frac{|\ker(\delta_A^0)|}{|\text{Im}(\delta_A^{-1})|} \right) = \log_4 (GSD_{\hat{A}}) = \log_4 \left(\left| \frac{4^{\partial A}}{4} \right| \right) = |\partial A - 1| \quad (5.19)$$

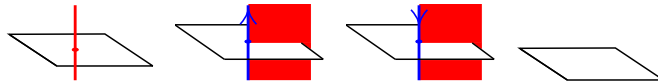


Figure 5.4: A plaquette in the boundary of region A : four types of configurations counted in the entanglement entropy calculation S_A in the 1,2,3-gauge model, with groups $G_1 = G_2 = G_3 = \mathbb{Z}_4$, presented in 5.4.4.

Chapter 6

Concluding Remarks and Outlook

In this final chapter we present a summary of the main topics of this work and we highlight the important aspects of what was discussed as well as some potential directions for further research that go beyond the scope of this work. This thesis was divided in three parts. We start by describing our object of interest namely, quantum states that exhibit long range entanglement or *topological order*; The Quantum Double Models and, in particular, the Toric Code[1] are the canonical examples of solvable Hamiltonian models with topological features. Specifically, the presence of topological order in the QDMs is probed by studying their ground state subspace since the dimension of this subspace is a topological invariant. In [85] this is precisely stated by relating the ground state degeneracy of the QDM to the Kuperberg's Invariant [109]. In 2D the QDMs and their generalizations known as *string-net* models [2] essentially account for all kinds of topological order in (2+1)D. This is no longer true for higher dimensions, since the 3D versions of the QDMs are not rich enough such as to include all kinds of three dimensional topological phases. This lack of understanding of higher dimensional topological phases has increased the motivation for the study of higher dimensional quantum models that go beyond usual gauge theories. The symmetry contents in such theories are given by n -groups [110, 111]. In [53], the first example of a *Higher Gauge Theory* in 3 + 1 dimensions is constructed using the notion of a 2-group as the gauge symmetry. A very similar approach to the one done independently by our group, see [67, 68]. See also, [100, 112] for a recent account on 3 + 1-dimensional models with topological ordered ground states.

In [3] we give a first step towards a systematic characterization of higher dimensional topological phases in their Abelian versions. These models are defined in arbitrary dimensions and thus can serve to exhibit the role of dimensionality in topologically ordered phases. Chapter 3 was intended as an introduction to these models. We constructed several 2 and 3 dimensional models that showcase the essential features of *quantum models* coming from *Abelian higher gauge theories*. In particular, we gave a description of their ground state subspace and calculated its degeneracy for specific examples. Whether these ground states have intrinsic topological order may be not so easy to answer by just looking at the *GSD*. In Chapter 5, we showed that entanglement entropy effectively measures the presence of long range entanglement by the appearance of a constant term called *topological entanglement entropy*.

Chapter 4 was dedicated to present the most general case of an Abelian higher gauge theory, this chapter is a synthetic version of [3]. We show how the language of Homological

Algebra turns out to be very natural to write the models in their most general form. Moreover, it allows for a straightforward analysis of the ground state subspace $\mathcal{H}_0 \subset \mathcal{H}$; specifically, we showcase the main result of [3] where we showed that the dimension of the ground state subspace \mathcal{H}_0 is a topological invariant whose value depends on both the model details and the topology of the manifold where it is defined.

In general, entanglement entropy of a quantum state is not easy to find analytically. However, in chapter 5 we calculated the bipartition entanglement entropy of the models constructed in [3] and presented in chapter 4. The language we use to describe the models and that allowed us to calculate their *GSD* shows itself suitable for the calculation of S_A . We show a general result in the form of Equation (5.15), namely

$$S_A = \log(GSD_{\tilde{A}}),$$

where $GSD_{\tilde{A}}$ counts the number of edge states, which live on the boundary of A . Furthermore, when calculating this number for specific examples, we show that S_A follows an *area law* and more importantly for our purposes, the universal contribution known as *topological entanglement entropy* appears whenever topological order is detected in the ground states of Abelian higher gauge theories. This fact confirms the topological nature of the models in a very clear way. Also showing that entanglement entropy can be used to probe topological order in all dimensions greater than 2.

Beyond Topological Order

The contents of this work represent a small part of the research activities in our group. Recently, there is an increasing interest in the so called *fracton phases* [55, 57, 60, 62], they are not topological in the sense of this work, this is, having a topologically dependent *GSD* and topological entanglement entropy contribution. These phases show extensive degeneracy of ground states, thus associated to the underlying geometry rather than topology itself. Entanglement entropy can sometimes be explicitly calculated [66] and these phases appear to break area law by non-universal terms. In this sense, we can list some potential lines of further research.

In [113] we show that considering Abelian higher gauge theories where some of the gauge fields (related to a particular dimension) are restricted to have *classical configurations* only, the otherwise purely topological models now become sensitive to geometrical quantities of the underlying manifold. This is showcased, as in the topological cases, in the ground state subspace of the Hamiltonian models. This fact suggests they may be, somehow, related to *fracton phases* that also have extensive ground state degeneracies. Whether entanglement entropy can be explicitly obtained for such models and if so, whether it carries information about the geometric nature of the ground state degeneracy remains an open question which can be tackled using a similar approach to that in chapter 5.

In [114] we found another way to systematically obtain Hamiltonian models that resemble the most famous fracton models, namely, the Haah code and the X-cube model. Our advantage with respect to other approaches for the study of such phases is that we provide a systematic procedure for obtaining such models in any dimension and for ar-

bitrary higher gauge fields. We show that the models have an extensive ground state degeneracy and a preliminary study of the excited states shows the geometrical nature of these models, similar to what is known in the literature [55, 56, 57, 58]. In this settings, entanglement entropy may give further information about the nature of long-range entanglement in these (non-) topological phases.

Finally, in [112] they construct, in detail, a *twisted* pure 2-gauge theory by extending the usual construction of the *Dijkgraaf-Witten* invariant [86] where the classifying space BG of the discrete group G is replaced by the n -th classifying space $B^n G$, which is an example of a *Eilenberg-MacLane* space. They propose further generalizations by considering *Postnikov Towers*. We believe this procedure can be systematically applied to our Abelian higher gauge theories to obtain their twisted versions, which remains an open problem.

Appendix A:

Review of Homology and Homological Algebra

In this appendix we intend to cover some basic notions regarding simplicial homology and cohomology. As we mentioned in the main text, the use of simplicial complexes arises only as a matter of simplicity. It is in this appendix that we review the basic concepts and notions that will ultimately help us understand the, slightly more abstract, topic of Homological Algebra discussed in Appendix C. We tried to leave this Appendix as self contained as possible. Nevertheless, for further details on these topics with focus on simplicial complexes we refer the reader to [106, 115, 116] and the references therein.

A.1 Simplices and Simplicial Complexes

The main ingredients of Simplicial Homology are the k -simplices. They correspond to the building blocks of a polyhedron. An *oriented k -simplex*, σ^k , is the smallest convex set of in \mathbb{R}^m containing $n + 1$ points $\{x_0, x_1, x_2, \dots, x_k\}$, with $k \leq m$. For instance, a 0-simplex is just a point, a 1-simplex is an oriented line segment, a 2-simplex is a triangle (including its interior), a 3-simplex is a tetrahedron as shown in Fig.(1). To denote a k simplex we use its $k + 1$ vertices to write $\sigma^k = [x_0, x_1, x_2, \dots, x_k]$. More importantly, the order in which the vertices appear defines the *orientation* of the k -simplex in question. Any even permutation of the order in which the vertices appear in a simplex gives another simplex with the same orientation, whereas an odd permutation of vertices will give a simplex with the opposite orientation.

At this point we can define what a *simplicial complex* is. In general terms, given a set V an abstract simplicial complex, K , is a collection of finite subsets of V such that if $\sigma^k = [x_0, \dots, x_k] \in K$ then every subset of σ^k (its faces) is also in K . More precisely,

Definition A.1.1 (Simplicial Complex). Let K be a set of finite number of simplices.

A finite **simplicial complex** K is a finite collection of simplices in \mathbb{R}^m for some m such

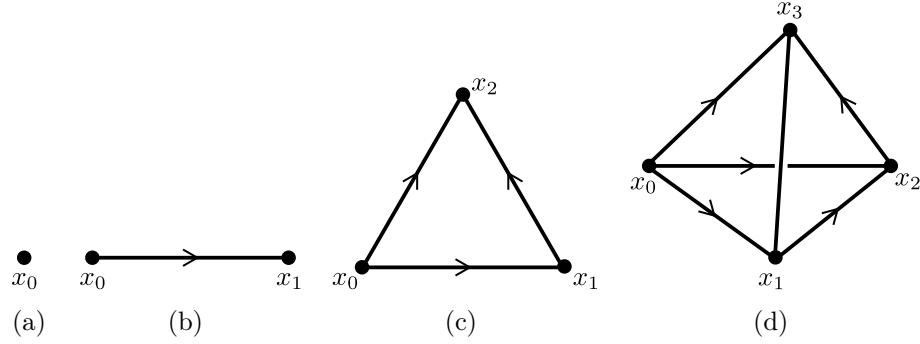


Figure 1: 0, 1, 2 and 3-simplices are respectively shown.

that:

- (i) if $\sigma \in K$ and $\sigma' \leq \sigma$ then $\sigma' \in K$.
- (ii) if $\sigma, \sigma' \in K$, then either $\sigma \cap \sigma' = \emptyset$ or is a common face of σ and of σ' .

Consider now, K , a finite simplicial complex. We denote by $K_n \subset K$ the set formed by n -simplices only. A finite simplicial complex can always ref needed!! be embedded in \mathbb{R}^m for some m . Such an embedding is called a *geometrical realization* of K . The subset occupied by K is denoted by $|K|$ and it is called a *polytope*. Generally speaking, when a topological space X is homeomorphic to a polytope $|K|$, it is called a *triangulable space* and the simplicial complex K is called a *triangulation* of X .

A.2 Simplicial Homology

In the previous section we defined all the basic ingredients that are needed for the study of Homology groups of a triangulable space X . Given a topological space X there might be several triangulations of X since a triangulation is not unique. However, the homology groups for any triangulation of the same space are isomorphic. In order to define the homology groups we will begin introducing the chain groups.

Definition A.2.1. (k -chain group) Let K be a simplicial complex. A simplicial k -chain, c , is a formal sum of k -simplexes with coefficients in \mathbb{Z} :

$$c = \sum_{x \in K_k} c_x x, \quad (\text{A.1})$$

where $c_x \in \mathbb{Z}$. The orientation is taken into account by saying that $[x_{p(0)}, \dots, x_{p(n)}] = \pm[x_0, \dots, x_n]$ where we have a plus sign if p has even parity and a negative sign otherwise.

Moreover, the set formed by simplicial k -chains, denoted $C_k(K)$, is the free abelian group with basis K_k and it is called the **n -chain group**.

The group operation in the above definition is an additive one, notice that $-\sigma^k$ is just σ^k with reversed orientation, which ensures the existence of inverse elements. Also, the group \mathbb{Z} is called the coefficient group and in general can be any Abelian group. Before we define the cycle and boundary groups we need to introduce the boundary operator. As its name might suggest it is an operator that acts on a n -simplex, σ^n , to produce its boundary. More precisely:

Definition A.2.2 (Boundary Operator). Define $\partial_n : C_n(K) \rightarrow C_{n-1}(K)$ by:

$$\partial_n([v_0, \dots, v_n]) = \sum_{i=0}^n (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n], \quad (\text{A.2})$$

where $[v_0, \dots, \hat{v}_i, \dots, v_n]$ denotes the $(n-1)$ -simplex obtained by removing the vertex v_i from $[v_0, \dots, v_n]$.

The above defines the boundary operator on the generators $\sigma^k \in K$. Its action on general n -chains is straightforwardly extended by linearity. More importantly, the fundamental property of the boundary operator is:

$$\partial_k \partial_{k+1} = 0. \quad (\text{A.3})$$

Definition A.2.3. Let $C = (C_n, \partial)$ be:

$$\dots \rightarrow C_n \xrightarrow{\partial_n^C} C_{n-1} \rightarrow \dots \quad (\text{A.4})$$

with the maps ∂_n defined by:

$$\partial_n([x_0, \dots, x_n]) = \sum_{i=0}^n (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_n]$$

where $[x_0, \dots, \hat{x}_i, \dots, x_n]$ denotes the $(n-1)$ -simplex obtained by removing the vertex x_i from $[x_0, \dots, x_n]$.

To gain some intuition on the above definition, in Fig.(2) we calculate the boundary of a 3-simplex.

The algebraic situation we have at this point is a sequence of homomorphisms between Abelian groups. If we let K be an n -dimensional simplicial complex, the sequence looks

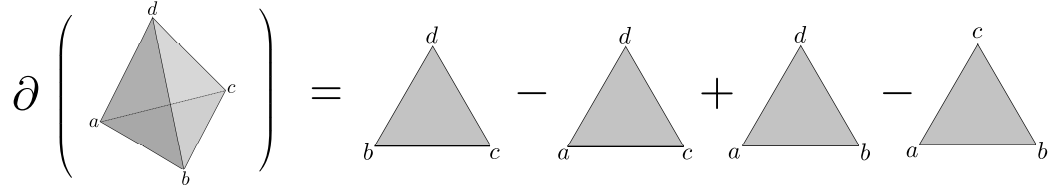


Figure 2: The boundary of 3-simplex is explicitly calculated.

like:

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0, \quad (\text{A.5})$$

where the 0's at the endpoints of the sequence stand for the trivial group and $i : 0 \hookrightarrow C_n(K)$ is an inclusion map. Such a sequence is called a **chain complex** and we will denote it as $C = (C_n, \partial_n)$. Now we will study the image and kernel of the ∂_n homomorphisms. This motivates the following definitions:

Definition A.2.4 (k -Cycles). If $c \in C_k(K)$ satisfies:

$$\partial_k(c) = 0, \quad (\text{A.6})$$

c is called an **n-cycle**. The set of n-cycles, denoted $Z_k(K)$ is a subgroup of $C_k(K)$ and is called the **k-cycle group**. In other words, $Z_k(K) = \ker \partial_k$.

Definition A.2.5 (k -Boundaries). Let $c \in C_k(K)$. If there exists an element $d \in C_{k+1}(K)$ such that:

$$\partial_{k+1}(d) = c, \quad (\text{A.7})$$

then c is called an **k-boundary**. The set of k-boundaries, denoted $B_k(K)$ is a subgroup of $C_k(K)$ and is called the **k-boundary group**. In other words, $B_k(K) = \text{Im } \partial_{k+1}$.

To gain a bit of geometrical intuition from the above definitions, consider the boundary operator ∂_k that takes the boundary of an n-chain, if c is an n-cycle it means that it has no boundary. Whereas, if $c = \partial_{k+1}d$ it means that it is the boundary of d . Intuitively we can infer that the boundary of a boundary is null. Hence $\partial_k \partial_{k+1} = 0$, from which it follows that $\text{Im } \partial_{k+1} \subset \ker \partial_k$. The elements of $Z_k(K)$ that are not boundaries play the central role in the theory of Homology groups. At this point we have all the ingredients we need to define the Homology groups of K .

Definition A.2.6 (k -th Homology group). Let K be an n dimensional finite simplicial complex. The **k -th Homology group**, $H_k(K)$, $0 \leq k \leq n$, associated to K is:

$$H_k(K) := \frac{Z_k(K)}{B_k(K)}.$$

The k -th Homology group $H_k(K)$ is the set of equivalence classes of k -cycles that do not bound any $(k+1)$ -chain, this is,

$$H_k(K) = \{[z], z \in Z_k(K)\},$$

where each equivalence class is called **Homology class**. Two k -cycles z and z' are said to be equivalent if and only if $z - z' \in B_k(K)$, this is, if they differ by a boundary. In this case they are said to be **homologous** to each other and denoted $z \sim z'$ or $[z] = [z']$. Finally, it can be proven that *Homology groups are topological invariants* (cf Theorem 4.10 in [117] or Chapter 2 of [118]). This can be, roughly stated by saying that if K and L are two simplicial complexes such that $|K| = |L| = X$ (we actually mean they are homeomorphic), then $H_k(K)$ and $H_k(L)$ are isomorphic.

A.3 Simplicial Cohomology

Let us now discuss the cohomology groups associated to a simplicial complex K . The cohomology groups are constructed by turning the chain groups $C_k(K)$ into groups of homomorphisms and the boundary operators ∂_k into their dual homomorphisms. We can choose an abelian group H in order to define a certain cochain complex, to do that let us begin by defining what a *k-cochain* actually is.

Definition A.3.1 (k -cochain group). A **k -cochain** with coefficients in the Abelian group H is a homomorphism $c^* : C_k(K) \rightarrow H$. Given a k -chain $c \in C_k(K)$ the k -cochain evaluates c by mapping it into H . We denote this evaluation by:

$$c^*(c) \equiv \langle c^*, c \rangle \in H.$$

Moreover, the set $C^k(K, H) \equiv \text{Hom}(C_k, H)$ has a natural group structure, namely, if $c^*, d^* \in C^k(K, H)$, their sum $c^* + d^*$ also belongs to $C^k(K, H)$. This group is called the **k -cochain group**.

Likewise, the boundary operator is dualized as follows.

Definition A.3.2 (Coboundary Operator). Let us consider the boundary operator $\partial_k : C_k(K) \rightarrow C_{k-1}(K)$. Canonically associated to it, there is a dual homomorphism that defines the coboundary operator,

$$d^{k-1} : \text{Hom}(C_{k-1}, H) \rightarrow \text{Hom}(C_k, H),$$

or simply $d^{k-1} : C^{k-1} \rightarrow C^k$. Let $\partial_k c \in C_{k-1}(K)$ and $c^* \in C^{k-1}$, the coboundary operator is defined by:

$$\langle c^*, \partial_k c \rangle = \langle d^{k-1} c^*, c \rangle.$$

From Eq.(A.3) it is easy to show that $d^k d^{k-1} = 0$. Notice that in this case the coboundary operator raises the index instead of lowering it, as the boundary operator does. Therefore, the algebraic situation consists on the following **cochain complex**:

$$\dots \xleftarrow{d^{k+1}} C^{k+1} \xleftarrow{d^k} C^k \xleftarrow{d^{k-1}} C^{k-1} \xleftarrow{d^{k-2}} \dots,$$

the cochain complex runs in the opposite direction. Once the coboundary operator is defined the analogs of $B_k(K)$ and $Z_k(K)$, this is, the k-boundaries and k-cocycles.

Definition A.3.3 (k-cocycles). If $c^* \in C^k$ satisfies:

$$d^k c^* = 0,$$

c^* is called a **k-cocycle**. The set of k-cocycles, denoted $Z^k = \ker d^k$ is a subgroup of C^k and is called the **k-cocycle group**.

Definition A.3.4 (k-coboundaries). Let $c^* \in C^k$, if there exists a $b^* \in C^{k-1}$ such that:

$$d^{k-1} b^* = c^*,$$

then c^* is called a **k-coboundary**. The set of k-coboundaries, $B^k = \text{Im } d^{k-1}$, is a subgroup of C^k and it is called the **k-coboundary group**.

The cochain complex gives rise to the cohomology of C with coefficients in H :

Definition A.3.5 (kth-Cohomology group). Let K be a finite dimensional simplicial complex, and consider C the chain complex of Eq.(A.5). The **k-th Cohomology group** of C with coefficients in H

$$H^k(C, H) := \frac{Z^k}{B^k} = \frac{\ker d^k}{\operatorname{Im} d^{k-1}}.$$

A.4 Homological Algebra

In what follows we intend to cover some basic notions regarding the topic of Homological Algebra. In particular, we are interested in defining the cochain complex that is obtained from considering maps between two chain complexes. These maps are very important in Chapter 4 and 5 since they are used to generalize the notions of gauge configurations, gauge transformations and holonomy measurements. This is why we dedicate an appendix to these topics. On our intention to write this appendix as self contained as possible we begin by reviewing some concepts already introduced in Appendix ?? for the simplicial case. Then we pass on to define what a Cohomology with coefficients on a chain complex is [73, 119]. We end the Appendix citing a Theorem by [73] that is of much help when interpreting the main result of [3], result that is presented in Chapter 4. For further details on these topics with focus on simplicial complexes we refer the reader to [106, 115, 116] and the references therein.

Definition A.4.1. An abstract chain complex $(C_\bullet, \partial_\bullet)$ is a sequence of abelian groups $\{C_k\}_{k \in \mathbb{Z}}$ and group morphisms $\partial_k : C_k \rightarrow C_{k-1}$ such that the composition of two such morphism is trivial i.e. $\partial_{k-1}\partial_k = 0$.

Similarly, an abstract cochain complex (C^\bullet, d^\bullet) is a sequence of abelian groups $\{C^k\}_{k \in \mathbb{Z}}$ and group morphisms $d^k : C^k \rightarrow C^{k+1}$ such that the composition of two consecutive morphism is trivial i.e. $d^{k+1}d^k = 0$.

The development of homological algebra was largely motivated by its applications to algebraic topology. Of particular importance are the homology and cohomology groups defined below.

Definition A.4.2 (Homology and Cohomology groups). Given a chain complex $(C_\bullet, \partial_\bullet)$, the homology groups $H_k(C)$ associated to it are defined by

$$H_k(C) = \ker(\partial_k) / \operatorname{im}(\partial_{k+1})$$

In a similar fashion, if (C^\bullet, d^\bullet) is a cochain complex then the cohomology groups $H^k(C)$ assigned to it are defined by

$$H^k(C) = \ker(d^k) / \text{im}(d^{k-1})$$

The importance of such groups arises due to the fact that they are a source of topological invariants. Roughly speaking, given a topological space X with suitable properties, it is possible to associate a chain complex $C(X)$ to X by some discretization procedure and define the homology $H_k(X)$ as $H_k(C(X))$. For instance, any manifold M can be realized as a simplicial complex and there is a standard procedure for building a chain complex from such simplicial complex. As discussed in Appendix ??, in general, there are different simplicial complexes that correspond to the same manifold, but all of them have the same homology groups, $H_k(M)$; As such, these groups are well-defined topological invariants.

There are also ways to assign a cochain complex to a manifold in order to obtain a topological invariant from the cohomology groups. One example of a topological invariant obtained this way is the De Rham Cohomology. Another procedure relies on the fact that, given a chain complex $(C_\bullet, \partial_\bullet)$ and an abelian group G , there is a corresponding cochain complex $(C^\bullet = \text{Hom}(C_k, G), d^\bullet)$ with $d^k(f) = f\partial_{k+1}$, where $f \in C_k$. The cohomology groups obtained this way, denoted $H^k(C, S)$ and are called cohomology groups with coefficients in G .

As usual, when introducing an algebraic structure, it is important to define what the correct notion of a morphism is. The usual definition for chain complexes is given by means of chain maps but we choose to start with a more flexible definition and later on specify to chain maps.

Definition A.4.3 (p-Maps). Given two chain complexes of Abelian groups $(C_\bullet, \partial_\bullet)$, $(C'_\bullet, \partial'_\bullet)$ a p -map $f : (C_\bullet, \partial_\bullet) \rightarrow (C'_\bullet, \partial'_\bullet)$ is a sequence of morphisms $f_n : C_n \rightarrow C'_{n-p}$.

The set of all p -maps, denoted $\text{hom}(C, C')^p$, is actually an abelian group under the binary operation defined by $(f + g)_n = f_n + g_n$. The unit of the group is the trivial p -map, denoted 0, defined by the trivial morphisms $0_n : C_n \rightarrow C'_{n-p}$. It is then straightforward to verify that

$$\text{hom}(C, C')^p = \prod_n \text{Hom}(C_n, C'_{n-p}) \quad (\text{A.8})$$

The abelian groups $\text{hom}(C, C')^p$ give rise to a cochain complex $(\text{hom}(C, C')^\bullet, \delta^\bullet)$ which we define below:

Definition A.4.4 (Coboundary). Let $\delta^p : \text{hom}(C, C')^p \rightarrow \text{hom}(C, C')^{p+1}$ be the group

morphism defined by

$$(\delta^p f)_n = f_{n-1} \partial_n - (-1)^p \partial'_{n-p} f_n$$

where $f \in \text{hom}(C, C')^p$. It is straightforward to check that $\delta^{p+1} \delta^p = 0$ so $(\text{hom}(C, C')^\bullet, \delta^\bullet)$ is a cochain complex. The situation is shown in the following diagram:

$$\cdots \rightarrow \text{hom}(C, C')^p \xrightarrow{\delta^p} \text{hom}(C, C')^{p+1} \rightarrow \cdots$$

The cohomology groups obtained from $(\text{hom}(C, C')^\bullet, \delta^\bullet)$ will be denoted by $\mathcal{H}^p(C, C')$ and referred to as the cohomology groups of C with coefficients in the chain complex C' . These groups are related to the usual cohomology groups of C with coefficients in $H_{n-p}(C')$ by the following theorem due to Ronald Brown [73]:

Theorem A.4.5. *Given chain complexes C, C' there is an isomorphism*

$$\prod_n H^n(C, H_{n-p}(C')) \rightarrow \mathcal{H}^p(C; C')$$

Note that an element of $\ker(\delta^0)$ corresponds to a sequence of group morphisms $f_n : C_n \rightarrow C'_n$ such that $(\delta^0 f)_n = f_{n-1} \partial_n - \partial'_{n-p} f_n = 0$. This condition is precisely what defines chain maps:

Definition A.4.6 (Chain Map). Given two chain complexes $(C_\bullet, \partial_\bullet), (C'_\bullet, \partial'_\bullet)$ a chain map $f : (C_\bullet, \partial_\bullet) \rightarrow (C'_\bullet, \partial'_\bullet)$ is a sequence of morphisms $f_n : C_n \rightarrow C'_n$ such that $f_n \partial'_{n-1} = \partial_n f_{n+1}$.

Chain maps can be represented diagrammatically as in Fig. (3) where the commuting squares correspond to the conditions each component must satisfy. Chain maps have the important property that they induce group morphism on the corresponding homology groups. It might happen that different chain maps induce the same morphism on homology groups. When that is the case, such maps are called chain homotopic. More precisely, two chain maps f, f' are homotopic whenever there is some $t \in \text{hom}(C; C')^{-1}$ such that $f' = f + \delta^{-1}$, when that is the case t is called a chain homotopy between f and f' . Since $\mathcal{H}^0(C, C') = \ker(\delta^0)/\text{im}(\delta^{-1})$ by definition, we see that $\mathcal{H}^0(C; C')$ is simply the group formed by homotopy classes of chain maps.

Additionally, we can construct a chain complex from the dual groups to those defined in Definition A.4.4. Before actually defining them, let us make a quick review on the dual groups of Abelian groups and how they are directly related to the characters of the group itself.

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\
 \cdots & \xrightarrow{\partial'_{n+2}} & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \xrightarrow{\partial'_{n-1}} \cdots
 \end{array}$$

Figure 3: Diagrammatic description of a chain map.

All irreducible representations of an abelian group S are one-dimensional and form an abelian group \hat{S} when S is finite. It is easier to understand \hat{S} by observing that any irreducible representation r of S is completely specified by a group morphism $\chi_r : S \rightarrow U(1)$ defined by $\chi_r(g) = \text{Tr}(r(g))$, the character of r . The binary operation of \hat{S} corresponds to the point wise multiplication of the corresponding characters i.e.

$$\chi_{r+r'}(g) = \chi_r(g)\chi_{r'}(g)$$

Therefore, one can think of \hat{S} as being the group $\hat{S} = \text{Hom}(S, U(1))$ due to the correspondence $r \leftrightarrow \chi_r$. This perspective makes it straightforward to check that, given a morphism f between finite abelian groups, there is a dual morphism \hat{f} defined by $\rho \mapsto \hat{f}(\rho) = \rho \circ f$, such that:

Proposition A.4.7. *The morphism \hat{f} is a group morphism and $\chi_{\hat{f}(\rho)} = \chi_\rho \circ f$.*

Proof. The following holds,

$$\chi_{\hat{f}(\rho)}(h) = \text{Tr}(\hat{f}(\rho)(h)) = \text{Tr}(\rho(f(h))) = \chi_\rho(f(h))$$

or $\chi_{\hat{f}(\rho)} = \chi_\rho \circ f$ as claimed. It follows also that,

$$\begin{aligned}
 \chi_{\hat{f}(\rho+\rho')} &= \chi_{\rho+\rho'} \circ f = \\
 &= \chi_\rho \chi_{\rho'} \circ f = (\chi_\rho \circ f)(\chi_{\rho'} \circ f) = \\
 &= \chi_{\hat{f}(\rho)} \chi_{\hat{f}(\rho')} = \chi_{\hat{f}(\rho)+\hat{f}(\rho')}
 \end{aligned}$$

and $\chi_{\hat{f}(0)} = \chi_0 \circ f = 1 = \chi_0$ which implies $\hat{f}(\rho + \rho') = \hat{f}(\rho) + \hat{f}(\rho')$ and $\hat{f}(0) = 0$. \square

This technique will be used extensively in order to move freely between groups and representations throughout this paper. For general references we derive the reader to [104, 105]

This is, consider the dual of $\text{hom}(C, G)^p$ as a group, it consists on maps from $\text{hom}(C, G)^p$ into $U(1)$. Namely,

Definition A.4.8 (Dual groups). Given the Abelian group $\text{hom}(C, G)^p$ and an arbitrary element f . Denote its dual group by $\text{hom}(C, G)_p$ which consists on maps:

$$\begin{aligned} s : \text{hom}(C, G)^p &\rightarrow U(1), \\ f &\mapsto s(f) \equiv \chi_s(f), \end{aligned}$$

where in the last step we used the fact that we can think of the dual group $\text{hom}(C, G)_p = \text{Hom}(\text{hom}(C, G)^p, U(1))$.

Furthermore, the pair $(\text{hom}(C, G)_p, \delta_p)$ defines a chain complex with the natural boundary operator δ_p . More precisely,

Definition A.4.9 (Boundary). The boundary operator

$$\delta_p : \text{hom}(C, G)_p \rightarrow \text{hom}(C, G)_{p-1},$$

is defined via the natural pairing between dual elements, this is, given $m \in \text{hom}(C, G)_p$ and $f \in \text{hom}(C, G)^{p-1}$:

$$\langle \delta_p m, f \rangle = \langle m, \delta^{p-1} f \rangle,$$

or equivalently,

$$\chi_{\delta_p m}(f) = \chi_m(\delta^{p-1} f).$$

To see that $(\text{hom}(C, G)_p, \delta_p)$ is a chain complex it is enough to note that $\delta_p \delta_{p-1} = 0$, which in turn comes from the fact that $(\text{hom}(C, G)^p, \delta^p)$ is a cochain complex.

Appendix B:

Auxiliary Isomorphism

In this appendix, we prove the equality $|\text{Im}(\delta^0)| = |\text{Im}(\delta_1)|$ that allowed us to relate the dimension of the Hilbert space \mathcal{H} and the ground state degeneracy GSD through:

$$GSD |\text{Im}(\delta^{-1})| |\text{Im}(\delta_1)| = \dim(\mathcal{H}) = \dim(\mathcal{H}_A) \dim(\mathcal{H}_B).$$

However, in order to do so, we will show that there is a well defined bijection between $\ker(\delta_1)$ and $\text{hom}(C, G)^1 / \text{Im}(\delta^0)$ from which the result follows.

So, in general, consider A, B two finite Abelian groups and $\phi : A \rightarrow B$ a homomorphism between them. Consider also $\hat{A} = \text{Hom}(A, U(1))$ and $\hat{B} = \text{Hom}(B, U(1))$ their corresponding irreducible representations, let $\hat{\phi} : \hat{B} \rightarrow \hat{A}$ be the homomorphism between representations induced by ϕ via:

$$\hat{\phi}(\beta) := \beta \circ \phi,$$

where $\beta \in \hat{B}$ is an irreducible representation of B , and $\hat{\phi}(\beta) \in \hat{A}$ as it is expected.

Proposition B.4.10. *The subgroups $\ker \hat{\phi}$ and $\frac{B}{\text{Im}(\phi)}$ are isomorphic.*

Proof. We will split the proof in two parts, in the first half of the proof we show that there is a well defined map between $\ker \hat{\phi}$ and $\frac{B}{\text{Im}(\phi)}$ and then we show that its inverse is also well defined, which turns the maps into a bijection.

1. Note that an irreducible representation $\beta \in \ker \hat{\phi}$, if and only if, $\hat{\phi}(\beta) = \beta \circ \phi = 1 \in U(1)$ which in turn is equivalent to say that $\text{Im} \phi \subset \ker \beta$. This allows us to

construct the following commuting diagram:

$$\begin{array}{ccc}
 B & \xrightarrow{\beta} & U(1) \\
 \pi \downarrow & \nearrow \beta' & \\
 \frac{B}{\text{Im}\phi} & &
 \end{array} \tag{B.9}$$

where $\pi : B \rightarrow \frac{B}{\text{Im}\phi}$ is the canonical projection sending $b \in B$ into its corresponding equivalence class $[b] \in \frac{B}{\text{Im}\phi}$. Furthermore, $\beta' \in \text{Hom}(\frac{B}{\text{Im}\phi}, U(1))$ is unique and defined as:

$$\beta'([b]) := \beta(b)$$

notice that β' is well defined within equivalence classes since $\text{Im}\phi \subset \ker\beta$. To see this, consider $b' \neq b \in [b]$, this means that $b - b' \in \text{Im}\phi \subset \ker\beta$, therefore:

$$\begin{aligned}
 \beta(b - b') &= 1, \quad \Rightarrow \beta(b)\beta(b')^{-1} = 1, \\
 &\Rightarrow \beta(b) = \beta(b') = \beta'([b]).
 \end{aligned}$$

This is, we have shown that given an irrep $\beta \in \ker\hat{\phi}$ then there is a unique morphism $\beta' \in \text{Hom}(\text{frac}B/\text{Im}\phi, U(1))$.

We now need to show that the converse also holds, to this intent, consider $\beta' : \frac{B}{\text{Im}\phi} \rightarrow U(1)$. Recall that $\text{Im}\phi \subset \ker\beta$. Observe also that β is the only map for which the diagram in B.9 commutes

Thus, we have shown that given a $\beta' \in \text{Hom}(\frac{B}{\text{Im}\phi}, U(1))$ there is a unique $\beta = \beta' \circ \pi \in \ker\hat{\phi}$.

2. Now we carry on showing that the map above is in fact a bijection and it defines an

isomorphism. Let ι be the map:

$$\begin{aligned}\iota : \ker \hat{\phi} &\longrightarrow \text{Hom} \left(\frac{B}{\text{Im} \phi}, U(1) \right), \\ \beta &\mapsto \beta',\end{aligned}$$

where $\beta'([b]) := \beta(b)$. Let now, κ , be the map:

$$\begin{aligned}\kappa : \text{Hom} \left(\frac{B}{\text{Im} \phi}, U(1) \right) &\longrightarrow \ker \hat{\phi}, \\ \beta' &\mapsto \beta := \beta' \circ \pi,\end{aligned}$$

where $\pi : B \rightarrow \frac{B}{\text{Im} \phi}$ is the canonical projection that sends $b \in B$ into its corresponding equivalence class $[b] \in \frac{B}{\text{Im} \phi}$. Notice that $\kappa = \iota^{-1}$, since:

$$(\kappa \circ \iota)(\beta)(b) = \kappa(\beta')(b) = (\beta' \circ \pi)(b) = \beta'([b]) = \beta(b).$$

Therefore, the map ι is a bijection. To prove that it defines an isomorphism we only need to check for its compatibility with the group operation in $\ker \hat{\phi}$; This is, given $\beta_1, \beta_2 \in \ker \hat{\phi}$, we want to show that $\iota(\beta_1 \cdot \beta_2) = \iota(\beta_1) \cdot \iota(\beta_2)$.

So, consider $b \in B$ and , $[b] \in \frac{B}{\text{Im} \phi}$:

$$\iota(\beta_1 \cdot \beta_2)([b]) = (\beta_1 \cdot \beta_2)'([b]) = (\beta_1 \cdot \beta_2)(b) = \beta_1(b) \cdot \beta_2(b) = \iota(\beta_1) \cdot \iota(\beta_2).$$

Hence, $\ker \hat{\phi} \simeq \text{Hom} \left(\frac{B}{\text{Im} \phi}, U(1) \right)$.

□

In particular, as a result of the above proposition, it is true that, for A, B finite groups:

$$|\ker \hat{\phi}| = \left| \text{Hom} \left(\frac{B}{\text{Im} \phi}, U(1) \right) \right| = \frac{|B|}{|\text{Im} \phi|}, \quad (\text{B.10})$$

where in the last step we used the fact that all groups are Abelian. We are one step away from our goal which can be stated as:

Proposition B.4.11. *Let $\phi : A \rightarrow B$ be a homomorphism between finite Abelian groups. Moreover, let $\hat{\phi} : \hat{B} \rightarrow \hat{A}$ its dual morphism. Then,*

$$|\text{Im } \phi| = |\text{Im } \hat{\phi}|.$$

Proof. From Prop. B.4.10, we know that: $|\ker \hat{\phi}| = \frac{|B|}{|\text{Im } \phi|}$. Now, applying the First Isomorphism Theorem [107] on $\hat{\phi} : \hat{B} \rightarrow \hat{A}$, we know that: $\hat{B}/\ker \hat{\phi} \simeq \text{Im } \hat{\phi}$, from which we can write:

$$\frac{|\hat{B}|}{|\ker \hat{\phi}|} = |\text{Im } \hat{\phi}|,$$

recall that $|\hat{B}| = |B|$ since we are dealing with Abelian groups. Replacing Eq. (B.10) into the above one, we get:

$$|\text{Im } \phi| = |\text{Im } \hat{\phi}|,$$

as it was to be shown. □

Appendix C:

Trace of Local Operators

In this appendix we show how taking the partial trace of the ground state projector, or any product of projection operators of the theory, implies in Eq.(5.10). This is, we show that the local projectors $A_{n,x}$ and $B_{n,x}$ are traceless unless they are trivial (equal to the identity operator).

We begin by writing the density matrix, ρ , using the local decomposition of \mathcal{A}_0 and \mathcal{B}_0 (see [3] for a detailed account on this). The local decomposition yields:

$$\mathcal{A}_0 = \prod_{n=0}^d \prod_{x \in K_n} A_{n,x}, \text{ and } \mathcal{B}_0 = \prod_{n=0}^d \prod_{x \in K_n} B_{n,x},$$

such that the density matrix of Eq.(5.7) can be written as:

$$\rho = \frac{1}{GSD} \left(\prod_{n=0}^d \prod_{x \in K_n} A_{n,x} \right) \left(\prod_{n=0}^d \prod_{x \in K_n} B_{n,x} \right),$$

this form is convenient for taking the partial trace as the operators are now localized at simplices $x \in K_n$ for $0 \leq n \leq d$, this allows for the identification of the operators that act exclusively on region A from the operators that act on both $\partial(A)$ and B , in order to get the terms that survive the partial trace. In this sense, the reduced density matrix is written as:

$$\rho_A = \text{Tr}_B(\rho) = \text{Tr}_B \left(\prod_n \prod_{x \in K_n} A_{n,x} \prod_{y \in K_n} B_{n,y} \right). \quad (\text{C.11})$$

Before going onto the calculation of the above partial trace, we will prove a property that will let us evaluate the partial trace rather straightforwardly.

Proposition C.4.12. *Let $x, y \in K_n$, be n -simplices for $0 \leq n \leq d$. The local operators,*

$A_{n,x}, B_{n,y} : \mathcal{H} \rightarrow \mathcal{H}$, are traceless unless they act trivially (as the identity operator $\mathbb{1}_{\mathcal{H}}$).

Proof. Let $\{|f\rangle\}$ be a basis of \mathcal{H} , with $f \in \text{hom}(C, G)^0$. We start by taking the trace of the local gauge transformations:

$$\text{Tr}(A_{n,x}) = \sum_f \langle f | A_{n,x} | f \rangle = \frac{1}{|G_{n+1}|} \sum_f \sum_{g \in G_{n+1}} \langle f | A_{e[g,n,x]} | f \rangle.$$

From Definition ??, the action of $A_{n,x}$ on a basis state consists in general on a shift of basis elements, which yields:

$$\text{Tr}(A_{n,x}) = \frac{1}{|G_{n+1}|} \sum_f \sum_{g \in G_{n+1}} \langle f | |f + \delta^{-1}(e[g,n,x])\rangle \rangle.$$

From the last expression it is clear that the only non-null term in the sum occurs only when $g = e \in G_{n+1}$, the identity element. Thus, we have:

$$\text{Tr}(A_{n,x}) = \frac{\text{Tr}(\mathbb{1})}{|G_{n+1}|} = \frac{\dim(\mathcal{H})}{|G_{n+1}|}.$$

Similarly, for the trace of local holonomy measurement operators, $B_{n,y}$, we have:

$$\text{Tr}(B_{n,y}) = \sum_f \langle f | B_{n,y} | f \rangle = \frac{1}{|G_{n-1}|} \sum_f \sum_{r \in \hat{G}_{n-1}} \langle f | B_{\hat{e}[r,n,y]} | f \rangle.$$

Using Definition ?? the above expression can be written as:

$$\text{Tr}(B_{n,y}) = \frac{1}{|G_{n-1}|} \sum_f \sum_{r \in \hat{G}_{n-1}} \chi_r(f_n(y)) \langle f | f \rangle = \frac{1}{|G_{n-1}|} \sum_f \sum_{r \in \hat{G}_{n-1}} \chi_r(f_n(y)) \chi_{\hat{e}}(f_n(y)) \langle f | f \rangle,$$

where in the last line we used the fact that $\chi_{\hat{e}}(g) = 1, \forall g \in G_{n-1}$ and $\hat{e} \in \hat{G}_{n-1}$, the trivial representation. From the orthogonality relations of characters [104, 120, 121, 122],

we note that:

$$\sum_f \chi_r(f_n(y)) \chi_e(f_n(y)) = \delta(e, f_n(y)),$$

which implies that the trivial representation term is the only one that has non-zero trace, since it acts as the identity operator.

$$\text{Tr}(B_{n,y}) = \frac{|\mathcal{H}|}{|G_{n-1}|}$$

□

This result can naturally be extended to products of such operators to show that the only term that survives the trace is the one that acts trivially. This allows us to express the reduced density matrix, ρ_A of Eq.(C.11) in terms of operators that act only in region A .

In this case, Proposition C.4.12 implies that any operator (or product of several) that is different from $\mathbb{1}_B$, the identity operator in \mathcal{H}_B , will have vanishing trace. This, in turn, tells us about the nature of the operators that survive the trace; In particular, local gauge transformations $A_{n,x}$ will survive the trace if and only if $x \in K_{n,\tilde{A}}$, where \tilde{A} is the interior of A ¹ as in Def. 5.1.1. On the other hand, local holonomy measurement operators $B_{n,y}$ will survive the trace if and only if $y \in K_{n,A}$ which corresponds to the entire region A . Consequently, the reduced density matrix is:

$$\rho_A = \text{Tr}_B(\mathbb{1}_B) \prod_n \prod_{x \in K_{n,\tilde{A}}} A_{n,x} \prod_{y \in K_{n,A}} B_{n,y}.$$

From which we write Eq. (5.10).

¹Local gauge transformations are labeled by simplices $x \in K_n$ and they act on the gauge fields at the co-boundary, $\partial^*(x)$. In particular, gauge transformations located at $x \in K_{n,\partial(A)}$, the boundary of A , also act on B . Thus, they do not contribute to the trace.

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