Razavi: Basic Circuit Theory I

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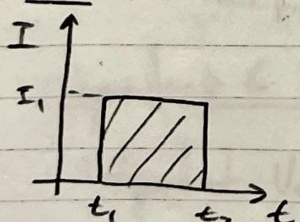
Outline
- Electric Quantities/Devices
- Circuit Laws
- Analysis Techniques
- Circuit theorems
- RC, RL, RLC
- Op-Amps

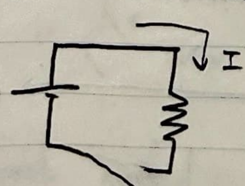
Electric Quantities
- Charge Q (Coulombs)
- Current occurs when charge moves
- $I = \frac{dQ}{dt}$
  1. Current has value & direction
  2. Open circuit $\rightarrow I = 0$
  3. $Q = \int_{t_1}^{t_2} I\,dt$ between times t_1, t_2

Ex



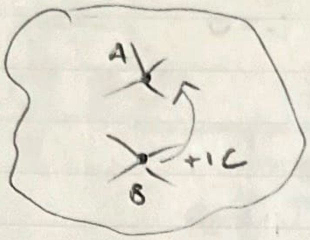
$$Q = \int_{t_1}^{t_2} I_1\,dt = (t_2 - t_1) I_1$$

Ex
Circuit that creates this current.

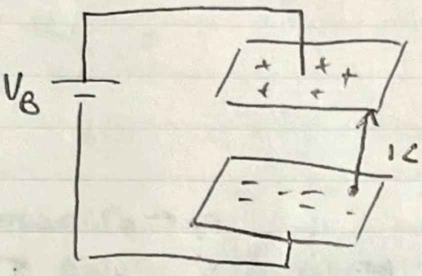


close switch @ t_1, open @ t_2.

- Voltage (v)



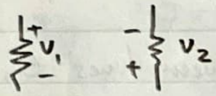
$V_A - V_B$ = the amount of energy necessary to take a positive unit of charge from B to A.



V_B = Energy needed to bring charge from lower plate to upper plate

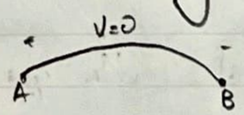
Important points

① A voltage has a value & a sign.



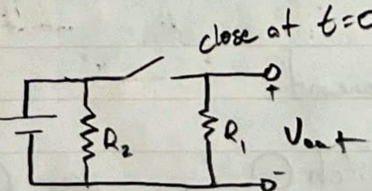
② Voltage is meaningful only between two points

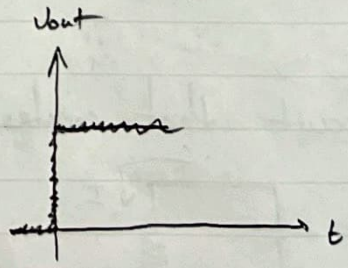
③ A short circuit guarantees $v = 0$.



Ex

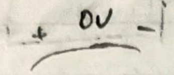
close at $t=0$





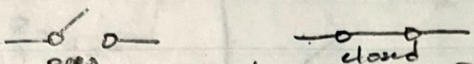
No change when R_c added.

Electric Devices

- wire  ideal $\Rightarrow R=0, C=0, L=0$

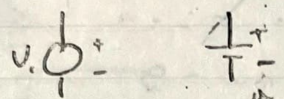
  $V = IR$

- switch



$\equiv$ open circuit $\quad\quad$ $\equiv$ short circuit

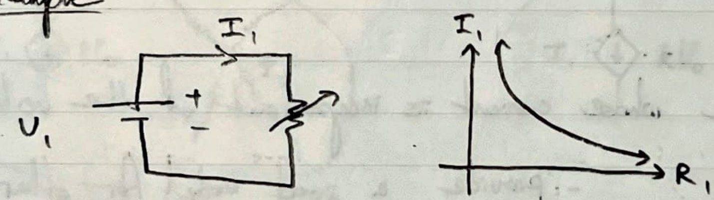
- voltage source



constant w/ time

<u>Defn.</u>
A two-terminal device whose voltage does not depend on the current through it

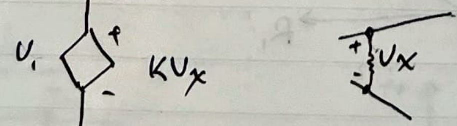
<u>Example</u>



<u>Independent & Dependent Voltage Sources</u>

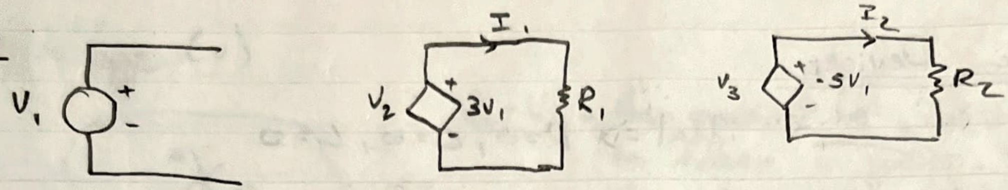
- Independent Voltage source - independent of other quantities in the circuit
- Dependent Voltage source - depends on other quantity in circuit

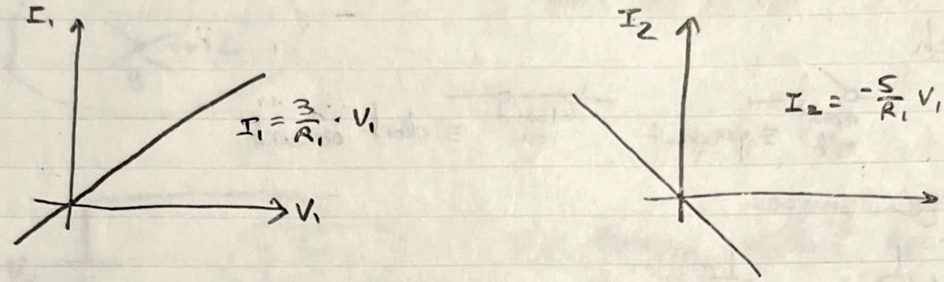
e.g.



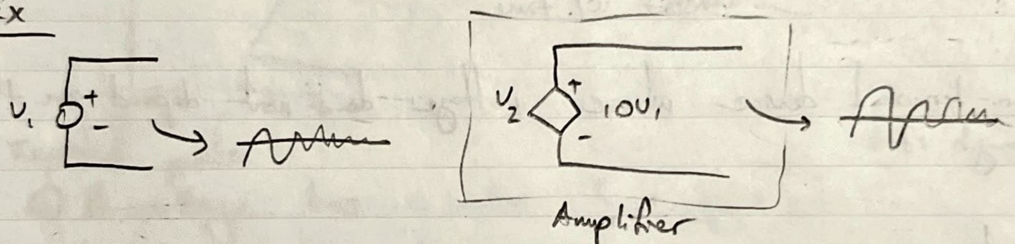
good model for complex circuits. (e.g. amplifiers)

Ex





Ex

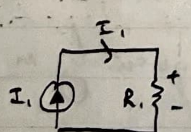


Current Source

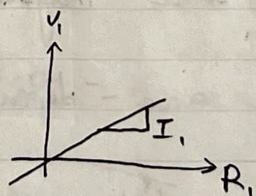
- A 2 terminal device whose current is independent of the voltage across it

- provide a good model for other devices such as transistors.

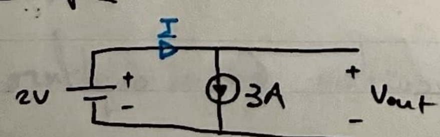
Example:



$V_1 = R_1 I_1$



Example

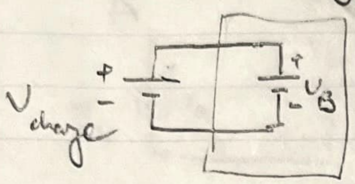


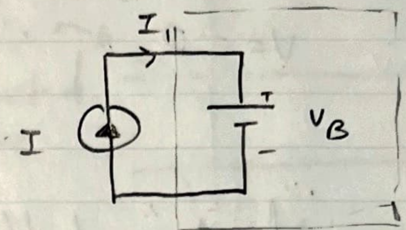
$V_{out} = 2\,V.$

$I = 3A$

Example

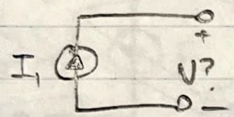
How do we charge a battery?





I is very large :(

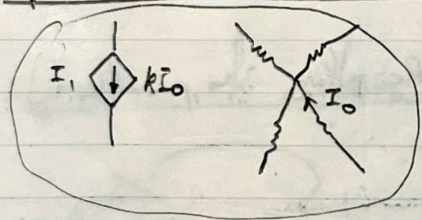
Example



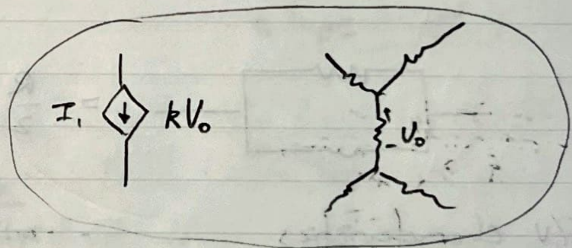
open circuit $\Rightarrow R = \infty$

$V = I_1 R \to \infty$ open current source :(bad

Dependent Current Sources

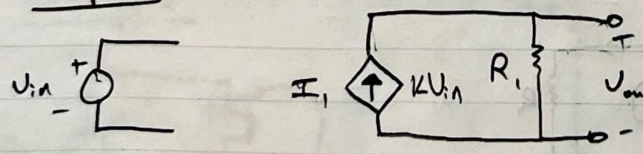


Multiple of current



Dependent of voltage

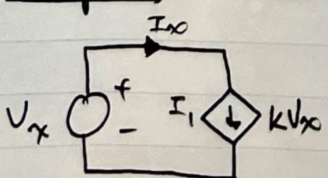
Example



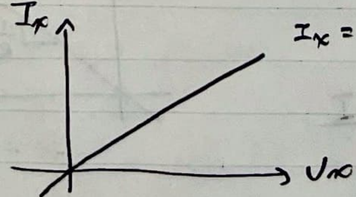
$$\underbrace{V_{out} = R_1 (k V_{in})}$$

Acts as amplifier

Example

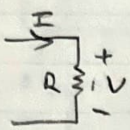


$I_x = kV_x$

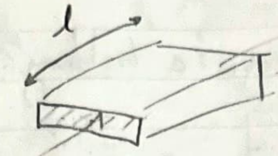


$$I_x = \frac{V_x}{R_{eq}}, \quad R_{eq} = \frac{1}{k}$$

Resistors & Ohm's Law



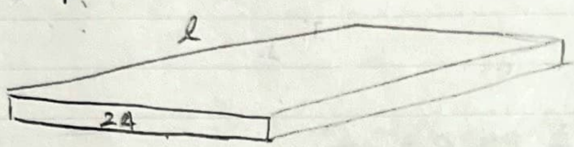
$V = IR$



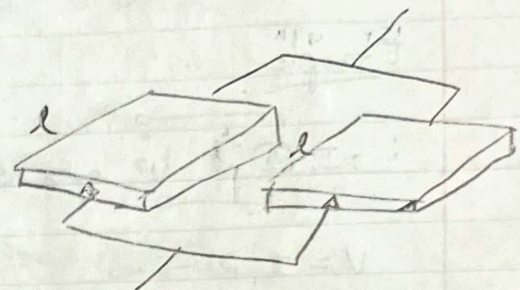
$R = \frac{\rho l}{A}$ ← resistivity

Example

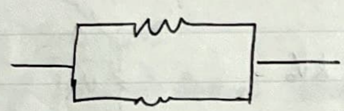
what happens if A is doubled?



$R \to \frac{R}{2}$



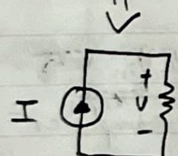
$R \to \frac{R}{2}$

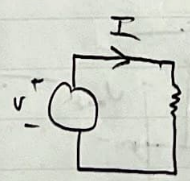


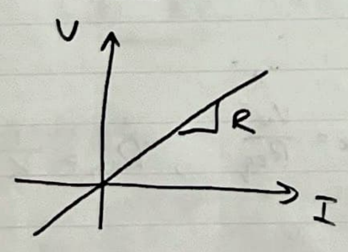
$= \frac{R}{2}$ (Resistors in parallel)

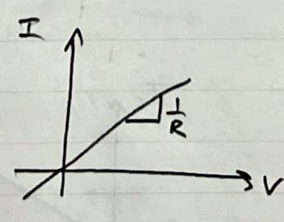
I/V characteristics

$V = IR$, $I = \frac{V}{R}$ Different causes & effects

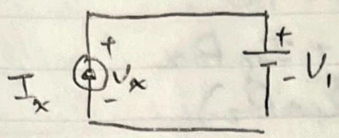




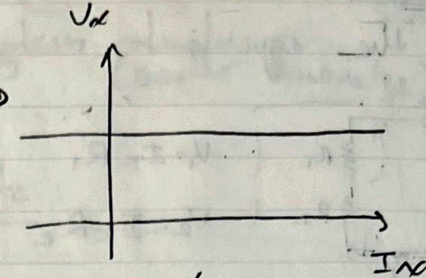




Example

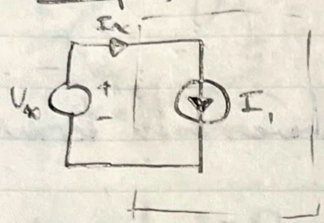


$V_x = V_1$ independent of I_x

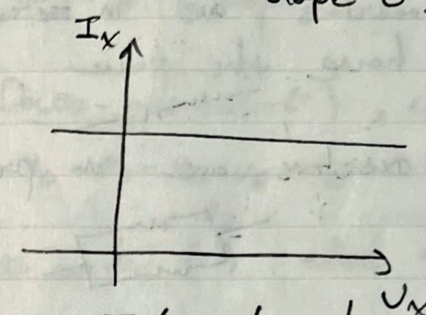


V/I characteristic slope 0.

Example

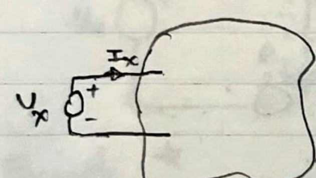


$I_x = I_1$



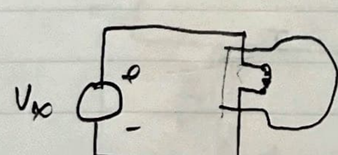
I/V characteristic slope 0.

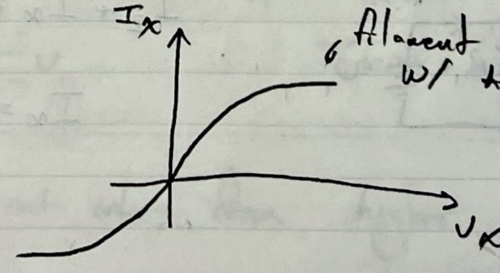
Application of Ohm's Law



$R_{eq} = \frac{V_x}{I_x}$

Example

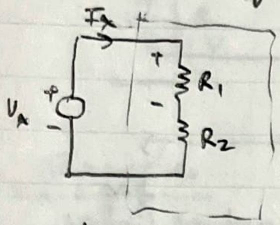




filament becomes more resistant w/ higher temperature

Example:

Find the equivalent resistance of this topology.



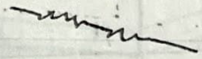
$V_1 = I_x R_1$

$V_2 = I_x R_2$

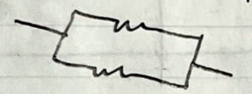
$$V_x = I_x R_1 + I_x R_2$$
$$= I_x(R_1 + R_2)$$
$$R_{eq} = R_1 + R_2 \quad \text{(resistors in series)}$$

Defns

Two resistors are in series if they share only one terminal and have the same current

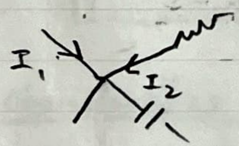


Two resistors are in parallel if they share both terminals.



KCL & KVL

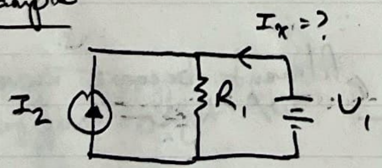
- KCL.



$$\sum_j I_j = 0 \quad \text{(Charge conservation)}$$

$$\sum I_{entering} = \sum I_{exiting}.$$

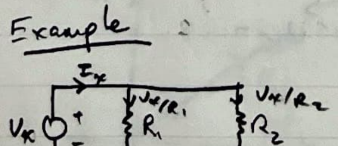
Example



$$I_2 + I_x = \frac{V_1}{R_1}$$

$$I_x = \frac{V_1}{R_1} - I_2$$

Example:



$$I_x = \frac{V_x}{R_1} + \frac{V_x}{R_2}$$

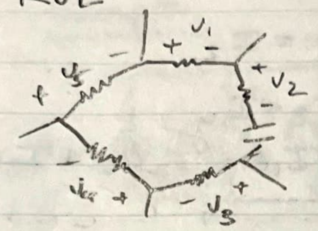
$$R_{eq} = \frac{V_x}{I_x} = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2}} = \frac{R_2 R_1}{R_1 + R_2}$$

<u>Observations</u>: $R_{eq} = \frac{R_1 R_2}{R_1 + R_2} < R_1, R_2$ (∥ → less resistance)

$R_1 \to 0 \Rightarrow R_{eq} \to 0$ (smaller dominates)

$R_1 \to \infty \Rightarrow R_{eq} = R_2$

- KVL



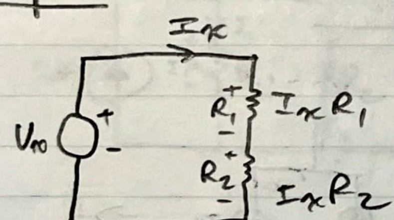
$\sum_j V_j = 0$ around a loop.

Approach 1: Down potential (−), else (+)

$-V_1 - V_2 - V_3 - V_4 - V_5 - V_6 = 0$.

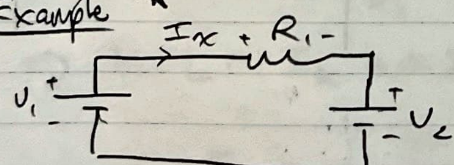
Approach 2: Opposite, see negative write (−), else (+).

<u>Example</u>



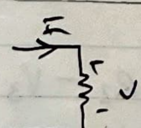
$V_{x0} - I_x R_1 - I_x R_2 = 0$

<u>Example</u> ☆



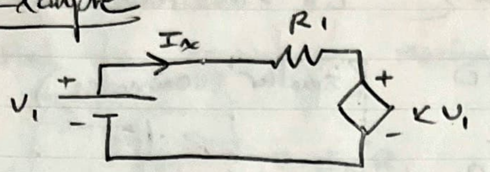
$V_1 - I_x R_1 - V_2 = 0$

$I_x = \frac{V_1 - V_2}{R_1}$



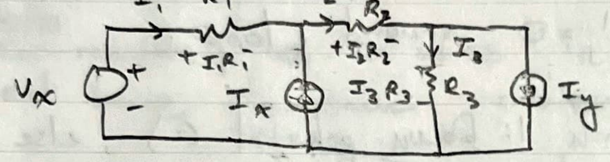
current enters from higher side.

Example



$$I_x = \frac{(1-k)V_1}{R_1}$$

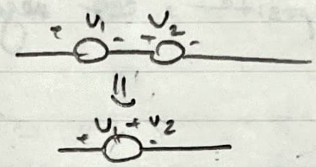
Example



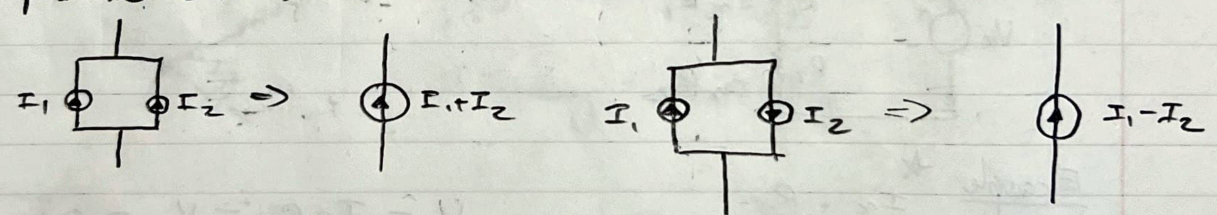
$$\begin{cases} I_1 + I_x = I_2 \\ I_2 = I_3 + I_y \\ V_x = I_1R_1 + I_2R_2 + I_3R_3 \end{cases}$$

Mechanics of KVL & KCL

- Series Voltage Sources

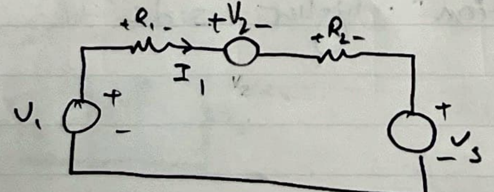


- parallel current sources



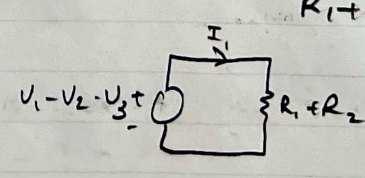
- Resistors in series/parallel

Example

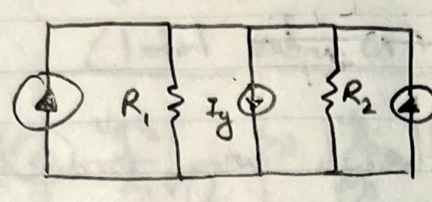


$$V_1 - I_1R_1 - V_2 - I_1R_2 - V_3 = 0$$

$$I_1 = \frac{V_1 - V_2 - V_3}{R_1 + R_2}$$



Example



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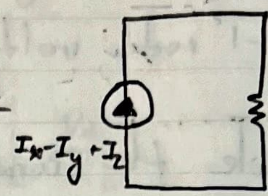
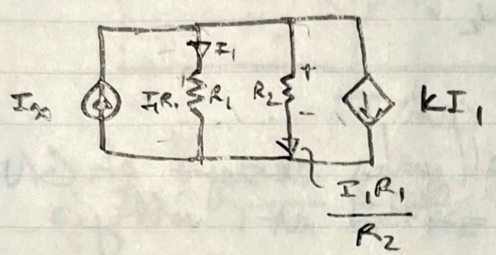


$R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$.

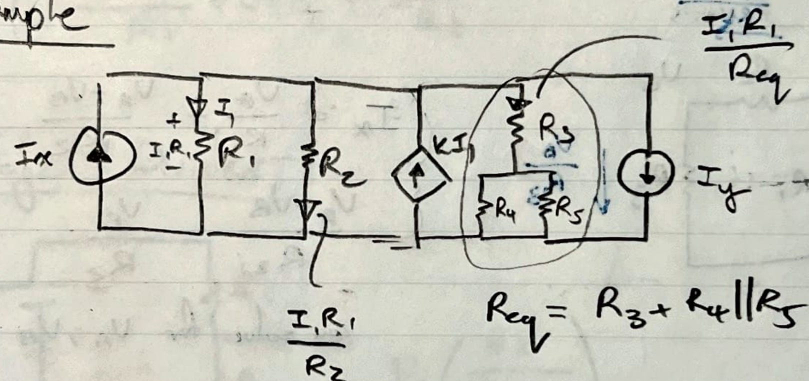
Example



$$I_x = I_1 + \frac{I_1 R_1}{R_2} + k I_1$$

$$I_1 = \frac{I_x}{1 + \frac{R_1}{R_2} + k}$$

Example



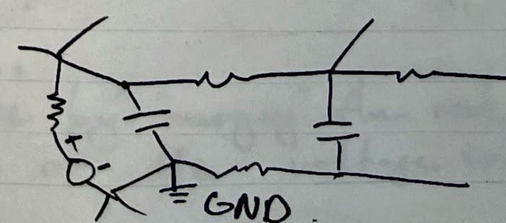
$$R_{eq} = R_3 + R_4 \| R_5$$

$$\Rightarrow \text{KCL: } I_x + k I_1 = I_1 + \frac{I_1 R_1}{R_2} + \frac{I_1 R_1}{R_3 + R_4 \| R_5} + I_y$$

$$I_1 = \frac{I_x - I_y}{1 + \frac{R_1}{R_2} + \frac{R_1}{R_3 + R_4 \| R_5} - k}$$

Node Analysis

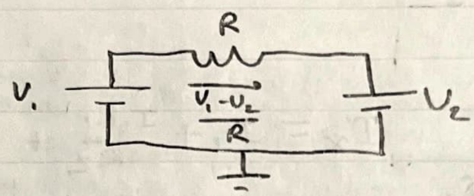
step 1: select reference node (the "ground")



Step 2: Assign voltages with respect to ground $\forall$ other nodes.

$\Rightarrow$ N-1 node voltages (N nodes total)

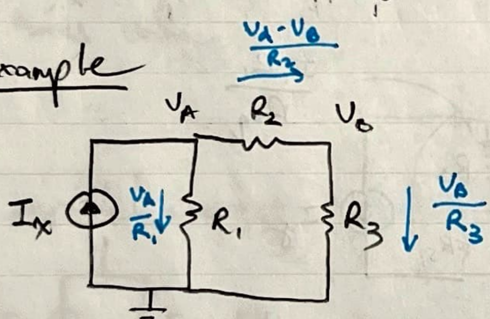
step 3: Write the current through every branch.



step 4: Write KCL @ all nodes except at GND

$\Rightarrow$ N-1 equations $\Rightarrow$ find N-1 voltages

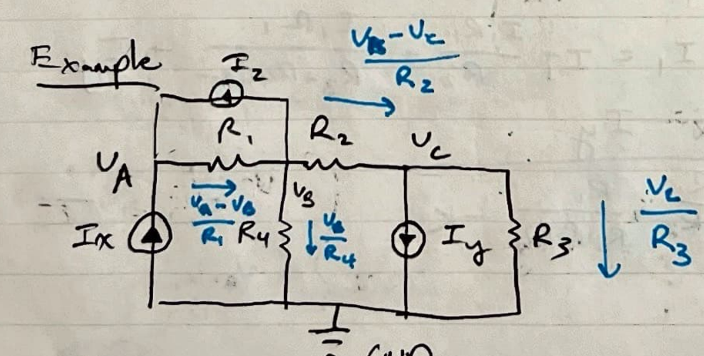
Example



$$\begin{cases} I_x = \frac{V_A}{R_1} + \frac{V_A - V_B}{R_2} \\ \frac{V_A - V_B}{R_2} = \frac{V_B}{R_3} \end{cases}$$

can solve for V_A ; V_B

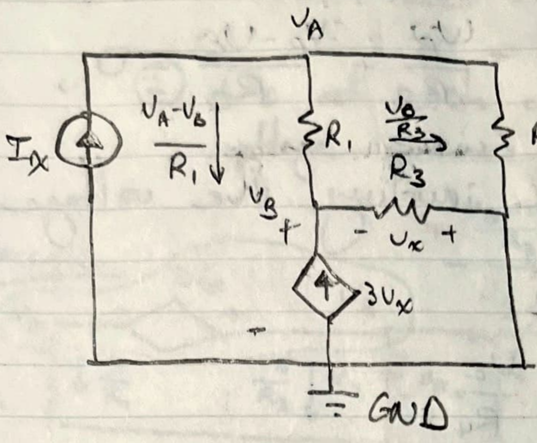
Example



$$\begin{cases} I_x + I_2 = \frac{V_A - V_B}{R_1} \\ \frac{V_A - V_B}{R_1} = \frac{V_B - V_C}{R_2} + \frac{V_B}{R_4} + I_2 \\ \frac{V_B - V_C}{R_2} = I_y + \frac{V_C}{R_3} \end{cases}$$

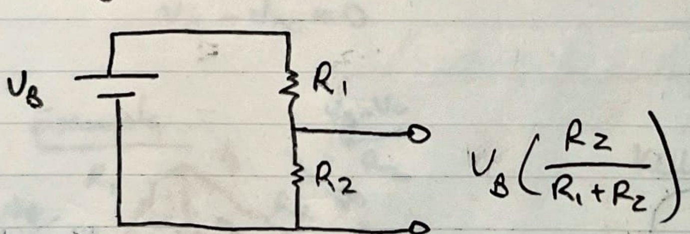
can solve V_A, V_B, V_C.

Example

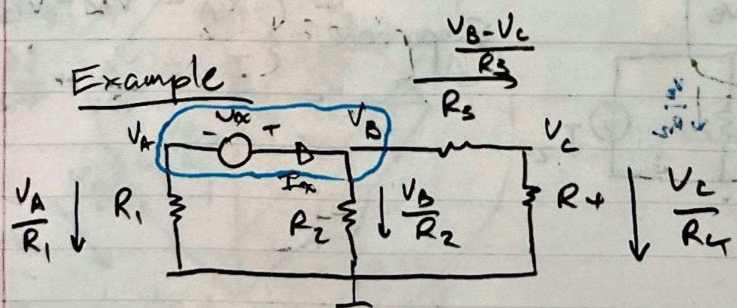


$$\text{KCL}\begin{cases} I_x = \dfrac{V_A - V_B}{R_1} + \dfrac{V_A}{R_2} \\ \dfrac{V_A - V_B}{R_1} + 3V_x = \dfrac{V_B}{R_3} \end{cases} \qquad \text{KVL}\begin{cases} V_B + V_x = 0. \end{cases}$$

Voltage Divider



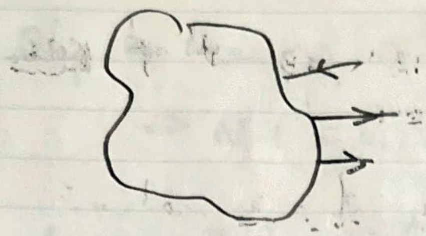
Example



$$\text{KCL}\begin{cases} \dfrac{V_B - V_C}{R_3} = \dfrac{V_C}{R_4} \\ \dfrac{V_B - V_C}{R_3} + \dfrac{V_B}{R_2} = I_x \\ I_x + \dfrac{V_A}{R_1} = 0. \end{cases}$$

With voltage source:

Step 5: Form a "supernode" by "merging" the node on the two sides of the voltage source



⇒ Write a KCL for this "supernode".

$$\frac{V_A}{R_1} + \frac{V_B}{R_2} + \frac{V_B - V_C}{R_3} = 0$$

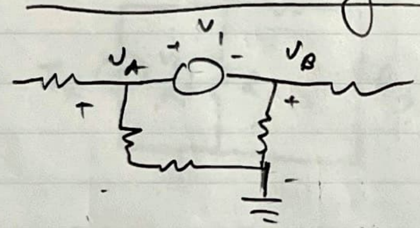
Step 6: Write a KVL involving the voltage source

$$V_A + V_x = V_B$$

So now:

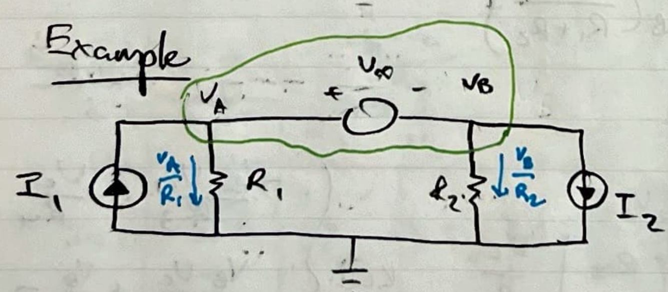
$$\begin{cases} \dfrac{V_B - V_C}{R_3} = \dfrac{V_C}{R_4} \\ \dfrac{V_A}{R_1} + \dfrac{V_B}{R_2} + \dfrac{V_B - V_C}{R_3} = 0 \\ V_A + V_x = V_B \end{cases}$$

A Note on writing KVLs



$$+V_A - V_1 - V_B = 0.$$

Example

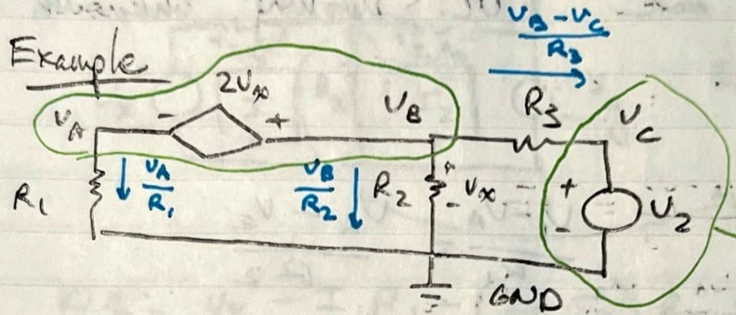


Supernode KCL
$$\begin{cases} V_A - V_x - V_B = 0 \\ I_1 = \dfrac{V_A}{R_1} + \dfrac{V_B}{R_2} + I_2 \end{cases}$$

Summary: ① Every voltage source reduces # of KCLs by 1.
⇒ Need to write KVL.

② Do not write a KCL at either terminal of a voltage source.

Example



⇒ No need for supernode because connected to ground.
⇒ causes repeat equation

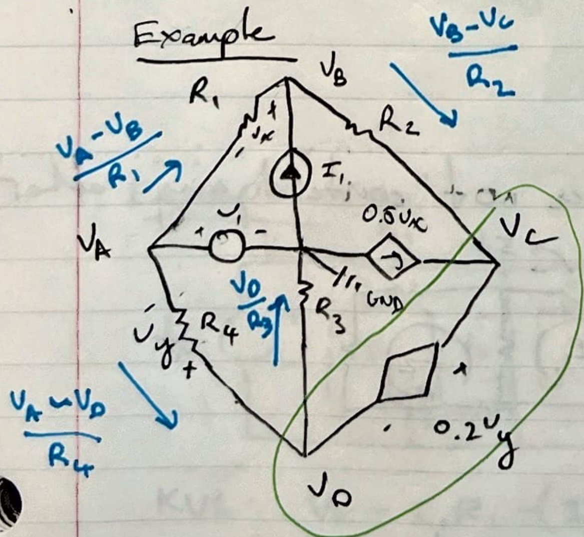
KCL: $\frac{V_A}{R_1} + \frac{V_B}{R_2} + \frac{V_B - V_C}{R_3} = 0$

KVL: $V_A + 2V_x - V_B = 0$

$V_C - V_2 = 0$

$V_B - V_x = 0$

Example



KCL: $\frac{V_A - V_B}{R_1} + I_1 = \frac{V_B - V_C}{R_2}$

$\frac{V_B - V_C}{R_2} + 0.5V_x + \frac{V_A - V_D}{R_4} = \frac{V_D}{R_3}$

KVL: $V_D + 0.2V_y - V_C = 0$

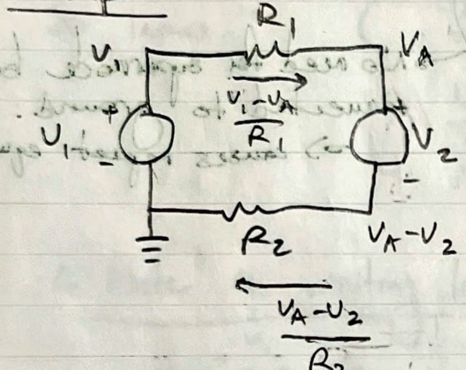
$V_A = V_1$

$V_A + V_x - V_B = 0$

$V_D + V_y - V_D = 0$

Summary: Started w/ 5 nodes
Called one ground → 4 nodes
Two voltage sources → Lose 2 KCL
2 KCL left
Write 2 KVL w/ voltage sources
Write 2 more KVL involving unknown voltages
V_x, V_y

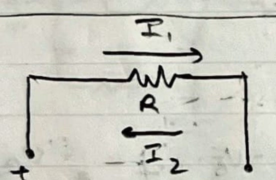
Example



$$\frac{V_1 - V_A}{R_1} = \frac{V_A - V_2}{R_2}$$

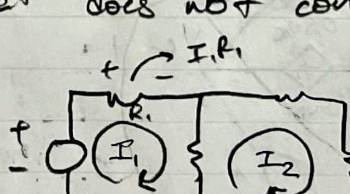
$$\Rightarrow V_A = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}$$

☆



$$V = (I_1 - I_2)R$$

- Mesh Analysis
  - A mesh is a loop that does not contain any other loops within it.



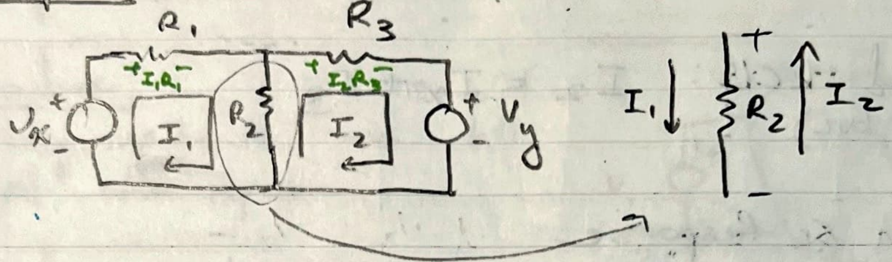
Three steps:

① Identify all of the meshes and assign a current to each.

② Write the voltage drops on the circuit.

③ Write KVL around every mesh.

Example



$V = (I_1 - I_2)R_2$

KVL: $V_x - I_1R_1 - (I_1 - I_2)R_2 = 0$

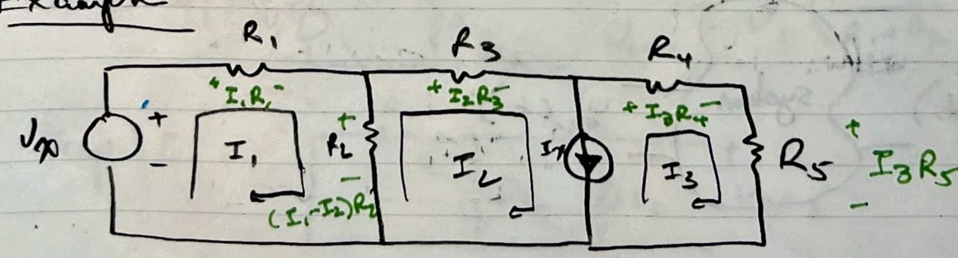
$(I_1 - I_2)R_2 - I_2R_3 - V_y = 0$

Two Observations

① Node Analysis is KCL centric
Has difficulty with voltage sources

② Mesh Analysis is KVL centric
Handles voltage sources easily
Has difficulty w/ current sources.

Example



KVL: $V_x - I_1R_1 - (I_1 - I_2)R_2 = 0$

$(I_1 - I_2)R_2 - I_2R_3 + ? = 0$

↳ current source does not satisfy Ohm's law!

Solution: Merge two meshes into one "supermesh"

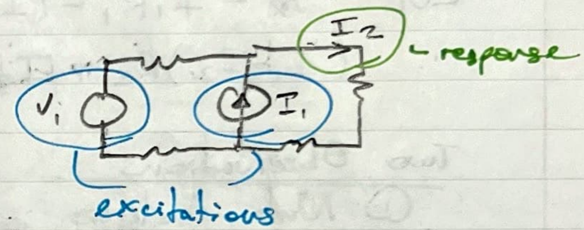
$$\Rightarrow \underbrace{(I_1 - I_2)R_2 - I_2 R_3 - I_3 R_4 - I_3 R_5}_{\text{Avoids current source}} = 0$$

Now need KCL: $I_2 = I_x + I_3$
(lost 1 KVL)

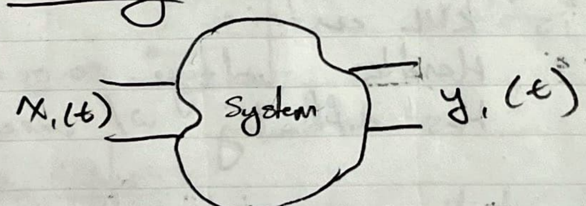
- Excitation & Response

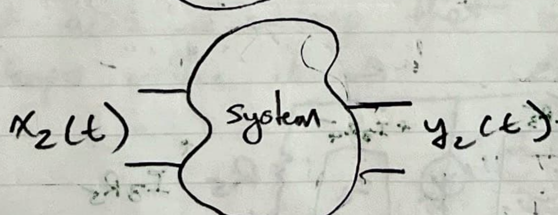
Independent sources: excitations, aka "inputs"

Other quantities:
Responses, aka "outputs"

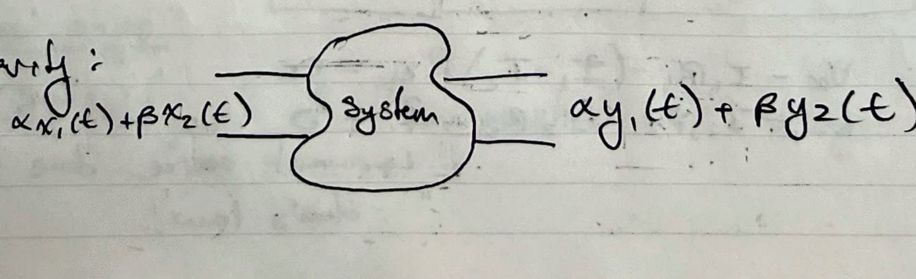


Linearity ☆





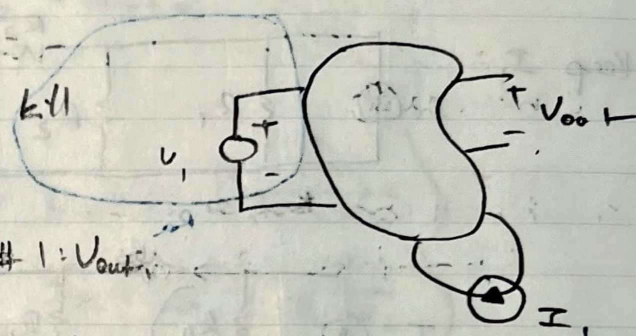
Linearity:



Superposition

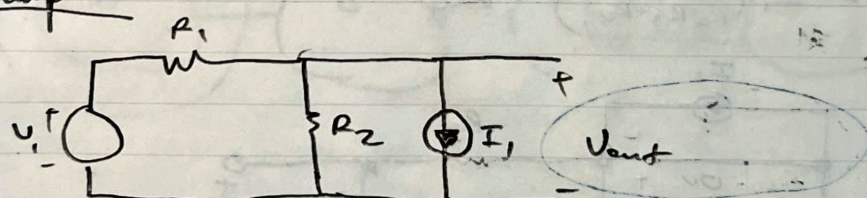
In a linear system, the total response to multiple excitations is equal to the sum of the corresponding responses.

Procedure

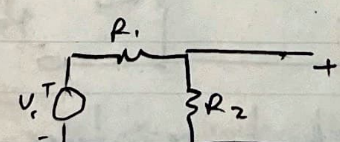


- Keep input # 1 and kill all other inputs.
- Find response to input # 1: V_{out_1}
- Repeat for each input: V_{out_2}, V_{out_3}, ...
- Total response $= V_{out_1} + V_{out_2} + \dots$

Example

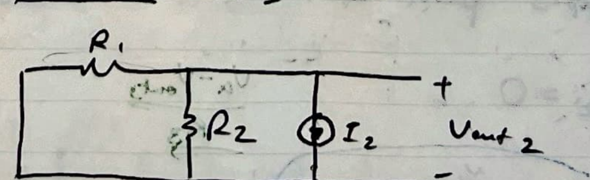


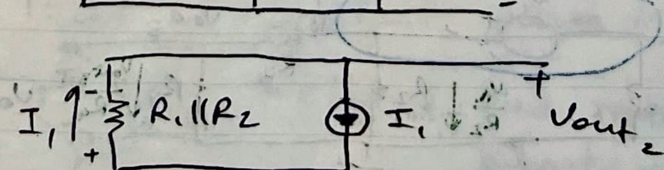
Keep V_1:



$V_{out_1} = V_1\left(\frac{R_2}{R_1+R_2}\right)$ (voltage divider)

Keep I_1:



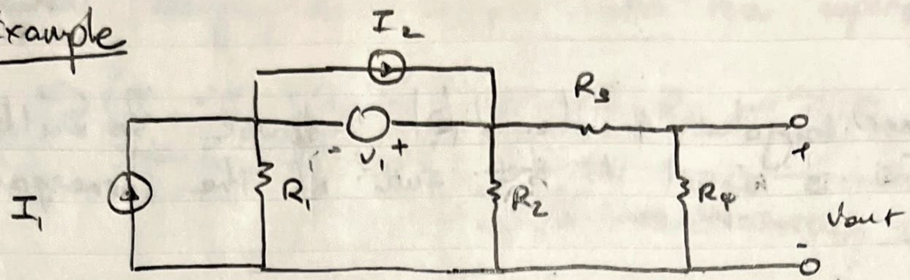


$V_{out_2} = -(R_1 \| R_2) I_1$

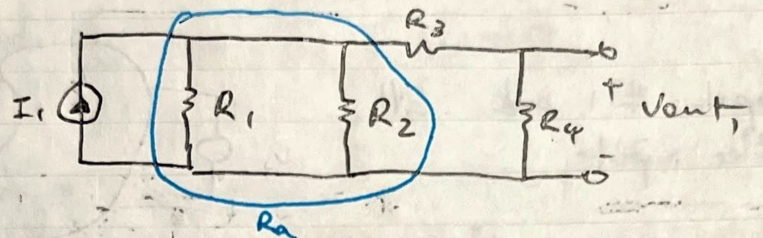
$= -I_1 \frac{R_1 R_2}{R_1+R_2}$

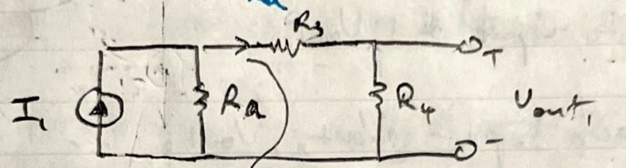
$V_{out} = V_{out_1} + V_{out_2} = V_1 \frac{R_2}{R_1+R_2} - I \frac{R_1 R_2}{R_1+R_2}$

Example



Keep I_1:

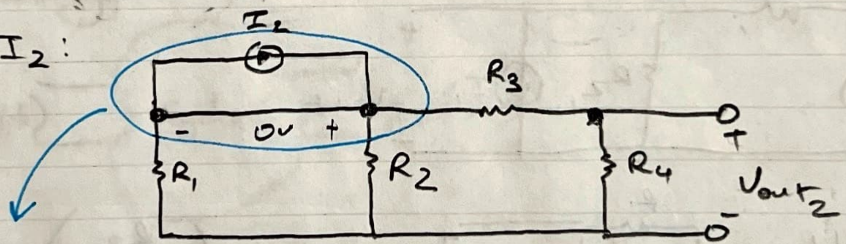




$I_1\left(\frac{R_a}{R_a+R_3+R_4}\right)$, current divider)

$V_{out_1} = I_1\left(\frac{R_a}{R_a+R_3+R_4}\right)\cdot R_4$

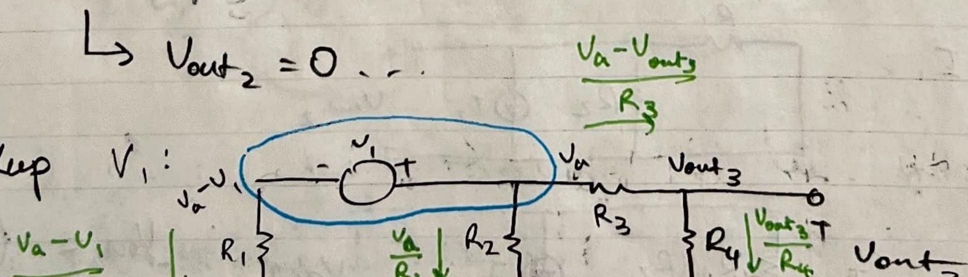
Keep I_2:



Current prefers path of least resistance

↳ $V_{out_2} = 0$ ..

Keep V_1:

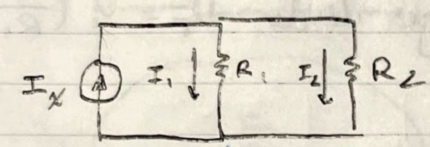


KCLs: $\dfrac{V_a - V_{out_3}}{R_3} = \dfrac{V_{out_3}}{R_4}$

$$\frac{V_a - V_1}{R_1} + \frac{V_a}{R_2} + \frac{V_a - V_{out_3}}{R_3} = 0.$$

$$V_{out} = V_{out_1} + V_{out_2} + V_{out_3}$$

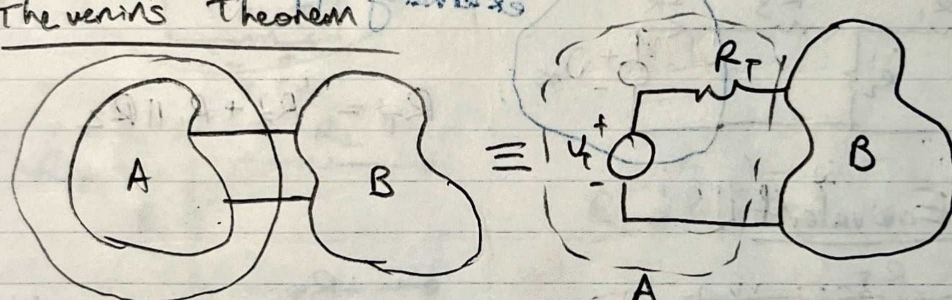
Current Divider



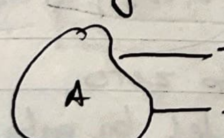
$$I_1 = I_x\left(\frac{R_2}{R_1 + R_2}\right)$$

$$I_2 = I_x\left(\frac{R_1}{R_1 + R_2}\right)$$

Thevenin's Theorem

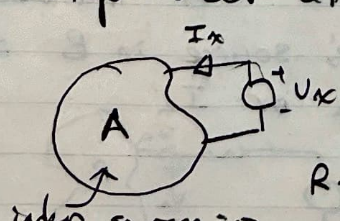


- Calculating v_T



v_T = "open-circuit" voltage

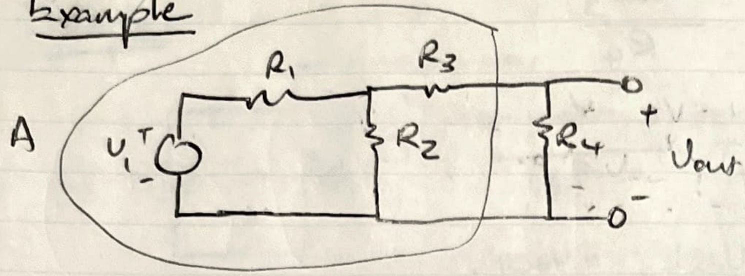
- Calculating R_T:
  - step 1: Set all independent sources in section A to zero.



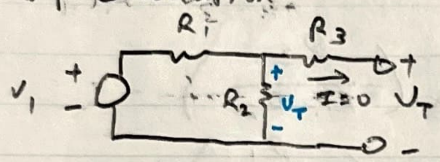
  - step 2: Find resistance between the output wires of section A. (Apply voltage, measure current)

$$R_T = \frac{V_x}{I_x}$$

Example

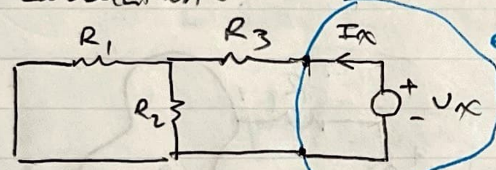


V_T calculation:



$I = 0 \Rightarrow$ voltage divider: $V_T = V_1 \left(\frac{R_2}{R_1+R_2}\right)$

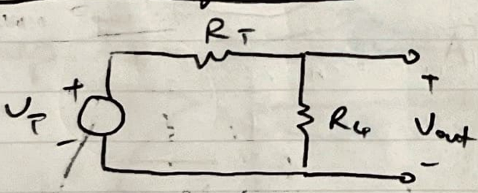
R_T calculation:



externally applied!

$$R_T = R_3 + R_1 \parallel R_2$$

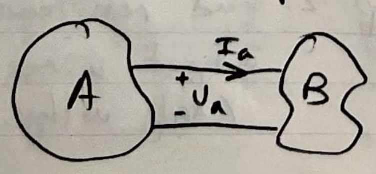
Thevenin Equivalent

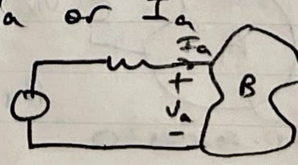


$$V_{out} = \frac{R_4}{R_4 + R_3 + R_1 \parallel R_2} \cdot V_T$$

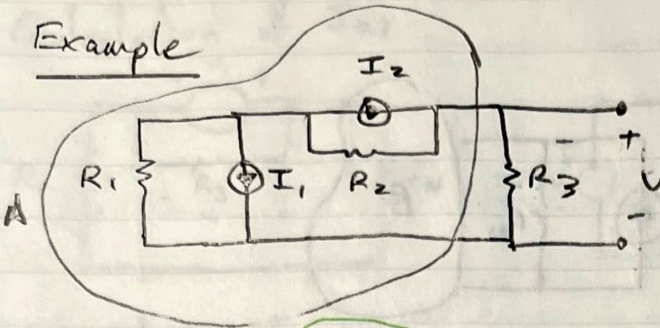
$$= V_1 \left(\frac{R_2}{R_1+R_2}\right)\left(\frac{R_4}{R_4+R_3+R_1 \parallel R_2}\right)$$

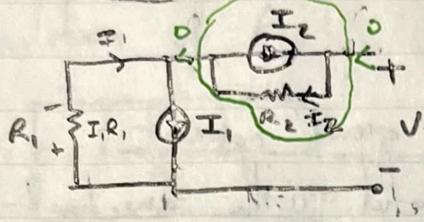
Notes:
1. Do not set dependent sources to zero.
2. Sections A & B must be connected by only 2 wires
3. Section A cannot control a dependent source in section B.
   - ↳ Thevenin's will lose that information
   - ↳ exception: if dep. source in B is a function of V_a or I_a





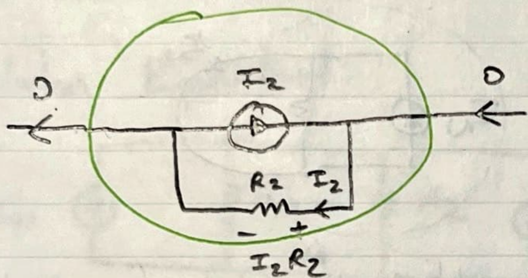
Example

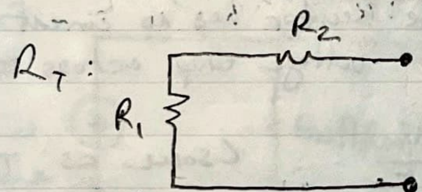




KVL: $V_T - I_2R_2 + I_1R_1 = 0$

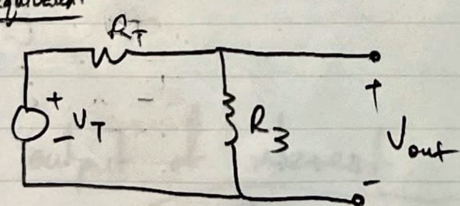
$$V_T = I_2R_2 - I_1R_1$$





$$R_T = R_1 + R_2$$

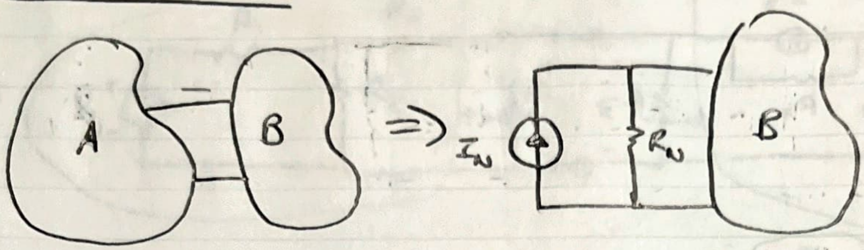
Thevenin Equivalent



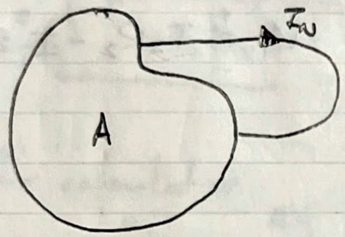
$$V_{out} = V_T\left(\frac{R_3}{R_3+R_T}\right)$$

$$= (I_2R_2 - I_1R_1)\left(\frac{R_3}{R_1+R_2+R_3}\right)$$

Norton's Theorem

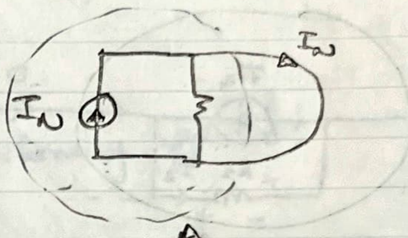


Calculation of I_N:



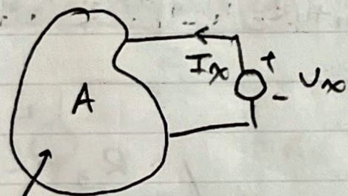
"short-circuit" current.

Be careful with direction!



Note: Resistor has no current b/c no voltage drop across it (shorted).

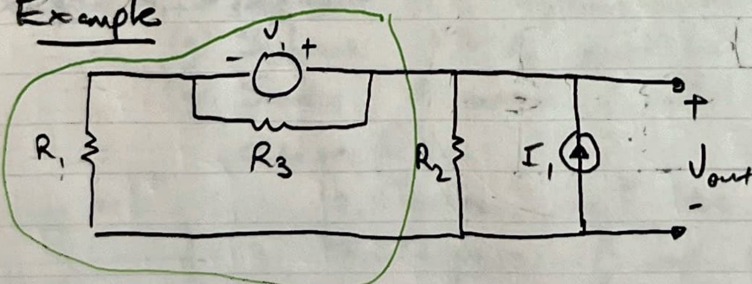
Calculation of R_N:



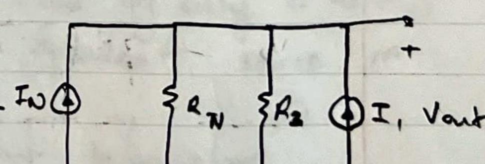
indep. sources = 0

$$R_N = \frac{V_x}{I_x}$$ (same as Thevenin's)

Example

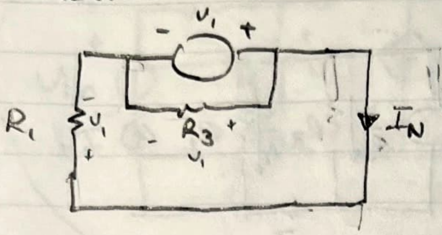


$\equiv$

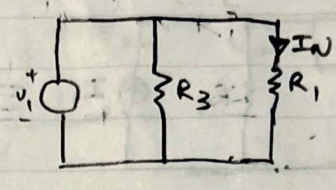


easy to handle so use Norton's.

Calculation of I_N:

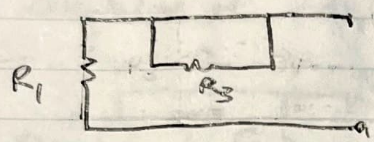


$\equiv$



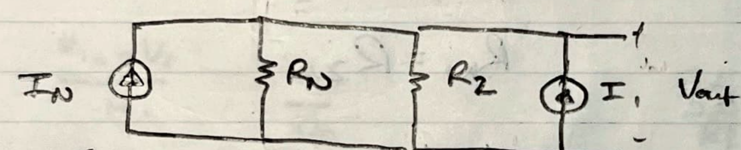
$$I_N = \frac{V_1}{R_1}$$

Calculation of R_N:



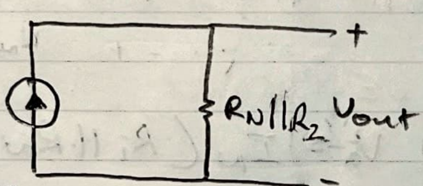
$$R_N = R_1$$

Norton Equivalent:



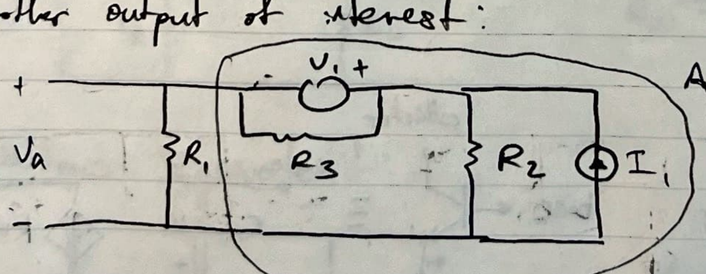
$I_1 + I_N$

(both same direction, entering same node)

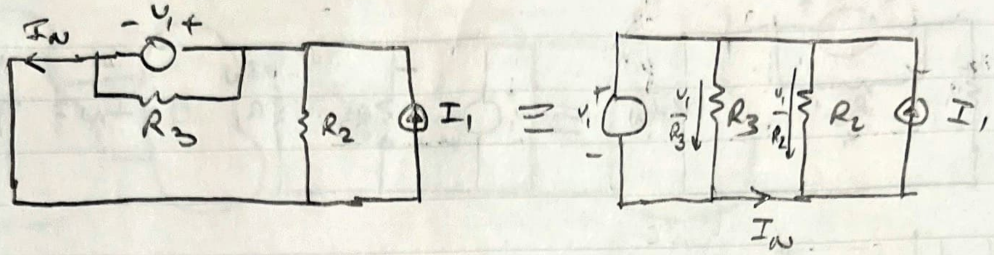


$$V_{out} = \left(I_1 + \frac{V_1}{R_1}\right)\left(\frac{R_1 R_2}{R_1 + R_2}\right)$$

Another output of interest:



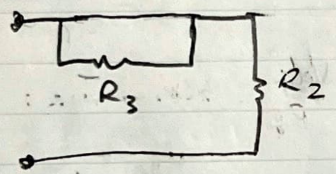
Calculation of I_N:



$$\frac{v_1}{R_2} + I_N = I_1$$

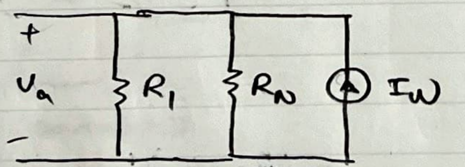
$$I_N = I_1 - \frac{v_1}{R_2}$$

Calculation of R_N:



$$R_N = R_2$$

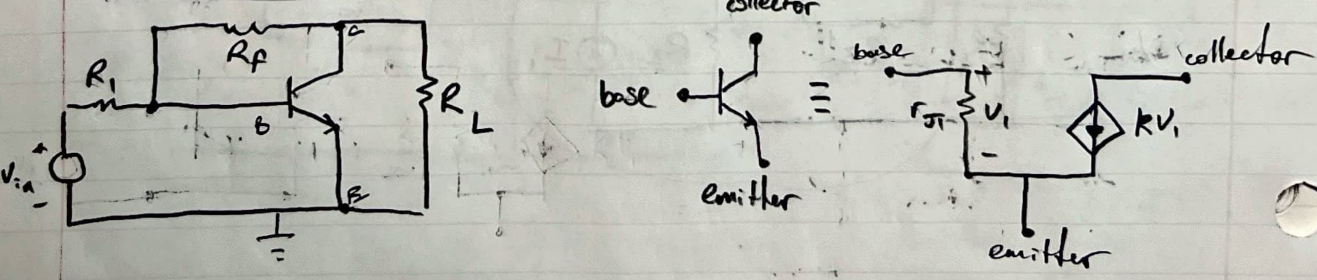
Norton Equivalent:



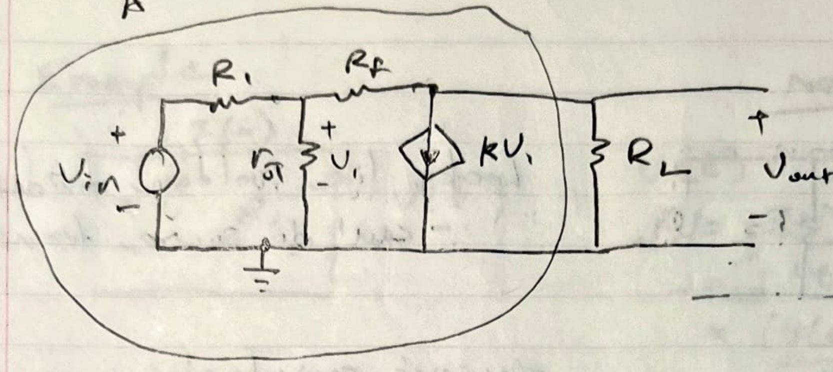
$$V_a = I_N\,(R_1 \,||\, R_N)$$

$$= \left(I_1 - \frac{v_1}{R_2}\right)\left(\frac{R_1 R_2}{R_1 + R_2}\right)$$

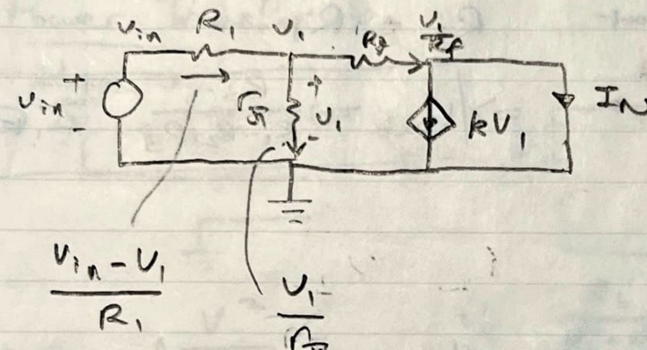
Amplifier Example



A



Calculation of I_N:



KCL: $\frac{v_{in} - V_1}{R_1} = \frac{V_1}{R_f} + \frac{V_1}{r_\pi}$

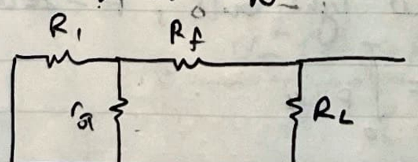
$$V_1 = \frac{v_{in}}{R_1}\left(\frac{1}{\frac{1}{R_1} + \frac{1}{r_\pi} + \frac{1}{R_f}}\right)$$

$$\frac{V_1}{R_f} = kV_1 + I_N$$

$$I_N = \frac{V_1}{R_f} - kV_1$$

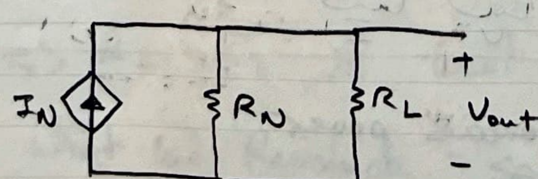
$$I_N = V_1\left(\frac{1}{R_f} - k\right)$$

Calculation of R_N:



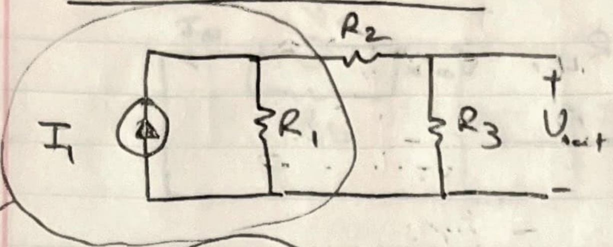
$$R_N = (R_f + R_L) \,||\, R_1 \,||\, r_\pi$$

Norton's Equivalent:



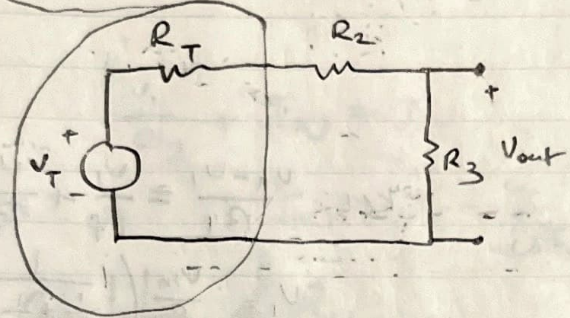
$$V_{out} = I_N\left(R_N \,||\, R_L\right)$$

Source Transformation



Looks like voltage divider
- can do source transformation

transform



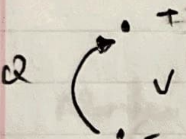
Thevenin's equivalent:

$V_T = I_1 R_1$

$R_T = R_1$

$$\Rightarrow \boxed{V_{out} = \frac{R_3}{R_1 + R_2 + R_3} \cdot I_1 R_1}$$

Energy



$\mathcal{E} \overset{?}{=} QV$ unsure voltage is constant for large Q.

$d\mathcal{E} = (dQ)V$.

$= V\,dQ$

Power: $P = \frac{d\mathcal{E}}{dt}$ e.g. 20W light bulb, $P = 20W$.

$20W = \frac{20J}{1 sec}$ $\mathcal{E} = \int_0^T P\,dt$

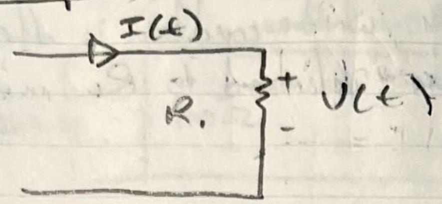
Power in Circuits:

$d\mathcal{E} = V\,dQ$

$$\boxed{P = \frac{d\mathcal{E}}{dt} = \underbrace{\frac{d\mathcal{E}}{dQ}}_{V} \underbrace{\frac{dQ}{dt}}_{I} = V(t)\,I(t)}$$

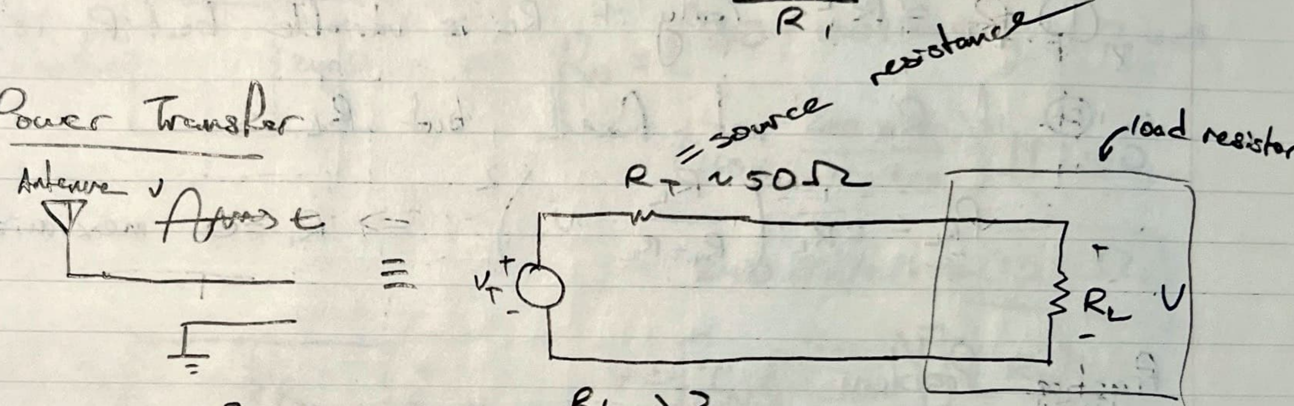
Instantaneous power

Example



$$v(t) = I(t)R,$$
$$P = v(t)\cdot I(t)$$
$$= [I(t)]^2 \cdot R,$$
$$= \frac{(v(t))^2}{R},$$

Power Transfer



$$P_{R_L} = \frac{V^2}{R_L} = \frac{\left(V_T \cdot \frac{R_L}{R_L+R_T}\right)^2}{R_L} = \frac{V_T^2 \cdot R_L}{(R_L+R_T)^2}$$

$$0 = \frac{dP_{R_L}}{dR_L} = \frac{(R_L+R_T)^2 \cdot V_T^2 - V_T^2 R_L(2(R_L+R_T))}{(R_L+R_T)^4}$$

$$= \frac{V_T^2 (R_L+R_T)\left[R_L+R_T-2R_L\right]}{(R_L+R_T)^4}$$

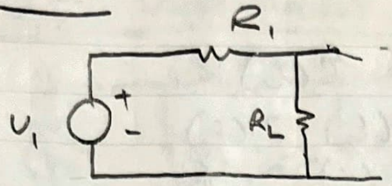
$$= \frac{V_T^2 (R_L+R_T)}{(R_L+R_T)^4} \cdot (R_T - R_L)$$

$\Rightarrow R_L = R_T$ is local max for P_{R_L}.

Want load Resistance = source Resistance for max power transfer.

$\Rightarrow$ "matched condition".

Caution



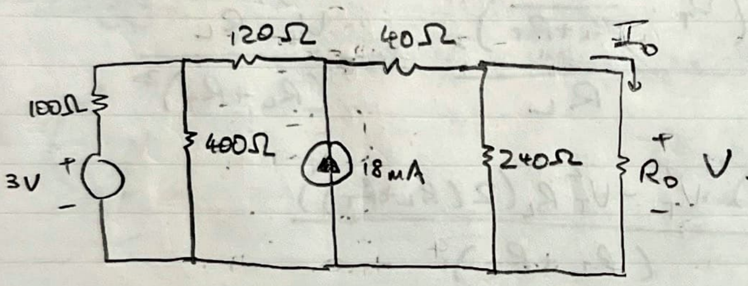
Under what condition is the power delivered to R_L maximum?

① $R_L \stackrel{?}{=} R_1$ (only if R_L is variable but R_1 is fixed

② if R_1 is not fixed, but R_L is.

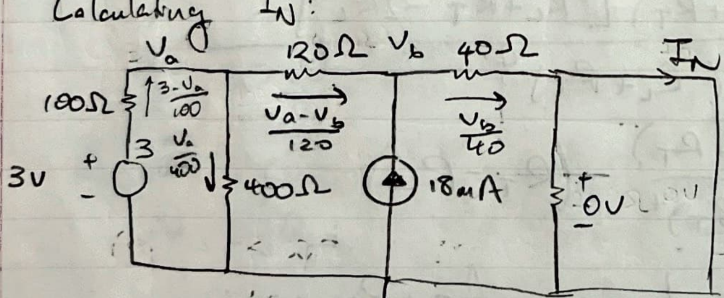
$$P_{R_L} = \frac{1}{R_L}\left(\frac{R_L}{R_1+R_L}\cdot V_1\right)^2 \Rightarrow R_1 = 0 \text{ maximizes}$$

Practice Problem



Solve for I_o, then find V in terms of Ro.

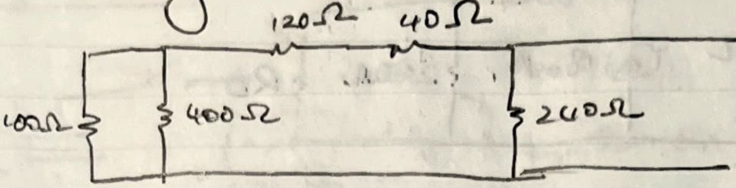
Calculating I_N:

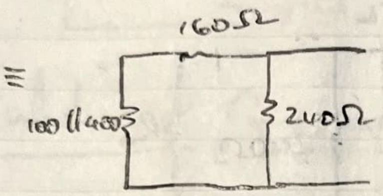


$$\begin{cases} \dfrac{3-V_a}{100} = \dfrac{V_a}{400} + \dfrac{V_a-V_b}{120} \\[2ex] \dfrac{V_a-V_b}{120} + \dfrac{18}{1000} = \dfrac{V_b}{40} \\[2ex] I_N = \dfrac{V_b}{40} \end{cases}$$

$$I_N = \frac{V_b}{40} = \frac{1}{40}\text{ A}$$

Calculating R_N:

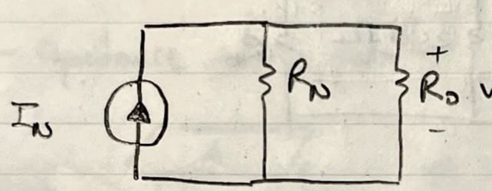




$$\Rightarrow R_N = (160 + 100 \| 400) \| 240$$

$$= \left(160 + \frac{100 \cdot 400}{100 + 400}\right) \| 240$$

$$= 240 \| 240 = \boxed{120\,\Omega}$$

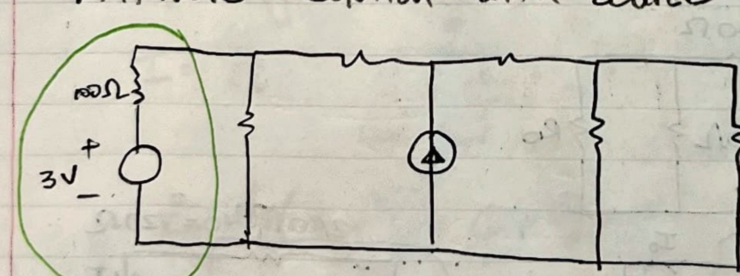


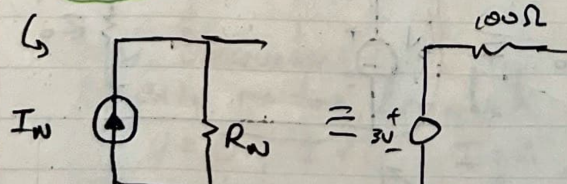
$$V = \left[I_N \left(\frac{R_N}{R_N + R_o} \right) \right] (R_o)$$

$$= 3 \left(\frac{R_o}{120 + R_o} \right)$$

$$I_o = 3 \left(\frac{1}{120 + R_o} \right)$$

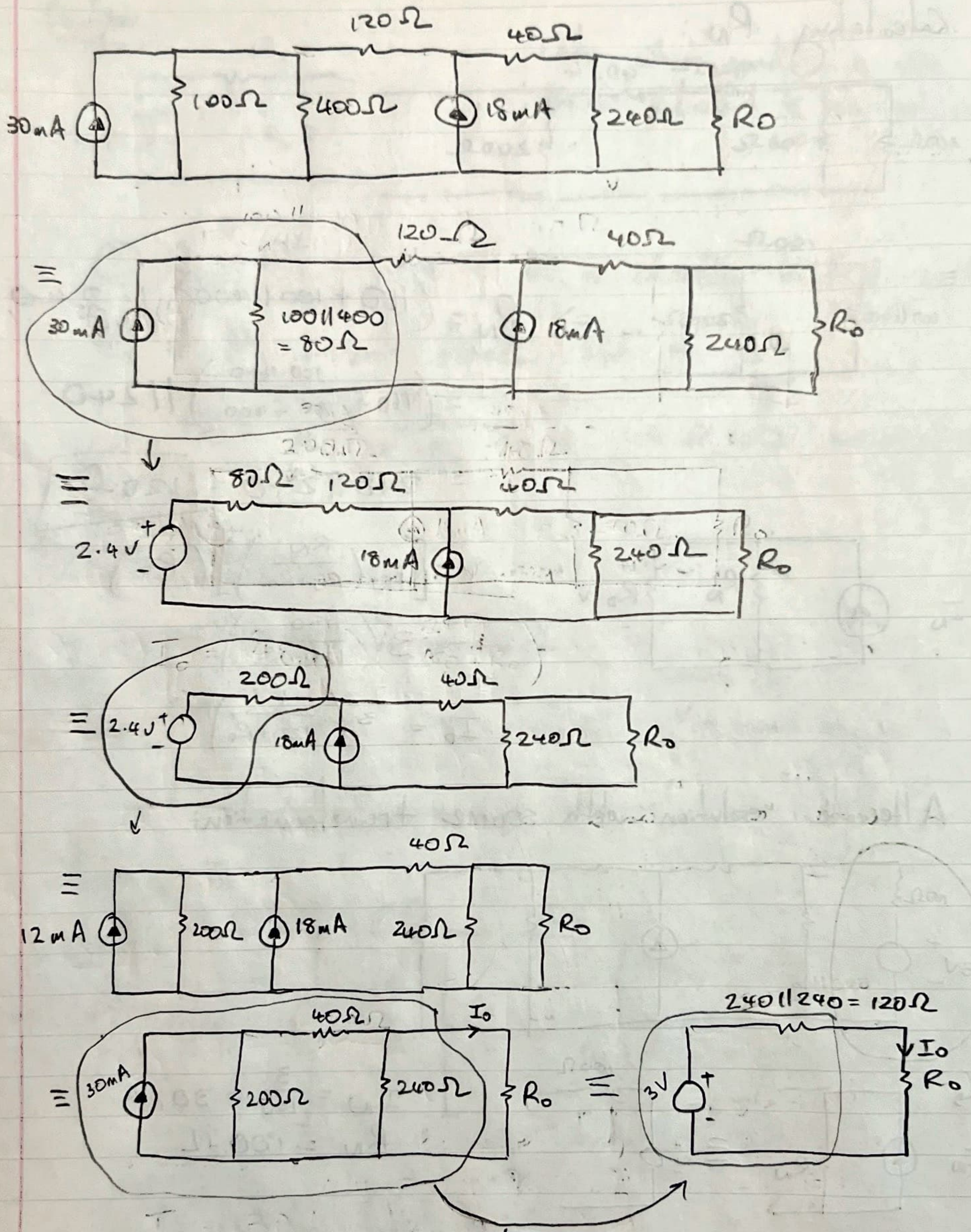
Alternate solution with source transformation:





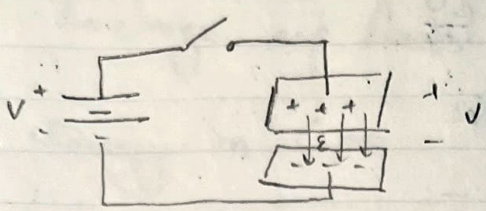
$$I_N = \frac{3}{100} = 30\,mA$$

$$R_N = 100\,\Omega.$$



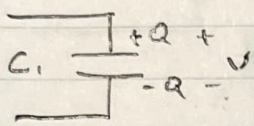
$$I_o = \frac{3}{120+R_o} \qquad V_o = \frac{3R_o}{120+R_o}$$

Capacitors



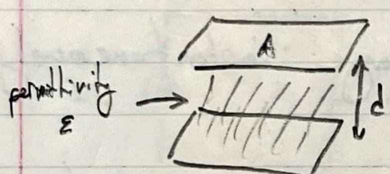
} capacitor: can charge / discharge

Definition of Capacitance



$$\Rightarrow C_1 = \frac{Q}{V}$$

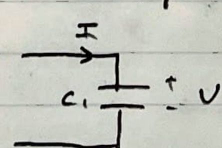
- Capacitance of a parallel-plate structure



$$C = \varepsilon \cdot \frac{A}{d}$$

- current-voltage relationships

$$I = \frac{dQ}{dt}$$



$$I = \frac{d}{dt}(C_1 v) = C_1 \frac{dV}{dt}$$

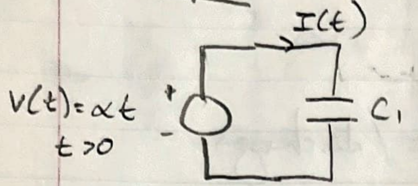
Some Observations:

① Unlike resistors for which $V=0 \Rightarrow I=0$, capacitors can have $V=0,\ I\neq 0;\ I=0,\ V\neq 0$

② Voltage can be found from the current

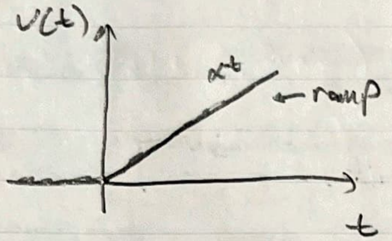
$$V = \frac{1}{C_1}\int_0^t I\,dt$$

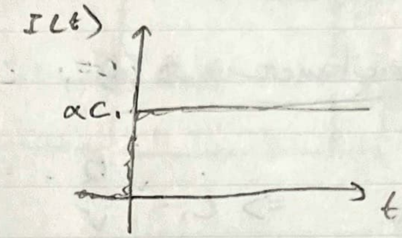
Example



$$I(t) = C_1 \frac{dV}{dt}$$

$$= C_1 \alpha$$





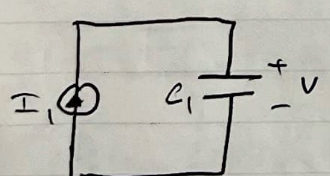
Important Points

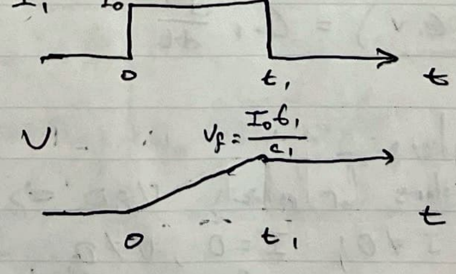
① If voltage on cap is constant, $I = C_1 \frac{dV}{dt} = 0$.

☆ ② For the voltage on the cap to change instantaneously, the current must be infinite.
↳ derives from $I(t) = C_1 \frac{dV}{dt}$.

⇒ If current available is $< \infty$, cap voltage cannot jump in zero time.

Example



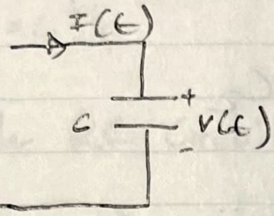


$$v(t) = \frac{1}{C_1} \int_0^t I \, dt = \frac{I_0 t}{C_1}$$

Voltage $\propto$ area under current vs. time waveform:

↳ charge cap faster with more current

• Energy in Caps

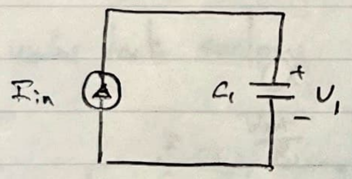


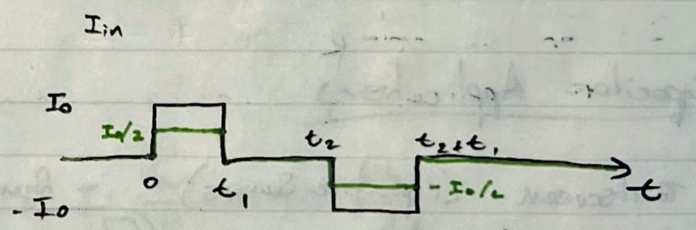
$$P = v(t)\,I(t)$$

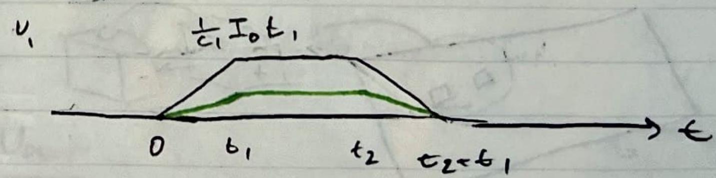
$$\mathcal{E} = \int P\,dt = \int v(t)\left[C\frac{dv}{dt}\right]dt$$

$$= \frac{1}{2}Cv^2$$

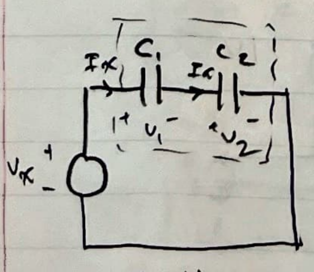
<u>Example</u>







• Series & Parallel Caps



$$v_x - v_1 - v_2 = 0$$

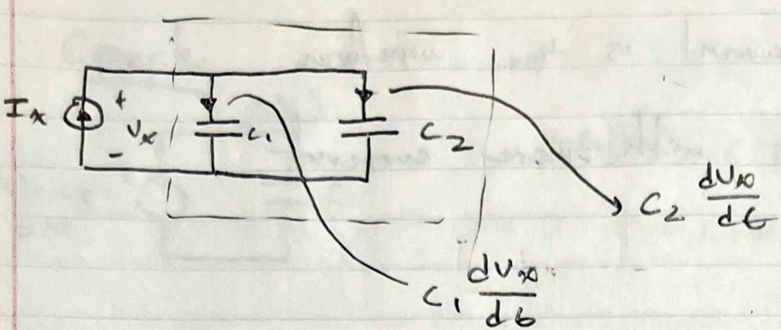
$$\Rightarrow v_x = v_1 + v_2$$

$$v_1 = \frac{1}{C_1}\int I_x\,dt$$

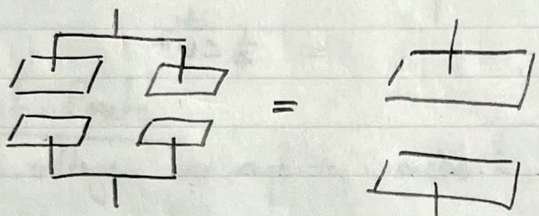
$$v_2 = \frac{1}{C_2}\int I_x\,dt$$

$$v_x = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int I_x\,dt$$

$$\Rightarrow \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}}$$



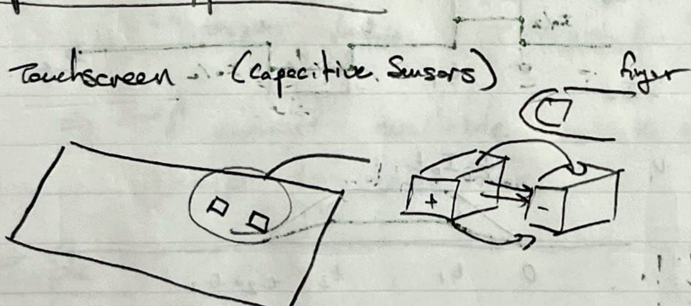
$$I_x = (C_1 + C_2)\frac{dv_x}{dt} \Rightarrow C_{eq} = C_1 + C_2$$



shorting the plates
they add

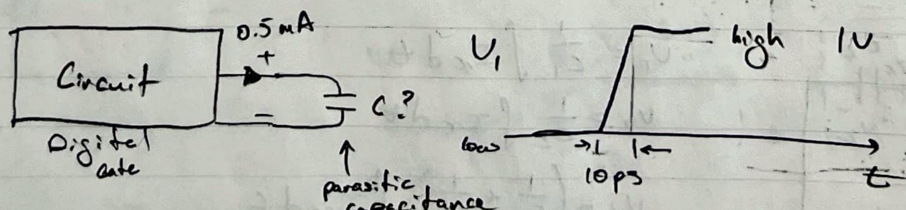
Capacitor Applications

- Touchscreen (capacitive Sensors)



change in capacitance

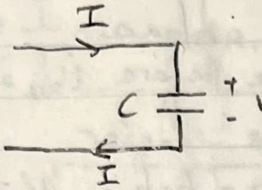
- Parasitic Caps



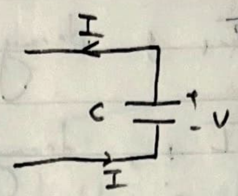
$$I = C\frac{dV}{dt}$$

$$0.5\,mA = C\frac{1V}{10ps} \Rightarrow C = 5\times10^{-15}\,F = \boxed{5\,fF}$$

· A Note on Caps

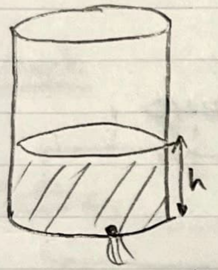


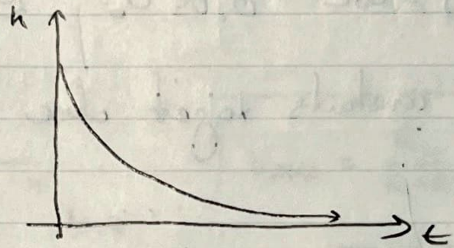
$I = C\frac{dv}{dt}$



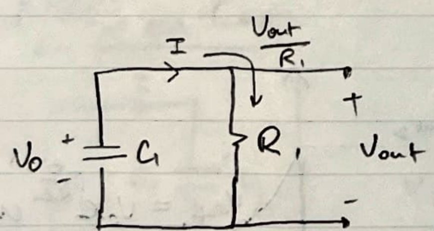
$I = -C\frac{dv}{dt}$

· Simple RC Circuit





water tank analogy



$$I = \frac{V_{out}}{R_1} = -C_1\frac{dV_{out}}{dt}$$

$$V_{out}(0) = V_0$$

$$\frac{V_{out}}{R_1} = -C_1\frac{dV_{out}}{dt}$$

$$\int_0^t \frac{dt}{-R_1C_1} = \int_{V_0}^{V_{out}} \frac{dV_{out}}{V_{out}}$$

$$\frac{-t}{R_1C_1} = \ln\left(\frac{V_{out}}{V_0}\right)$$

$$\boxed{V_{out} = V_0 e^{-\frac{t}{R_1C_1}}}$$

① The decay rate depends on $R_1 C_1 = \tau$ ("time constant")

For $\tau_2 > \tau_1$, decay becomes slower
(more time before $V_{out} = V_0 e^{-1}$)

$\tau_3 < \tau_1$, decay becomes faster
(less time before $V_{out} = V_0 e^{-1}$)

② $V_{out}(t=\tau) = V_0 e^{-1} = 0.37 V_0$

$V_{out}(t=2\tau) = V_0 e^{-2} = 0.14 V_0$

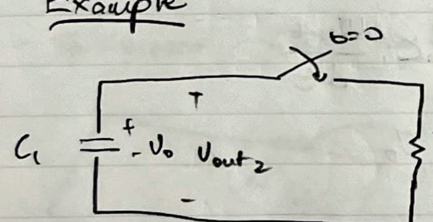
$V_{out}(t=3\tau) = V_0 e^{-3} = 0.05 V_0 \approx 0$

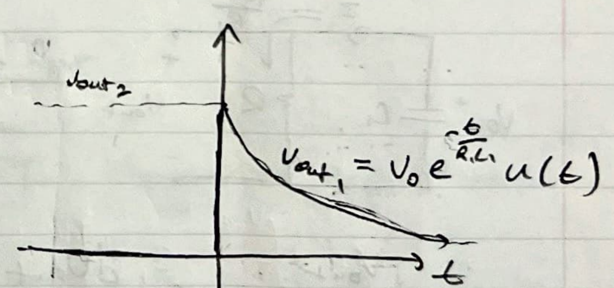
Three time constants to get close to zero.

Current:

$$I = \frac{V_{out}}{R_1} = \frac{V_0}{R_1} e^{-\frac{t}{R_1 C_1}}$$

Example

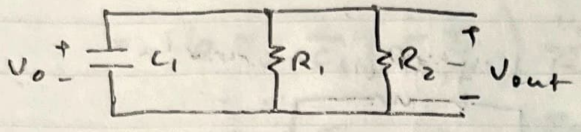


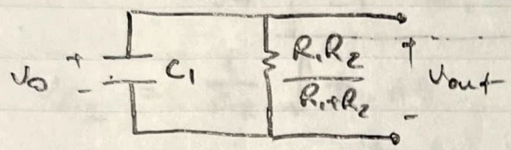


$V_{out\,1}$ jumps; $V_{out\,2}$ does not.

↳ Need to multiply by $u(t)$ (unit step function)

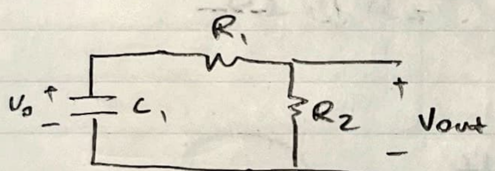
Example





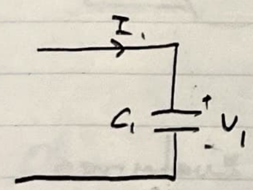
$$\boxed{V_{out} = V_o e^{-\frac{t}{\tau}} \qquad \tau = \frac{R_1R_2}{R_1+R_2} C_1}$$

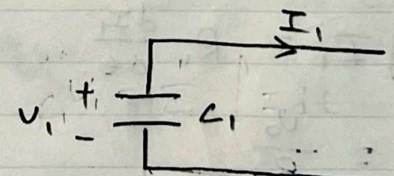
Example



$$\boxed{V_{out} = \frac{R_2}{R_1+R_2}\left(V_o e^{-\frac{t}{\tau}}\right) \qquad \tau = (R_1+R_2)C_1}$$

Notes on Caps



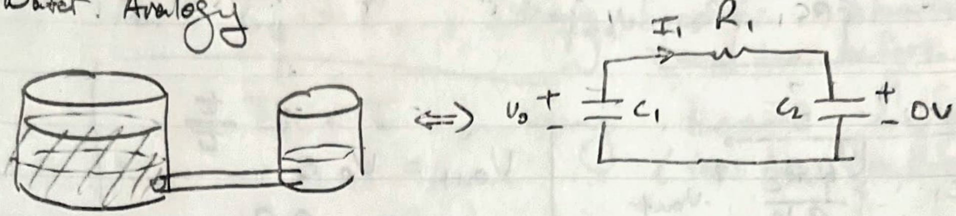


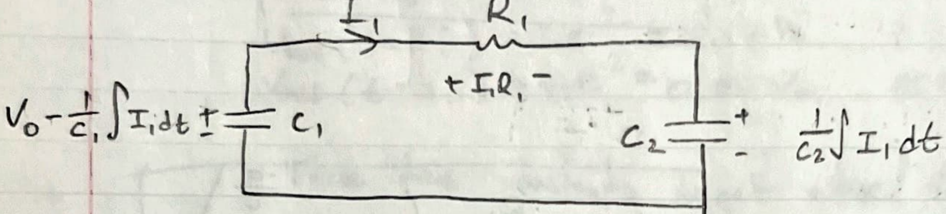
$$\Delta V_1 = \frac{1}{C_1}\int I_1 dt \qquad \Delta V_1 = -\frac{1}{C_1}\int I_1 dt$$

After integration we must include a constant and, based on the initial condition(s), find its value.

Two-Capacitor Problem

Water Analogy





$$V_0 - \frac{1}{C_1}\int I_1 dt = I_1 R_1 + \frac{1}{C_2}\int I_1 dt$$

$$V_0 - \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int I_1 dt = I_1 R_1$$

$$-\left(\frac{1}{C_1} + \frac{1}{C_2}\right) I_1 = R_1 \frac{dI_1}{dt}$$

$$I_1(t=0) = \frac{V_0}{R_1}$$

$$\Rightarrow \int_0^t -\frac{1}{R_1}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) dt = \int_{\frac{V_0}{R_1}}^{I(t)} \frac{dI_1}{I_1}$$

$$-\frac{1}{R_1}\left(\frac{1}{C_1} + \frac{1}{C_2}\right) t = \ln\left(\frac{I(t)}{\frac{V_0}{R_1}}\right)$$

$$I(t) = \frac{V_0}{R_1} e^{-\frac{t}{\tau}}$$

$$\tau = R_1\left(\frac{C_1 C_2}{C_1 + C_2}\right)$$

$\Rightarrow C_1, C_2$ in series

$$V_{C_1} = V_0 - \frac{1}{C_1}\int_0^t \frac{V_0}{R_1} e^{-\frac{t}{\tau}}\, dt$$

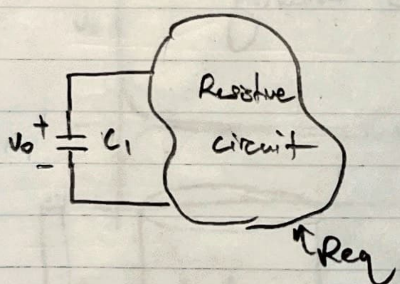
$$= V_0 - \frac{1}{C_1}(-\tau)\frac{V_0}{R_1} e^{-\frac{t}{\tau}} + \frac{1}{C_1}(-\tau)\frac{V_0}{R_1}$$

$$= V_0 + \frac{\tau}{C_1}\cdot\frac{V_0}{R_1} e^{-\frac{t}{\tau}} - \frac{\tau}{C_1}\cdot\frac{V_0}{R_1}$$

$$V_{C_2} = \frac{1}{C_2}\int I_1\, dt$$

$$= \frac{1}{C_2}\cdot\int_0^t \frac{V_0}{R_1} e^{-\frac{t}{\tau}}\, dt$$

$$= \frac{\tau}{C_2}\left(\frac{V_0}{R_1}\right)\left(1 - e^{-\frac{t}{\tau}}\right)$$



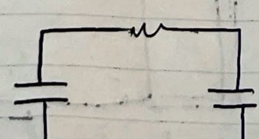
$\tau = R_{eq} C_1$

Observations

① Is charge conserved? Yes.

② Is energy conserved? No.

e.g.

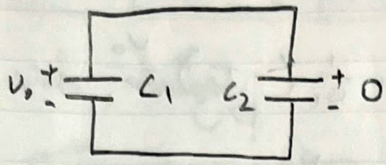


At $t=0$, $\mathcal{E} = \frac{1}{2} C_1 V_0^2$

$t=\infty$, $\mathcal{E} = \frac{1}{2} C_1 \left(\frac{V_0 C_1}{C_1+C_2}\right)^2 + \frac{1}{2} C_2 \left(\frac{V_0 C_1}{C_1+C_2}\right)^2$

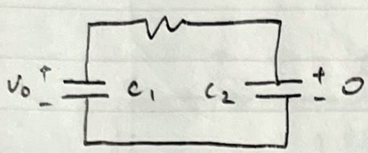
$= \frac{1}{2}\left(\frac{V_0^2 C_1^2}{C_1+C_2}\right)$

Energy lost to resistor.



Two capacitor paradox.

Due to conservation of Q:



At $t=0$, $Q = C_1 V_0$

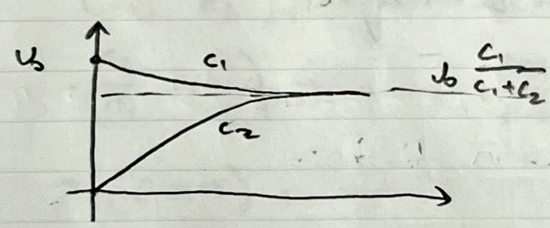
$t=\infty$, $Q = C_1 V_\infty + C_2 V_\infty$.

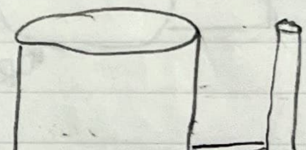
$$V_\infty (C_1 + C_2) = C_1 V_0$$

$$V_\infty = V_0 \left(\frac{C_1}{C_1+C_2}\right).$$

③ Effect of Cap values. "Charge Sharing"

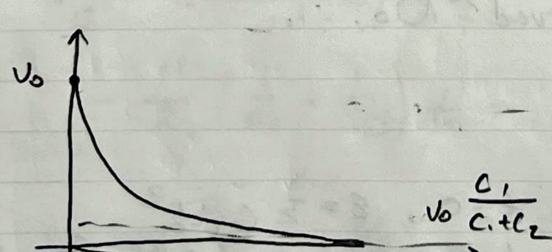
C_1 : large C_2 : small

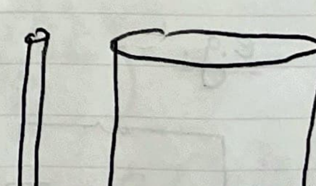




water analogy
(don't need much water to fill C_2)

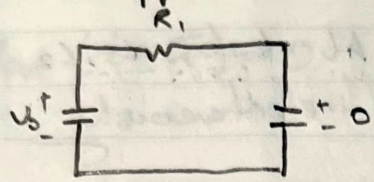
C_1 : small C_2 : large



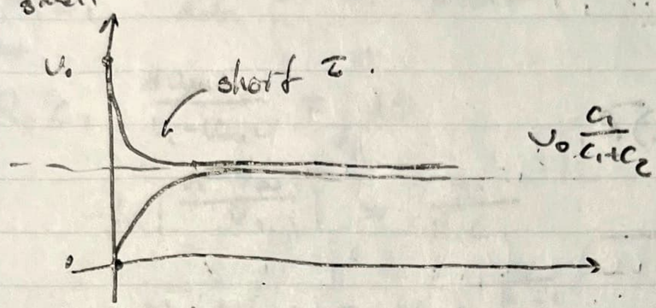


(Need lots of water to fill C_2)

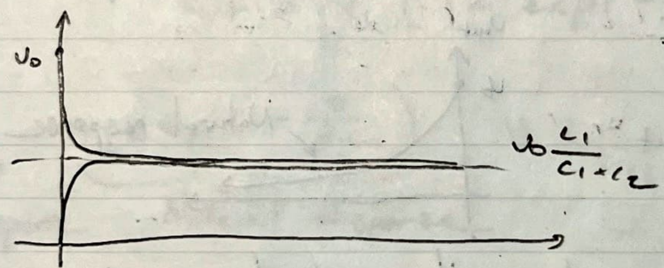
④ what happens as $R_1 \downarrow$?



R_1 small

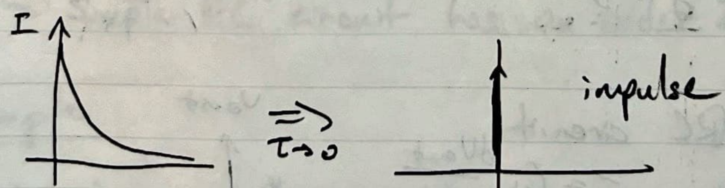


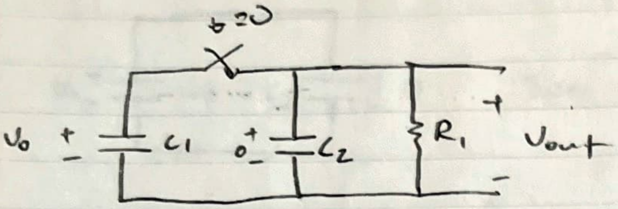
$R_1 \to 0, \ \tau \to 0, \ \frac{v_0}{R_1} = I \to \infty$



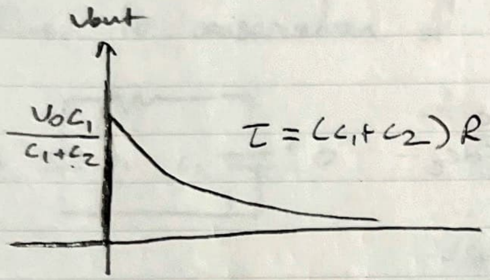
$\Rightarrow$ Instantaneous change in V_{c_1} & V_{c_2}

$I = \frac{v_0}{R_1} e^{-\frac{t}{\tau}}$

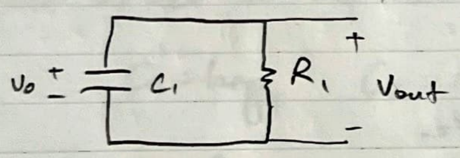


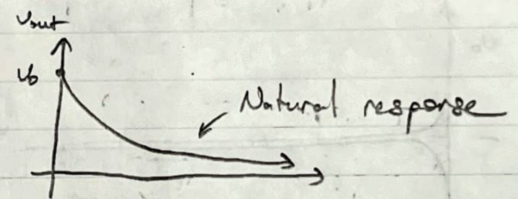


At $t=0$; C_1, C_2 share charge instantaneously

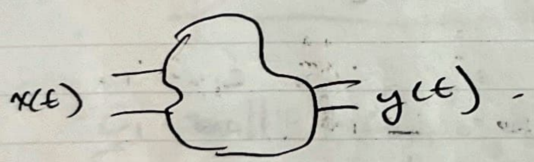


- Source-Free Circuits (Circuits with no explicit inputs)

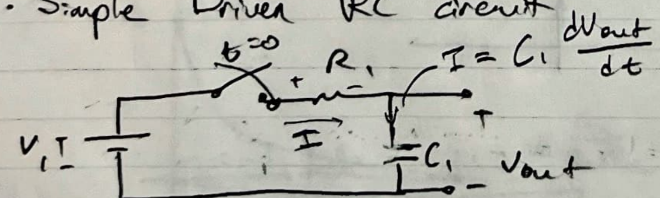


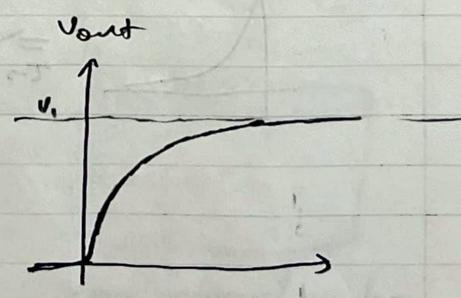


- Driven Circuits: (Circuits with explicit inputs)



- Simple Driven RC circuit





KVL: $V_1 - IR_1 - V_{out} = 0$

$$\begin{cases} V_1 = R_1 C_1 \frac{dV_{out}}{dt} + V_{out} \\ V_{out}(t=0^+) = 0 \end{cases}$$ (Cap voltage cannot jump because $I < \infty$)

$$R_1 C_1 \frac{dV_{out}}{dt} = V_1 - V_{out}$$

$$\int_0^{V_{out}} R_1 C_1 \frac{dV_{out}}{V_1 - V_{out}} = \int_0^t dt$$

$$-\ln\left|\frac{V_1 - V_{out}}{V_1}\right| = \frac{t}{R_1 C_1}$$

$$V_1 - V_{out} = V_1 e^{-\frac{t}{R_1 C_1}}$$

$$V_{out} = \left[V_1\left(1 - \exp\left(-\frac{t}{R_1 C_1}\right)\right)\right] u(t)$$

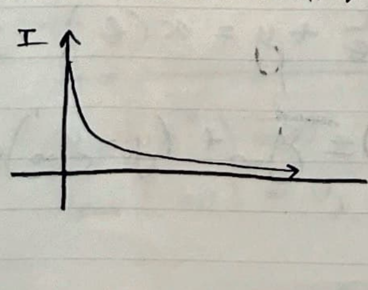
$$\underbrace{V_1}_{\text{Forced Response}} - \underbrace{V_1 \exp\left(-\frac{t}{R_1 C_1}\right)}_{\text{Natural Response}}$$

Forced Response: Response that is left at $t = \infty$.

- "Order" of a Circuit = Order of a diff. equation
  - → Simple RC circuit has an order of 1.

Example

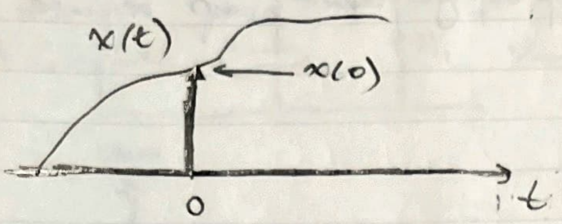
Plot the current.



$$I(t) = \frac{1}{R_1 C_1} V_1 \exp\left(-\frac{t}{R_1 C_1}\right) \cdot C_1 \cdot u(t)$$

$$= \frac{V_1}{R_1} \exp\left(-\frac{t}{R_1 C_1}\right) u(t)$$

Property of Impulse Function:



$$x(t)\,\delta(t) = x(0)\,\delta(t)$$

(Remember $\delta(t) = 0 \;\forall t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$)

$$V_{out} = V_1\left(1 - \exp\frac{-t}{\tau}\right)u(t)$$

$$\frac{d}{dt}V_{out} = \frac{V_1}{\tau}\exp\frac{-t}{\tau}\,u(t) + V_1\left(1 - \exp\frac{-t}{\tau}\right)\delta(t)$$

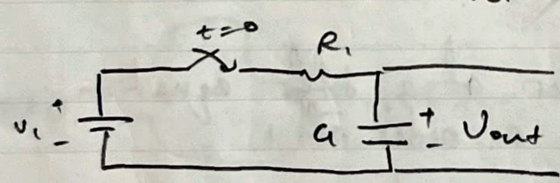
because $\frac{d}{dt}u(t) = \delta(t)$

- derivative of step = impulse.

$$\frac{d}{dt}V_{out} = \frac{V_1}{\tau}\exp\frac{-t}{\tau}\,u(t) + V_1(1 - \exp 0)\,\delta(t)$$

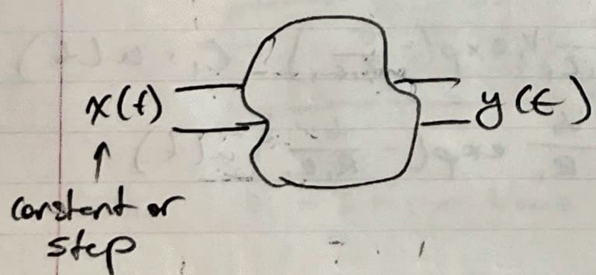
$$\frac{dV_{out}}{dt} = \frac{V_1}{\tau}\left(\exp\frac{-t}{\tau}\right)u(t) \quad \text{as expected.}$$

- Shortcut Method for First-Order Circuits



$$\begin{cases} R_1 C_1 \dfrac{dV_{out}}{dt} + V_{out} = V_1 \\ V_{out}(t=0^+) = 0 \end{cases}$$

General First-Order System:



$$\alpha\frac{dy}{dt} + y = x(t)$$

$$y(t) = y_\infty + (y_0 - y_\infty)\exp\frac{-t}{\tau}$$

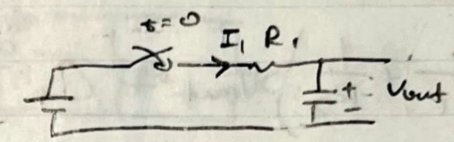
$y_\infty = y(t=\infty)$

$y_0 = y(t=0)$

$\tau = \alpha$

Necessary that coefficient of y is 1.

<u>Example</u>



$$V_{out} = V_{out,\infty} + (V_{out,0} - V_{out,\infty}) \exp^{-\frac{t}{\tau}}$$

$$= V_1 + (0 - V_1) \exp^{-\frac{t}{\tau}}$$

$$= \left(V_1 - V_1 \exp^{-\frac{t}{\tau}}\right) u(t)$$

<u>Example</u>

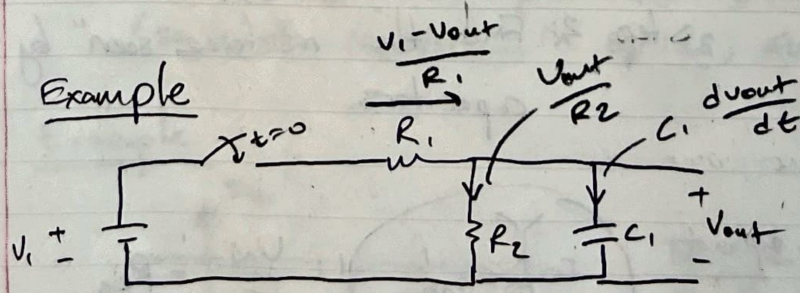
Finding $I_1(t) = ?$

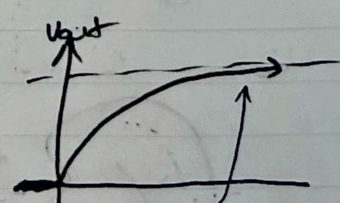
$$I_1(t) = I_{1,\infty} + (I_{1,0} - I_{1,\infty}) \exp^{-\frac{t}{\tau}}$$

$$= 0 + \left(\frac{V_1}{R_1} - 0\right) \exp^{-\frac{t}{\tau}}$$

$$= \frac{V_1}{R_1} \exp^{-\frac{t}{\tau}} u(t)$$

<u>Example</u>



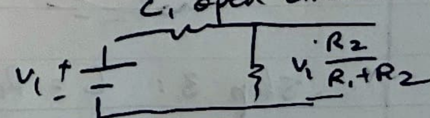


$V_{out}(0^+) = 0$

$V_{out}(\infty) = V_1 \dfrac{R_2}{R_1 + R_2}$

V_{out} = constant

C_1 open circuit.



To find τ: write diff. eq.

KCL: $\dfrac{V_1 - V_{out}}{R_1} = \dfrac{V_{out}}{R_2} + C_1 \dfrac{dV_{out}}{dt}$

$$\frac{V_1}{R_1} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_{out} + C_1 \frac{dV_{out}}{dt}$$

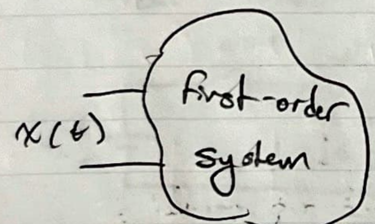
$$\frac{V_1}{R_1} \cdot \frac{R_1 R_2}{R_1 + R_2} = V_{out} + C_1 \left(\frac{R_1 R_2}{R_1 + R_2}\right) \frac{dV_{out}}{dt}$$

$\Rightarrow \tau = \dfrac{R_1 R_2}{R_1 + R_2} C_1$

Using $y(t) = y_\infty + (y_0 - y_\infty) \exp^{-\frac{t}{\tau}}$ as the solution:

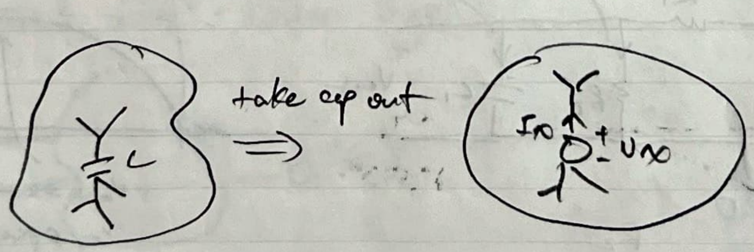
$$\boxed{V_{out} = \frac{R_2}{R_1 + R_2} V_1 \left(1 - \exp^{-\frac{t}{\tau}}\right) u(t)}$$

Finding τ by inspection:



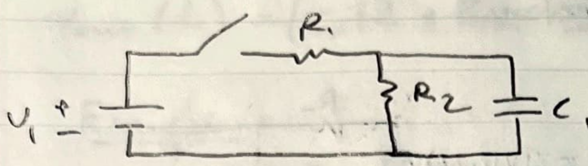
Step 1: Set all independent sources and initial conditions to 0.

Step 2: Find the resistance "seen" by the capacitors.

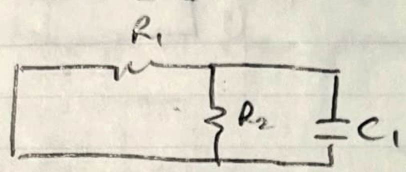


$\dfrac{V_{x}}{I_{x}} = R_{eq}$.

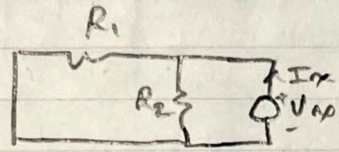
Step 3: $\tau = R_{eq} C$.



step 1 ⇒



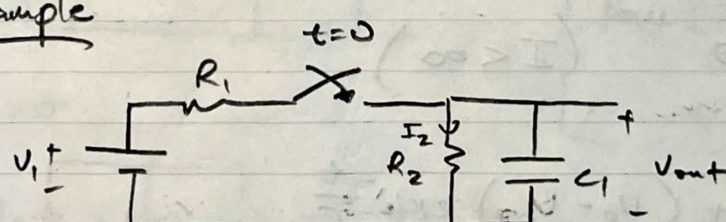
step 2 ⇒



$$R_{eq} = R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

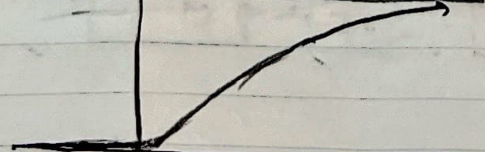
$$\boxed{\tau = C_1 \frac{R_1 R_2}{R_1 + R_2}}$$

Example



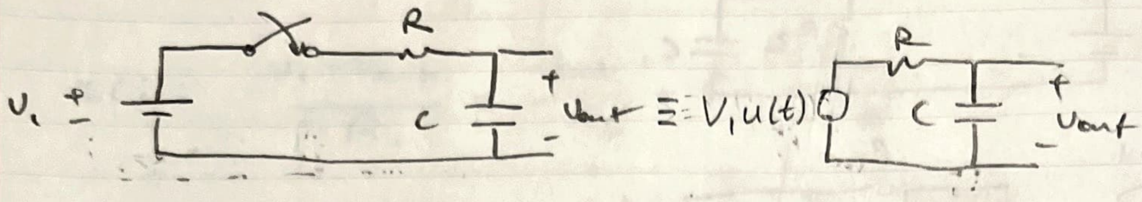
Vout same as in previous example
(switch & resistor in series, swap did nothing)

Example

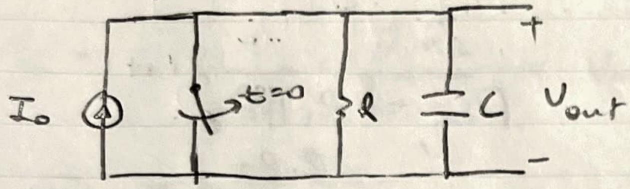


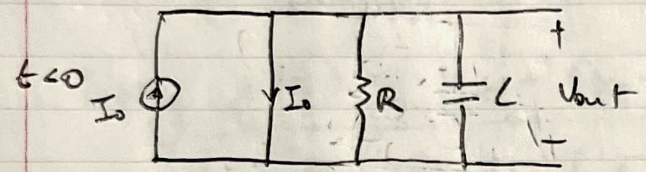
$$\frac{V_1 R_2}{R_1 + R_2}$$

- Note on Switches & Step Functions



- Example

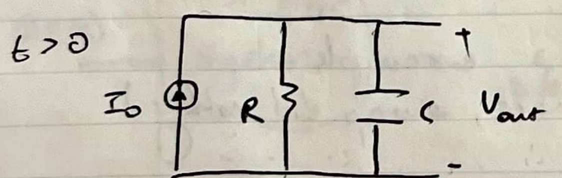




I_o flows entirely through the switch ($V_{out} = 0$).

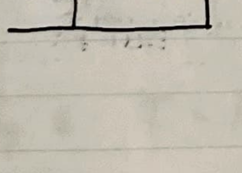
$V_{out}(t = 0^+) = 0 \qquad (I < \infty)$

$$V_{out}(t) = V_\infty + (V_0 - V_\infty)\exp\frac{-t}{\tau}$$



$V_{out}(\infty) = I_0 R$.
↑
constant

τ calculation:

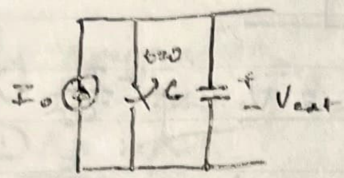


$\Rightarrow \tau = RC$.

$$V_{out}(t) = \left(I_o R + (0 - I_o R)\exp\frac{-t}{RC}\right) u(t)$$

<u>Example</u>

what happens in the previous example if $R \to \infty$?

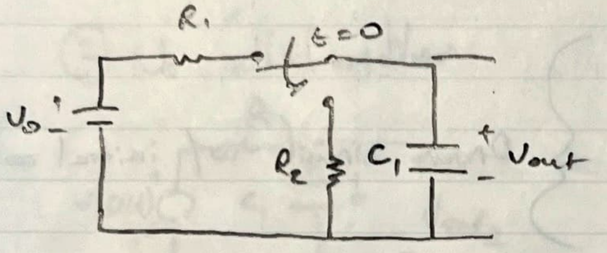


$V_{out}(0^-) = 0$

$V_{out}(0^+) = 0$

$V_{out}(t) = \frac{1}{C}\int_0^t I_o\,dt = \frac{I_o t}{C} u(t)$

<u>Example</u>

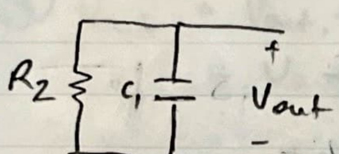


$$\boxed{V_{out}(t<0) = V_o}$$

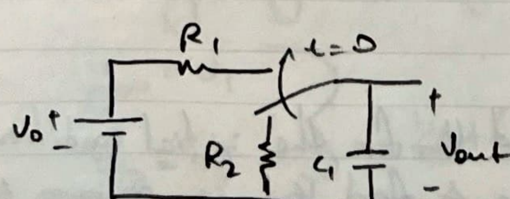
$V_{out}(t=0^+) = V_o$

$$\boxed{\begin{aligned} V_{out}(t>0) &= V_o \exp\left(-\frac{t}{\tau}\right) \\ &= V_o \exp\left(-\frac{t}{R_2 C_1}\right) \end{aligned}}$$

$t > 0$



<u>Example</u>



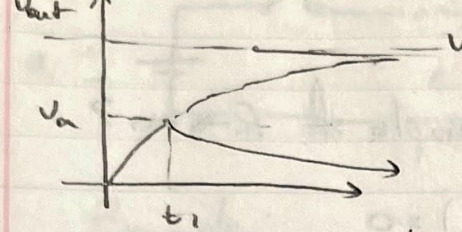
$V_{out}(t<0) = 0$

$V_{out}(t=0^+) = 0$

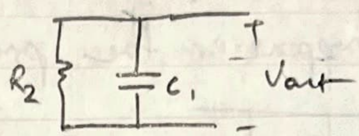
$V_{out}(t>0) = V_o\left(1 - \exp\left(\frac{-t}{R_1 C_1}\right)\right)$

$$\boxed{V_{out}(t) = V_o\left(1 - \exp\left(\frac{-t}{R_1 C_1}\right)\right) u(t)}$$

At $t = t_1$, switch goes back to position 1.



$t > t_1$:

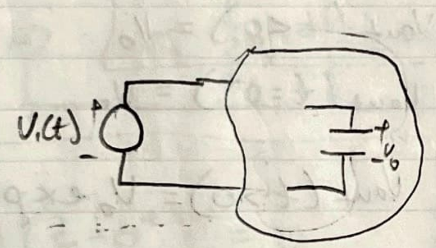


$$V_a = V_0\left(1 - \exp^{-\frac{t_1}{R_1 C_1}}\right)$$

$$V_{out} = V_a \exp^{-\frac{t}{\tau_2}}$$

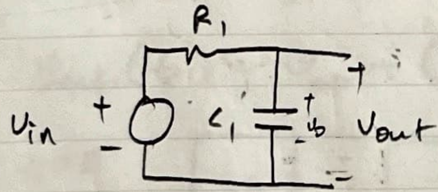
$$= V_a \exp^{-\frac{t}{R_2 C_1}}$$

How do we analyze a driven circuit w/ an initial condition?



Driven circuit w/ initial condition

Method 1: Solve the diff. eq. w/ the initial condition(s)



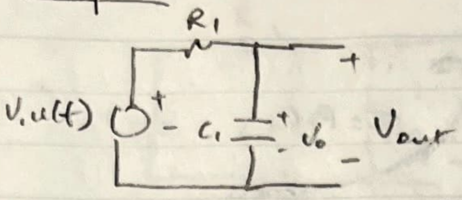
$$I = C_1 \frac{dV_{out}}{dt}$$

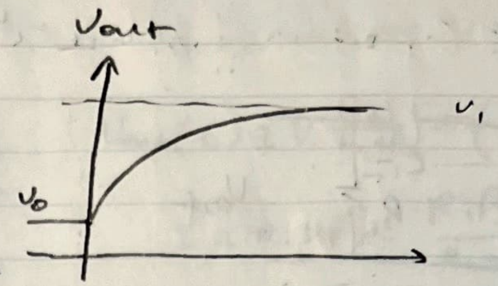
$$\begin{cases} V_{in} = R_1 C_1 \dfrac{dV_{out}}{dt} + V_{out} \\ V_{out}(0^+) = V_0 \end{cases}$$

Method 2: Use superposition

1. Assume no input & find V_{out} for the initial condition
2. Assume no initial condition & find V_{out} in response to input
3. Add the two responses.

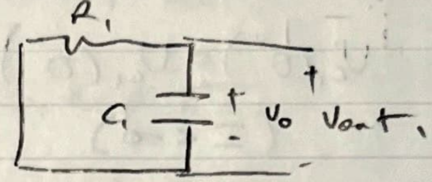
Example





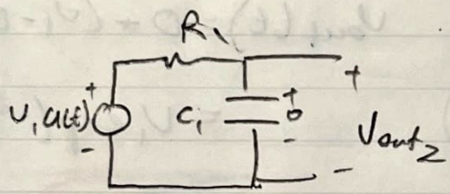
Apply superposition:

① No input:



$$V_{out_1} = V_0 \exp\left(-\frac{t}{R_1 C_1}\right)$$

② No initial condition



$$V_{out_2} = V_1\left(1 - \exp\frac{-t}{R_1 C_1}\right)$$

③ $V_{out} = V_{out_1} + V_{out_2}$

$$= V_0 \exp\left(-\frac{t}{\tau}\right) + V_1\left(1 - \exp\frac{-t}{R_1 C_1}\right)$$

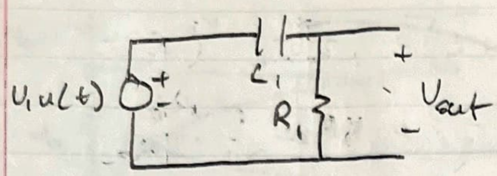
$$\tau = R_1 C_1, \quad t > 0.$$

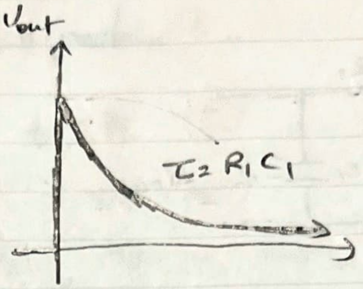
Alternatively: $y(t) = y_\infty + (y_0 - y_\infty) \exp -\frac{t}{\tau}$

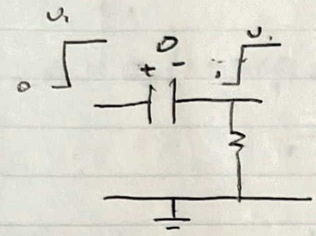
$$\left.\begin{aligned} y_\infty &= V_1 \\ y_0 &= V_0 \end{aligned}\right\} \quad V_{out}(t) = V_1 + (V_0 - V_1) \exp -\frac{t}{\tau}$$

$$\tau = R_1 C_1$$

Example



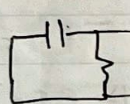




$$V_{C_1}(0^+) = V_{C_1}(0^-) = 0$$
$$(I < \infty)$$

$y_\infty = 0$, $y_0 = V_1$

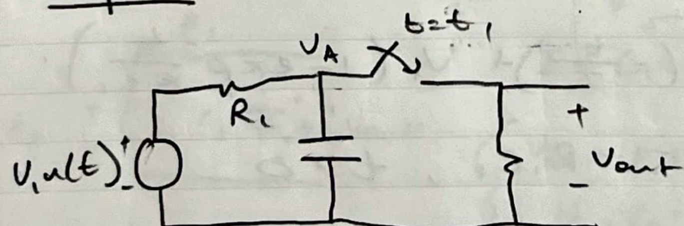
τ:



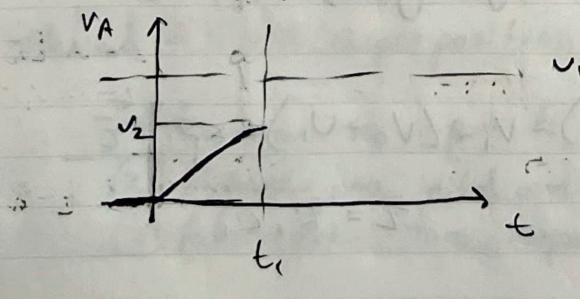
$\tau = R_1 C_1$

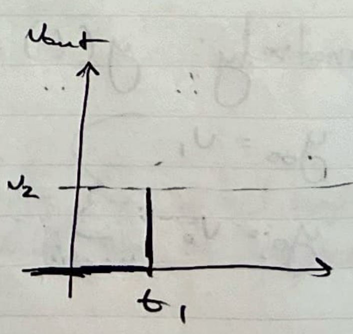
$$V_{out}(t) = 0 + (V_1 - 0)\exp\left(-\frac{t}{\tau}\right)$$
$$= V_1 \exp\left(-\frac{t}{\tau}\right) u(t).$$

Example

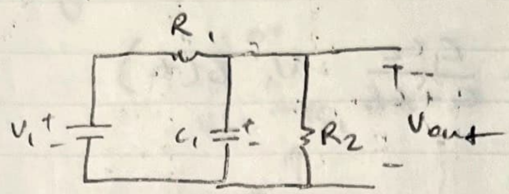


$V_{out}(t<0) = 0$
$V_{out}(0^+) = 0$





$t > t_1$



$$V_{out}(t) = V_1 \frac{R_2}{R_1+R_2} + \left(V_2 - V_1 \frac{R_2}{R_1+R_2}\right) \exp \frac{-t}{\tau}$$

$$\tau = R_1 \| R_2 = \frac{R_1 R_2}{R_1 + R_2}$$

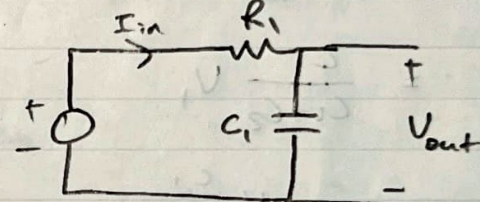
$$t < t_1 : \; V_A = \left(V_1 - V_1 \exp \frac{-t}{R_1 C_1}\right) u(t)$$

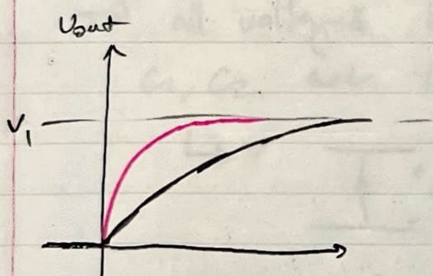
$$\Rightarrow V_2 = V_A(t_1) = V_1 - V_1 \exp \frac{-t_1}{R_1 C_1}$$

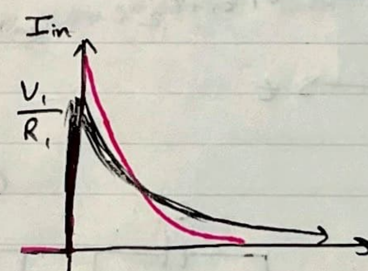
Example

Part a)

$V_{in} = V_1 u(t)$

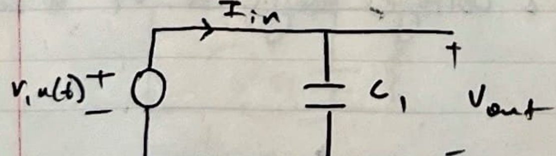


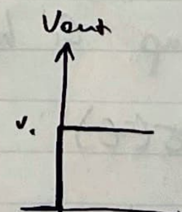


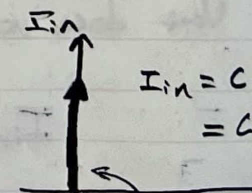


$R_1 \to 0$

Part b)



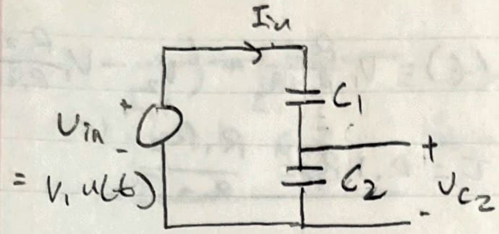




$$I_{in} = C_1 \frac{dV_{out}}{dt} = C_1 V_1 \delta(t)$$

$C_1 V_1 \delta(t)$

Part c) (Capacitive Voltage divider)

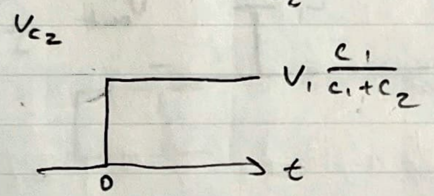


$$I_{in} = \frac{C_1 C_2}{C_1 + C_2} \cdot V_1 \, \delta(t)$$

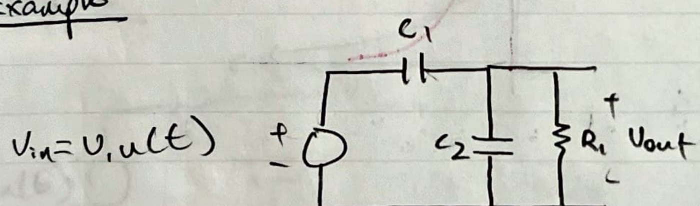
$$V_{C_2} = \frac{1}{C_2} \int_0^t I \, dt$$

$$= \frac{1}{C_2} \int_{0^-}^{0^+} \frac{C_1 C_2}{C_1 + C_2} V_1 \, \delta(t)$$

$$= \frac{C_1}{C_1 + C_2} V_1$$



Example



If V_{out} does not jump: C_1 has a voltage change of V_1 in zero time.

$$I_{in} \propto \delta(t)$$

* I_{in} cannot flow through R_1, otherwise it would create infinite voltage

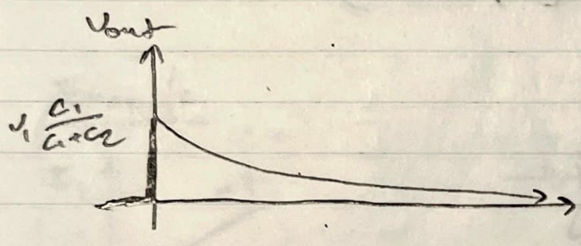
$\Rightarrow$ impulse must flow through C_2

Charge delivered to R_1 from 0^- to 0^+

$$\int_{0^-}^{0^+} \underbrace{I_{R_1}}_{\text{finite}} \, dt = 0$$

$\Rightarrow$ Only delivered to C_1, C_2

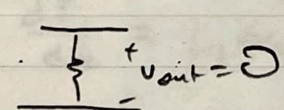
$$V_{C_2}(0^+) = V_1 \frac{C_1}{C_1+C_2}$$



If all voltages become constant at $t=\infty$,

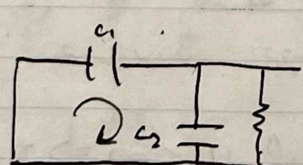
C_1, C_2 are open

$\hookrightarrow$



Why is system of first order?

$t<0 \Rightarrow v_{in} = 0$.

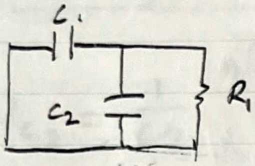


$\hookrightarrow$ Only one initial condition can be applied independently.

$\Rightarrow$ first order system.

$y_{\infty} = 0, \quad y_{0^+} = V_1 \frac{C_1}{C_1 + C_2}$

τ : set indep. sources to 0.

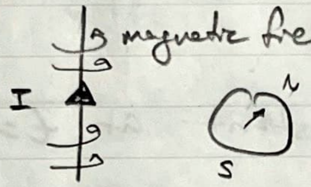


$\tau = R_1 (C_1 + C_2)$

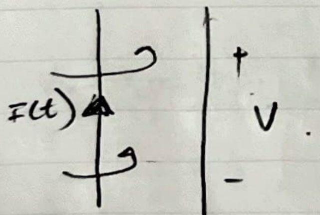
$V_{out}(t) = V_1 \frac{C_1}{C_1 + C_2} \exp\left(-\frac{t}{\tau}\right) u(t)$

Inductors

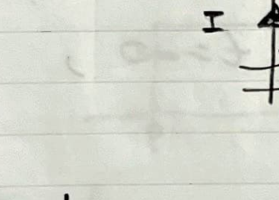
Oersted:



Faraday:



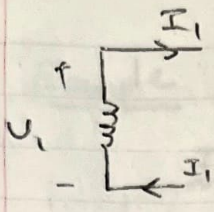
Inductor Basics:



$V_1 = L_1 \frac{dI_1}{dt}$

if $V_1 < \infty \Rightarrow I_1$ cannot jump in zero time

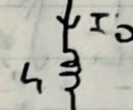
If I_1 is constant $\Rightarrow V_1 = 0$. $\Rightarrow L_1 \equiv$ short



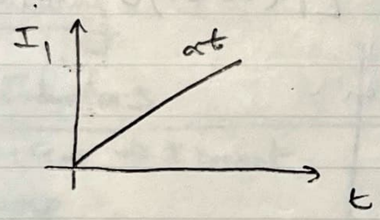
$$V_1 = -L\frac{dI_1}{dt}$$

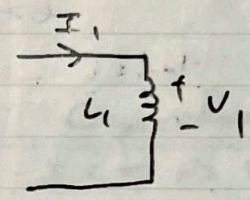
To find the current of an inductor:

$$I_1 = \frac{1}{L_1}\int V_1\,dt$$

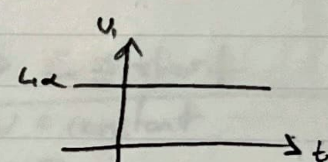
One more point: Initial condition: 

Example

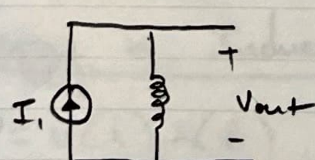


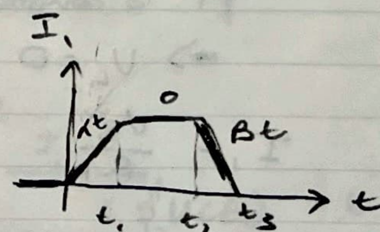


$$V_1 = L_1\frac{dI_1}{dt} = L_1\alpha$$

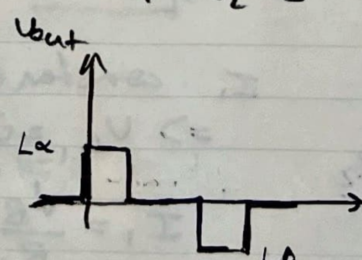


Example

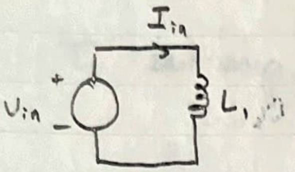


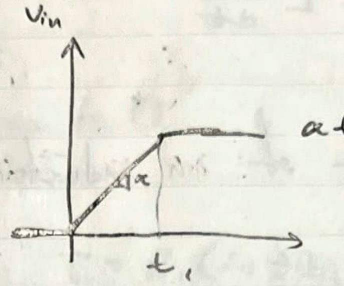


$$V_{out} = L\frac{dI}{dt}$$



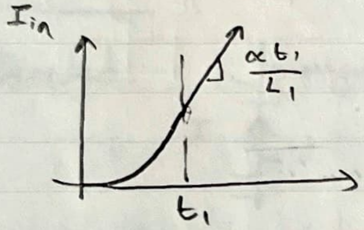
Example





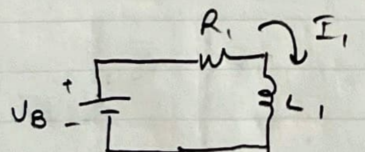
$$I_{in} = \frac{1}{L_1}\int_0^t \alpha t \, dt \qquad t \in (0, t_1)$$

$$= \frac{\alpha t^2}{2L_1}$$



$t > t_1$: $\frac{1}{L_1}(\alpha t_1)t$

Example

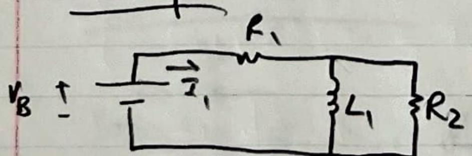


I_1 is constant

$\Rightarrow V_{L_1} = 0$

$$I_1 = \frac{V_B}{R_1}$$

Example

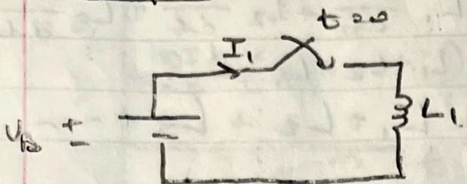


I_1 constant

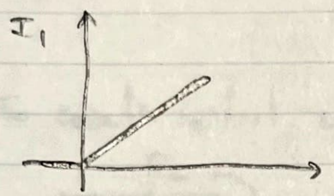
$\Rightarrow V_{L_1} = 0 \Rightarrow V_{R_2} = 0$

$I_1 = \frac{V_B}{R_1}$ (all of I_1 flows through L_1)

Example



$$t<0, \quad I_1 = 0$$
$$t=0^+, \quad I_1 = 0$$
$$t>0, \quad I_1 = \frac{1}{L_1}\int V_B\, dt = \frac{1}{L_1} V_B t$$

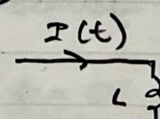


Summary

Inductors	Capacitors
If $V<\infty \Rightarrow I$ cannot jump $I=0 \Rightarrow L \equiv$ open	If $I<\infty \Rightarrow V$ can't jump $V=0 \Rightarrow C \equiv$ short
If $I =$ constant $\Rightarrow V_L = 0$ $\Rightarrow L \equiv$ short	If $V =$ constant $\Rightarrow I_L = 0$ $\Rightarrow C \equiv$ open

Energy in Inductors

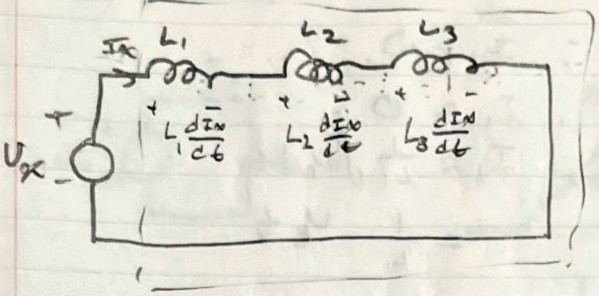
$$P = V(t)I(t)$$



$$V(t) = L\frac{dI}{dt}$$

$$\mathcal{E} = \int_0^t P(t)\,dt$$
$$= \int_0^t L\frac{dI}{dt}\cdot I\,dt$$
$$= \boxed{\frac{1}{2}LI^2(t)}$$

Inductors in series

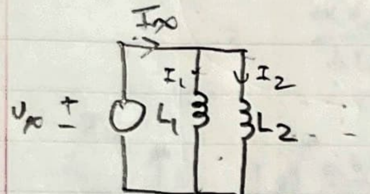


$$V_x = L_1 \frac{dI_x}{dt} + L_2 \frac{dI_x}{dt} + L_3 \frac{dI_x}{dt}$$

$$V_x = (L_1 + L_2 + L_3) \frac{dI_x}{dt}$$

$$L_{eq} = L_1 + L_2 + L_3 + \dots$$

Inductors in parallel



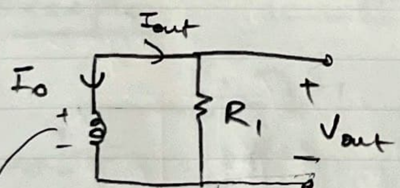
$$I_1 = \frac{1}{L_1} \int V_x \, dt$$

$$I_2 = \frac{1}{L_2} \int V_x \, dt$$

$$I_x = \frac{1}{L_{eq}} \int V_x \, dt$$

$$I_x = I_1 + I_2 \Rightarrow \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \dots$$

Simple RL circuits (Source-free)



$$I_{out}(\infty) = 0$$

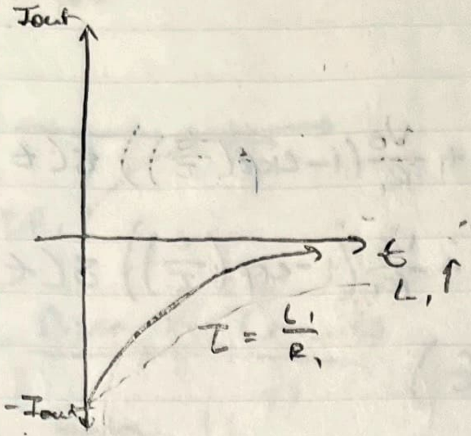
$$V_{out}(\infty) = 0$$

Find I_{out} first:

$$\begin{cases} -L_1 \frac{dI_{out}}{dt} = I_{out} R_1 \\ I_{out}(0^+) = -I_{out} \end{cases}$$

$$\frac{L_1}{R_1} \frac{dI_{out}}{dt} + I_{out} = 0.$$

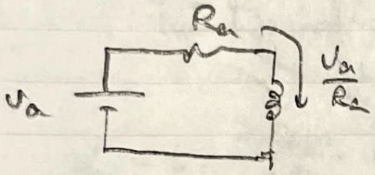
$$I_{out}(t) = -I_0 \exp\left(-\frac{t}{\tau}\right), \quad \tau = \frac{L_1}{R_1}$$

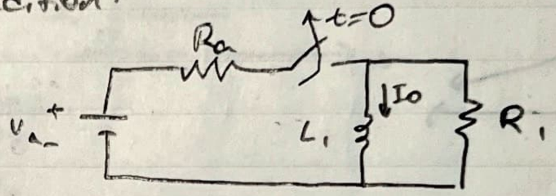


$$V_{out} = R_1 I_{out}$$

$$= -R_1 I_0 \exp\left(-\frac{t}{\tau}\right) \quad t>0$$

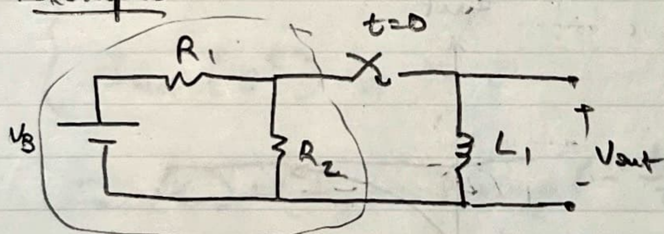
To create initial condition:





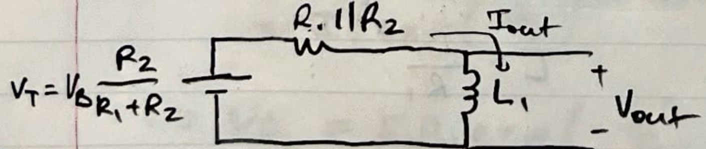
$$I_0 = \frac{V_a}{R_a}$$

Example



$t<0$, $V_{out}=0$ because switch is off.

$I_{L_1} = 0$ at $t=0^-$



$$I_{out}(0^+) = 0$$

$$I_{out}(\infty) = \frac{V_B}{R_1}$$

↳ L_1 shorts R_2.

$$I_{out} = \frac{V_B}{R_1}\left(1-\exp\left(\frac{t}{\tau}\right)\right)u(t)$$

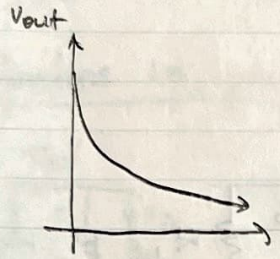
$$\tau = \frac{L_1}{R_1 \| R_2}$$

$$V_{out} = L_1 \frac{dI_{out}}{dt}$$

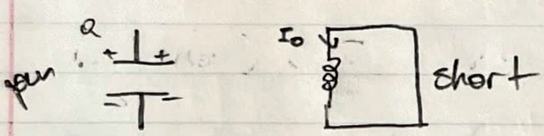
$$= L_1 \frac{V_B}{R_1}\left(\frac{1}{\tau}\right)\left(\exp{-\frac{t}{\tau}}\right)u(t) + \frac{V_B}{R_1}\left(1-\exp\left(-\frac{t}{\tau}\right)\right)\delta(t)$$

$$= L_1 \frac{V_B}{R_1}\left(\frac{1}{\tau}\right)\left(\exp{-\frac{t}{\tau}}\right)u(t) + \frac{V_B}{R_1}\left(1-\exp\left(\frac{0}{\tau}\right)\right)\delta(t)$$

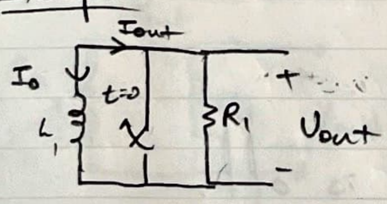
$$= L_1 \frac{V_B}{R_1}\left(\frac{1}{\tau}\right)\left(\exp{-\frac{t}{\tau}}\right)u(t)$$

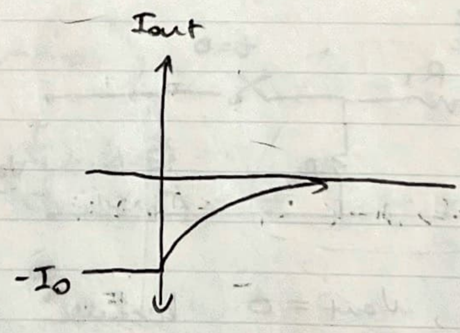


Example



Example



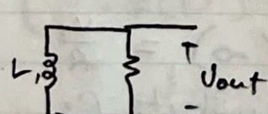


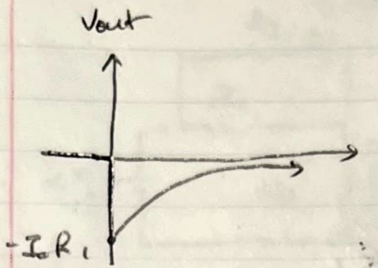
$t<0$: $I_{out} = -I_0$

$$I_{out}(t) = -I_0 \exp\left(-\frac{t}{\tau}\right) \quad t>0$$

$$\tau = \frac{L_1}{R_1}$$

$t>0$:





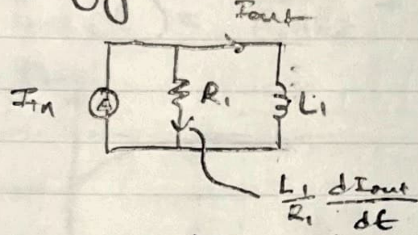
$$V_{out}(0^+) = -I_0 R_1$$

$$V_{out}(t) = I_{out} R_1$$

$$= -R_1 I_0 \exp\left(\frac{t}{\tau}\right) u(t)$$

Driven RL Circuits

Topology 1:



$$I_{in} = I_1 u(t)$$

$$V_{L_1} = L_1 \frac{dI_{out}}{dt} = V_{R_1}$$

$$\begin{cases} \frac{L_1}{R_1}\cdot\frac{dI_{out}}{dt} + I_{out} = I_1 u(t) \\ I_{out}(0^+) = 0 \end{cases} \qquad I_{out}(\infty) = I_1$$

$$I_{out}(t) = \left(I_1 - I_1 \exp\left(-\frac{t}{\tau}\right)\right) u(t)$$

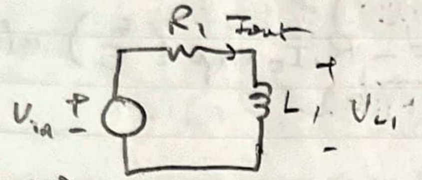
$$\tau = \frac{L_1}{R_1}$$

$$I_{R_1} = I_1 u(t) - I_{out}(t)$$

$$= I_1 \exp\left(-\frac{t}{\tau}\right)$$

$$\Rightarrow V_{R_1} = I_1 R_1 \exp\left(-\frac{t}{\tau}\right)$$

Topology 2:

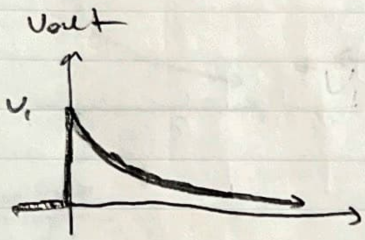


$V_{in} = V_1 u(t)$

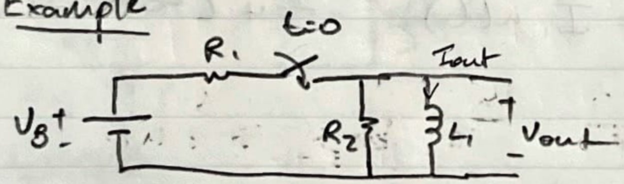
$$I_{out}(t<0) = 0$$
$$I_{out}(0^+) = 0$$
$$I_{out}(\infty) = \frac{V_{in}}{R_1}$$

$$I_{out}(t) = \frac{V_{in}}{R_1}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)u(t)$$

$$\tau = \frac{L_1}{R_1}$$



Example

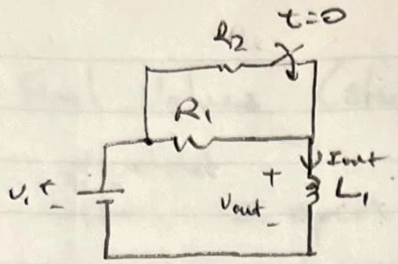


$$I_{out}(0^+) = 0$$
$$I_{out}(\infty) = \frac{V_B}{R_1}$$
$$\tau = \frac{L_1}{R_1 \| R_2}$$

$$I(t) = \frac{V_B}{R_1}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)u(t)$$

$$V_{out} = L_1 \frac{dI}{dt}$$

$$= L_1 \cdot \frac{V_B}{R_1} \cdot \left(\frac{1}{\tau}\right)\exp\left(-\frac{t}{\tau}\right)u(t)$$



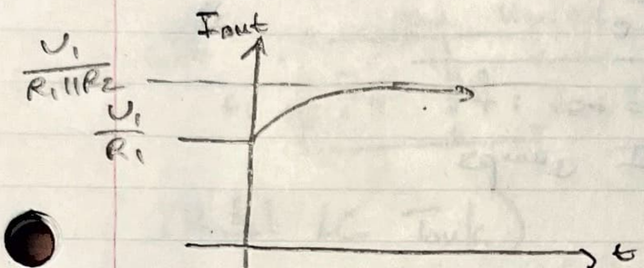
$$I_{out}(0^-) = \frac{V_1}{R_1}$$

$$I_{out}(0^+) = \frac{V_1}{R_1}$$

$$I_{out}(\infty) = \frac{V_1}{R_1 \| R_2}$$

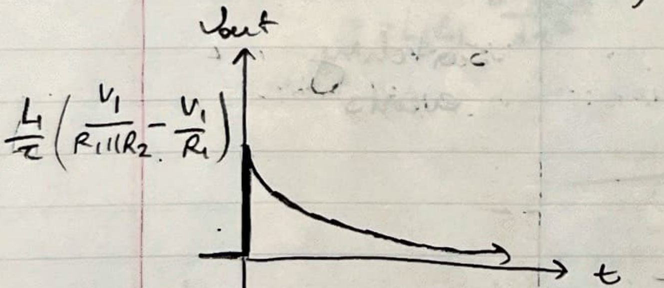
$$t > 0: \quad \tau = \frac{L_1}{R_1 \| R_2}$$

$$I_{out}(t) = \frac{V_1}{R_1 \| R_2} + \left(\frac{V_1}{R_1} - \frac{V_1}{R_1 \| R_2}\right) \exp\left(-\frac{t}{\tau}\right), \quad t > 0$$



$$V_{out}(t) = L_1 \frac{dI_{out}}{dt}$$

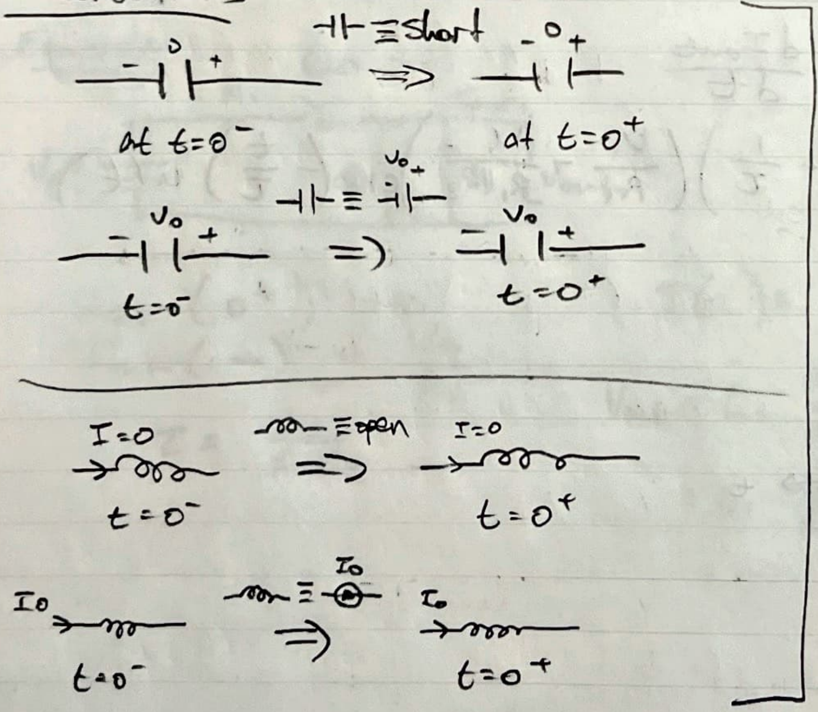
$$= L_1 \left(-\frac{1}{\tau}\right)\left(\frac{V_1}{R_1} - \frac{V_1}{R_1 \| R_2}\right) \exp\left(-\frac{t}{\tau}\right) u(t)$$



Summary

	R	C	L
Basic eq.	$V = IR$	$I = C\frac{dV}{dt}$	$V = L\frac{dI}{dt}$
Energy stored	—	$\frac{1}{2}CV^2$	$\frac{1}{2}LI^2$
Initial condition	—	V_0	I_0
Singularities	—	$I=\infty$ if V jumps	$V=\infty$ if I jumps

Observations



Switching events

Final Values (steady state)

V = constant

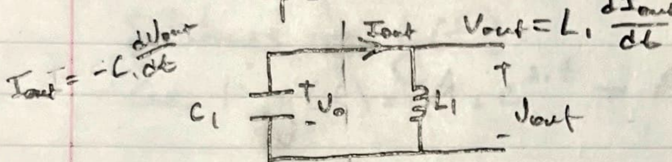
$\dashv\vdash \quad \equiv$ open

I = constant

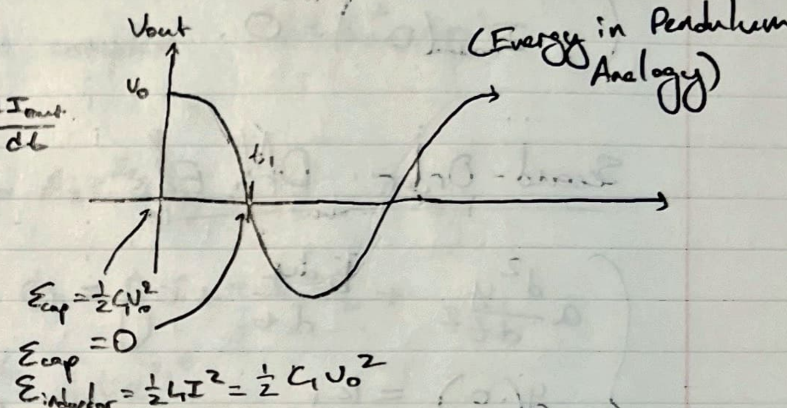
$\equiv$ short

RLC circuits

- A simple LC circuit

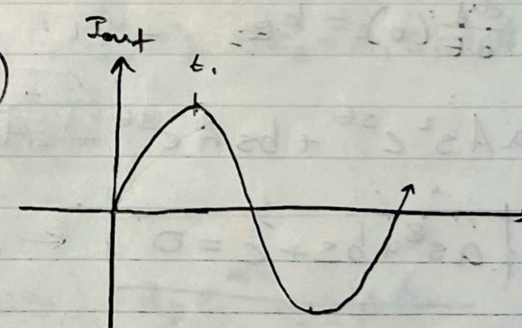


(Parallel LC Tank)



$$I_{out} = -C_1 \left(L_1 \frac{d^2 I_{out}}{dt^2} \right)$$

$$= -L_1 C_1 \frac{d^2 I_{out}}{dt^2}$$



Observations

- At $t = t_1$, $\frac{dV_{out}}{dt} < 0 \Rightarrow I_{out} > 0$.

$\frac{dI_{out}}{dt} = 0 \Rightarrow V_{out} = 0$

$$V_{out} = L_1 \frac{dI_{out}}{dt}$$

$$= L_1 \left(-C_1 \frac{dV_{out}}{dt}\right)$$

$$\begin{cases} L_1 C_1 \frac{d^2 V_{out}}{dt^2} + V_{out} = 0 \\ V_{out}(0^+) = V_0 \\ I_{out}(0^+) = 0. \end{cases}$$

<u>Second-Order Diff. Eqs.</u>

$$\begin{cases} a\frac{d^2 y}{dt^2} + b\frac{dy}{dt} + cy = 0 \\ y(0) = k_1 \\ \frac{dy}{dt}(0) = k_2 \end{cases} \qquad y(t) = Ae^{st}$$

$$aAs^2 e^{st} + bsAe^{st} + cAe^{st} = 0$$

$$\boxed{as^2 + bs + c = 0} \leftarrow \text{characteristic equation}$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Guess:

$$y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$y(0) = k_1 = A_1 + A_2$$

$$\frac{dy}{dt}(0) = k_2 = s_1 A_1 + s_2 A_2$$

$as^2 + bs + c = 0$, $s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Three cases:

① $s_1 \neq s_2$, both are real $(b^2 - 4ac > 0)$

② $s_1 = s_2 = \frac{-b}{2a}$ $(b^2 - 4ac = 0)$

③ $s_1 \neq s_2$, both are complex $(b^2 - 4ac < 0)$

Solutions for $y(t)$:

Case 1: $y(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$,

$$k_1 = A_1 + A_2 \;;\; k_2 = A_1 s_1 + A_2 s_2$$

Case 2: $y(t) = A_1 e^{s_1 t} + A_2 t e^{s_1 t}$

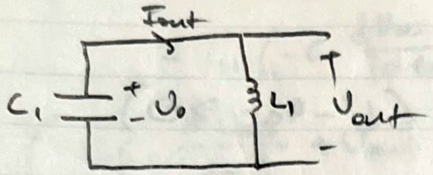
$$y(0) = k_1 = A_1 \;;\; \frac{dy}{dt}(0) = k_2 = A_1 s_1 + A_2$$

Case 3: $s_{1,2} = \underbrace{\frac{-b}{2a}}_{\alpha} \pm j \underbrace{\frac{\sqrt{4ac - b^2}}{2a}}_{\omega}$

$$y(t) = A_1 e^{\alpha t} \cos(\omega t) + A_2 e^{\alpha t} \sin \omega t$$

$$y(0) = k_1 = A_1 \;;\; \frac{dy}{dt}(0) = k_2 = A_1 \alpha + A_2 \omega$$

· Ideal LC Tank



$V_{out}(0^+) = V_0$

$I_L(0^+) = 0$

$$I_{out} = -C_1 \frac{dV_{out}}{dt}, \quad V_{out} = L_1 \frac{dI_{out}}{dt}$$

$$V_{out} = -L_1 C_1 \frac{d^2 V_{out}}{dt^2}$$

$$L_1 C_1 \frac{d^2 V_{out}}{dt^2} + V_{out} = 0$$

characteristic $\longrightarrow$ $L_1 C_1 s^2 + 1 = 0$

$$s_{1,2} = \pm \sqrt{\frac{-1}{L_1 C_1}}$$
$$= \pm j \sqrt{\frac{1}{L_1 C_1}}$$

$$V_{out}(t) = A_1 \cos \omega_1 t + A_2 \sin \omega_1 t \quad \left(\omega_1 = \sqrt{\frac{1}{L_1 C_1}}\right)$$

$$V_{out}(0) = A_1 = V_0$$

$$0 = I_L(0^+) = I_{out}(0) = -C_1 \frac{dV_{out}}{dt}(0)$$

$$A_2 \omega = \frac{dV_{out}}{dt}(0) = 0$$

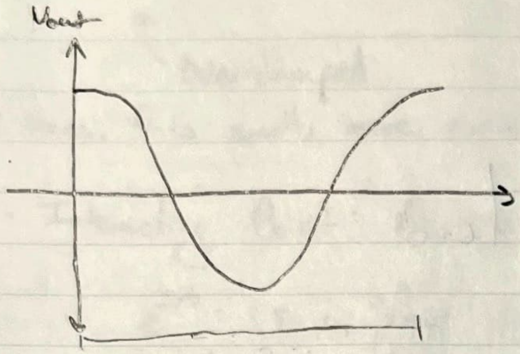
$$A_2 = 0$$

$$\Rightarrow V_{out}(t) = V_0 \cos \omega_1 t \quad \left(\omega_1 = \sqrt{\frac{1}{L_1 C_1}}\right)$$

$$T = 2\pi \sqrt{L_1 C_1}$$

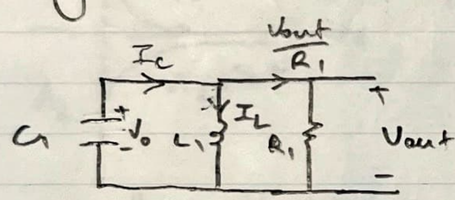
$$\omega_1 = \frac{1}{\sqrt{L_1 C_1}} \quad \text{"radian frequency"}$$

$$f_1 = \frac{1}{2\pi\sqrt{L_1 C_1}} \text{ Hz}$$



$$T = 2\pi\sqrt{L_1 C_1}$$

- Lossy LC Tank



$$I_C = -C_1 \frac{dV_{out}}{dt}$$

$$V_{out} = L_1 \frac{dI_L}{dt}$$

$$I_L = I_C - \frac{V_{out}}{R_1} \Rightarrow V_{out} = L_1\left(-C_1 \frac{d^2 V_{out}}{dt} - \frac{1}{R_1}\frac{dV_{out}}{dt}\right)$$

$$\begin{cases} L_1 C_1 \frac{d^2 V_{out}}{dt} + \frac{L_1}{R_1}\frac{dV_{out}}{dt} + V_{out} = 0 \\ V_{out}(0^+) = V_0 \\ I_L(0^+) = 0 \end{cases}$$

$$L_1 C_1 s^2 + \frac{L_1}{R_1} s + 1 = 0$$

$$s^2 + \frac{1}{R_1 C_1} s + \frac{1}{L_1 C_1} = 0$$

$$s_{1,2} = \frac{-1}{2R_1 C_1} \pm \frac{\sqrt{\left(\frac{1}{R_1 C_1}\right)^2 - 4 \cdot \frac{1}{L_1 C_1}}}{2}$$

Observation: always negative, dimension: Hz.

$s_{1,2} < 0 \Rightarrow$ Decay

Case 1: $\frac{1}{R_1^2 C_1^2} > \frac{4}{L_1 C_1}$ $\left(R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \right)$

$$\Rightarrow V_{out} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$V_{out}(0^+) = V_0 = A_1 + A_2$$

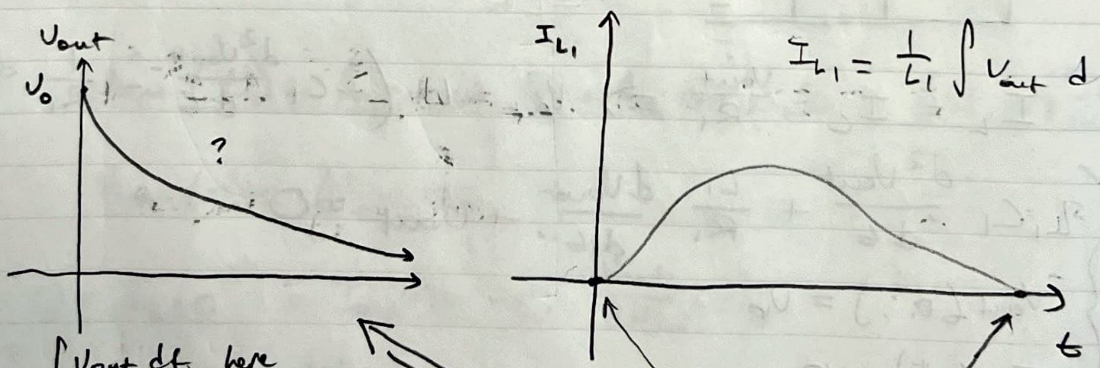
At $t = 0^+$:

$$\frac{V_0}{R_1} = I_C = -C_1 \left.\frac{dV_{out}}{dt}\right|_{t=0^+}$$

$$\left.\frac{dV_{out}}{dt}\right|_{t=0^+} = \frac{-V_0}{R_1 C_1}$$

$$\frac{dV_{out}}{dt}(0) = -\frac{V_0}{R_1 C_1} = A_1 s_1 + A_2 s_2$$

$$\Rightarrow \begin{cases} A_1 + A_2 = V_0 \\ A_1 s_1 + A_2 s_2 = \frac{-V_0}{R_1 C_1} \end{cases}$$



$I_{L_1} = \frac{1}{L_1}\int V_{out}\, dt$

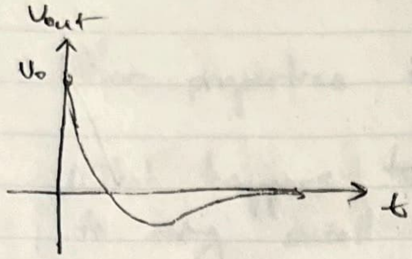
$\int V_{out}\, dt$ here increases monotonically

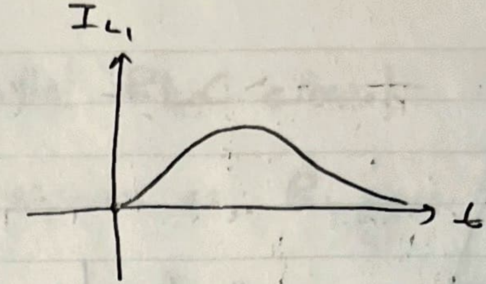
Inconsistent

know $I_{L_1}(0) = 0$

$I_{L_1}(\infty) = 0$

So must peak somewhere and go back down





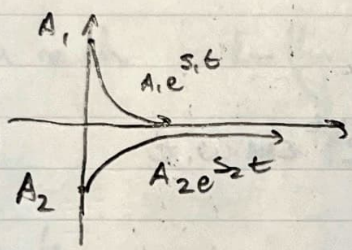
Overdamped

(Since R is small, more current leaks from LC Tank)

- Interesting Point: $A_1 > 0$, $A_2 < 0$.

  $e^{s_1 t}$: is a "fast" exp $\left(s_1 = -\frac{1}{2R_1C_1} - \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1C_1}}\right)$

  $e^{s_2 t}$: is a "slow" exp $\left(s_2 = -\frac{1}{2R_1C_1} + \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1C_1}}\right)$

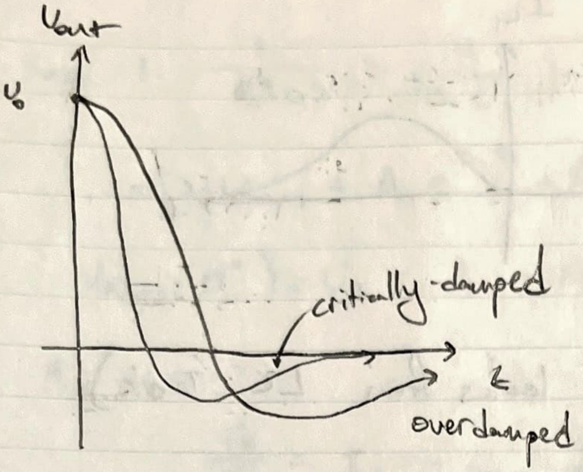


Case 2: $R_1 = \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Rightarrow V_{out} = A_1e^{s_1t} + A_2e^{s_2t}t$

critically damped

$s_1 = -\frac{1}{2R_1C_1}$

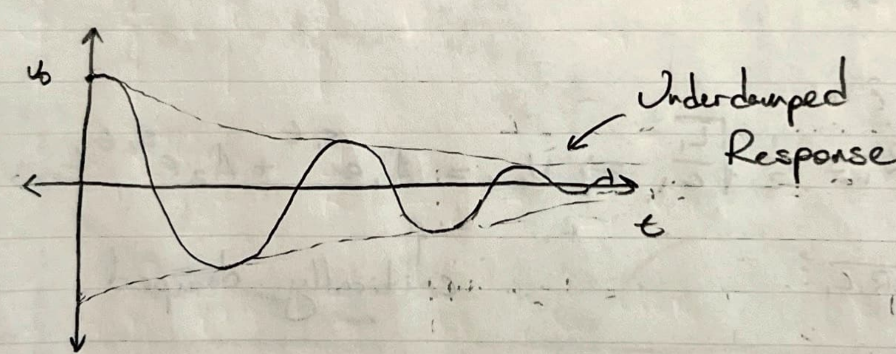
$$\begin{cases} A_1 = V_0 \\ A_1 s_1 + A_2 = -\frac{V_0}{R_1C_1} \end{cases}$$



Case 3: $R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}} \Rightarrow$ roots are complex

$$s_{1,2} = \underbrace{-\frac{1}{2R_1C_1}}_{\alpha} \pm j\underbrace{\frac{1}{2}\sqrt{\frac{4}{L_1C_1} - \frac{1}{R_1^2C_1^2}}}_{\omega_1}$$

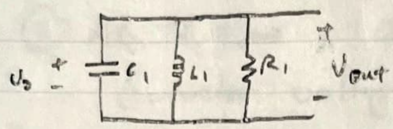
$$V_{out}(t) = A_1 e^{\alpha t}\cos\omega_1 t + A_2 e^{\alpha t}\sin\omega_1 t$$

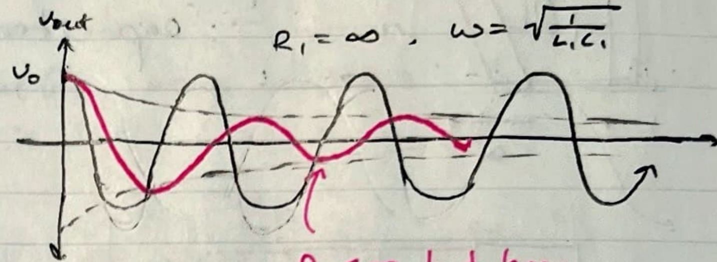


$$\begin{cases} A_1 = V_0 \\ A_1\alpha + A_2\omega_1 = -\frac{V_0}{R_1C_1} \end{cases}$$

• Other properties of Parallel RLC circuit

What happens to the response as R_1 goes from ∞ to very small values?





$R_1 < \infty$ but large

$\left(R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}}\right)$

$\left(\omega = \frac{1}{2}\sqrt{\frac{4}{L_1C_1} - \frac{1}{R_1^2C_1^2}}\right)$

$\left(\omega < \sqrt{\frac{1}{L_1C_1}}\right)$

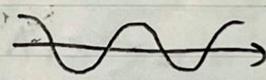
Before $R_1 \leq \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$ ... oscillations damped more and more ...

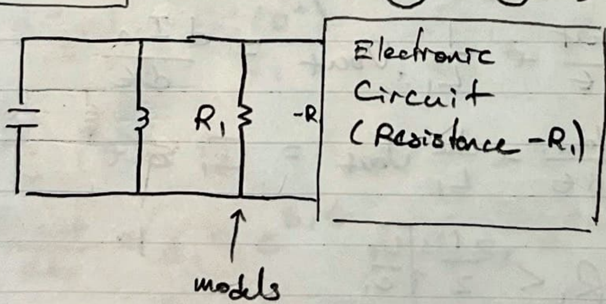
Then reach critically damped & overdamped

reach 0 faster

Example

Oscillator — Periodic

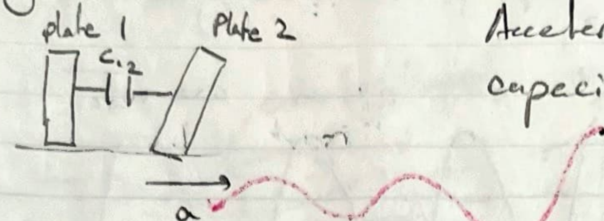




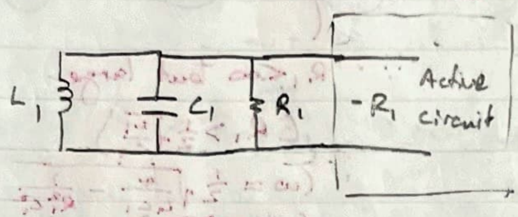
$$R_{eq} = \frac{R_1(-R_1)}{R_1 - R_1} \rightarrow -\infty \Rightarrow \text{open circuit.}$$

- How do we measure capacitance?

e.g. Accelerometer

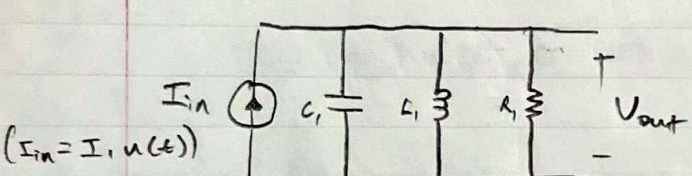


Acceleration causes change in capacitance...



Measure the change in frequency $\left(\omega = \frac{1}{\sqrt{L_1 C_1}}\right)$.

- Driven Parallel RLC circuit



$$\frac{V_{out}}{R_1} + \frac{1}{L_1}\int V_{out}\,dt + C_1 \frac{dV_{out}}{dt} = I_{in}$$

$$C_1 \frac{d^2 V_{out}}{dt^2} + \frac{1}{R_1}\frac{dV_{out}}{dt} + \frac{1}{L_1} V_{out} = \frac{d I_{in}}{dt}$$

$$C_1 \frac{d^2 V_{out}}{dt^2} + \frac{1}{R_1}\frac{dV_{out}}{dt} + \frac{1}{L_1} V_{out} = I_1 \delta(t)$$

Case 1: Overdamped $\quad R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$

$$\left(s_{1,2} = -\frac{1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2 C_1^2} - \frac{4}{L_1 C_1}} \right)$$

$$V_{out} = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t}\right) u(t)$$

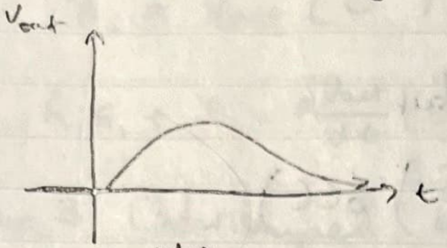
Observations:

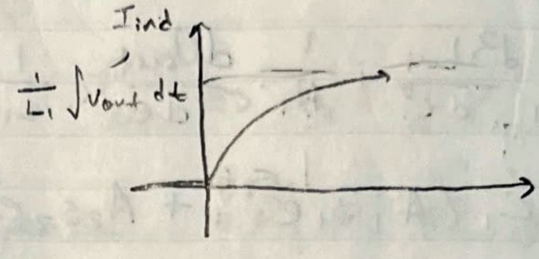
① $V_{out}(0^+) = 0$

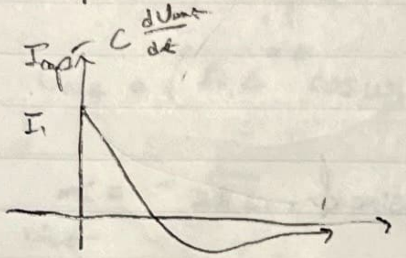
$I_L(0^+) = 0$

② All of $I_1 u(t)$ flows through C_1 at $t = 0^+$

③ At $t = \infty$, $I_1 u(t)$ flows through L_1 (inductor voltage is 0).







$$\begin{cases} V_{out}(0^+) = 0 = A_1 + A_2 \\ \left.\frac{dV_{out}}{dt}\right|_{t=0} = \frac{I_{C_1}(0^+)}{C_1} = \frac{I_1}{C_1} = A_1 s_1 + A_2 s_2 \end{cases}$$

Verify diff. eq.:

$$\frac{dV_{out}}{dt} = \left(A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}\right)u(t) + (A_1 + A_2)\,\delta(t)$$

$$= \left(A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}\right)u(t)$$

$$\frac{d^2 V_{out}}{dt^2} = \left(A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t}\right)u(t) + (A_1 s_1 + A_2 s_2)\,\delta(t)$$

$$= \left(A_1 s_1^2 e^{s_1 t} + A_2 s_2^2 e^{s_2 t}\right)u(t) + \left(\frac{I_1}{C_1}\right)\delta(t)$$

Note that when evaluating

$$C_1 \frac{d^2 V_{out}}{dt^2} + \frac{1}{R_1} \frac{d V_{out}}{dt} + \frac{1}{L_1} V_{out},$$

All parts of $\frac{d^2 V_{out}}{dt^2}$, $\frac{dV_{out}}{dt}$, V_{out} with $u(t)$ will add to 0. (They add up to characteristic equation)

$$C_1 \frac{d^2 V_{out}}{dt^2} + \frac{1}{R_1} \frac{dV_{out}}{dt} + \frac{1}{L_1} V_{out}$$

$$= C_1 (A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}) \delta(t)$$

$$= C_1 \left(\frac{I_1}{C_1}\right) \delta(t)$$

$$= I_1 \delta(t) \quad \checkmark$$

As desired.

$$I_{cap}(t) = C_1 (A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}) u(t)$$

$$I_L(t) = \frac{1}{L_1} \int_0^t V_{out} \, dt$$

$$= \frac{1}{L_1} \left(\frac{A_1}{s_1} e^{s_1 t} + \frac{A_2}{s_2} e^{s_2 t} \right) u(t) - \frac{1}{L_1} \left(\frac{A_1}{s_1} + \frac{A_2}{s_2} \right)$$

$$\frac{A_1}{s_1} + \frac{A_2}{s_2} = A_1 \left(\frac{1}{s_1} - \frac{1}{s_2} \right)$$

$$= A_1 \left(\frac{s_2 - s_1}{s_1 s_2} \right)$$

$s_1 s_2 = \frac{1}{L_1 C_1}$ by Vieta's

$A_1 (s_1 - s_2) = \frac{I_1}{C_1}$

$$= \left(\frac{-\frac{I_1}{C_1}}{\frac{1}{L_1 C_1}} \right)$$

$$= -I_1 L_1$$

$$\Rightarrow \boxed{I_L(t) = \frac{1}{L}\left(\frac{A_1}{s_1}e^{s_1 t} + \frac{A_2}{s_2}e^{s_2 t}\right)u(t) + I_1}$$

Case 2: Critically-Damped Response $\left(R_1 = \frac{1}{2}\sqrt{\frac{L_1}{C_1}}\right)$

$$V_{out} = \left[A_1 e^{s_1 t} + A_2 t e^{s_1 t}\right]u(t) \quad \left(s_1 = \frac{-1}{2R_1C_1}\right)$$

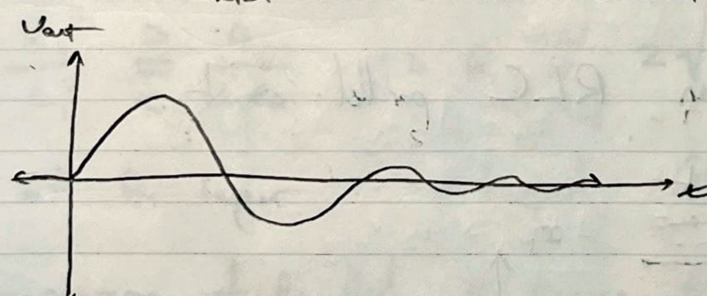
$$A_1 = V_{out}(0^-) = 0$$

$$A_1 s_1 + A_2 = \frac{dV_{out}}{dt}(0^+) = \frac{I_1}{C_1} \Rightarrow A_2 = \frac{I_1}{C_1}$$

Case 3: Underdamped Response $\left(R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}}\right)$

$$V_{out} = \left(A_1 e^{\alpha t}\cos\omega_1 t + A_2 e^{\alpha t}\sin\omega_1 t\right)u(t)$$

$$\alpha = -\frac{1}{2R_1C_1} \qquad \omega = \frac{1}{2}\sqrt{\frac{1}{4L_1C_1} - \frac{1}{R_1^2C_1^2}}$$

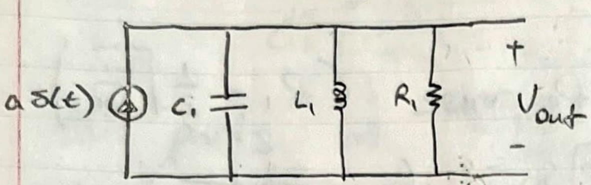


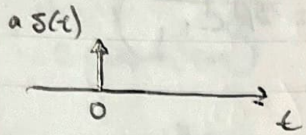
$$A_1 = V_{out}(0^+) = 0$$

$$A_1\alpha + A_2\omega_1 = \left.\frac{dV_{out}}{dt}\right|_{0^+} = \frac{I_1}{C_1}$$

$$A_2\omega_1 = \frac{I_1}{C_1}$$

Impulse Response





$$V_{out} = L_1 \frac{dI_L}{dt}$$

↳ derivative of impulse ∴

(Vout still governed by C_1)

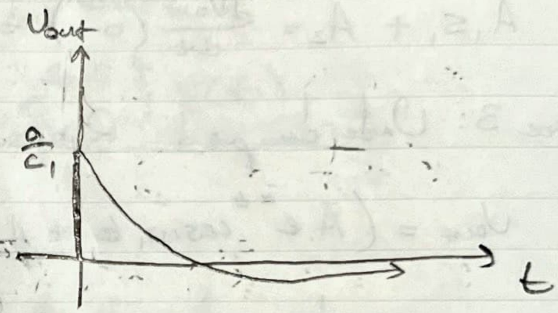
⇒ All current through C_1

$$I_{cap} = C_1 \frac{dV_{out}}{dt}$$

$$V_{out} = \frac{1}{C_1} \int_{0^-}^{0^+} I_{cap}\, dt$$

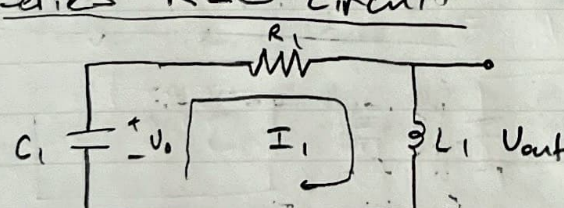
$$= \frac{1}{C_1} \int_{0^-}^{0^+} a\,\delta(t)\, dt$$

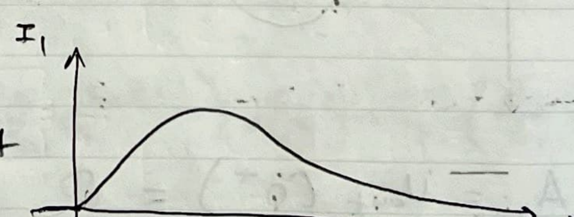
$$= \frac{a}{C_1}$$



⇒ Same as undriven RLC parallel circuit

Series RLC circuit

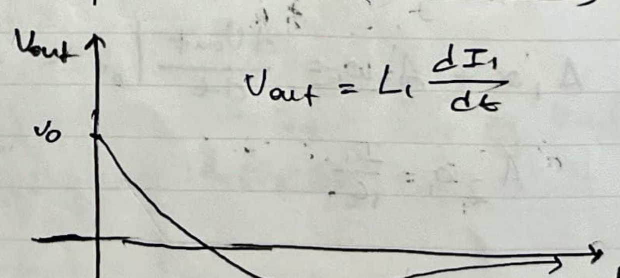




$$I_{L_1}(0^+) = 0$$

$$I_1(\infty) = 0$$

$$V_{out}(0^+) = V_0$$



$$V_{out} = L_1 \frac{dI_1}{dt}$$

• Quantitative Analysis

$$V_0 - \frac{1}{C_1}\int I_1 dt - I_1 R_1 - L_1 \frac{dI_1}{dt} = 0$$

$$\begin{cases} \frac{1}{C_1} I_1 + \frac{dI_1}{dt} R_1 + L_1 \frac{d^2 I_1}{dt^2} = 0 \\ I_1(0^+) = 0 \\ V_{cap}(0^+) = V_0 \end{cases}$$

$$L_1 s^2 + R_1 s + \frac{1}{C_1} = 0$$

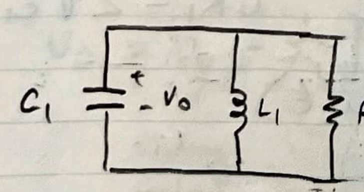
$$s^2 + \frac{R_1}{L_1} s + \frac{1}{L_1 C_1} = 0$$

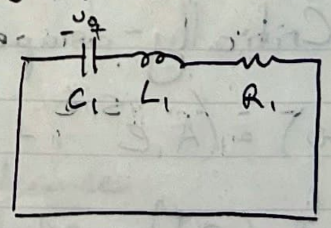
$$s_{1,2} = -\frac{R_1}{2L_1} \pm \frac{1}{2}\sqrt{\left(\frac{R_1}{L_1}\right)^2 - \frac{4}{L_1 C_1}}$$

$$\frac{R_1^2}{L_1^2} \begin{matrix} > \\ = \\ < \end{matrix} \frac{4}{L_1 C_1} \Rightarrow R_1 \begin{matrix} > \\ = \\ < \end{matrix} 2\sqrt{\frac{L_1}{C_1}}$$

If R_1 larger $\Rightarrow$ overdamped (since, intuitively, in series)

• Comparison of Parallel & series RLC Circuits





$R_1 = \infty \Rightarrow LC$ ideal oscillates

$R_1 = 0 \Rightarrow LC$ ideal oscillates

$R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$ dampens the oscillation (underdamped)

$R_1 > 2\sqrt{\frac{L_1}{C_1}}$ dampens the oscillation (overdamped)

$$V_0 = V_{cap}(0^+) = V_L(0^+) = L_1 \frac{dI_1}{dt}\bigg|_{t=0^+}$$

$$\Rightarrow \frac{dI_1}{dt}\bigg|_{t=0^+} = \frac{V_0}{L_1}$$

$$\begin{cases} L_1 \frac{d^2 I_1}{dt} + R_1 \frac{dI_1}{dt} + \frac{1}{C_1} I_1 = 0 \\ I_1(0^+) = 0 \\ \frac{dI_1}{dt}(0^+) = \frac{V_0}{L_1} \end{cases}$$

Three Natural Responses:

Case 1: Overdamped response $\left(R_1 > 2\sqrt{\frac{L_1}{C_1}}\right)$

$$\Rightarrow I_1(t) = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t}\right) u(t)$$

$$0 = I_1(0^+) = A_1 + A_2$$

$$\frac{V_0}{L_1} = \frac{dI_1}{dt}(0^+) = (A_1 s_1 + A_2 s_2) u(0)$$

$$= A_1 s_1 + A_2 s_2$$

Case 2: Critically-damped response $\left(R_1 = 2\sqrt{\frac{L_1}{C_1}}\right)$

$$\Rightarrow I_1(t) = \left(A_1 e^{s_1 t} + A_2 t e^{s_1 t}\right) u(t)$$

$$0 = I_1(0^+) = A_1$$

$$\frac{V_0}{L_1} = \frac{dI_1}{dt}(0^+) = A_1 s_1 + A_2$$

$$\Rightarrow A_2 = \frac{V_0}{L_1}$$

Case 3: Underdamped response ($R_1 < 2\sqrt{\frac{L_1}{C_1}}$)

$$\Rightarrow I_1 = \left(A_1 e^{\alpha t}\cos(\omega_1 t) + A_2 e^{\alpha t}\sin(\omega_1 t)\right)u(t)$$

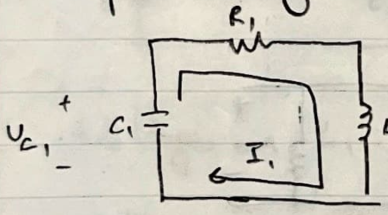
$$s_{1,2} = \underbrace{-\frac{R_1}{2L_1}}_{\alpha} \pm j\underbrace{\frac{1}{2}\sqrt{\frac{4}{L_1C_1} - \left(\frac{R_1}{L_1}\right)^2}}_{\omega_1}$$

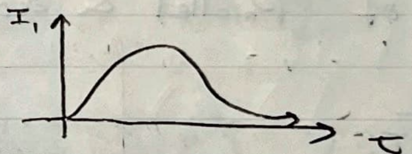
$$0 = I_1(0^+) = A_1$$

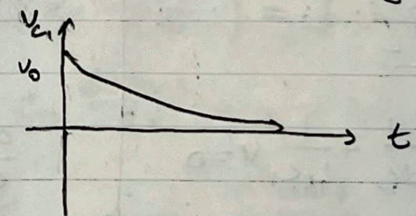
$$\frac{V_0}{L_1} = \frac{dI_1}{dt}\Big|_{0^+} = A_1\alpha + A_2\omega_1$$

$$\Rightarrow A_2\omega_1 = \frac{V_0}{L_1}$$

• Cap voltage





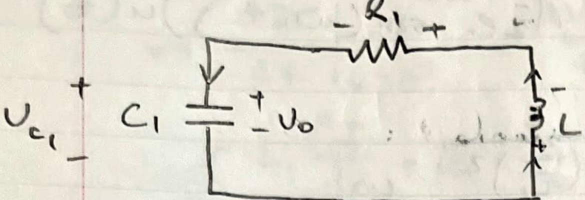


$$V_{C_1} = V_0 - \frac{1}{C_1}\int_0^t I_1(t)\,dt$$

starting voltage (V_0)

voltage lost because of lost charge (between 0 and t) ($\frac{1}{C_1}\int_0^t I_1(t)\,dt$)

Alternative Method: Find V_{c_1} directly (w/o I_1)



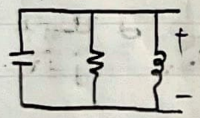
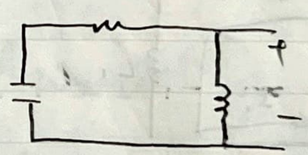
$C_1 \frac{dV_{c_1}}{dt}$: current through C_1

$R_1 C_1 \frac{dV_{c_1}}{dt}$: voltage across R_1

$$L_1 \frac{dI}{dt} = L_1 \frac{d}{dt}\left(C_1 \frac{dV_{c_1}}{dt}\right) = L_1 C_1 \frac{d^2 V_{c_1}}{dt^2} \text{ : voltage across } L_1$$

$$\begin{cases} L_1 C_1 \frac{d^2 V_{c_1}}{dt^2} + R_1 C_1 \frac{dV_{c_1}}{dt} + V_{c_1} = 0 \quad \text{(KVL)} \\ V_{c_1}(0^+) = V_0 \\ I_{L_1}(0^+) = 0 \Rightarrow I_{c_1}(0^+) = 0 \Rightarrow C_1 \left.\frac{dV_{c_1}}{dt}\right|_{t=0^+} = 0 \end{cases}$$

• Summary of Parallel & Series RLC circuits.

Parallel	Series
	
$\frac{d^2V}{dt^2} + \frac{1}{R_1C_1}\frac{dV}{dt} + \frac{1}{L_1C_1}V = 0$	$\frac{d^2V}{dt^2} + \frac{R_1}{L_1}\frac{dV}{dt} + \frac{1}{L_1C_1}V = 0$
$s_{1,2} = -\frac{1}{2R_1C_1} \pm \frac{1}{2}\sqrt{\frac{1}{R_1^2C_1^2} - \frac{4}{L_1C_1}}$	$s_{1,2} = -\frac{R_1}{2L_1} \pm \frac{1}{2}\sqrt{\frac{R_1^2}{L_1^2} - \frac{4}{L_1C_1}}$
$R_1 < \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$: Overdamped	$R_1 > 2\sqrt{\frac{L_1}{C_1}}$: Overdamped
$R_1 = \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$: Critically-damped	$R_1 = 2\sqrt{\frac{L_1}{C_1}}$: Critically-damped
$R_1 > \frac{1}{2}\sqrt{\frac{L_1}{C_1}}$: Underdamped	$R_1 < 2\sqrt{\frac{L_1}{C_1}}$: Underdamped

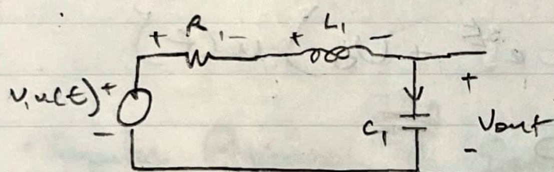
- A Note on Diff. Eqs.
  - First-Order Eqs. $\frac{dy}{dt} + \alpha y = \beta$

    $y(\infty) = \frac{\beta}{\alpha}$ (if y becomes constant)

  - Second-Order Eqs. $\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + \beta y = \gamma$

    $y(\infty) = \frac{\gamma}{\beta}$ (if y becomes constant)

- Step Response



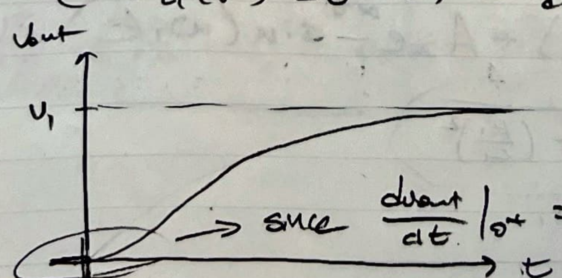
Current through cap: $C_1 \frac{dV_{out}}{dt}$

Voltage on L_1: $L_1 \frac{d}{dt}\left(C_1 \frac{dV_{out}}{dt}\right) = L_1 C_1 \frac{d^2V_{out}}{dt^2}$

Voltage on $R_1 = R_1 C_1 \frac{dV_{out}}{dt}$

$$V_1 u(t) - R_1 C_1 \frac{dV_{out}}{dt} - L_1 C_1 \frac{d^2V_{out}}{dt^2} - V_{out} = 0$$

$$\begin{cases} L_1 C_1 \frac{d^2V_{out}}{dt^2} + R_1 C_1 \frac{dV_{out}}{dt} + V_{out} = V_1 u(t) \\ V_{out}(0^+) = 0 \\ I_{L_1}(0^+) = 0 \Rightarrow C \left.\frac{dV_{out}}{dt}\right|_{0^+} = 0 \Rightarrow \left.\frac{dV_{out}}{dt}\right|_{0^+} = 0 \end{cases}$$



$\rightarrow$ since $\left.\frac{dV_{out}}{dt}\right|_{0^+} = 0$.

Characteristic Eq: $s^2 + \frac{R_1}{L_1}s + \frac{1}{L_1C_1} = 0$

Case 1: Overdamped $\quad R_1 > 2\sqrt{\frac{L_1}{C_1}}$

$$V_{out} = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_0\right) u(t)$$

$$A_1 + A_2 + V_0 = V_{out}(0^+) = 0$$

$$(A_1 s_1 + A_2 s_2)\, u(t) + V_{out}(0^+)\,\delta(t) = \frac{dV_{out}}{dt}\Big|_{0^+} = 0$$

$$A_1 s_1 + A_2 s_2 = 0$$

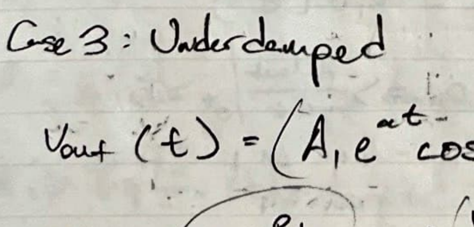
Case 2: Critically-damped $\quad R_1 = 2\sqrt{\frac{L_1}{C_1}}$

$$V_{out}(t) = \left(A_1 e^{s_1 t} + A_2 t e^{s_2 t} + V_0\right) u(t)$$

$$A_1 + V_0 = V_{out}(0^+) = 0$$

$$(A_1 s_1 + A_2)\, u(t) + V_{out}(0^+)\,\delta(t) = \frac{dV_{out}}{dt}\Big|_{0^+} = 0$$

$$A_1 s_1 + A_2 = 0$$



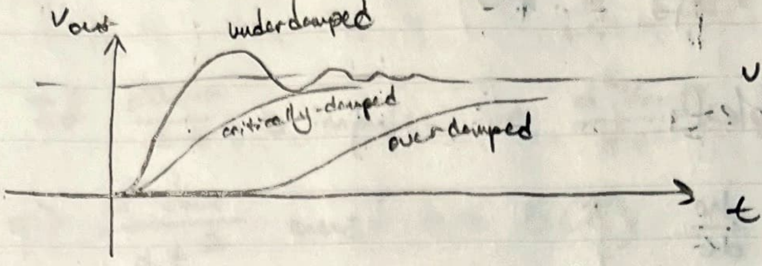
Case 3: Underdamped $\quad R_1 < 2\sqrt{\frac{L_1}{C_1}}$

$$V_{out}(t) = \left(A_1 e^{\alpha t}\cos(\omega_1 t) + A_2 e^{\alpha t}\sin(\omega_1 t) + V_0\right) u(t)$$

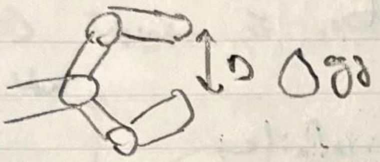
$$s_{1,2} = \underbrace{-\frac{R_1}{2L_1}}_{\alpha} \pm j\, \underbrace{\frac{1}{2}\sqrt{\frac{4}{L_1C_1} - \left(\frac{R_1}{C_1}\right)^2}}_{\omega_1}$$

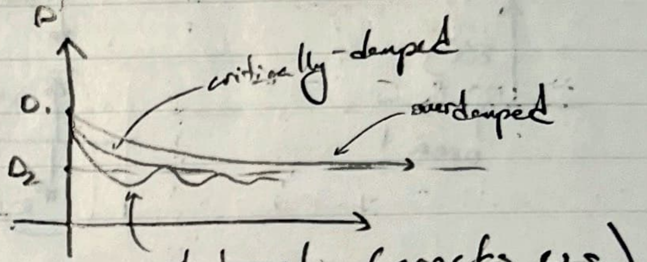
$$A_1 + V_0 = V_{out}(0^+) = 0$$

$$A_1 \alpha + A_2 \omega = \frac{dV_{out}}{dt}\Big|_{0^+} = 0$$



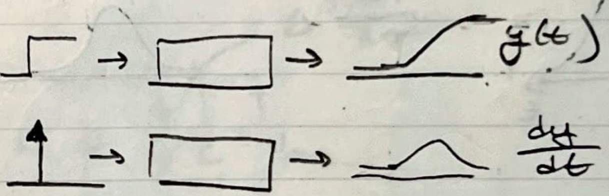
· Robotics Analogy





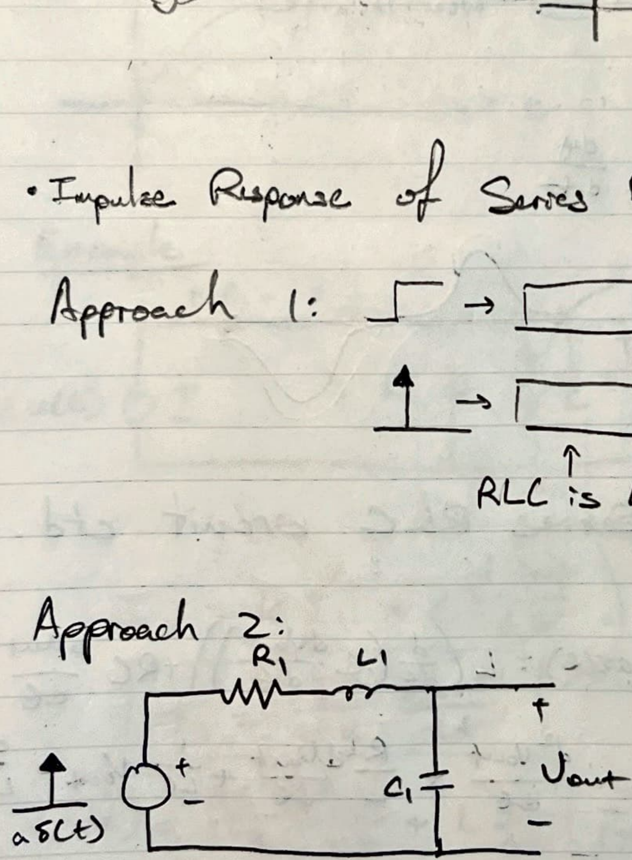
· Impulse Response of Series RLC circuit

Approach 1:



RLC is Linear Time Invariant system.

Approach 2:



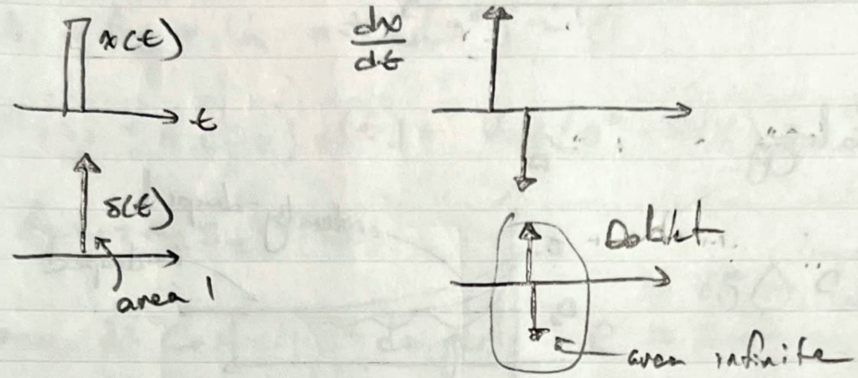
Voltage on C_1 can't jump because requires ∞ current

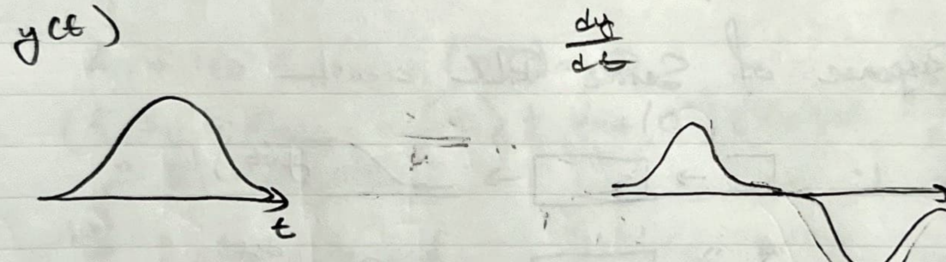
↳ can only jump current here because on L_1

($I_{L_1} \not\to \infty \Rightarrow V_{out}$ cannot jump)

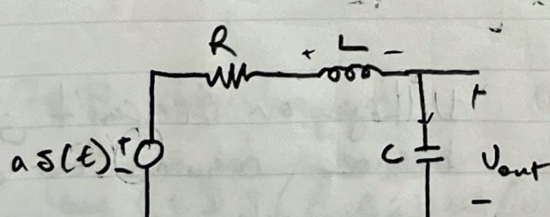
Since $\delta(t)$ jumps to ∞ and resistor/cap do not
$\Rightarrow$ All of impulse through inductor

- Impulses & Doublets





- Impulse Response of Series RLC circuit ctd.



$$a\delta(t) = L\left(\frac{d}{dt}\left(C\frac{dV_{out}}{dt}\right)\right) + RC\frac{dV_{out}}{dt} + V_{out}$$

$$\frac{d^2V_{out}}{dt} + \frac{R}{L}\frac{dV_{out}}{dt} + \frac{1}{LC}V_{out} = \frac{a}{LC}\delta(t)$$

If V_{out} is impulse → $\frac{dV_{out}}{dt}$ is doblet

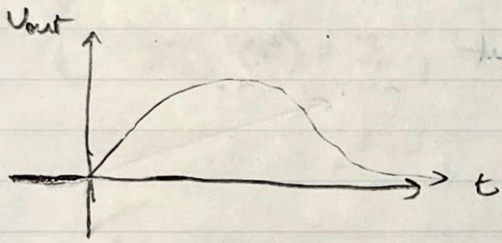
→ $\frac{a}{LC}\delta(t)$ cannot match doublet

If $\frac{dV_{out}}{dt}$ is impulse → $\frac{d^2V_{out}}{dt^2}$ is doublet

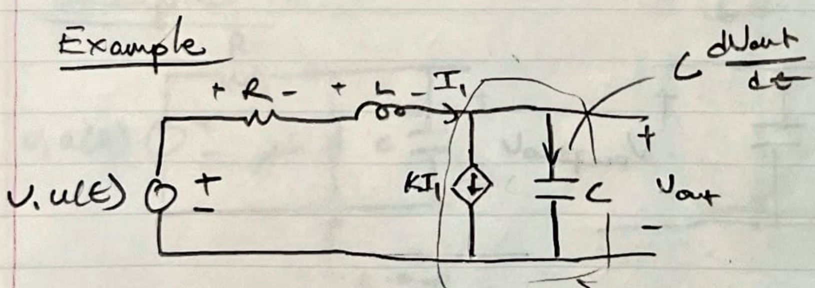
⇒ $\frac{d^2V_{out}}{dt^2}$ must have $\delta(t)$

⇒ $\frac{dV_{out}}{dt}$ has a step

⇒ V_{out} is continuous.



Example



$C_{eq} = \frac{C}{1-K}$

$$I_1 = KI_1 + C\frac{dV_{out}}{dt}$$

$$\boxed{I_1 = \frac{C}{1-K}\frac{dV_{out}}{dt}}$$

$$V_1u(t) = \frac{RC}{1-K}\frac{dV_{out}}{dt} + L\frac{d}{dt}\left(\frac{C}{1-K}\frac{dV_{out}}{dt}\right) + V_{out}$$

$$\frac{LC}{1-K}\frac{d^2V_{out}}{dt^2} + \frac{RC}{1-K}\frac{dV_{out}}{dt} + V_{out} = V_1u(t)$$

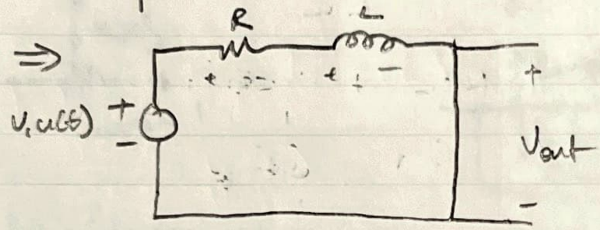
$$\frac{d^2 V_{out}}{dt^2} + \frac{R}{L}\frac{dV_{out}}{dt} + \frac{1}{L\frac{C}{1-k}} V_{out} = \frac{V_1}{L\frac{C}{1-k}} u(t)$$

$\Rightarrow$ Only C changed to $\frac{C}{1-k}$

What if $k=1$?

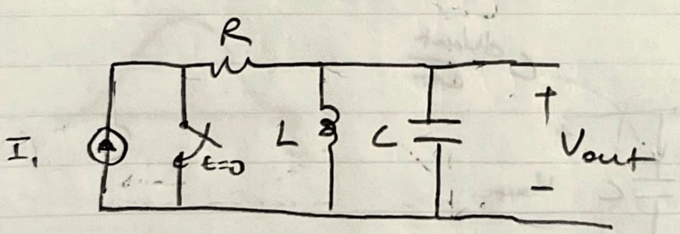
$\Rightarrow$ cap is shorted

$\Rightarrow$

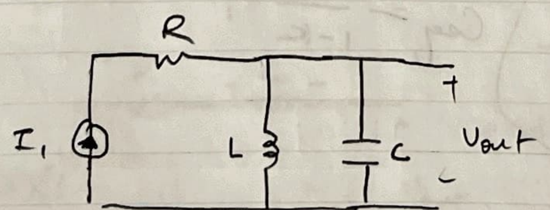


$V_{out} = 0$.

Example



$t<0$



Since all quantities are constant

$\Rightarrow I_{cap} = 0$

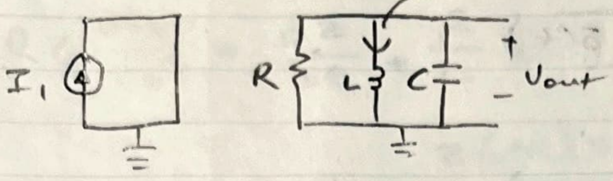
$\Rightarrow I_L = I_1$

Since I_L is constant $\Rightarrow V_L = 0$ $\Rightarrow V_{out} = 0$.

$t > 0$

$V_{out}(0) = 0$

I_1: initial condition



$\Rightarrow$ Parallel RLC circuit

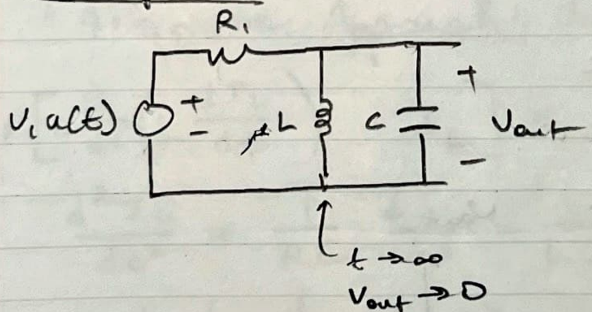
$$\begin{cases} \dfrac{d^2 V_{out}}{dt^2} + \dfrac{1}{RC}\dfrac{dV_{out}}{dt} + \dfrac{1}{LC} V_{out} = 0 \\ V_{out}(0) = 0 \\ I_{Ind}(0) = I_1 \end{cases}$$

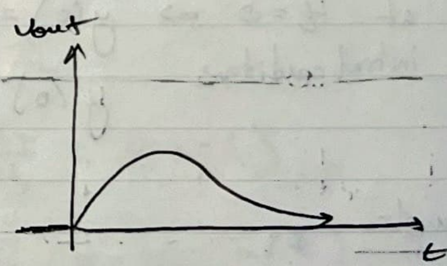
KCL at $t = 0^+$:

$$I_1 + C\left.\frac{dV_{out}}{dt}\right|_{0^+} = 0$$

$$\Rightarrow \left.\frac{dV_{out}}{dt}\right|_{0^+} = -\frac{I_1}{C_1}$$

Example





- Diff Eqs. with Impulses

$$\alpha \frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + \gamma y = \delta(t)$$

Procedure:

① Replace $\delta(t)$ with $u(t)$

② Create a new diff. eq:

$$\alpha \frac{d^2z}{dt^2} + \beta \frac{dz}{dt} + \gamma z = u(t) \Rightarrow z(\infty) = \frac{1}{\gamma}$$

e.g., Overdamped: $z(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \frac{1}{\gamma}$

add this so $z(\infty) = \frac{1}{\gamma}$

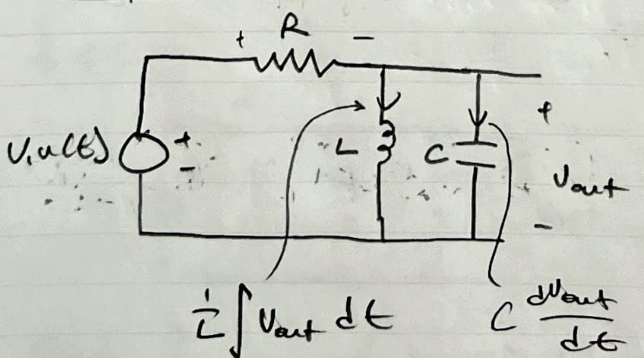
③ $\frac{dz}{dt} = y(t)$

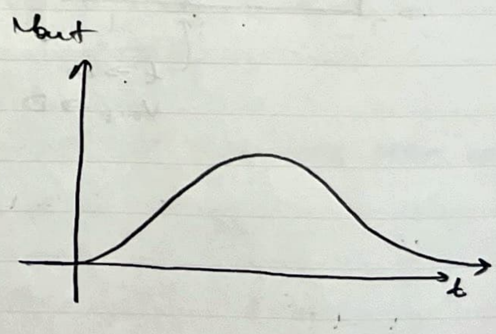
e.g., Overdamped: $y(t) = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$

at $t=0$ → $y(0) = k_1$

initial conditions $y'(0) = k_2$

Example





$$R\left(\frac{1}{L}\int V_{out}\,dt + C\frac{dV_{out}}{dt}\right) + V_{out} = V_1 u(t)$$

$$RC\frac{d^2 V_{out}}{dt^2} + \frac{dV_{out}}{dt} + \frac{R}{L}V_{out} = V_1\,\delta(t)$$

$$RC\frac{d^2 z}{dt^2} + \frac{dz}{dt} + \frac{R}{L}z = V_1\,u(t)$$

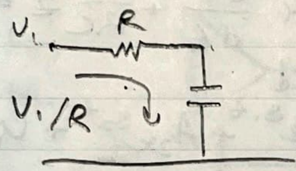
$$z(\infty) = V_1 \cdot \frac{L}{R}$$

e.g. Overdamped $z(t) = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_1\frac{L}{R}\right)u(t)$.

$$\frac{dz}{dt} = V_{out}(t) = \left(A_1 s_1 e^{s_1(t)} + A_2 s_2 e^{s_2 t}\right)u(t)$$

Find initial conditions: $V_{out}(0^+) = 0$

$$C\left.\frac{dV_{out}}{dt}\right|_{0^+} = \frac{V_1}{R} \Rightarrow \left.\frac{dV_{out}}{dt}\right|_{0^+} = \frac{V_1}{CR}$$



Alternative Approach: Using I_L

$$R\left[C\frac{d}{dt}\left(L\frac{dI_L}{dt}\right) + I_L\right] + L\frac{dI_L}{dt} = V_1\,u(t)$$

$$\frac{d^2 I_L}{dt^2} + \frac{1}{RC}\frac{dI_L}{dt} + \frac{1}{LC}I_L = \frac{V_1}{RCL}u(t)$$

Avoids the $\delta(t)$...

e.g. Overdamped $I_L(t) = \left(A_1 e^{s_1 t} + A_2 e^{s_2 t} + \frac{V_1}{R}\right)u(t)$

$$I_L(0^+) = 0$$

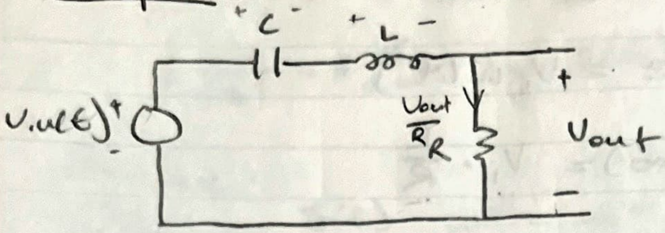
$$L\left.\frac{dI_L}{dt}\right|_{0^+} = V_{out}(0^+) = 0$$

$$\Rightarrow \left.\frac{dI_L}{dt}\right|_{0^+} = 0$$

$$\frac{1}{C}\int \frac{V_{out}}{R}\,dt$$

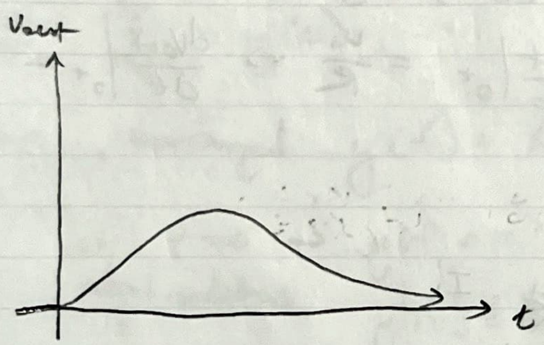
Example

$$L\frac{d}{dt}\left(\frac{V_{out}}{R}\right)$$



$$\frac{1}{C}\int \frac{V_{out}}{R}\,dt + \frac{L}{R}\frac{dV_{out}}{dt} + V_{out} = V_1 u(t)$$

$$\frac{L}{R}\frac{d^2V_{out}}{dt^2} + \frac{dV_{out}}{dt} + \frac{1}{RC}V_{out} = V_1\,\delta(t)$$



$$\frac{L}{R}\frac{d^2z}{dt^2} + \frac{dz}{dt} + \frac{1}{RC}z = V_1 u(t)$$

e.g. Overdamped

$$z(t) = A_1e^{s_1t} + A_2e^{s_2t} + V_1RC$$

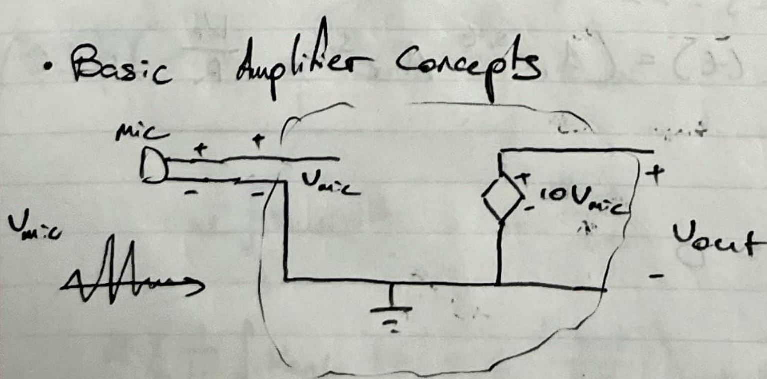
$$V_{out}(t) = A_1s_1e^{s_1t} + A_2s_2e^{s_2t}$$

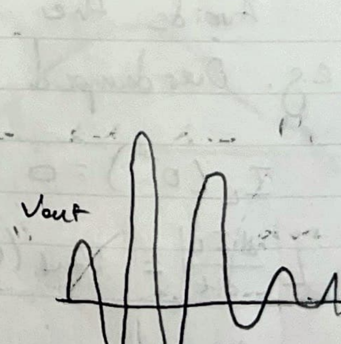
Initial conditions: $V_{out}(0^+) = 0$

$L\frac{d}{dt}\left(\frac{V_{out}}{R}\right)\Big|_{0^+} = V_1$ (Voltage on cap & resistor = 0)

$\Rightarrow \frac{d}{dt}(V_{out})\Big|_{0^+} = V_1\cdot\frac{R}{L}$

- Basic Amplifier Concepts

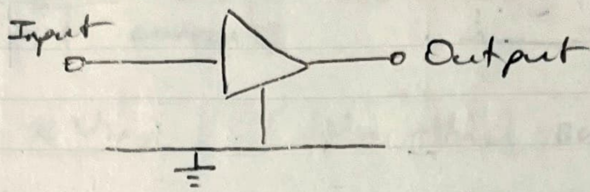




↳ zero input current

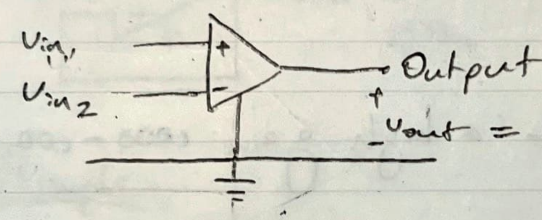
Notes: ① Voltage gain (A_o)

Output / Input

② 

③ Amplifier has a zero input current

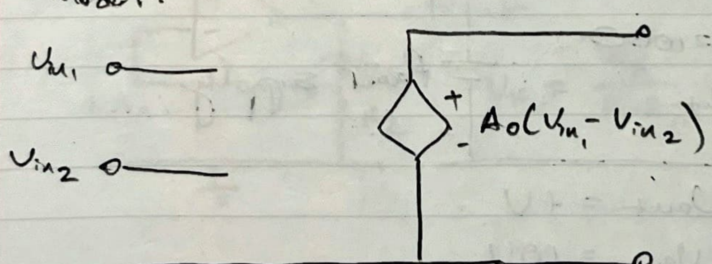
- Operational Amplifier



$V_{out} = A_o(V_{in_1} - V_{in_2})$

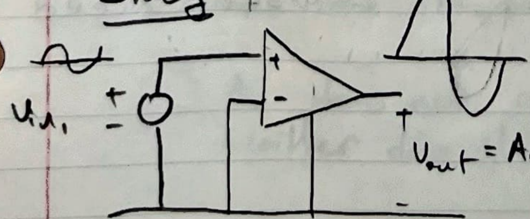
↑ Voltage Gain

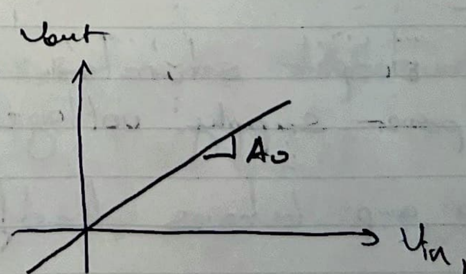
Model:

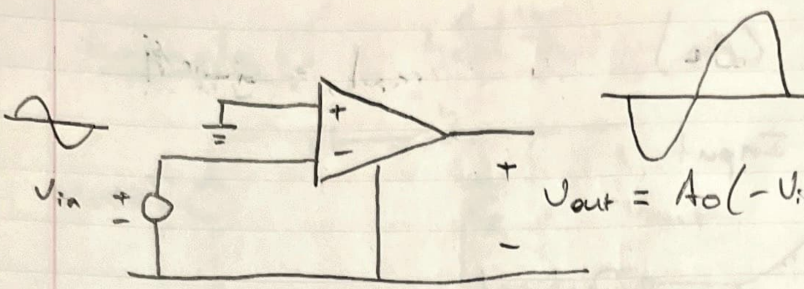


∴ Op Amp has zero input currents.

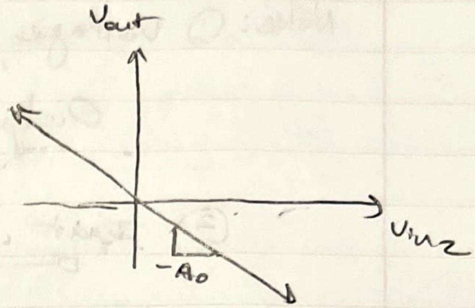
<u>Study</u>

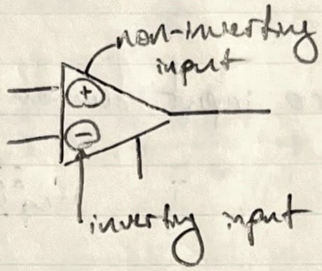






$V_{out} = A_0(-V_{in_2})$

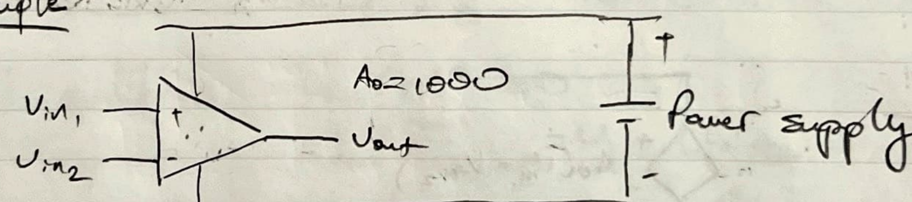




• Observation

Op amps typically have a high gain: 1000 – 100,000.

<u>Example</u>



$V_{in_1} - V_{in_2} = 1mV \Rightarrow V_{out} = 1V$

$V_{in_1} - V_{in_2} = 10mV \Rightarrow V_{out} = 10V$

$V_{in_1} - V_{in_2} = 100mV \Rightarrow V_{out} = 100V$!!

↳ cannot generate because Power supply <100V

Op amp output saturates (i.e. it cannot exceed its power supply voltage)

⇒ Op amp behaves badly.

In a well-designed circuit $|V_{in_1} - V_{in_2}|$ is small

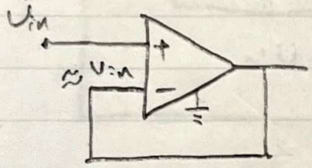
Ideal Op Amp

① No input currents

② $V_{in_1} \approx V_{in_2}$ $\quad (|V_{in_1} - V_{in_2}|$ small$)$

$\Rightarrow$ Simpler analysis

Example



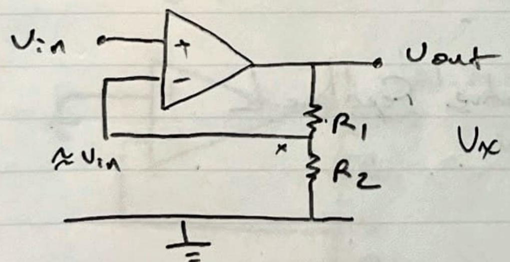
$V_{out} = ?$

$\approx V_{in}$

Unity Gain Buffer

Example

Non-inverting Amplifier



$$V_x = \frac{R_2}{R_1 + R_2} V_{out} \approx V_{in}$$

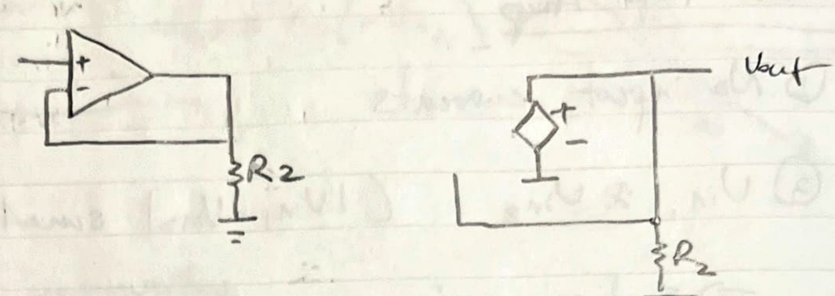
$$V_{out} = \left(1 + \frac{R_1}{R_2}\right) V_{in} \qquad \text{Gain} = 1 + \frac{R_1}{R_2}$$

Observations

① V_{out} has the same sign as $V_{in} \Rightarrow$ Noninverting

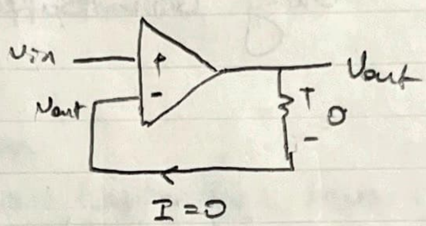
② A_0 does not appear in the result
Neither does the op amp equivalent

3. what happens if $R_1 \to 0$?



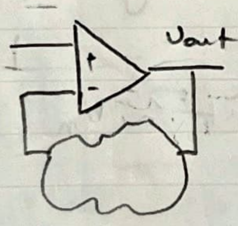
$V_{out} = V_{in}$

4. what happens if $R_2 \to \infty$



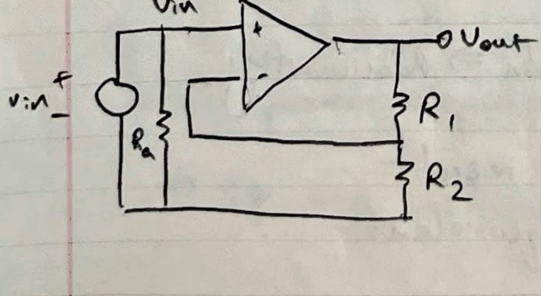
$V_{out} = V_{in}$

5. Feedback

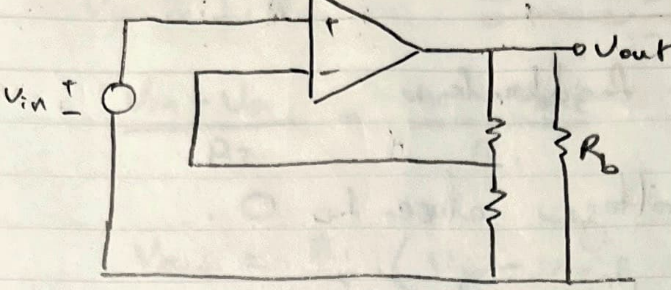


⇒ Negative Feedback

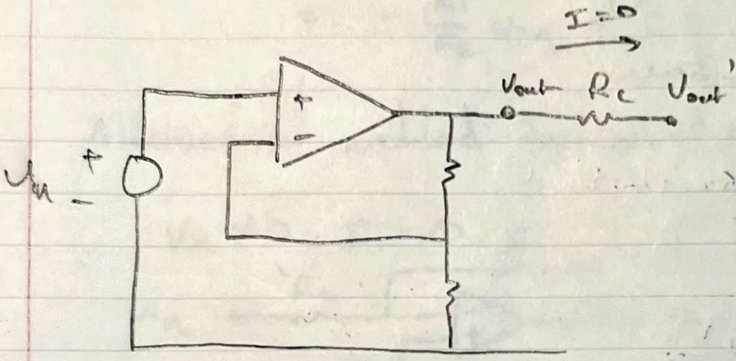
<u>Twists on the Non-inverting amplifier</u>



R_a makes no difference
(Voltage at non-inv. input still v_{in})

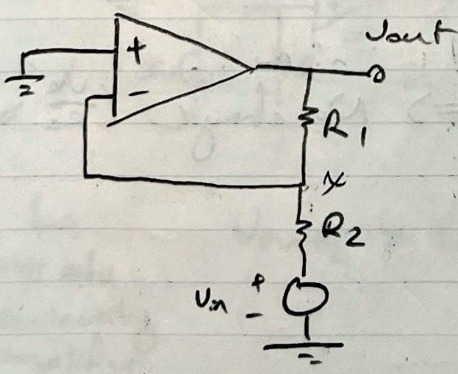


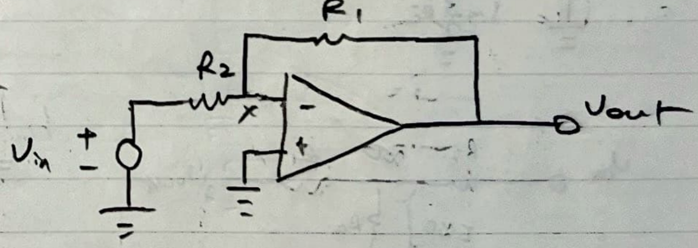
R_b makes no difference because V_{out} is always fixed by dependent voltage source (in model)



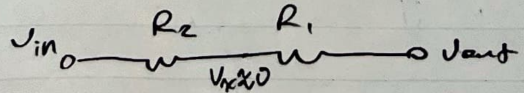
R_c makes no difference ($V_{out}' = V_{out}$) because $I = 0$.

· Inverting Amplifier





$V_x \approx 0$.



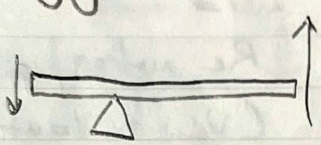
$$\frac{V_{in} - 0}{R_2} = \frac{0 - V_{out}}{R_1} \Rightarrow \boxed{V_{out} = \frac{-R_1}{R_2} V_{in}}$$

Observations

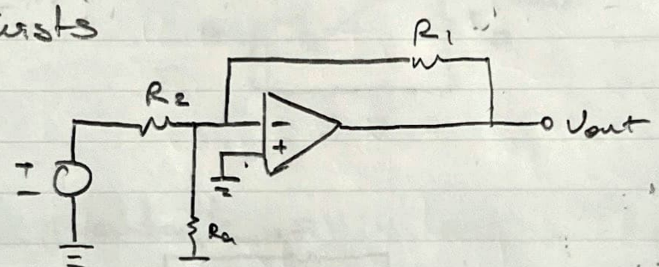
① Circuit has negative feedback.

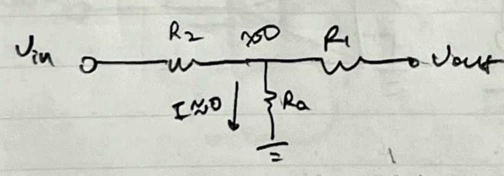
② Node x has a voltage close to 0.
↳ called a "virtual" ground

③ Analogy with see saw



· Twists

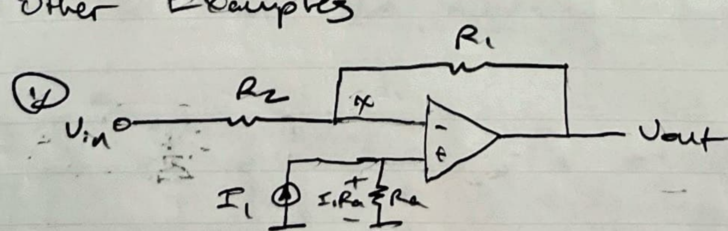




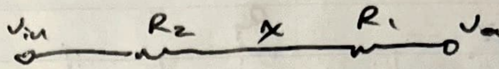
$\Rightarrow$ No change ($\frac{V_x}{R_a}$ is small)

· Other Examples

①



$V_x \approx I_1 R_a$



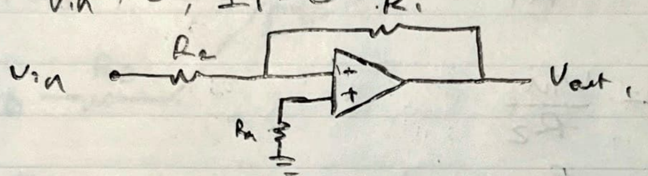
$$\frac{V_{in} - V_x}{R_2} = \frac{V_x - V_{out}}{R_1}$$

$$V_{out} = \frac{R_1}{R_2}\left(V_x - V_{in}\right) + V_x$$

$$= -\frac{R_1}{R_2} V_{in} + \left(1 + \frac{R_1}{R_2}\right)\left(I_1 R_a\right)$$

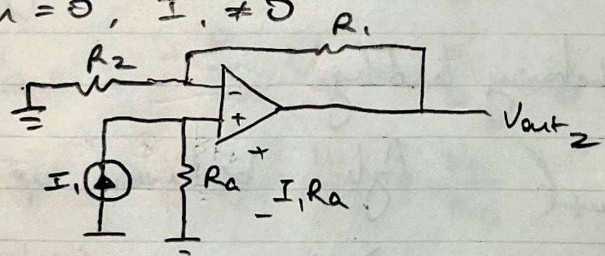
Alternative method: Superposition

$V_{in} \neq 0, \; I_1 = 0$



$$V_{out_1} = -\frac{R_1}{R_2} V_{in}$$

$V_{in} = 0, \; I_1 \neq 0$

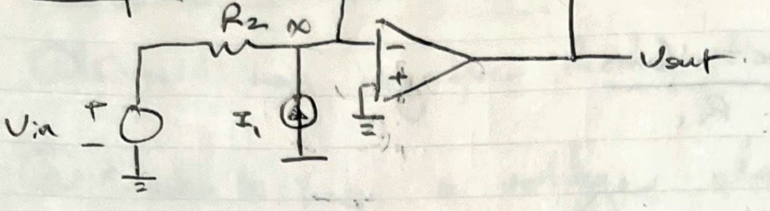


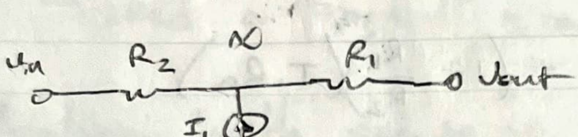
↳ maps onto non-inverting amplifier

$$V_{out_2} = \left(1 + \frac{R_1}{R_2}\right)\left(I_1 R_a\right)$$

$$V_{out} = V_{out_1} + V_{out_2} = -\frac{R_1}{R_2} V_{in} + \left(1 + \frac{R_1}{R_2}\right)\left(I_1 R_a\right)$$

Example





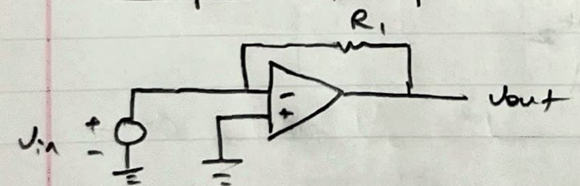
$V_x \approx 0$

$$\frac{V_{in} - 0}{R_2} + I_1 = \frac{0 - V_{out}}{R_1}$$

$$\frac{V_{out}}{R_1} = -I_1 - \frac{V_{in}}{R_2}$$

$$V_{out} = -I_1 R_1 - \frac{R_1}{R_2} V_{in}$$

Example Op amps behaving badly:

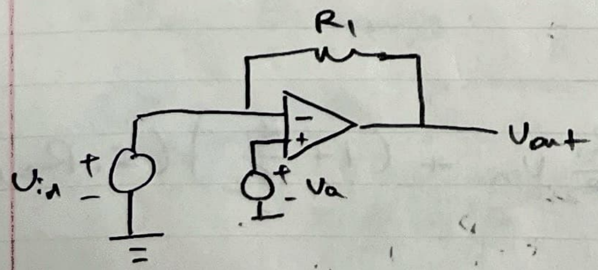


fight between op amp & V_{in}.

$\Rightarrow$ V_{in} wins (ideal)

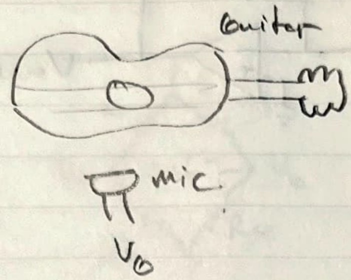
$\Rightarrow$ V_{out} huge

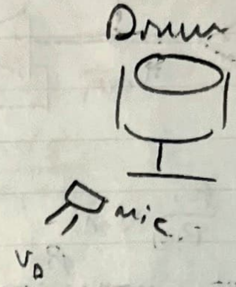
$\rightarrow$ saturates



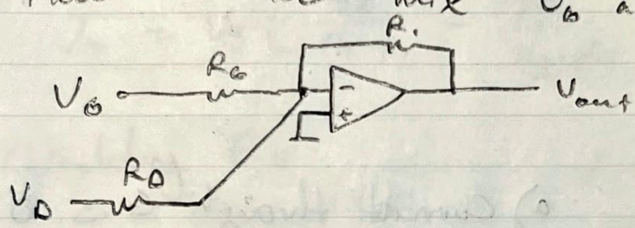
$\Rightarrow$ op amp saturates

Adder (Summer)





How do we "mix" V_G and V_D? $(V_G + V_D)$.

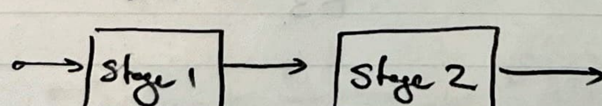


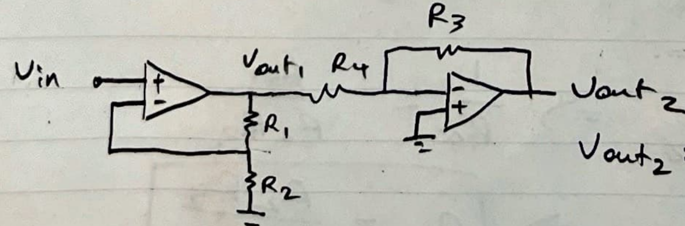
Easy to see w/ superposition:

$$V_{out} = -\frac{R_1}{R_G} V_G - \frac{R_1}{R_D} V_D$$

$$= -R_1\left(\frac{V_G}{R_G} + \frac{V_D}{R_D}\right)$$

Cascaded Stages



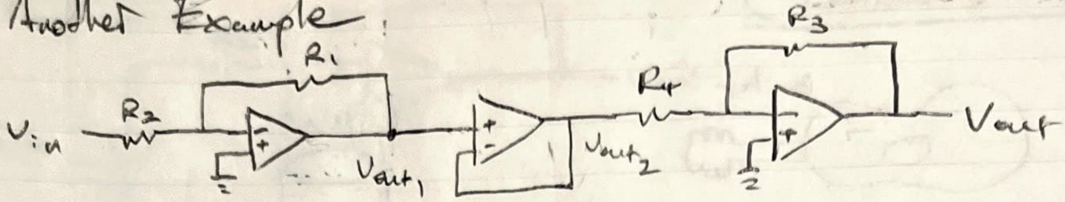


$$V_{out_2} = -\frac{R_3}{R_4} V_{out_1}$$

$$= -\frac{R_3}{R_4}\left(1 + \frac{R_1}{R_2}\right) V_{in}$$

$$V_{out_1} = V_{in}\left(1 + \frac{R_1}{R_2}\right)$$

Another Example



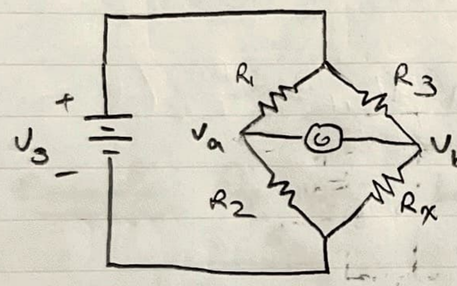
$V_{out_1} = -\frac{R_1}{R_2} V_{in}$

$V_{out_2} = V_{out_1}$

$V_{out} = -\frac{R_3}{R_4} V_{out_2} = -\frac{R_3}{R_4}\left(-\frac{R_1}{R_2} V_{in}\right)$

Practice Problems

1)



a) Current through G is 0. $R_x = ?$

$\Rightarrow$ No voltage difference btwn V_a & V_b.

$\Rightarrow \quad V_a = V_b = V$

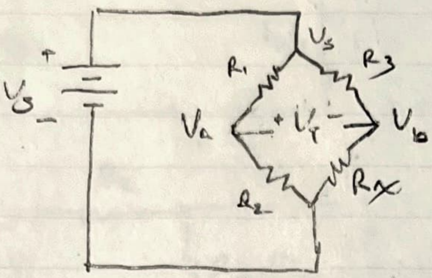
$\Rightarrow \quad \frac{V_s - V}{R_1} = \frac{V}{R_2}$ and $\frac{V_s - V}{R_3} = \frac{V}{R_x}$

$V_s - V = \frac{R_1}{R_2} V$

$\Rightarrow \quad \frac{\frac{R_1}{R_2} V}{R_3} = \frac{V}{R_x} \Rightarrow \frac{1}{R_x} = \frac{R_1}{R_2 R_3}$

$\Rightarrow \boxed{R_x = \frac{R_2 R_3}{R_1}}$

b) Find the Thevenin equivalent seen by G.



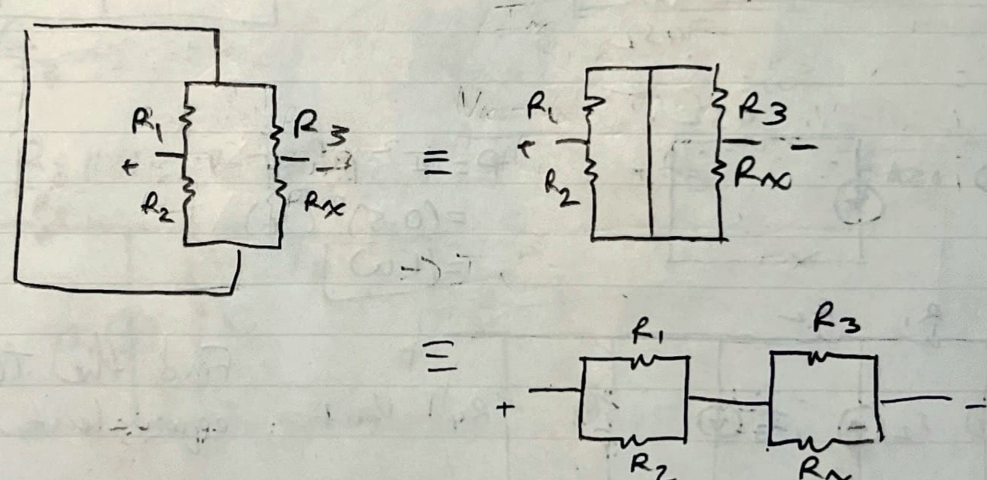
$$V_a - V_b = V_T$$

$$V_a = \frac{V_s}{R_1+R_2} \cdot R_2$$

$$V_b = \frac{V_s}{R_3+R_x} \cdot R_x$$

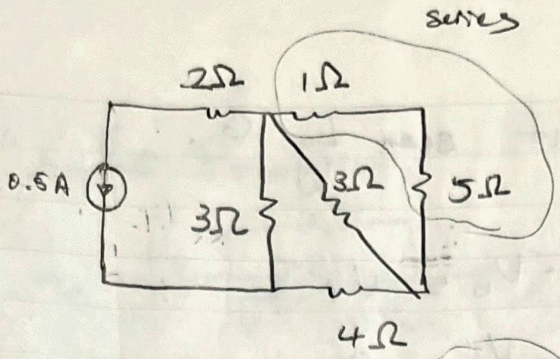
$$V_T = V_s\left(\frac{R_2}{R_1+R_2} - \frac{R_x}{R_3+R_x}\right)$$

Calculating R_T:

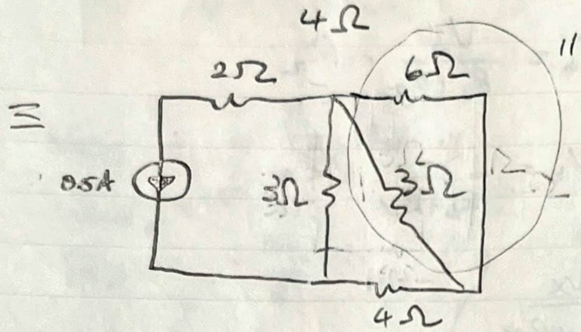


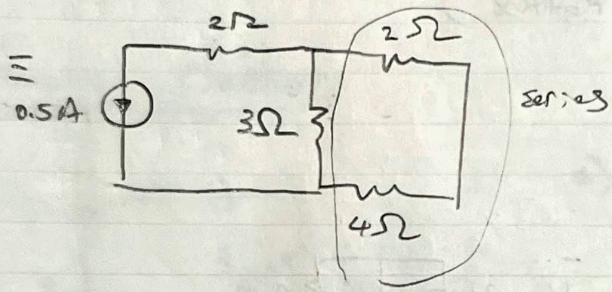
$$\Rightarrow R_T = \frac{R_1 R_2}{R_1+R_2} + \frac{R_3 R_x}{R_3+R_x}$$

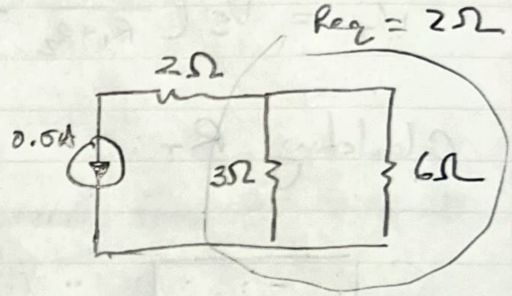
2)

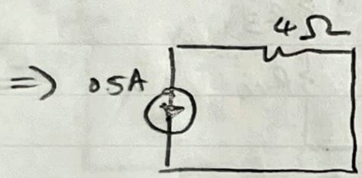


Power delivered by current source =?







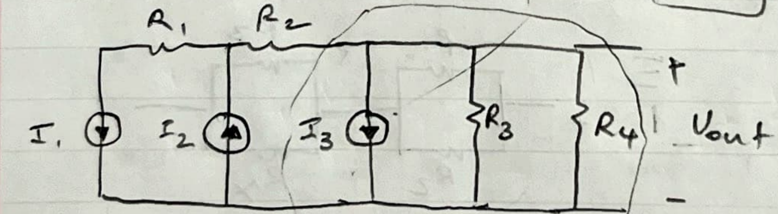


$P = I^2 R$

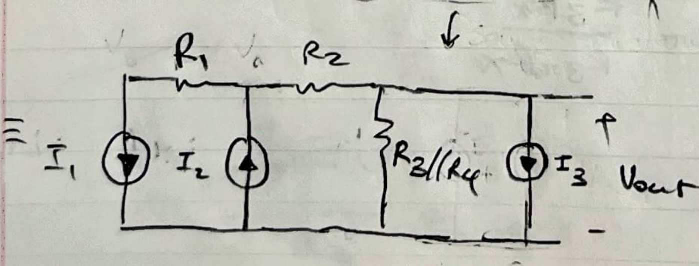
$= (0.5)^2 (4)$

$= \boxed{1\,W}$

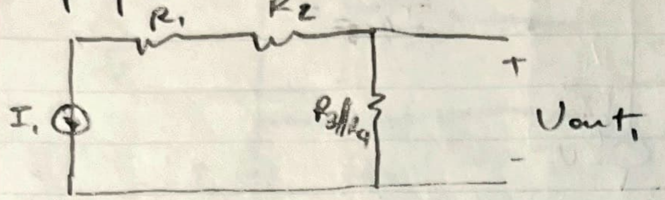
3)



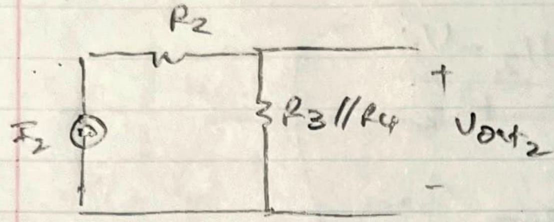
Find the Thevenin equivalent.



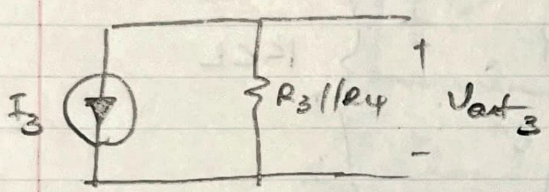
Superposition:



$$V_{out_1} = -I_1(R_3 \| R_4)$$



$$V_{out_2} = I_2(R_3 \| R_4)$$

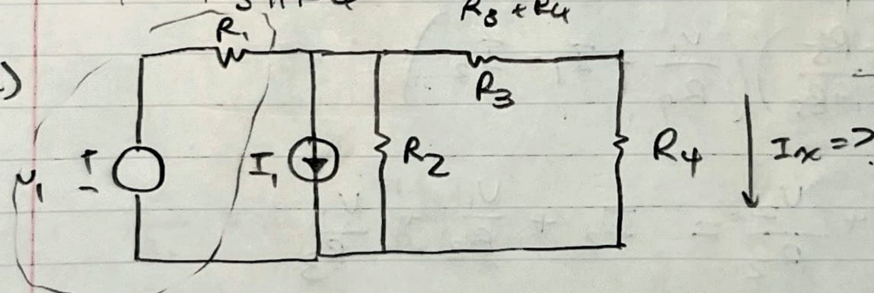


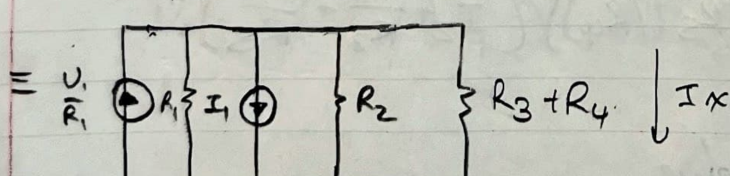
$$V_{out_3} = -I_3(R_3 \| R_4)$$

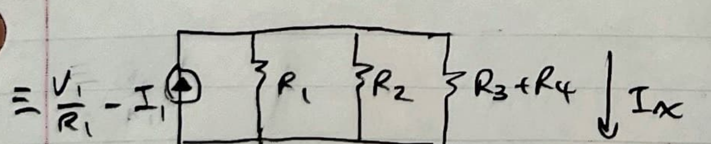
$$V_T = (I_2 - I_1 - I_3)\left(\frac{R_3 R_4}{R_3 + R_4}\right)$$

$$R_T = R_3 \| R_4 = \frac{R_3 R_4}{R_3 + R_4}$$

4)





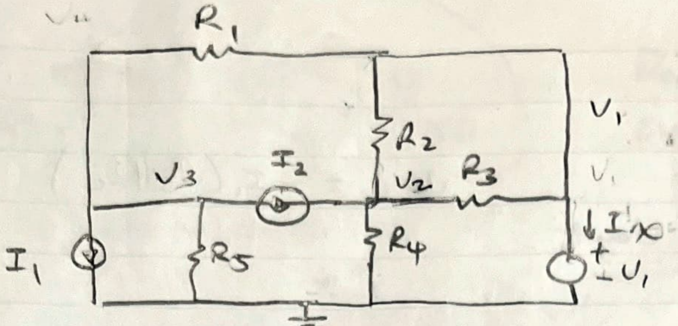


↳ current divider

$$I_x = \left(\frac{V_1}{R_1} - I_1\right)\left(\frac{R_1 \| R_2}{R_1 \| R_2 + R_3 + R_4}\right)$$

$$= \left(\frac{V_1}{R_1} - I_1\right)\left(\frac{R_1 R_2}{R_1 R_2 + (R_3 + R_4)(R_1 + R_2)}\right)$$

5)



$I_x = ?$

$$① \; I_x = \frac{V_2 - V_1}{R_3} + \frac{V_3 - V_1}{R_1} + \frac{V_2 - V_1}{R_2}$$

$$② \; I_2 + \frac{V_1 - V_2}{R_2} + \frac{V_1 - V_2}{R_3} = \frac{V_2}{R_4}$$

$$③ \; \frac{V_3}{R_5} + I_1 + I_2 = \frac{V_1 - V_3}{R_1}$$

KCL

$$③ \; \frac{V_3}{R_5} + \frac{V_3}{R_1} = \frac{V_1}{R_1} - I_1 - I_2$$

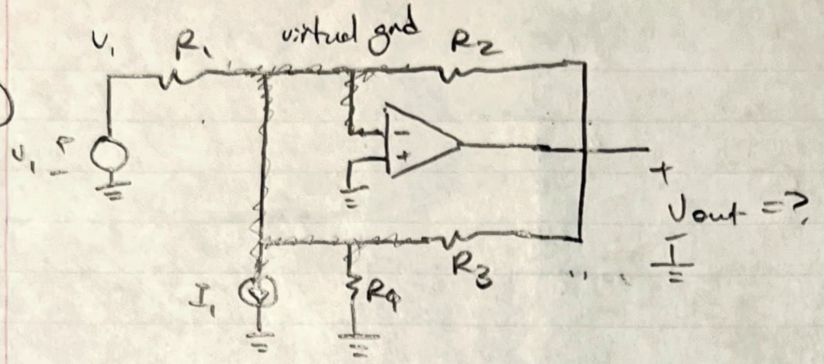
$$V_3 = \left(\frac{R_1 R_5}{R_1 + R_5}\right)\left(\frac{V_1}{R_1} - I_1 - I_2\right)$$

$$② \; \frac{V_2}{R_4} + \frac{V_2}{R_3} + \frac{V_2}{R_2} = I_2 + \frac{V_1}{R_2} + \frac{V_1}{R_3}$$

$$V_2 = (R_2 \| R_3 \| R_4)\left(I_2 + \frac{V_1}{R_2} + \frac{V_1}{R_3}\right)$$

① $I_x = ?$ plug & solve.

6)



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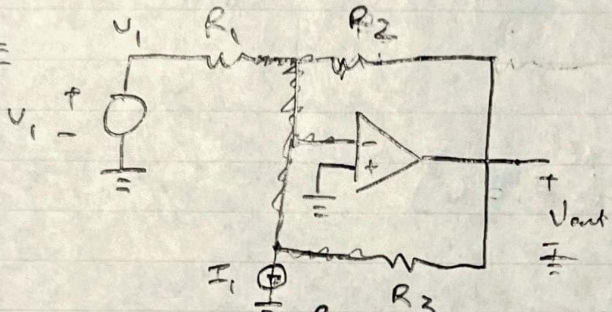

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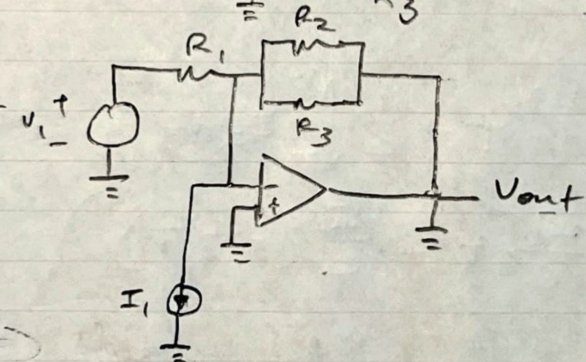

$$\frac{v_1}{R_1} = I_1 - \frac{V_{out}}{R_2 \| R_3}$$

$$V_{out} = \left(I_1 - \frac{v_1}{R_1} \right) \left(\frac{R_2 R_3}{R_2 + R_3} \right)$$