习题4-4.

$-1 = C - B\frac{1}{2}$

3. $\int \frac{\frac{1}{2}(2x-2)+2}{x^2-2x+5}\,dx = \frac{1}{2}\ln|x^2-2x+5| + \cancel{2}\arctan\frac{x-1}{2} + C$

6. $\cancel{\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{(x+1)^2} + \frac{B}{x-1}}$ $\quad \frac{x^2+1}{(x+1)^2(x-1)} = \frac{Ax+C}{(x+1)^2} + \frac{B}{x-1}$ $\quad \begin{cases} B = \frac{1}{2} \\ A = \frac{1}{2} \\ C = -\frac{1}{2} \end{cases}$

$\int \frac{x^2+1}{(x+1)^2(x-1)}\,dx = \int \frac{\frac{1}{2}x-\frac{1}{2}}{(x+1)^2} + \frac{\frac{1}{2}}{x-1}\,dx = \frac{1}{4}\int \frac{2(x+1)-4}{(x+1)^2}\,dx + \frac{1}{2}\int \frac{1}{x-1}\,dx$

$= \frac{1}{2}\ln(x+1) + \frac{1}{x+1} + \frac{1}{2}\ln(x-1) + C$

8. $\int \frac{x^5+x^4-8}{x^3-x}\,dx = \int \cancel{x^5+x^4-x}\; x^2+x+1+\frac{1}{x}\cancel{\frac{x^2+x-8}{x^3-x}} + \frac{x-7}{x^3-x}\,dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln x + \int \frac{\overset{x-7}{\cancel{x^2+x-8}}}{x(x+1)(x-1)}\,dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln x + \int \frac{-4}{x+1} + \frac{7}{x} + \frac{-3}{x-1}\,dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + 8\ln x - 4\ln(x+1) - 3\ln(x-1) + C$

9. $\frac{1}{(x^2+1)(x^2+x)} = \frac{A}{x^2+1} + \frac{B}{x} + \frac{C}{x+1} \Rightarrow \begin{cases} A = -1 \\ B = 1 \\ C = -\frac{1}{2} \end{cases}$

$\therefore \int \frac{dx}{(x^2+1)(x^2+x)} = -\arctan x + \ln x - \frac{1}{2}\ln(x+1) + C$

13. $\int \frac{-x^2-2}{(x^2+x+1)^2}\,dx = \int \frac{-x^2-x-1\ \cancel{+x-1}\ \frac{1}{2}(2x+1)-\frac{3}{2}}{(x^2+x+1)^2}\,dx$

$\cancel{= -\ln(x^2+x+1) - \int \frac{1}{x^2+x+1}}$

$= -\frac{2}{\sqrt{3}}\arctan\left[\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right] - \frac{1}{2}\cancel{\ln}\frac{1}{x^2+x+1} - \frac{3}{2}\int \frac{dx}{(x^2+x+1)^2}$

$\cancel{\int \frac{d(x+\frac{1}{2})}{(x^2+x+1)^2} = \frac{x+\frac{1}{2}}{(x^2+x+1)^2} + \int \frac{(2x+1)(x+\frac{1}{2})}{(x^2+x+1)^3}\,dx = \frac{x+\frac{1}{2}}{(x^2+x+1)^2} + 2\int \frac{x^2+x+1-\frac{3}{4}}{(x^2+x+1)^3}}$

$= -\frac{2}{\sqrt{3}}\arctan\left[\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right] - \frac{1}{2}\cdot\frac{1}{x^2+x+1} - \frac{3}{2}\left[\frac{x+\frac{1}{2}}{\frac{3}{2}(x^2+x+1)} + \frac{2}{3}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right] + C$

1.

$\therefore$ 原式 $= -\frac{4}{\sqrt{3}}\arctan\left[\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right] - \frac{x+1}{x^2+x+1} + C$

15. $\int \frac{dx}{3+\cos x} = \cancel{\int \frac{dx}{2\cos^2\frac{x}{2}+2}} = \int \frac{1}{3+\frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2}\,du.$

$= \int \frac{2}{4+2u^2}\,du = \int \frac{1}{u^2+2}\,du.$

$= \frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan\frac{x}{2}}{\sqrt{2}}\right)$

17. $\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2}{1+u^2}du}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}} = \int \frac{2}{1+u^2+2u+1-u^2}\,du.$

$= \int \frac{1}{u+1}\,du = \ln|u+1| + C = \ln|\tan\frac{x}{2}+1| + C$

18. $\int \frac{dx}{2\sin x-\cos x+5} = \int \frac{\frac{2}{u^2+1}du}{\frac{4u}{u^2+1}+\frac{u^2-1}{u^2+1}+5} = \int \frac{2\,du}{4u+u^2-1+5u^2+1}$

$= \int \frac{du}{3u^2+2u} = \frac{1}{2}\int \frac{1}{u} + \frac{-3}{3u+2}\,du$

$= \frac{1}{2}\ln|u| - \frac{1}{2}\ln|3u+2| + C.$

$= \frac{1}{2}\ln|\tan\frac{x}{2}| - \frac{1}{2}\ln|3\tan\frac{x}{2}+2| + C.$

20 $\int \frac{(\sqrt{x})^3-1}{\sqrt{x}+1}\,dx = \int \frac{(\sqrt{x})^3-1}{\sqrt{x}+1} \frac{d\sqrt{x}}{\frac{1}{2\sqrt{x}}}$，令 $\sqrt{x}=u$，则 $x=u^2$.

$m=\sqrt{x}+1$

原 $= \int \frac{2u(u^3-1)}{u+1}\,du = \int \frac{2(m-1)((m-1)^3-1)}{m}\,dm =$

$= 2\int m^3-4m^2+6m-5+\frac{2}{m}\,dm$

$= \frac{1}{2}m^4 - \frac{8}{3}m^3 + 6m^2 - 5m + 4\ln m$

21. $$\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\,dx = \int \frac{x+2-2\sqrt{x+1}}{x}\,dx = x+2\ln x - 2\int \frac{\sqrt{x+1}}{x}\,dx$$

① 又 $$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{\sqrt{x+1}}{x}\,\frac{d\sqrt{x+1}}{\frac{1}{2\sqrt{x+1}}}$$ 令 $\sqrt{x+1}=u$. 则 $$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{2u^2}{u^2-1}\,du$$

$$= \int 2+\frac{2}{u^2-1}\,du = 2u + \int \frac{2}{u^2-1}\,du = 2u + 2\cdot\left[\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right] + C$$

则原式 $$= x+2\ln x - 2\left[2\sqrt{x+1} + \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C\right]$$

② 又 $$\int \frac{\sqrt{x+1}}{x}\,dx = \int \frac{2u^2}{u^2-1}\,du = \int 2+\frac{1}{u-1}-\frac{1}{u+1}\,du = 2u+\ln|u-1|-\ln|u$$

$$\frac{t^2}{(1-t^2)^3} = \frac{1}{(1-t^2)^3} - \frac{1}{(1-t^2)^2}$$

$$\int x^2\sqrt{x^2+t} = \frac{x}{8}(2x^2+t)\sqrt{x^2+t} - \frac{t^2}{8}\ln|x+\sqrt{x^2+t} + C$$

$$\frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^2}\cancel{\int \frac{dx}{x^2+a^2}}\ \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\ln(x+\sqrt{x^2+1}) \rightarrow \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{2x}{2\sqrt{x^2+1}}\right.$$

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习题5-1

4.(3) $\int_{-1}^{2}|x|dx = \int_{-1}^{0}|x|dx + \int_{0}^{2}|x|dx$

$\int_{-1}^{0}|x|dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\Delta x_i$，其中 $\lambda = \max\{\Delta x_i\}$ $i=1,2\cdots,n$

$= \lim_{\lambda\to 0}\sum_{i=1}^{n}\frac{n-i}{n}\cdot\frac{1}{n} = \cancel{\sum_{n\to\infty}}\lim_{n\to\infty}\frac{n^2-\frac{n(n+1)}{2}}{n^2} = \lim_{n\to\infty}\frac{\frac{n^2}{2}-\frac{n}{2}}{n^2}$

$= \frac{1}{2}$

同理 $\int_{0}^{2}|x|dx = 2$ ，原式 $=\frac{5}{2}$

(4) $\int_{-3}^{3}\sqrt{9-x^2}\,dx$，~~将[-3,3]分为n份~~，

由于 $\sqrt{9-x^2}$ 为偶函数 $\sqrt{9-x^2} = \sqrt{9-(-x)^2}$

$\therefore \int_{-3}^{3}\sqrt{9-x^2}dx = 2\int_{0}^{3}\sqrt{9-x^2}\,dx$ ，将[0,3]分为n份

则 $\int_{0}^{3}\sqrt{9-x^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\cdot\sqrt{9-(\frac{3i}{n})^2} = \lim_{n\to\infty}\frac{9}{n^2}\sum_{i=1}^{n}\sqrt{n^2-i^2}$

根据几何意义，设 $y=\sqrt{9-x^2}$，$y^2 = 9-x^2$

$x^2+y^2=9$ 其中 $y\geq 0$

$\therefore$ 函数为一个圆的一半，半径为3.

$\int_{-3}^{3}\sqrt{9-x^2}\,dx = \frac{9}{2}\pi$

8.

(3) $\int_{3}^{-1}g(x)dx = -\int_{-1}^{3}g(x)dx = -3$.

(4) $\int_{-1}^{3}[\frac{4}{5}f(x)+\frac{3}{5}g(x)]dx = \frac{4}{5}\int_{-1}^{3}f(x) + \frac{3}{5}\int_{-1}^{3}g(x)dx$

$= \frac{4}{5}\cdot 4 + \frac{3}{5}\cdot 3 = 5$

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13. (1) 当 $x \in [0,1]$，$x^2 > x^3$

$\therefore \int_0^1 x^2 dx > \int_0^1 x^3 dx$

(5) 当 $x \in (0,1)$，$e^x > x+1$

$\therefore \int_0^1 e^x dx > \int_0^1 (1+x) dx$