习题5-2

3. $\int_0^y e^t dt + \int_0^x \cos t dt = 0$

$e^y - 1 + \cos x - 1 = 0$

$e^y \cdot \frac{dy}{dx} - \sin x = 0$

$$\frac{dy}{dx} = \frac{\sin x}{e^y} = \frac{\sin x}{2-\cos x}$$

4. $I(x) = -\frac{1}{2}\int_0^x -2te^{-t^2} dt = -\frac{1}{2}(e^{-x^2} - 1)$

$I'(x) = x e^{-x^2} = 0$ 时 $x = 0$

$I'' = e^{-x^2} + x e^{-x^2}(-2x)$. 当 $x=0$, $I'' = 1$

$\therefore$ $x=0$ $I(x)\min = 0$

5.(3) $\frac{d}{dx}\int_{\sin x}^{\cos x} \cos(\pi t^2) dt = \frac{d\int_0^{\cos x} \cos(\pi t^2) dt}{dx} - \frac{d\int_0^{\sin x} \cos(\pi t^2)}{dx}$

$= \cos(\pi \cos^2 x) - \cos(\pi \sin^2 x)$

8(8) $\int_{-1}^0 \frac{3x^4+3x^2+1}{x^2+1} dx = \int_{-1}^0 3x^2 + \frac{1}{x^2+1} dx$

$F(x) = x^3 + \arctan x$.

$\therefore \int_{-1}^0 \frac{3x^4+3x^2+1}{x^2+1} = 1 + \frac{\pi}{4}$

(11) $\int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} -\sin x \, dx = 4$

(12) $\int_0^2 f(x) dx$, $f(x) = \begin{cases} x+1 , & x \le 1 \\ \frac{1}{2}x^2 , & x > 1 \end{cases}$

$\therefore \int_0^2 f(x) dx = \int_0^1 (x+1) dx + \int_1^2 \frac{1}{2}x^2 dx = \frac{8}{3}$

10.(2) 证 $\int_{-\pi}^{\pi} \sin kx\,dx = 0 \quad (k \in N_+)$

证明：$\sin kx$ 为奇函数

$\therefore \int_{-\pi}^{\pi} \sin kx\,dx = \int_{0}^{\pi} \sin kx\,dx - \int_{0}^{-\pi} \sin kx\,dx$

$= \int_{0}^{\pi} \sin kx\,dx - \int_{0}^{\pi} \sin(-kx)\,d(-x)$

$= \int_{0}^{\pi} \sin kx\,dx - \int_{0}^{\pi} \sin ku\,du \qquad u = (-x)$

$= \int_{0}^{\pi} \sin kx\,dx - \int_{0}^{\pi} \sin kx\,dx = 0.$

11. (1) $\lim\limits_{x\to 0} \dfrac{\int_0^x \cos t^2\,dt}{x}$

$\because \lim\limits_{x\to 0} \int_0^x \cos t^2\,dt = 0$

$\therefore$ 原式 $= \lim\limits_{x\to 0} \dfrac{\cos x^2}{1} = 1$

(2) $\lim\limits_{x\to 0} \dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x t e^{2t^2}\,dt}$

$\because$ 由题，分子分母趋于0

$\therefore$ 原式 $= \lim\limits_{x\to 0} \dfrac{2\int_0^x e^{t^2}\,dt \times e^{x^2}}{x e^{2x^2}} = \lim\limits_{x\to 0} \dfrac{2\int_0^x e^{t^2}\,dt}{x e^{x^2}}$

$= \lim\limits_{x\to 0} \dfrac{2 e^{x^2}}{e^{x^2} + 2x^2 e^{x^2}} = 2.$

12.

设 $f(x) = \begin{cases} x^2, & x \in [0,1), \\ x, & x \in [1,2] \end{cases}$

$\because f(1) = \lim\limits_{x\to 1^-} f(x)$

$\therefore f(x)$ 连续

$\therefore \Phi(x)$ 存在且连续

习题5-3

1.(4) $f(\theta)=(1-\sin^3\theta)d\theta$ $\quad F(x)=\int f(\theta)d\theta$

$$F(x)=\int 1-\sin^3\theta = \int d\theta+\frac{1}{4}\int \cancel{4\sin^3\theta}\ 3\sin\theta-4\sin^3\theta d\theta-\int 3\sin\theta d\theta$$

$$=\theta+\frac{1}{4}\int \sin 3\theta d\theta-\int 3\sin\theta d\theta$$

$$=\theta+3\cos\theta \cancel{+}-\frac{1}{12}\cos 3\theta+C$$

$$\int_0^{\pi}(1-\sin^3\theta)d\theta=\cancel{X}[F(x)]_0^{\pi}=\cancel{\frac{35}{12}}+\pi-3+\frac{1}{12}+3-\frac{1}{12}=\pi$$

(10) $f(x)=\frac{1}{x^2\sqrt{1+x^2}}$ $\quad \int f(x)dx=F(x)$

$$F(x)=\int\frac{1}{x^2\sqrt{1+x^2}}dx=2\int\frac{1}{x^2}d\sqrt{1+x^2}$$ ，设 $u=\sqrt{1+x^2}$

$$F(x)=2\int\frac{1}{u^2-1}du.=-\int\frac{1}{1+u}\ \cancel{+}\ \frac{1}{\cancel{1-u}\ u-1}du$$

$$=-\ln(1+u)+\ln(\frac{u-1}{\cancel{1-u}})\ \cancel{dx}+C$$

$$=\ln(\frac{\cancel{1-u}\ u-1}{1+u})+C$$

$$\int_1^{\sqrt{3}}f(x)dx=[F(x)]_1^{\sqrt{3}}=-\ln 3-\ln(\frac{\sqrt{2}-1}{\sqrt{2}+1})$$

当 $x=\sqrt{3}$, $u=2$

$x=1$ $\quad u=\sqrt{2}$.

(16) $\int_1^{e^2}\frac{dx}{x\sqrt{1+\ln x}}=F(x)$ ， $f(x)=\frac{1}{x\sqrt{1+\ln x}}$

$$F(x)=2\sqrt{1+\ln x}+C$$

$$\therefore \int_1^{e^2}f(x)dx=[F(x)]_1^{e^2}=2\sqrt{3}-2$$

7. $f(x)=xe^{x^2}$. $x=\cancel{f}(\varphi(f(x)))$. $F(x)=\int_0^{x}\varphi(y)dy$

$$F(x)=\int x\,dxe^{x^2}=\int x(e^{x^2}+2x^2e^{x^2})dx$$

3 $$=\int 2xe^{x^2}+2x^3e^{x^2}dx-\frac{1}{2}\int 2xe^{x^2}dx=x^2e^{x^2}-\frac{1}{2}e^{x^2}+C$$

$$\int_0^e \varphi(y)dy = [F(x)]_0^e = e^{e^2+2} - \frac{1}{2}e^{e^2} + \frac{1}{2}$$

8.(4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}dx$, $f(x) = \frac{x}{\sin^2 x}$ $F(x) = \int \frac{x}{\sin^2 x}dx$

$\cancel{F(x) = \int \frac{x}{\sin^2 x}dx = \int \frac{2x}{1-\cos 2x}dx}$

$\cancel{\because d\tan x = \frac{1}{\cos^2 x}dx \quad \therefore dx = \frac{1}{\tan^2 x+1}d\tan x}$

$\cancel{又 \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}}$

$F(x) = \int \frac{x}{\sin^2 x}dx = \int x\, d\cot x = x\cot x - \int \cot x\, dx$

$= x\cot x - \ln|\sin x| + C$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}dx = F(x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{3\sqrt{3}} - \ln\frac{\sqrt{3}}{2} - \frac{\pi}{4} + \ln\frac{\sqrt{2}}{2}$.

8.(9) $f(x) = (x\sin x)^2$ $F(x) = \int f(x)dx$

$F(x) = \int x^2\sin^2 x\, dx = \int x^2 \frac{1-\cos 2x}{2}dx = \frac{1}{6}x^3 - \frac{1}{2}\int x^2\cos 2x\, dx$

$= \frac{1}{6}x^3 - \frac{1}{16}\int (2x)^2\cos 2x\, d2x$. 设 $2x = u$

$\int u^2\cos u\, du = u^2\sin u + 2u\cos u - 2\sin u$

$\therefore F(x) = \frac{1}{6}x^3 - \frac{1}{16}(4x^2\sin 2x + 4x\cos 2x - 2\sin 2x)$

$\int_0^{\pi} (x\sin x)^2 dx = \frac{\pi^3}{6} - \frac{\pi}{4}$

(10) $f(x) = \sin(\ln x)$, $F(x) = \int \sin(\ln x)dx = \int e^{\ln x}\sin(\ln x)d\ln x$

设 $\ln x = u$. $\int e^u \sin u\, du = e^u \cdot (-\cos u) - e^u \cdot (-\sin u) + \int e^u(-\sin u)du$

$\therefore \int e^u\sin u\, du = \frac{e^u(\sin u - \cos u)}{2}$

$\int_1^e f(x)dx = \frac{e(\sin 1 - \cos 1) + 1}{2}$

习题5-4

$\cancel{\frac{1}{2}+\frac{1}{2}x^2+ \quad 1-x^2}$

1.

(4) 收敛

$$\int_0^{+\infty}\frac{dx}{(1+x)(1+x^2)}=\int_0^{+\infty}\frac{-\frac{1}{2}x+\frac{1}{2}}{1+x^2}+\frac{\frac{1}{2}}{1+x}\,dx$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{1}{1+x^2}+\frac{1}{1+x}-\frac{x}{1+x^2}\,dx$$

$$=\frac{1}{2}\left[\arctan x+\ln(1+x)-\frac{1}{2}\ln|x^2+1|\right]_0^{+\infty}$$

$$=\frac{\pi}{4}$$

(5) $\int_0^{+\infty}e^{-pt}\sin wt\,dt\ (p>0,w>0)$

$$=\lim_{m\to+\infty}\int_0^m e^{-pt}\sin wt\,dt$$

$$=\lim_{m\to+\infty}[g(m)]$$

$$=\frac{w}{w^2+p^2}$$

又 $\int_0^m e^{-pt}\sin wt\,\frac{dwt}{w}=$

$$\frac{1}{w}\left[-e^{-pt}\cos wt\right]_0^m-\frac{1}{w}\int_0^m pe^{-pt}\cos wt\,dt$$

$$=\frac{1}{w}\left[1-e^{-pm}\cos wm\right]-\frac{p}{w^2}\left[[e^{-pt}\sin wt]_0^m+\int_0^m pe^{-pt}\sin wt\,dt\right]$$

$$=\frac{1}{w}\left[1-e^{-pm}\cos wm\right]-\frac{p}{w^2}\left[e^{-pm}\sin wt\right]-\frac{p^2}{w^2}\int_0^m e^{-pt}\sin wt\,dt$$

$$\therefore 原式=\frac{1}{w^2+p^2}\left[w-we^{-pm}\cos wm-pe^{-pm}\sin wt\right]$$

$$=g(m)$$

(6) 由于 $x^2+2x+2>0$

$\therefore \frac{1}{x^2+2x+2}$ 连续

$\cancel{\lim_{t\to+\infty}\int_0^t\frac{dx}{x^2+2x+2}}$

$$\int\frac{dx}{x^2+2x+2}=F(x)=\arctan(x+1)+C$$

$\therefore \lim_{t\to+\infty}\int_0^t\frac{dx}{x^2+2x+2}$ 和 $\lim_{t\to-\infty}\int_t^0\frac{dx}{x^2+2x+2}$ 存在

$\therefore$ 原式收敛

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+2x+2}=[F(x)]_{-\infty}^{+\infty}=\pi$$

(9) ~~$f(x)=\int$~~ $f(x)=\frac{x}{\sqrt{x-1}}$

$F(x)=\int\frac{x}{\sqrt{x-1}}dx=\cancel{\int\frac{dx}{\sqrt{x-1}}+\int\sqrt{x}+}\ 2\int x\,d\sqrt{x-1}$

设 $u=\sqrt{x-1}$ $\quad F(x)=2\int(u^2+1)du=\frac{2}{3}u^3+2u+C$

$=\frac{2}{3}(x+2)\sqrt{x-1}$ ，又∵ 1 为 $f(x)$ 瑕点、

$\lim\limits_{t\to1^+}\int_t^2\frac{x\,dx}{\sqrt{x-1}}=\lim\limits_{t\to1^+}\left[\frac{8}{3}-\frac{2}{3}(t+2)\sqrt{t-1}\right]=\frac{8}{3}$ 存在.

∴ 原式 $=\frac{8}{3}$

(10)．$f(x)=\frac{1}{x\sqrt{1-(\ln x)^2}}$ ，$F(x)=\cancel{\int\frac{}{x\sqrt{1-\ln(x}}}\int f(x)dx$

$F(x)=\int\frac{1}{\sqrt{1-u^2}}du\quad(u=\ln x)=\arcsin(\ln x)+C$

其中 e 为 $f(x)$ 瑕点、

$\lim\limits_{t\to e^-}\int_1^t\frac{dx}{x\sqrt{1-(\ln x)^2}}=\lim\limits_{t\to e^-}\arcsin(\ln t)=\frac{\pi}{2}$

5．$f(x)=\ln x$ ，$F(x)=\int f(x)dx=x\ln x-x+C$

$\int_0^1\ln x\,dx=\left[x\ln x-x\right]_0^1=\lim\limits_{x\to0}(x\ln x-x)+1=1$