习题6-2.

(1) $ds = [\sqrt{8-x^2} - \frac{1}{2}x^2]dx$

$$S = \int_{-2}^{2}[\sqrt{8-x^2} - \frac{1}{2}x^2]dx = 2\int_{0}^{2}[\sqrt{8-x^2} - \frac{1}{2}x^2]dx$$

$$g(x) = \sqrt{8-x^2}$$

$$G(x) = \int g(x)dx = \int\sqrt{8-x^2}\,dx = \int\sqrt{8-8\cos^2 t}\,d\,2\sqrt{2}\cos t$$

又∵ $x \in [-2\sqrt{2}, 2\sqrt{2}]$，则 $t \cancel{x} = \arccos\frac{x}{2\sqrt{2}} \in [0, \pi]$

∴ $\sin t > 0$

$$\therefore G(x) = \int\sqrt{8}\sin t \cdot 2\sqrt{2}(-\sin t)dt = \int -8\sin^2 t\,dt$$

$$= \int -8\cdot\frac{1-\cos 2t}{2}dt = \int(4\cos 2t - 4)dt$$

$$= 2\sin 2t - 4t$$

$$\therefore \int_{0}^{2}[\sqrt{8-x^2} - \frac{1}{2}x^2]dx = \left[2\sin(2\arccos\frac{x}{2\sqrt{2}}) - 4\arccos\frac{x}{2\sqrt{2}} - \frac{1}{6}x^3\right]_0^2$$

$$= (2 - \pi - \frac{8}{6}) - (-2\pi)$$

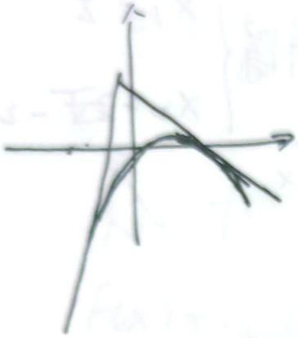
$$= \pi + \frac{2}{3}$$

$$S = 2\pi + \frac{4}{3}$$

(3) $y=e^x,\ y=e^{-x}$ 与 $x=1$

$S=\int_0^1(e^x-e^{-x})dx = e+\frac{1}{e}-2$

3. $y=-x^2+4x-3$ 及其在点 $(0,-3)$ 和 $(3,0)$ 处的切线



$-(x-3)(x-1)$

$f(x)=-x^2+4x-3.$

l_{t_1} $f(-3)=-24$, $f'(-3)=10$, $y=10x+6$

l_{t_2} $f(3)=0$, $f'(3)=-2$, $y=-2x+6$

$\therefore$ l_{t_1} 与 l_{t_2} 交于 $D(0,6)$ ， l_{t_1} 与 x 轴交于 $-\frac{3}{5}$

设切点 A B， $S_{\triangle ABD}=$ $(\frac{3}{5}+3)\cdot(6+24)\times\frac{1}{2}=54$

又 AB: $\frac{y+6}{2}=\cancel{-x}\ \frac{4x}{2}-3$ ， $y=4x-12$

抛物线与 AB 围成面积 ΔS.

$\Delta S=\int_{-3}^{3}-x^2+4x-3-4x+12=\int_{-3}^{3}-x^2+9.$

$=[-\frac{1}{3}x^3+9x]_{-3}^{3} = -9+27-9+27=36$

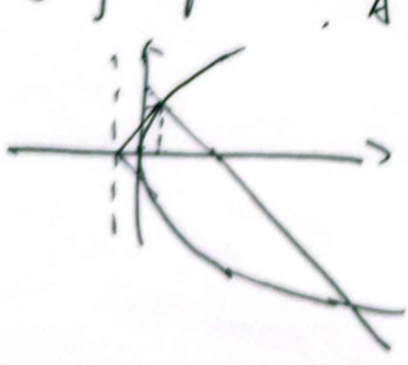
$\therefore S=S_{\triangle ABD}-\Delta S=18.$

4. $y=\cancel{-x+\frac{p}{2}}\ x+\frac{p}{2}$

~~$(y+3p)(y-p)$~~

~~$\frac{y^2}{2p}+2py-\frac{3}{2}p^2$~~

$C: y^2=2px$



$A(\frac{p}{2},p)$ ， A 处法线 $l: y=-x+\frac{3}{2}p$

$\Delta S=(x_2-x_1)\Delta y$

另一交点 $B(\frac{9p}{2},-3p)$

$x_2=\frac{3}{2}p-y$, $x_1=\frac{y^2}{2p}$

$\Delta S=(\frac{3}{2}p-y-\frac{y^2}{2p})\Delta y$

$S=\int_{-3p}^{p}(\frac{3}{2}p-y-\frac{y^2}{2p})dy$

$2\ \ \frac{-9}{2}\ \ -\frac{9}{2}\ \ +\frac{27p}{6}$

$=[\frac{3}{2}py-\frac{1}{2}y^2-\frac{y^3}{6p}]_{-3p}^{p}=[\frac{5}{6}p^2--\frac{9}{2}p^2]=\frac{16}{3}$

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$，曲线围成面积 $S = \frac{1}{2}\int_{t_1}^{t_2} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$

$\longrightarrow$ (简单闭合). (逆向)

证： $S = \int_{t_1}^{t_2} x\frac{dy}{dt}dt = \frac{1}{2}\int_{t_1}^{t_2} x\,dy + \frac{1}{2}\left[xy\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} y\,dx\right]$

若曲线闭合，则 $xy\Big|_{t_1}^{t_2} = f(t_2)g(t_2) - f(t_1)g(t_1) = 0$

$\therefore S = \frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt.$

应用该公式于 6.(2)

$\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $\begin{cases} \dot{x} = -3a\sin t\cos^2 t \\ \dot{y} = 3a\cos t\sin^2 t \end{cases}$

$S = \frac{1}{2}\int_0^{2\pi}(a\cos^3 t \cdot 3a\cos t\sin^2 t + a\sin^3 t \cdot 3a\sin t\cos^2 t)\,dt$

$= \frac{1}{2}\int_0^{2\pi} 3a^2(\cos^2 t\sin^2 t)\,dt$

$= \frac{3}{4}a^2\int_0^{4\pi}\frac{1+\cos 2t}{2}\cdot\frac{1-\cos 2t}{2}\,d2t$

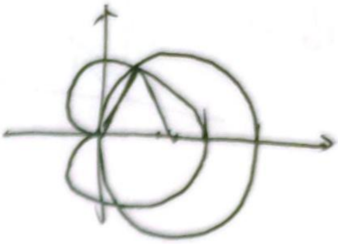
$= \frac{3a^2}{\cancel{16}\,32}\int_0^{8\pi} 1 - \left(\frac{1+\cos 4t}{2}\right)d4t$

$= \frac{3a^2}{32}\left[\frac{1}{2}\cdot 4t\right]_0^{8\pi} = \frac{3}{8}a^2\pi$

半B

9(1) $\rho=3\cos\theta$ 及 $\rho=1+\cos\theta$





$\rho_1=3\cos\theta$ 为一个 圆心为 $\frac{3}{2}$，半径为 $\frac{3}{2}$ 的圆

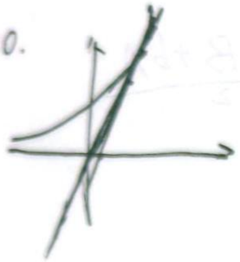
$\rho_2=1+\cos\theta$ 为一个 心形线，设交曲线为 $\rho=f(\theta)$

求交点A. A: $\theta=\frac{\pi}{3}$, $\rho=\frac{3}{2}$

$\therefore S=\int_0^{2\pi} f(\theta)d\theta = \int_0^{\frac{\pi}{3}} \rho_2 \, d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \rho_1 \, d\theta$

$= [\theta+\sin\theta]_0^{\frac{\pi}{3}} + [3\sin\theta]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = \frac{\pi}{3} + 3 - \frac{3\sqrt{3}}{2}$

10.



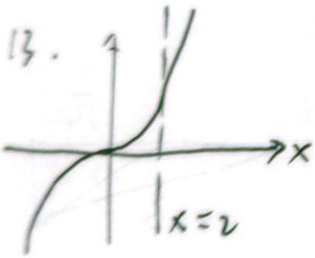
$y=e^x$ 过原点的切线为 $y=ex$，切点 $(1, e)$

$S=\int_{-\infty}^0 e^x dx + \int_0^1 e^x - ex \, dx$

$= \int_{-\infty}^1 e^x dx - \int_0^1 ex \, dx = [e^x]_{-\infty}^1 - [\frac{ex^2}{2}]_0^1$

$= \frac{e}{2}$

13.



法1. $y=x^3$, $x\in[0,2]$ 绕x轴时

$\Delta S = \pi \cdot y^2 dx$ $\quad S=\int_0^2 \pi \cdot x^6 dx = \pi[\frac{1}{7}x^7]_0^2 = \frac{128}{7}\pi$

~~法2. $y=x^3$, $x\in[0,2]$ 绕x轴时~~

~~$S=\int_0^{2\pi}(\int_0^2 x^3 dx)d\theta$~~

$y=x^3$, $x\in[0,2]$ 绕y轴时

$\Delta S_2 = \cancel{\pi x^2 y^2 dx y}$ $\quad S_1 = S - S_2$

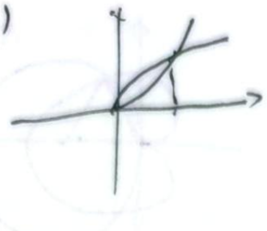
$\pi x^2 dy$

$S_2=\int_0^8 \pi x^2 dy = \int_0^2 3\pi x^4 dx = [\frac{3}{5}\pi x^5]_0^2$

$= \cancel{\frac{3}{5}\pi} \frac{96\pi}{5}$, $S=\pi \cdot 2^2 \cdot 8 = 32\pi$

$\therefore S_1 = \frac{64}{5}\pi$

16 (1)



交点：(1, 1)

$\therefore \Delta S = \cancel{\int (\sqrt{y} - y^2)\, dy}$ $\quad S_1 = \int_0^1 \pi y\, dy$

$\Delta S_1 - \Delta S_2 \quad S_2 = \int_0^1 \pi y^4 dy$

$\Delta S_1 = \pi (\sqrt{y})^2 dy . \ \Delta S_2 = \pi \cdot y^4 dy$

$S = \pi \int_0^1 y - y^4 dy = \pi \cdot [\frac{1}{2}y^2 - \frac{1}{5}y^5]_0^1 = \frac{3}{10}\pi$

18.

$S = \frac{1}{3}(S_上 + S_下 + \sqrt{S_上 S_下}) \cdot h$

$\because$ 截锥体

$S_上 = \pi ab . \ S_下 = \pi AB$

$\therefore \frac{A}{a} = \frac{B}{b} = k$

$\therefore S = \frac{1}{3}\pi h (ab + AB + \sqrt{abAB})$ $\quad \therefore \sqrt{abAB} = \frac{aB + bA}{2}$

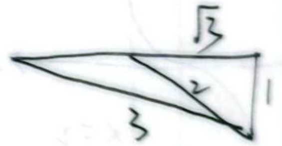
23. $\cancel{dc = \sqrt{x^2 + \ln^2 x}\, dx}$ $\quad x \in [\sqrt{3}, 2\sqrt{2}]$

$\cancel{C = \int_{\sqrt{3}}^{2\sqrt{2}} \sqrt{x^2 + \ln^2 x}\, dx}$, 设 $\cancel{F(x) = \int \sqrt{x^2 + \ln^2 x}\, dx}$

$f(x) = \sqrt{x^2 + \ln^2 x}$

$\begin{cases} \cancel{x = \sqrt{t}} \\ \cancel{y = \frac{1}{2}\ln t} \end{cases}$ $\quad \cancel{F(x) = \int x \sqrt{1 + (\frac{\ln x}{x})^2}\, dx}$

令



$ds = \sqrt{1 + \frac{1}{x^2}}\, dx \quad x \in [\sqrt{3}, 2\sqrt{2}]$

令 $x = \tan t$. $ds = \frac{1}{\tan t}\sqrt{\tan^2 t + 1}\, d\tan t,\ t \in [\frac{\pi}{3}, \arctan 2\sqrt{2}]$

$= \frac{1}{\tan t} \cdot \frac{1}{\cos^2 t}\, dt = \frac{1}{\sin t \cos t}\, dt$

$= \frac{\sin t\, dt}{\sin^2 t \cos^2 t} = -\frac{1}{(1-\cos^2 t)\cos^2 t}\, d\cos t$, $\cos t \in [\frac{1}{2}, \frac{1}{3}]$

$\cos t = u$

$\therefore ds = \frac{1}{(u^2-1)u^2}\, du = (\frac{1}{u^2-1} - \frac{1}{u^2})\, du = \frac{1}{2u-2} - \frac{1}{2u+2} - \frac{1}{u^2}\, du$

$S = \int_{\frac{1}{2}}^{\frac{1}{3}} \frac{1}{2-2u} - \frac{1}{2u+2} - \frac{1}{u^2}\, du$

$= [\frac{-1}{2}\ln|\frac{u-1}{u+1}| + \frac{1}{u}]_{\frac{1}{2}}^{\frac{1}{3}} = [-\frac{1}{2}\ln\frac{8}{9} + 3 + \frac{1}{2}\ln\frac{3}{4} - 2]$

5

26. 计算星形线 $x=a\cos^3 t$ $y=a\sin^3 t$ 的全长.

$\dot{x}=-a\sin t\cos^2 t$

$\dot{y}=a\cos t\sin^2 t$

$$C=\int_0^{2\pi}\sqrt{a^2\sin^2 t\cos^4 t+a^2\cos^2 t\sin^4 t}\,dt$$

$$=a\int_0^{2\pi}|\sin t\cos t|\,dt$$

$$=a\int_0^{\pi}\sin 2t\,d2t=-a\cdot[\cos 2t]_0^{\pi}=2a$$

29. $\rho=e^{a\theta}$ 相应于 $0\le\theta\le\varphi$ 的一段弧长

$$ds=\rho\theta\,d\theta=\theta e^{a\theta}d\theta$$

$$S=\frac{1}{a}\left[(\theta-\frac{1}{a})e^{a\theta}\right]_0^{\varphi}=\frac{1}{a}\left((\varphi-\frac{1}{a})e^{a\varphi}+\frac{1}{a}\right)$$