习题4-2

2\. (4) $\int \frac{dx}{\sqrt[3]{2-3x}}$ $\because d(2-3x)=-3$

☆ $f'(x)dx = df(x)$

$\therefore dx = \frac{df(x)}{f'(x)}$

令 $u=2-3x$

$\therefore -\frac{1}{3}\int \frac{-3dx}{\sqrt[3]{2-3x}} = -\frac{1}{3}\int \frac{d(2-3x)}{\sqrt[3]{2-3x}} = -\frac{1}{3}\int \frac{du}{\sqrt[3]{u}}$

又 $du^{\frac{2}{3}} = \frac{2}{3}u^{-\frac{1}{3}}du$

$\therefore$ 原式 $= -\frac{1}{2}u^{\frac{2}{3}} = -\frac{1}{2}(2-3x)^{\frac{2}{3}}$

(5)

$\int (\sin ax - e^{\frac{x}{b}})\,dx = \int \sin ax\,dx - \int e^{\frac{x}{b}}dx$

$= \frac{1}{a}\int \sin ax\,d\,ax - \int e^{\frac{x}{b}} \frac{d\frac{x}{b}}{\frac{1}{b}}$

$= -\frac{\cos ax}{a} - be^{\frac{x}{b}} + C$

(8)

$\int x\cos(x^2)\,dx = \int x\cos(x^2)\frac{dx^2}{2x} = \frac{\sin(x^2)}{2} + C$

(9) $\int \frac{x}{\sqrt{2-3x^2}}dx = \int \frac{x}{\sqrt{2-3x^2}}\frac{dx^2}{2x} = \frac{1}{2}\int \frac{dx^2}{\sqrt{2-3x^2}}$

$= \frac{1}{2}\int \frac{dx^2}{\sqrt{2-3x^2}} = \frac{1}{2}\int \frac{1}{\sqrt{2-3x^2}} \frac{d\cancel{x^2}(2-3x^2)}{-3} = -\frac{1}{3}\sqrt{2-3x^2} + C$

(11)

$\int \frac{x+1}{x^2+2x+5}dx = \int \frac{x+1}{x^2+2x+5}\frac{d(x^2+2x+5)}{2x+2} = \frac{1}{2}\ln(x^2+2x+5) + C$

(16)

$\int \frac{dx}{x\ln x\ln\ln x} = \int \frac{1}{x\ln x\ln\ln x}\frac{d\ln x}{\frac{1}{x}} = \int \frac{1}{\ln x\ln\ln x}\frac{d\ln\ln x}{\frac{1}{\ln x}}$

$= \int \frac{1}{\ln\ln x}\cdot \frac{d\ln\ln\ln x}{\frac{1}{\ln\ln x}} = \ln\ln\ln x$

(18) $\int \frac{10^{2\arccos x}}{\sqrt{1-x^2}} dx = \int \frac{10^{2\arccos x}}{\sqrt{1-x^2}} \frac{d\arccos x}{-\frac{1}{\sqrt{1-x^2}}}$

$= - \int 10^{2\arccos x} d\arccos x = - \int 10^{2\arccos x} \frac{d 10^{2\arccos x}}{\cancel{2 10^{2x} \ln 10} \; 2 \cdot \ln 10 \cdot 10^{2\arccos x}}$

$= - \frac{10^{2\arccos x}}{2\ln 10} \cancel{x} + C$

(19) $\int \tan\sqrt{1+x^2} \cdot \frac{x\,dx}{\sqrt{1+x^2}} = \int \tan\sqrt{1+x^2} \cdot \frac{x}{\sqrt{1+x^2}} \frac{d\sqrt{1+x^2}}{\frac{2x}{2\sqrt{1+x^2}}}$

$= \int \tan\sqrt{1+x^2} \; d \cancel{\tan}\sqrt{1+x^2}$

$= \ln|\cos\sqrt{1+x^2}| + C$

(21) $\int \frac{1+\ln x}{(x\ln x)^2} dx = \int \frac{1+\ln x}{(x\ln x)^2} \frac{dx\ln x}{1+\ln x} = -\frac{1}{x\ln x} + C$

(25) $\int \cos^2(wt+\varphi)\, dt = \int \cos^2(wt+\varphi) \frac{d(wt+\varphi)}{w}$

$= \frac{1}{2w} \int \frac{\cos(2wt+2\varphi)+1}{2} d(2wt+2\varphi)$

$= \frac{1}{4w}(\sin(2wt+2\varphi) + 2wt+2\varphi + C$

(28) $\int \sin 5x \sin 7x \, dx$

$= \frac{1}{2}\int (\cos 2x - \cos 12x)\, dx$

$= \frac{1}{2} \int \cos 2x \frac{d2x}{2} - \int \cos 12x \frac{d12x}{12} = \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 12x}{12}\right) \cancel{x} + C$

(29) $\int \tan^3 x \sec x \cdot dx = \int \tan^3 x \sec x \dfrac{d\frac{1}{\cos x}}{\frac{\sin x}{\cos^2 x}} = \int \tan^2 x \, d\dfrac{1}{\cos x}$

又 $\tan^2 x + 1 = \dfrac{1}{\cos^2 x}$

∴ 原式 $= \int \dfrac{1}{\cos^2 x} - 1 \; d\dfrac{1}{\cos x} = \dfrac{1}{3}\dfrac{1}{\cos^3 x} - \dfrac{1}{\cos x} + C$

(30) $\int \dfrac{dx}{e^x + e^{-x}} = \int \dfrac{1}{e^x + e^{-x}} \dfrac{de^x}{e^x} = \int \dfrac{1}{e^{2x}+1} de^x$

令 $e^x = \tan t$，则 $\dfrac{1}{e^{2x}+1} = \cos^2 t = \dfrac{\cos 2t + 1}{2}$

原式 $= \int \cancel{\dfrac{\cos 2t + 1}{2}} \cos^2 t \; d\tan t = \int \cos^2 t \dfrac{1}{\cos^2 t} dt$

$= t + C = \arctan e^x + C.$

(32) $\int \dfrac{x^3}{9+x^2} dx = \int \dfrac{x^3}{9+x^2} \dfrac{dx^2}{2x} = \dfrac{1}{2}\int (1 - \dfrac{9}{9+x^2}) dx^2$

$= \dfrac{1}{2}\left(x^2 - 9\ln(9+x^2) + C\right)$

(33) $\cancel{\int \dfrac{dx}{2x^2-1} = \int \dfrac{-\sin t}{2\cos^2 t - 1}}$

$\int \dfrac{dx}{2x^2-1} dx = \dfrac{1}{2}\int \left(\dfrac{1}{\sqrt{2}x - 1} - \dfrac{1}{\sqrt{2}x + 1} \right) \dfrac{d\sqrt{2}x}{\sqrt{2}}$

$= \dfrac{1}{2\sqrt{2}}\left[\ln(\sqrt{2}x - 1) - \ln(\sqrt{2}x + 1)\right] + C.$

(35) $\int \dfrac{x}{x^2-x-2} dx = \dfrac{1}{2}\int \dfrac{2x-1}{x^2-x-2} dx + \dfrac{1}{2}\int \dfrac{1}{x^2-x-2} dx$

$= \dfrac{1}{2}\int \dfrac{2x-1}{x^2-x-2} \dfrac{d(x^2-x-2)}{2x-1} + \dfrac{1}{2}\dfrac{1}{3}\int \left(\dfrac{1}{x-2} - \dfrac{1}{x+1}\right) dx$

$= \dfrac{1}{2}\ln(x^2-x-2) + \dfrac{1}{6}\ln(x-2) - \dfrac{1}{6}\ln(x+1) + C$

习题4-2.

(36) $\int \frac{x^2dx}{\sqrt{a^2-x^2}}$

令 $x=a\cos t$ ，$t=\arccos\frac{x}{a}$

原式 $=-\int \frac{a^2\cos^2 t}{\sqrt{a^2-a^2\cos^2 t}}\cdot a\sin t\, dt$

$$=-a^2\int \cos^2 t\, dt$$

$$=-a^2\int \frac{\cos 2t+1}{2}dt$$

$$=-a^2\left(\frac{1}{4}\int 2\cos 2t\, dt+\int \frac{1}{2}dt\right)$$

$$=-a^2\left(\frac{1}{4}\sin 2t+\frac{t}{2}+C\right)$$

$$=-\frac{a^2}{4}\cdot 2\sin t\cos t-\frac{a^2}{2}\arccos\frac{x}{a}+C$$

$$=-\frac{1}{2}x\sqrt{a^2-x^2}-\frac{a^2}{2}\arccos\frac{x}{a}+C$$

(37) $\cancel{\int \frac{dx}{x\sqrt{x^2-1}}=\int \frac{2x\,dx}{2x^2\sqrt{x^2-1}}=\frac{1}{2}\int \frac{1}{2(x^2-1+1)\sqrt{x^2-1}}d(x^2-1)}$

令 $x^2-1=u$

$\cancel{\text{原式}=\frac{1}{2}\int \frac{1}{(u+1)\sqrt{u}}du}$

$$\int \frac{dx}{x\sqrt{x^2-1}}=\int \frac{dx}{x^2\sqrt{1-\frac{1}{x^2}}}=-\int \frac{1}{\sqrt{1-\frac{1}{x^2}}}d\left(\frac{1}{x}\right)=\arccos\frac{1}{x}+C$$

(38) $\int \frac{dx}{\sqrt{(x^2+1)^3}}=\int \frac{1}{\sqrt{x^2+1}}\cdot\frac{dx}{x^2+1}$ 令 $u=\arctan x$

则原式 $=\int \frac{1}{\sqrt{\tan^2 u+1}}du=\int \cos u\, du=\sin u+C=\sin(\arctan x)+C$

(40)

$\int \frac{dx}{1+\sqrt{2x}}=\int \frac{\sqrt{2x}}{1+\sqrt{2x}}\cdot\frac{1}{\sqrt{2x}}dx=\int \frac{\sqrt{2x}}{1+\sqrt{2x}}d\sqrt{2x}$ ，令 $\sqrt{2x}=u$ ，则原式 $=$

$$\int 1-\frac{1}{1+u}\,du=u-\ln(1+u)+C=\sqrt{2x}-\ln(\sqrt{2x}+1)+C$$

(41) $\int \frac{dx}{1+\sqrt{1-x^2}} = \int \frac{\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-x^2}} dx = \int \frac{u}{1+u} du$，其中 $u = \arcsin x$

$= u - \ln(1+u) + C = \arcsin x - \ln(1+\arcsin x) + C$

(43) $\int \frac{x-1}{x^2+2x+3} dx = \int \frac{x+1-2}{(x+1)^2+2} d(x+1)$，令 $x+1=u$

原式 $= \int \frac{u-2}{u^2+2} du$

A5

习题 4-3.

4. $\int xe^{-x}dx = -\int x\,de^{-x} = -[xe^{-x} - \int \cancel{dx}\cdot e^{-x}\cdot dx] = (1-x)e^{-x}$

$\cancel{= \int(-x)de^{-x} =}$

5. $\int x^2\ln x\,dx = \int x^2\,d\,x\ln x = x^3\ln x - \int \cancel{dx}^2\, x\ln x\cdot dx^2\ \cancel{= x^3}$

$\therefore\ x^3\ln x = 3\int x^2\ln x\,dx$

$\therefore$ 原式 $= \frac{x^3\ln x}{3}$

9. $\int x^2\arctan x\,dx = \int \arctan x\,d\frac{1}{3}x^3 = \frac{x^3}{3}\arctan x - \int \frac{1}{3}x^3\,d\arctan x$

$= \frac{x^3}{3}\arctan x - \int\frac{x^3}{3x^2+3}dx = \frac{x^3}{3}\arctan x - \frac{1}{2}\int\frac{x^2}{3x^2+3}dx^2$

$= \frac{x^3}{3}\arctan x - \frac{1}{2}\int(\frac{1}{3} - \frac{1}{3x^2+3})dx^2$

$= \frac{x^3}{3}\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1) + C$

14. $\int x\sin x\cos x\,dx = x\sin^2x - \int \sin x\,d\,x\cancel{\cos}\sin x = x\sin^2x - \int\sin x(\sin + x\cos x)dx$

$\Rightarrow\ 2\int x\sin x\cos x\,dx = x\sin^2x - \int\frac{1-\cos2x}{2}dx = x\sin^2x - \frac{x}{2} + \frac{\sin2x}{4} + C$

$\therefore$ 原式 $= \frac{x\sin^2x}{2} - \frac{x}{4} + \frac{\sin2x}{8} + C.$

18. $\int\frac{\ln^3x}{x^2}dx = -\int\ln^3x\,d\frac{1}{x} = \int\ln^3\frac{1}{x}\,d\frac{1}{x}$

[illegible]展开: $\int\ln^nx\,dx = x\ln^nx - nx\ln^{n-1}x + xn(n-1)\ln^{n-2}x \cdots \pm ?\,n!\cdot x\ln$

$= \cancel{\sum_{i=0}^{n-1}}\ x\cdot\left[\sum_{i=0}^{n-1}\frac{n!\ln^nx}{(n-i)!\ln^ix}\right]$

6 $\therefore\ \int\ln^3\frac{1}{x}d\frac{1}{x} = \frac{1}{x}\ln^3\frac{1}{x} - 3\frac{1}{x}\ln^2\frac{1}{x} + 6\cdot\frac{1}{x}\ln\frac{1}{x} + C$

20. $\int \cos\ln x\,dx = \int x\cos\ln x\,d\ln x$. 令 $u=\ln x$

原 $= \int e^u\cos u\,du = e^u\sin u - \int e^u \sin u\,du = e^u\sin u + e^u\cos u - \int e^u\cos u\,du$

$\int e^u\cos u\,du = \dfrac{e^u\cos u + e^u\sin u}{2}$

原 $= \dfrac{x\cos\ln x + x\sin\ln x}{2}$

21. $\int (\arcsin x)^2 dx$ 令 $\arcsin x = t$, $x=\sin t$.

原 $= \int t^2 \cancel{\cos t\,dt}\, d\sin t = t^2\sin t - \int \sin t\cdot dt^2$

$= t^2\sin t - \int 2t\sin t\,dt$

$\int t\sin t\,dt = \int t\,d(-\cos t) = -t\cos t - \int -\cos t\,dt$

$= -t\cos t + \sin t$

原 $= t^2\sin t + 2t\cos t - 2\sin t$.

22. $\int e^x\sin^2 x\,dx \cancel{= \int e^x \sin x\,d(-\cos x) = -e^x\sin x\cos x + \int \cos x\,d e^x\sin x}$

$\cancel{= -e^x\sin x\cos x + \int \cos x\cdot e^x\cdot(\sin x+\cos x)dx}$

$= \int e^x\dfrac{1-\cos 2x}{2}dx = \dfrac{1}{2}\left[e^x - \int e^x\cos 2x\,dx\right]$

$\int e^x\cos 2x\,dx = \int e^x d\dfrac{\sin 2x}{2} = \dfrac{e^x\sin 2x}{2} + \int e^x d\dfrac{\cos 2x}{4}$

$= \dfrac{e^x\sin 2x}{2} + \dfrac{1}{4}e^x\cos 2x + \int e^x\dfrac{\cos 2x}{4}dx$

$\therefore \int e^x\sin^2 x\,dx = \dfrac{1}{2}\left[e^x - \left[\dfrac{4}{3}\cdot\left(\dfrac{e^x\sin 2x}{2} + \dfrac{1}{4}e^x\cos 2x\right)\right]\right]$

$= \dfrac{1}{2}e^x - \dfrac{1}{3}e^x\sin 2x - \dfrac{1}{6}e^x\cos 2x$