8.

不妨设 $y=kx^2$

11. (1) $f(x)=x^3\ln x$, $f'(x)=x^2+3x^2\ln x$ $f''(x)=2x+3x+6x\ln x$

$f'''(x)=5+6+6\ln x$ $f^{(4)}(x)=\frac{6}{x}$ $f^{(5)}(x)=\frac{-6}{x^2}$

$$f(x)=(x-1)+\frac{5}{2}(x-1)^2+\frac{11}{6}(x-1)^3+\frac{1}{4}(x-1)^4+R_4(x)$$

$$R_4(x)=\frac{-1}{20(\theta x)^2}(x-1)^5$$

(3) $f(x)=e^{\sin x}$ $f'(x)=e^{\sin x}\cos x$ $f''(x)=e^{\sin x}\cos^2 x-e^{\sin x}\sin x$

$f'''(x)=e^{\sin x}\cos^3 x-2e^{\sin x}\cos x\sin x-e^{\sin x}\cos x\sin x-e^{\sin x}\cos x$

$f^{(4)}(x)=e^{\sin x}\cos^4 x-3e^{\sin x}\cos^2 x\sin x-3e^{\sin x}\cos^2 x\sin x-3e^{\sin x}\cos^2 x$

$+3e^{\sin x}\sin^2 x-e^{\sin x}\cos^2 x+e^{\sin x}\sin x$

$f(0)=1$ $f'(0)=1$ $f''(0)=1$ $f'''(0)=0$

$f(x)=1+x+\frac{x^2}{2}+o(x^3)$

12.

(3) 证：当 $e<a<b<e^2$，$\ln^2 b-\ln^2 a>\frac{4}{e^2}(b-a)$

由拉氏中值，$\frac{\ln^2 b-\ln^2 a}{b-a}=\frac{2\ln t}{t}$，$t\in(a,b)$

设 $f(x)=\frac{2\ln x}{x}$，$f'(x)=\frac{2-2\ln x}{x^2}$，当 $x>e$ 时，$f'(x)<0$

$\therefore f(x)\downarrow$

$\therefore f(t)>f(e^2)=\frac{4}{e^2}$，原式成立

17 $y''=-\sin x$，$y'=\cos x$

$K=\frac{|\sin x|}{(1+\cos^2 x)^{\frac{3}{2}}}=\frac{1}{r}\leq 1$，当 $x=\frac{\pi}{2}$ 取等

此时 $r=1$

20 琴生不等式

习题3-7

4. $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$ $\begin{cases} \dot{x} = -3a\cos^2 t\sin t \\ \dot{y} = 3a\sin^2 t\cos t \end{cases}$ $\begin{cases} \ddot{x} = -3a\cos^3 t + 6a\cos t\sin^2 t \\ \ddot{y} = -3a\sin^3 t + 6a\sin t\cos^2 t \end{cases}$

$$K = \frac{9a^2|\cos^2 t\sin^4 t - 2\cos^4 t\sin^2 t + \sin^2 t\cos^4 t - 2\cos^2 t\sin^4 t|}{27a^3[\cos^4 t\sin^2 t + \sin^4 t\cos^2 t]^{\frac{3}{2}}}$$

$$= \frac{|\cos^2 t\sin^4 t + \sin^2 t\cos^4 t|}{3a[\cos^4 t\sin^2 t + \sin^4 t\cos^2 t]^{\frac{3}{2}}}$$

$$= \frac{1}{3a[\cos^2 t\sin^2 t]^{\frac{1}{2}}} = \left|\frac{1}{3a\cos t\sin t}\right|$$

$$K|_{t=t_0} = \left|\frac{1}{3a\cos t_0\sin t_0}\right|$$

5. $y = \ln x$ $y' = \frac{1}{x}$, $y'' = -\frac{1}{x^2}$

$$K = \frac{\frac{1}{x^2}}{(1+\frac{1}{x^2})^{\frac{3}{2}}} = \frac{x}{(x^2+1)^{\frac{3}{2}}}$$

$$r = \frac{1}{K} = \frac{(x^2+1)^{\frac{3}{2}}}{x}$$

$$r' = 0 \Longrightarrow \frac{3}{2}(x^2+1)^{\frac{1}{2}}\cdot 2x^2 = (x^2+1)^{\frac{3}{2}}$$

$$\Longleftrightarrow 3x^2 = x^2+1$$

$$x = \frac{\sqrt{2}}{2} \text{ 或 } -\frac{\sqrt{2}}{2}\text{(舍)}.$$

$\therefore$ 当 $x = \frac{\sqrt{2}}{2}$ 时，$r_{min} = \frac{(\sqrt{\frac{3}{2}})^3}{\frac{\sqrt{2}}{2}} = \frac{3\sqrt{3}}{2}$

7. $G = mg = 700N/m$ $y' = \frac{x}{5000}$ $y'' = \frac{1}{5000}$ $K = \frac{\frac{1}{5000}}{[1+(\frac{x}{5000})^2]^{\frac{3}{2}}}$

$F_N - G = m\frac{v^2}{r}$

$$F_N = 700 + 70\times 40000\cdot\frac{\frac{1}{5000}}{[1+(\frac{x}{5000})^2]^{\frac{3}{2}}}$$

$$= 700 + \frac{560\cdot(5000)^3}{[5000^2+x]^{\frac{3}{2}}}$$，当 $x=0$，$F_N = 1260N$

8

5. $f'(x) = \frac{1}{x}$

$f(x) = \int f'(x)dx = \ln|x| + C$

$f(e^3) = 3 \Rightarrow f(x) = \ln|x| + 1$

总习题三

5. $f(x)=x^3-3x+a$,

若 $\exists x_1<x_2\in[0,1]$ $f(x_1)=f(x_2)=0$

则 由罗尔中值定理，$\exists t\in(x_1,x_2)\subset(0,1)$

$f'(t)=0$

而 $f'(x)=3x^2-3$. 当 $x\in(0,1)$，$f'(x)<0$，矛盾

9. 由拉氏中值定理 ☆

$$\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f'(t)}{g'(t)}$$

$\because$ ~~$g'(t)$~~ $\forall x\ g'(x)>0$

$x>a \Rightarrow g(x)-g(a)>0$

$$\therefore \frac{|f(x)-f(a)|}{g(x)-g(a)}=\frac{|f'(t)|}{g'(t)}<1$$

$\therefore$ 原命题成立

10. 求下列极限

(2) $\lim\limits_{x\to0}\left[\frac{1}{\ln(1+x)}-\frac{1}{x}\right]=\lim\limits_{x\to0}\left[\frac{x-\ln(1+x)}{x\ln(1+x)}\right]$

由洛必达，上式 $=\lim\limits_{x\to0}\left[\frac{1-\frac{1}{1+x}}{\frac{x}{1+x}+\ln(1+x)}\right]=\lim\limits_{x\to0}\left[\frac{x}{x+(1+x)\ln(1+x)}\right]$

$$=\lim_{x\to0}\left[\frac{1}{1+\ln(1+x)+1}\right]=\frac{1}{2}$$

(3) $\lim\limits_{x\to+\infty}\left(\frac{2}{\pi}\arctan x\right)^x$

令 $t=\arctan x$

~~tant~~ 原式 $=\lim\limits_{t\to\frac{\pi}{2}}\left(\frac{2}{\pi}t\right)^{\tan t}=\lim\limits_{t\to\frac{\pi}{2}} e^{\frac{\ln\frac{2}{\pi}t}{\frac{1}{\tan t}}}$

洛必达 $\lim\limits_{t\to\frac{\pi}{2}} e^{\frac{\frac{\pi}{2}\cdot\frac{2}{\pi}\cdot\frac{1}{t}}{\frac{-1}{\sin^2 t}}}=e^{-\frac{2}{\pi}}$

P9 习题4-1

(5) $\int \frac{dx}{x\sqrt[3]{x}} = \int x^{-\frac{5}{2}} dx = -\frac{2}{3} x^{-\frac{3}{2}} + C$

(9) $\int \frac{dh}{\sqrt{2gh}} = \frac{1}{\sqrt{2g}} \int h^{-\frac{1}{2}} dh = \cancel{-\sqrt{\frac{2h}{g}} + C} \quad \sqrt{\frac{2h}{g}} + C$

(12) $\int \frac{(1-x)^2}{\sqrt{x}} dx = \int (x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{2}}) dx$

$= \frac{2}{5} x^{\frac{5}{2}} - \frac{4}{3} x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

(14) $\int \left(\frac{3}{1+x^2} - \frac{2}{\sqrt{1-x^2}} \right) dx = \int \frac{3}{1+x^2} dx - \int \frac{2}{\sqrt{1-x^2}} dx$

$= 3\arctan x + 2\arccos x + C$

(20) $\int \frac{dx}{1+\cos 2x} = \int \frac{dx}{2\cos^2 x} = \frac{\tan x}{2} + C$

(23) $\int \cot^2 x \, dx = \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = \int \frac{dx}{\sin^2 x} - \int dx$

$= -\cot x - x + C$

(25) $\int \frac{x^2}{x^2+1} dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = \int dx - \int \frac{1}{x^2+1} dx$

$= x - \arctan x + C$

(26) $\int \frac{3x^4+2x^2}{x^2+1} dx = \int \frac{3x^4+6x^2+3-4x^2-4+1}{x^2+1} dx$

$= \int (3x^2+3) dx + \int -4 \, dx + \int \frac{dx}{x^2+1}$

$= x^3 + 3x - 4x + \arctan x + C$

$= x^3 - x + \arctan x + C$

4. (1) $s = \cancel{\frac{k}{2} t(20-t) + C} \quad 20t - \frac{k}{2} t^2 \cancel{+C}$

(2) $\frac{ds}{dt} = 20 - kt = 0, \quad t = \frac{20}{k}, \quad s = \cancel{100} \frac{200}{k}$

$s = 50, \ k = 4$

8.

不妨设 $y=kx^2$ $\quad y'=2kx$ $\quad y''=2k$

$$K=\frac{2k}{(1+4k^2x^2)^{\frac{3}{2}}}$$

又对于 $y=kx^2$，当 $x=5$ 时，$y=0.25$

$\therefore k=\frac{1}{100}$ $\quad$ 当 $x=0$，$K=\frac{1}{50}=\frac{1}{r}$

$\cancel{F_N=G}$ $F_a=m\frac{v^2}{r}=G-F_N$

$$F_N=G-5000\cdot\frac{6^2}{50}$$

$$=50000-3600=46400N$$