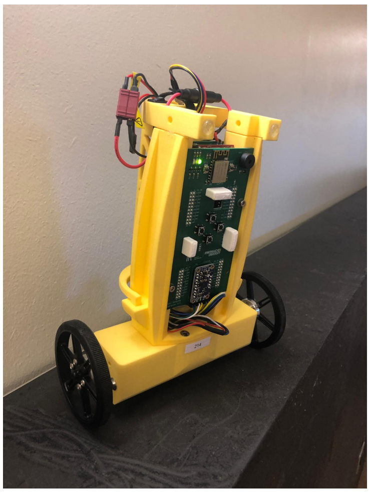
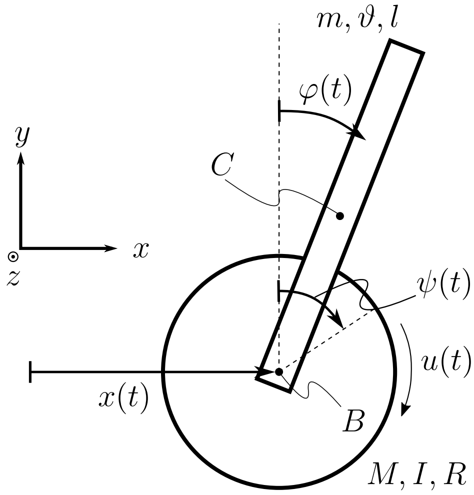
Lagrange – Segway Case Study



(a) IDSC segway prototype.



(b) 2D model of the segway.

Figure 1: Graphic representations of the analyzed system.

Assumptions

- The system is considered as planar (no movement in z-axis).
- The wheel rolls without slipping.
- There is no friction between the two bodies.

Lagrange Approach

- Degrees of freedom (DOF)? Generalized coordinates $\vec{q} = [q_1, \dots, q_{3n-k}]^\intercal$?
- Kinetic and potential energies for each body $j = 1, \dots, n$?
- Lagrange function

$$L\left(\vec{q}, \dot{\vec{q}}\right) = \sum_{j=1}^{n} T\left(\vec{q}, \dot{\vec{q}}\right) - \sum_{j=1}^{n} U\left(\vec{q}\right) \tag{1.1}$$

- Generalized forces $\vec{Q} = [Q_1, \dots, Q_{3n-k}]^\intercal$?

$$Q_i = \sum_{j=1}^{n} Q_{i,j} \qquad i = 1, \dots, 3n-k \tag{1.2}$$

- Lagrange formalism

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \qquad i = 1, \dots, 3n-k \tag{1.3}$$

Remark. Note that the index i goes from 1 to $3n-k$ because we assume that the number of generalized coordinates chosen is equal to the number of degrees of freedom. They are therefore called **minimal** coordinates.

Procedure

We start with the computation of the DOF to obtain the minimum number of generalized coordinates needed. The amount of DOF is

$$\#\text{DOF} = 3 \cdot n - k,$$

where n is the number of bodies in the system (in this case $n = 2$) and k is the amount of holonomic constraints. The holonomic constraints are

$$\begin{aligned}
y_\text{B} &= R \\
x_\text{B} &= x = \psi \cdot R \\
y_\text{C} &= y_\text{B} + \frac{l}{2}\cos(\varphi) \\
x_\text{C} &= x_\text{B} + \frac{l}{2}\sin(\varphi),
\end{aligned}$$

resulting in $k = 4$ holonomic constraints and therefore leading to 2 DOF.
The generalized coordinates that we choose are

$$\vec{q} = \begin{bmatrix} x \\ \varphi \end{bmatrix}. \tag{1.4}$$

Please note that the choice of generalized coordinates is not unique. In general it is common practice to choose an angle when modeling rotating parts or objects (e.g., pendulum, disks, ...). In this case we choose the coordinate x for the wheel, given that it only moves in that direction, and φ since we describe a pendulum. Note that we could have also chosen ψ for the wheel, given the relationship imposed by the no-slip condition.
Once the generalized coordinates are defined, we can proceed with the computation of the kinetic and potential energies of each body.
The kinetic energy for the wheel is

$$T_\text{wheel} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\dot{\psi}^2 = \frac{1}{2}\left(M + \frac{I}{R^2}\right)\dot{x}^2, \tag{1.5}$$

where we used the time derivative of the no-slip condition to eliminate $\dot{\psi}$, i.e.,

$$\dot{x} = \dot{\psi} \cdot R.$$

The potential energy of the wheel is

$$U_\text{wheel} = 0, \tag{1.6}$$

as we assume that the no-potential baseline is at the height of the wheel's center of gravity, i.e., $U(y_\text{B} = R) = 0$. Note that this baseline can be chosen arbitrarily, since a constant offset in the potential energy would disappear when the Langrangian function is derived.
For the second body we know that the general form of the kinetic energy is

$$T_\text{bar} = \frac{1}{2}m|\vec{v}_\text{C}|^2 + \frac{1}{2}\vartheta\dot{\varphi}^2. \tag{1.7}$$

The velocity in point C is

$$\begin{aligned}\vec{v}_{\mathrm{C}} &= \vec{v}_{\mathrm{B}} + \vec{\omega}_{\mathrm{bar}} \times \vec{r}_{\mathrm{BC}} \qquad \text{(Velocity transfer formula)}\\ &= \dot{x}\vec{e}_{\mathrm{x}} - \dot{\varphi}\vec{e}_{\mathrm{z}} \times \left[\frac{l}{2}\sin(\varphi)\vec{e}_{\mathrm{x}} + \frac{l}{2}\cos(\varphi)\vec{e}_{\mathrm{y}}\right]\\ &= \dot{x}\vec{e}_{\mathrm{x}} + \dot{\varphi}\frac{l}{2}\cos(\varphi)\vec{e}_{\mathrm{x}} - \dot{\varphi}\frac{l}{2}\sin(\varphi)\vec{e}_{\mathrm{y}}\\ &= \left[\dot{x} + \dot{\varphi}\frac{l}{2}\cos(\varphi)\right]\vec{e}_{\mathrm{x}} - \dot{\varphi}\frac{l}{2}\sin(\varphi)\vec{e}_{\mathrm{y}},\end{aligned} \tag{1.8}$$

and its squared norm is

$$\begin{aligned}|\vec{v}_{\mathrm{C}}|^2 &= \left(\dot{x} + \dot{\varphi}\frac{l}{2}\cos(\varphi)\right)^2 + \dot{\varphi}^2\frac{l^2}{4}\sin^2(\varphi)\\ &= \dot{x}^2 + \dot{x}\dot{\varphi}l\cos(\varphi) + \dot{\varphi}^2\frac{l^2}{4}.\end{aligned} \tag{1.9}$$

Inserting Equation (1.9) into Equation (1.7), we obtain the final expression for the kinetic energy of the bar

$$T_{\mathrm{bar}} = \frac{1}{2}\left(\dot{x}^2 + \dot{x}\dot{\varphi}l\cos(\varphi) + \dot{\varphi}^2\frac{l^2}{4}\right) + \frac{1}{2}\vartheta\dot{\varphi}^2. \tag{1.10}$$

The potential energy of the bar is

$$U_{\mathrm{bar}} = mg\frac{l}{2}\cos(\varphi) \tag{1.11}$$

Once the energies of all the bodies are known, we can formulate the Lagrange function according to Equation (1.1) by simply plugging in Equations (1.5), (1.6), (1.10) and (1.11):

$$L = \frac{1}{2}\left(M + \frac{I}{R^2}\right)\dot{x}^2 + \frac{1}{2}\left(\dot{x}^2 + \dot{x}\dot{\varphi}l\cos(\varphi) + \dot{\varphi}^2\frac{l^2}{4}\right) + \frac{1}{2}\vartheta\dot{\varphi}^2 - mg\frac{l}{2}\cos(\varphi) \tag{1.12}$$

The last components needed before being able to formulate the complete Lagrangian formalism are the generalized forces. As we can see in Figure 1b, there is a torque applied on the wheel. In fact, the motor is mounted on the hub that connects wheel and bar. Thus, there is a reaction torque on the bar, as shown in Figure 2.

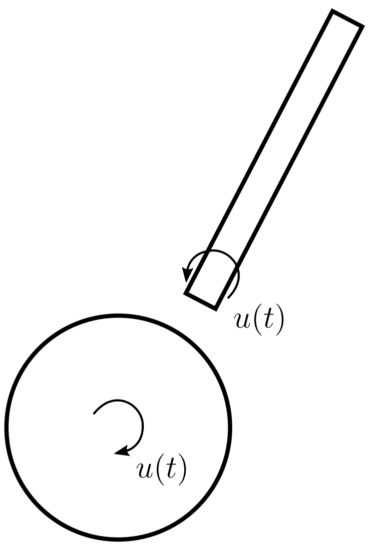


Figure 2: Schematic of the separated bodies. The torque $u(t)$ acts on both of them but in opposite direction.

To obtain the generalized forces we need to rewrite the bodies' angular velocities according to

$$\vec{\omega}_j = \bar{J}_{\mathrm{R},j} \cdot \dot{\vec{q}} + \vec{\xi}_j \qquad j = 1, \dots, n, \tag{1.13}$$

where $\bar{J}_{\mathrm{R},j}$ is the rotational Jacobian of the j-th body and $\vec{\xi}_j$ is its offset velocity. The generalized forces for the j-th body can be computed as

$$\vec{Q}_j = \bar{J}_{\mathrm{R},j}^{\intercal} \vec{M}_j \qquad j = 1, \dots, n, \tag{1.14}$$

and the final generalized force vector $\vec{Q}$ is then computed using Equation (1.2).
Equation (1.13) for the wheel reads as

$$\vec{\omega}_{\text{wheel}} = -\dot{\psi}\vec{e}_{\mathrm{z}} = -\frac{1}{R}\dot{x}\vec{e}_{\mathrm{z}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\frac{1}{R} & 0 \end{bmatrix}}_{\bar{J}_{\mathrm{R,wheel}}} \begin{bmatrix} \dot{x} \\ \dot{\varphi} \end{bmatrix} + \vec{0}, \tag{1.15}$$

whereas for the bar it is

$$\vec{\omega}_{\text{bar}} = -\dot{\varphi}\vec{e}_{\mathrm{z}} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}}_{\bar{J}_{\mathrm{R,bar}}} \begin{bmatrix} \dot{x} \\ \dot{\varphi} \end{bmatrix} + \vec{0}. \tag{1.16}$$

Plugging the obtained Jacobians into Equation (1.14) leads to

$$\vec{Q}_{\text{wheel}} = \begin{bmatrix} 0 & 0 & -\frac{1}{R} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -u \end{bmatrix} = \begin{bmatrix} \frac{u}{R} \\ 0 \end{bmatrix} \tag{1.17}$$

for the wheel, and

$$\vec{Q}_{\text{bar}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ -u \end{bmatrix} \tag{1.18}$$

for the bar. According to Equation (1.2), the final generalized force vector reads as

$$\vec{Q} = \begin{bmatrix} Q_x \\ Q_\varphi \end{bmatrix} = \begin{bmatrix} \frac{u}{R} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -u \end{bmatrix} = \begin{bmatrix} \frac{u}{R} \\ -u \end{bmatrix}. \tag{1.19}$$

At this point we are ready to proceed with the computation of the derivatives necessary to construct the Lagrangian function, see Equation (1.3).
We start with the generalized coordinate x:

$$\frac{\partial L}{\partial \dot{x}} = \left(M + \frac{I}{R^2}\right)\dot{x} + m\dot{x} + \frac{1}{2}m\dot{\varphi}l\cos(\varphi) \tag{1.20}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m + M + \frac{I}{R^2}\right)\ddot{x} + \frac{1}{2}m\ddot{\varphi}l\cos(\varphi) - \frac{1}{2}m\dot{\varphi}^2 l\sin(\varphi) \tag{1.21}$$

$$\frac{\partial L}{\partial x} = 0, \tag{1.22}$$

leading to the final equation

$$\boxed{\left(m + M + \frac{I}{R^2}\right)\ddot{x} + \frac{1}{2}m\ddot{\varphi}l\cos(\varphi) - \frac{1}{2}m\dot{\varphi}^2 l\sin(\varphi) = \frac{u}{R}}. \tag{1.23}$$

For the generalized coordinate φ:

$$\frac{\partial L}{\partial \dot{\varphi}} = m\dot{\varphi}\frac{l^2}{4} + \vartheta\dot{\varphi} + \frac{1}{2}m\dot{x}l\cos(\varphi) \tag{1.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \left(m\frac{l^2}{4} + \vartheta\right)\ddot{\varphi} + \frac{1}{2}m\ddot{x}l\cos(\varphi) - \frac{1}{2}m\dot{x}\dot{\varphi}l\sin(\varphi) \tag{1.25}$$

$$\frac{\partial L}{\partial \varphi} = -\frac{1}{2}m\dot{x}\dot{\varphi}l\sin(\varphi) + mg\frac{l}{2}\sin(\varphi), \tag{1.26}$$

leading to the final equation

$$\boxed{\left(m\frac{l^2}{4} + \vartheta\right)\ddot{\varphi} + \frac{1}{2}m\ddot{x}l\cos(\varphi) - mg\frac{l}{2}\sin(\varphi) = -u}. \tag{1.27}$$

Finally, as shown in the lecture, this can be rewritten in matrix form as follows:

$$\underbrace{\begin{bmatrix} m + M + \frac{I}{R^2} & \frac{1}{2}ml\cos(\varphi) \\ \frac{1}{2}ml\cos(\varphi) & m\frac{l^2}{4} + \vartheta \end{bmatrix}}_{M(\vec{q})}\underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\varphi} \end{bmatrix}}_{\ddot{\vec{q}}} = \underbrace{\begin{bmatrix} \frac{u}{R} + \frac{1}{2}m\dot{\varphi}^2 l\sin(\varphi) \\ -u + mg\frac{l}{2}\sin(\varphi) \end{bmatrix}}_{f\left(\vec{q},\dot{\vec{q}},\vec{u}\right)} \tag{1.28}$$

The mass matrix $M(\vec{q})$ is regular $(\det(M(\vec{q})) \neq 0)$ and can therefore be inverted. To obtain the usual representation used in control theory, we rewrite

$$\ddot{\vec{q}} = M(\vec{q})^{-1} \cdot f\left(\vec{q}, \dot{\vec{q}}, \vec{u}\right). \tag{1.29}$$