Signals and Systems

Lecture 10: Filtering: Applied Concepts

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Outline

Recall and introduction

In the previous two lectures, we discussed:

- finite-impulse-response (FIR)
- and infinite-impulse-response (IIR) filters.

In these lectures, we introduced the concepts of filtering in the context of low-pass filtering a signal.

In this lecture, we will consider:

1. other types of filters: namely high-pass, band-pass, band-stop and notch.
2. tools to design these filters using the low-pass filter design methods, seen previously.
3. application of the bilinear transform to a discrete-time filtering problem highlighting the importance of *prewarping* frequencies when converting between discrete time and continuous time.
4. practical considerations that motivate the selection of FIR or IIR filters.

Outline

High-Pass Filter Design

Consider the ideal frequency response of low-pass and high-pass filters in CT:

$$H_{\mathsf{LPI}}(\omega) = \begin{cases} 1 & 0 \le \omega \le \omega_c \\ 0 & \omega_c < \omega \end{cases}$$

$$H_{\mathsf{HPI}}(\omega) = \begin{cases} 0 & 0 \le \omega \le \omega_c \\ 1 & \omega_c < \omega. \end{cases}$$

One can readily relate the two as

$$H_{\mathsf{HPI}}(\omega) = 1 - H_{\mathsf{LPI}}(\omega),$$

and may therefore be inclined to calculate the transfer function of a high-pass filter from the transfer function of a low-pass filter as:

$$H_{\mathsf{HP}}(s) = 1 - H_{\mathsf{LP}}(s).$$

High-pass Filter Design

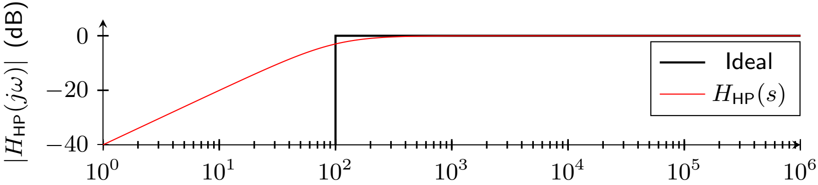
Let us consider this transformation in more detail. Take, for example, a first-order low-pass filter with CT transfer function

$$H_{\text{LP}}(s) = \frac{\omega_c}{s + \omega_c}.$$

Using the above method, the corresponding high-pass filter is calculated as

$$H_{\text{HP}}(s) = 1 - \frac{\omega_c}{s + \omega_c} = \frac{s}{s + \omega_c},$$

and has, for $\omega_c = 100$, the following magnitude response:



High-pass filter from a low-pass Butterworth filter

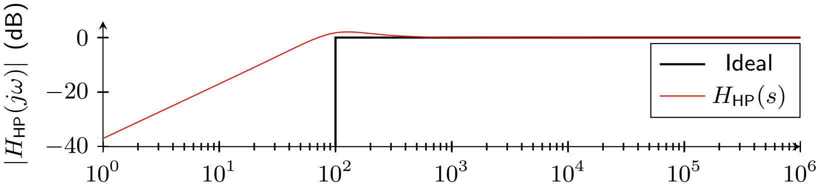
Consider the second-order, low-pass Butterworth filter with CT transfer function

$$H_{\mathrm{LP}}(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}.$$

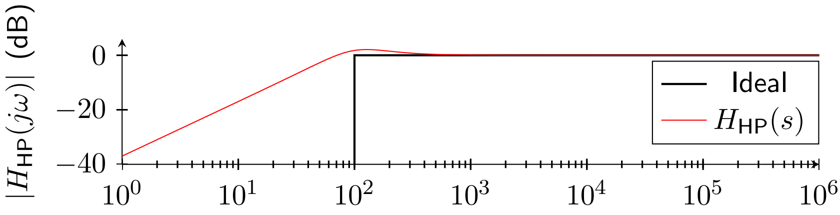
The corresponding high-pass filter is calculated as

$$H_{\mathrm{HP}}(s) = 1 - H_{\mathrm{LP}}(s) = \frac{s^2 + \sqrt{2}\omega_c s}{s^2 + \sqrt{2}\omega_c s + \omega_c^2},$$

and, with $\omega_c = 100$, has the magnitude response:



High-pass filter from a low-pass Butterworth filter



⚠ **Conclusion:** this response

- is not flat,
- and does not roll-off at 40 dB/decade, as one would expect from a second-order Butterworth filter. This is due to the relationship $H_{\text{HP}}(s) = 1 - H_{\text{LP}}(s)$ only holding when considering the ideal responses $H_{\text{HPI}}(\omega)$ and $H_{\text{LPI}}(\omega)$;
- will cause undesired behavior.

Outline

Designing a high-pass filter in CT

We now consider an alternative method to calculate the transfer function of a high-pass filter from the transfer function of a low-pass filter by performing a transformation in the frequency domain.

This transformation should

1. preserve stability – the open left halfplane should be mapped to the open left halfplane;
2. map the $j\omega$ axis to the $j\omega$ axis; and
3. map $\omega = 0$ to $\omega = \infty$ and $\omega = \infty$ to $\omega = 0$.

One such transformation is $s \to s^{-1}$, which satisfies the above properties in the following ways:

Designing a high-pass filter in CT

1. Let $s = a + jb$. Then

$$\frac{1}{s} = \frac{1}{a+jb} = \frac{a-jb}{a^2+b^2},$$

therefore

$$\text{Re}\,(s) = a\ , \quad \text{Re}\left(s^{-1}\right) = \frac{a}{a^2+b^2}$$

and $\text{Re}\,(s) < 0 \Leftrightarrow \text{Re}\left(s^{-1}\right) < 0$.

Thus the transformation preserves stability, since it maps the open left halfplane to the open left halfplane.

2. Let $s = j\omega$. Then

$$\frac{1}{s} = -j\frac{1}{\omega},$$

thus the $j\omega$ axis is mapped to the $j\omega$ axis.

Note that positive frequencies are mapped to negative frequencies.

3. Is trivial to see.

Designing a high-pass filter in CT

We can see what this transformation does to the ideal response, where we now include negative frequencies because of item 2 above (although, as we shall see, it makes no difference).

$$H_{\mathsf{LPI}}(\omega) = \begin{cases} 1 & 0 \leq |\omega| \leq 1/\omega_c \\ 0 & 1/\omega_c < |\omega| \end{cases}$$

$$H_{\mathsf{HPI}}(\omega) = H_{\mathsf{LPI}}(-1/\omega) = \begin{cases} 0 & 0 \leq |\omega| \leq \omega_c \\ 1 & \omega_c < |\omega|. \end{cases}$$

We therefore have the following technique:

1. choose desired corner frequency ω_c for a high-pass filter,
2. design a low-pass filter $H_{\mathsf{LP}}(s)$ with corner frequency $1/\omega_c$,
3. calculate the transfer function of the corresponding high-pass filter as $H_{\mathsf{HP}}(s) = H_{\mathsf{LP}}(s^{-1})$.

Example : Designing a CT high-pass filter

Let

$$H_{\mathrm{LP}}(s) = \frac{\frac{1}{\omega_c}}{s + \frac{1}{\omega_c}},$$

then

$$H_{\mathrm{HP}}(s) = H_{\mathrm{LP}}(s^{-1}) = \frac{\frac{1}{\omega_c}}{\frac{1}{s} + \frac{1}{\omega_c}} = \frac{s}{s + \omega_c},$$

the same as we saw before. Now consider a second-order low-pass Butterworth filter with transfer function

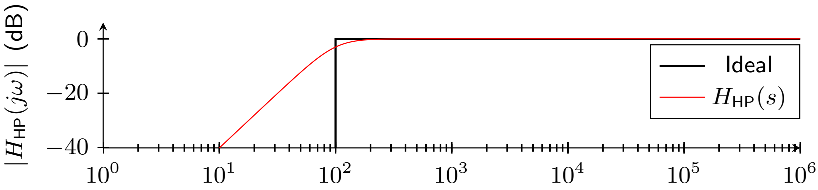
$$H_{\mathrm{LP}}(s) = \frac{\frac{1}{\omega_c^2}}{s^2 + \frac{\sqrt{2}s}{\omega_c} + \frac{1}{\omega_c^2}},$$

then

$$H_{\mathrm{HP}}(s) = H_{\mathrm{LP}}(s^{-1}) = \frac{\frac{1}{\omega_c^2}}{\frac{1}{s^2} + \frac{\sqrt{2}}{\omega_c s} + \frac{1}{\omega_c^2}} = \frac{s^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2},$$

Designing a high-pass filter in CT

This is a second-order, high-pass Butterworth filter with corner frequency ω_c. For $\omega_c = 100$, this has the desired magnitude response:



Conclusion : Because we are using a frequency transformation, rather than operating on the transfer function, we obtain the desired behavior:

- A flat response ✓
- and a roll-off of 40 dB/decade ✓ (as expected from a second-order Butterworth filter.)

Outline

Designing a high-pass filter in DT

An alternative frequency-domain transformation exists in DT.

This transformation should

1. preserve stability – the inside of the unit circle should be mapped to the inside of the unit circle;
2. map the unit circle to the unit circle;
3. for $z = e^{j\Omega}$, map $\Omega = 0$ to $\Omega = \pi$, and $\Omega = \pi$ to $\Omega = 0$.

One such transformation is $z \to -z$, which satisfies the above properties in the following ways:

Designing a high-pass filter in DT

1. $|z| < 1 \quad \Leftrightarrow \quad |-z| = |z| < 1$
2. $|z| = 1 \quad \Leftrightarrow \quad |-z| = |z| = 1$
3. $z = e^{j0} = 1 \quad \Rightarrow \quad -z = -1 = e^{j\pi}$

Note that

$$z \to -z \quad \Rightarrow \quad e^{j\Omega} \to -e^{j\Omega} = e^{j\pi}e^{j\Omega} = e^{j(\Omega+\pi)}.$$

Conclusion: this transformation causes the frequency response to be shifted by π.

Designing a high-pass filter in DT

We therefore have the following technique:

1. Choose Ω_c be the desired corner frequency of a high-pass filter, where $0 < \Omega_c < \pi$ without loss of generality.
2. Since the ideal frequency response at negative frequencies is the same as at positive frequencies:
   we design a low-pass filter $H_{\text{LP}}(z)$ with corner frequency at
   $$-(\pi + \Omega_c) \equiv -(\pi + \Omega_c) + 2\pi = \pi - \Omega_c,$$
   where
   $$0 < \pi - \Omega_c < \pi$$
3. and calculate the transfer function of the corresponding high-pass filter as
   $$H_{\text{HP}}(z) = H_{\text{LP}}(-z)$$

Example: designing a high-pass filter in DT

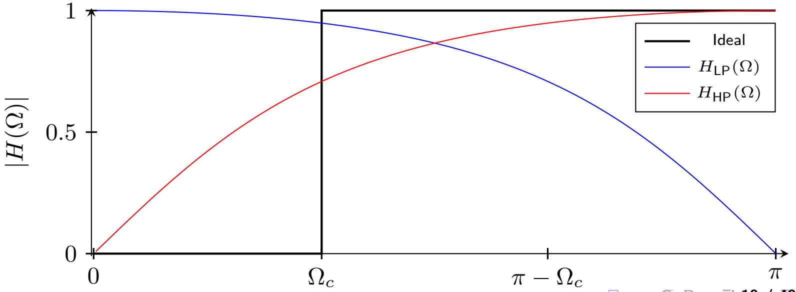
Let

$$H_{\mathrm{LP}}(z) = \frac{(1-\alpha)(\frac{1+z^{-1}}{2})}{1-\alpha z^{-1}},$$

where α is chosen such that the desired roll-off is at $\pi - \Omega_c$. Then

$$H_{\mathrm{HP}}(z) = H_{\mathrm{LP}}(-z) = \frac{(1-\alpha)(\frac{1-z^{-1}}{2})}{1+\alpha z^{-1}},$$

which has magnitude response:



Outline

Band-Pass Filter Design

The ideal frequency response of a CT band-pass filter with corner frequencies ω_0 and $\omega_1 > \omega_0$ is given by:

$$H_{\mathsf{BPI}}(\omega) = \begin{cases} 0 & 0 \le \omega \le \omega_0 \\ 1 & \omega_0 < \omega \le \omega_1 \\ 0 & \omega_1 < \omega \end{cases}$$

and can be achieved by multiplying an ideal low-pass response with corner frequency ω_1 and an ideal high-pass response with corner frequency ω_0:

$$H_{\mathsf{BPI}}(\omega) = H_{\mathsf{LPI}}(\omega) H_{\mathsf{HPI}}(\omega).$$

One might therefore expect to obtain a band-pass filter as follows:

$$H_{\mathsf{BP}}(s) = H_{\mathsf{LP}}(s) H_{\mathsf{HP}}(s).$$

As we now illustrate with an example, this method of designing a band-pass filter leads to the expected result, provided that $\omega_1/\omega_0 \gg 1$.

Example

Consider the second-order, low-pass Butterworth filter with corner frequency ω_1

$$H_{\mathsf{LP}}(s) = \frac{\omega_1^2}{s^2 + \sqrt{2}\omega_1 s + \omega_1^2}$$

and the second-order, high-pass Butterworth filter with corner frequency ω_0

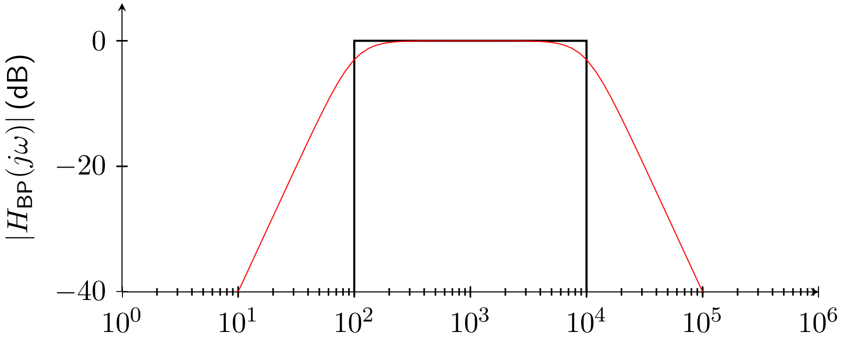
$$H_{\mathsf{HP}}(s) = \frac{s^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2}.$$

Using the proposed method, we write the transfer function of the band-pass filter as

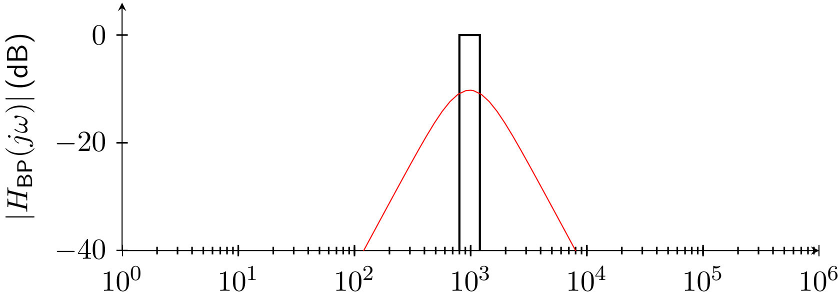
$$H_{\mathsf{BP}}(s) = \frac{s^2\omega_1^2}{(s^2 + \sqrt{2}\omega_0 s + \omega_0^2)(s^2 + \sqrt{2}\omega_1 s + \omega_1^2)}.$$

Example

For $\omega_1/\omega_0 \gg 1$, we obtain the desired magnitude response:



If, however, ω_0 and ω_1 are relatively close (e.g. $\omega_0 = 800$ and $\omega_1 = 1200$), the magnitude response exhibits undesirable characteristics: strong attenuation in the pass band.



We now present a technique that works for all ratios of $\omega_1/\omega_0 > 1$.

Outline

Low-pass to band-pass filter transformation in CT

Similar to the design of a high-pass filter, we can perform a transformation in the frequency domain to obtain a band-pass filter from a low-pass filter.

Method:

1. In order to obtain a band-pass with corner frequencies ω_0 and ω_1, we first design a low-pass filter with corner frequency $\omega_c = \omega_1 - \omega_0$.
2. We then transform it using the following transformation

$$s \to \frac{s^2 + \omega_s^2}{s},$$

where $\omega_s = \sqrt{\omega_0 \omega_1}$, the geometric mean of ω_0 and ω_1. ω_s is therefore at the center of the passband when viewed on a Bode plot.

Low-pass to band-pass filter transformation in CT

Let us analyze the effect of such a transformation on the ideal response

$$H_{\mathsf{LPI}}(\omega) = \begin{cases} 1 & 0 \leq |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega|. \end{cases}$$

Discussion:

- **Low-frequencies** of the band-pass are mapped to high-frequencies of the low-pass

$$\lim_{s \to 0} \left(\frac{s^2 + \omega_s^2}{s} \right) = \infty$$

and therefore

$$H_{\mathsf{BPI}}(0) = H_{\mathsf{LPI}}(\infty) = 0.$$

- **High-frequencies** of the band-pass are mapped to high-frequencies of the low-pass

$$\lim_{s \to \infty} \left(\frac{s^2 + \omega_s^2}{s} \right) = \infty$$

and therefore

$$H_{\mathsf{BPI}}(\infty) = H_{\mathsf{LPI}}(\infty) = 0.$$

Low-pass to band-pass filter transformation in CT

- **The frequency** ω_s of the band-pass is mapped to a frequency of 0 on the low-pass
$$\left.\frac{s^2+\omega_s^2}{s}\right|_{s=j\omega_s}=0$$
and therefore
$$H_{\mathsf{BPI}}(\omega_s)=H_{\mathsf{LPI}}(0)=1.$$
- **The corner frequencies** of the band-pass are mapped to the corner frequency of the low-pass. For example,
$$\left.\frac{s^2+\omega_s^2}{s}\right|_{s=j\omega_1}=-j\frac{-\omega_1^2+\omega_0\omega_1}{\omega_1}=j(\omega_1-\omega_0)=j\omega_c.$$
- It can be shown that the transformation preserves stability.

We can then conclude that,
$$H_{\mathsf{BPI}}(\omega)=H_{\mathsf{LPI}}\left(\frac{-\omega^2+\omega_s^2}{-\omega}\right)=\begin{cases}0 & 0\le|\omega|<\omega_0\\ 1 & \omega_0\le|\omega|\le\omega_1\\ 0 & \omega_1<|\omega|.\end{cases}$$

Example

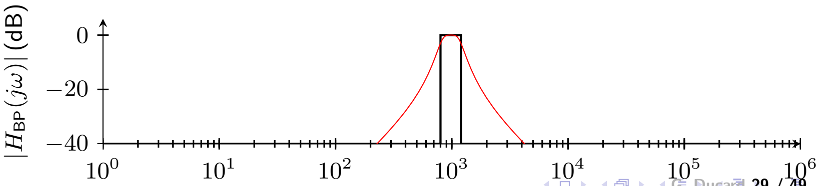
Consider a second-order, low-pass Butterworth filter with transfer function

$$H_{\text{LP}}(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2},$$

where $\omega_c = \omega_1 - \omega_0$. By applying the frequency-domain transformation, we obtain

$$H_{\text{BP}}(s) = H_{\text{LP}}\left(\frac{s^2 + \omega_s^2}{s}\right),$$

where $\omega_s = \sqrt{\omega_0 \omega_1}$. The resulting transfer function $H_{\text{BP}}(s)$ has the desired magnitude response, even if ω_0 and ω_1 are close together. For example, the magnitude response of $H_{\text{BP}}(s)$ for $\omega_0 = 800$ and $\omega_1 = 1200$ is shown below.



Outline

Low-pass to band-pass filter transformation in DT

Alternative frequency-domain transformations also exist in DT. One such transformation is

$$z \rightarrow -z^2,$$

which has the following properties:

1. $z = -1$ maps to $z = -1$ and $z = 1$ maps to $z = -1$. Therefore

$$H_{\mathsf{BPI}}(\Omega = \pi) = H_{\mathsf{BPI}}(\Omega = 0) = H_{\mathsf{LPI}}(\Omega = \pi) = 0.$$

2. $z = j$ maps to $z = 1$. Therefore

$$H_{\mathsf{BPI}}(\Omega = \pi/2) = H_{\mathsf{LPI}}(\Omega = 0) = 1.$$

Conclusion :

1. This **transformation results in a band-pass filter** centered at the DT frequency $\pi/2$.
2. **More complicated transformations** must be used to center the filter at arbitrary frequencies.

Outline

The ideal frequency response of a CT band-stop filter with corner frequencies ω_0 and $\omega_1 > \omega_0$ is given by:

$$H_{\mathsf{BSI}}(\omega) = \begin{cases} 1 & 0 \leq \omega \leq \omega_0 \\ 0 & \omega_0 < \omega \leq \omega_1 \\ 1 & \omega_1 < \omega \end{cases}$$

and can be achieved by adding an ideal low-pass response with corner frequency ω_0 to an ideal high-pass response with corner frequency ω_1:

$$H_{\mathsf{BSI}}(\omega) = H_{\mathsf{LPI}}(\omega) + H_{\mathsf{HPI}}(\omega).$$

One might therefore expect to obtain a band-stop filter as follows:

$$H_{\mathsf{BS}}(s) = H_{\mathsf{LP}}(s) + H_{\mathsf{HP}}(s).$$

As was the case for the band-pass filter, it turns out that this way of designing a band-stop filter leads to the desired result, provided that $\omega_1/\omega_0 \gg 1$. We illustrate this with an example.

Example

Consider the second-order, low-pass Butterworth filter with corner frequency ω_0

$$H_{\mathsf{LP}}(s) = \frac{\omega_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2}$$

and the second-order, high-pass Butterworth filter with corner frequency ω_1

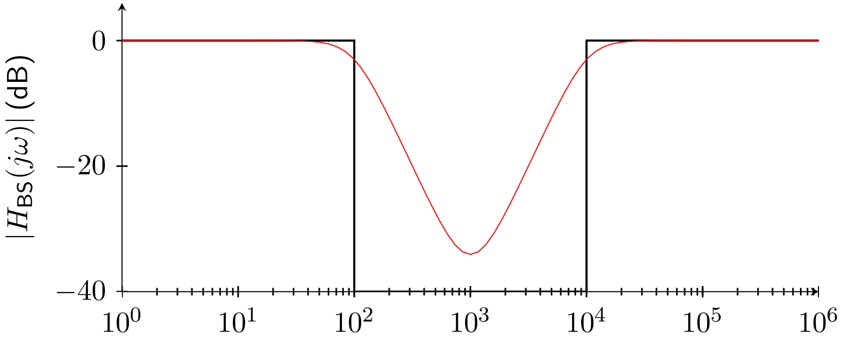
$$H_{\mathsf{HP}}(s) = \frac{s^2}{s^2 + \sqrt{2}\omega_1 s + \omega_1^2}.$$

Using the proposed method, we write the transfer function of the band-stop filter as

$$H_{\mathsf{BS}}(s) = \frac{\omega_0^2}{s^2 + \sqrt{2}\omega_0 s + \omega_0^2} + \frac{s^2}{s^2 + \sqrt{2}\omega_1 s + \omega_1^2}$$

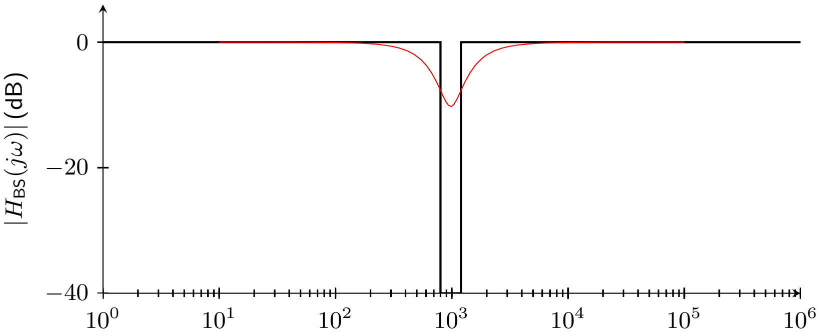
Band-pass filters

For $\omega_1/\omega_0 \gg 1$, we obtain the desired magnitude response:



Band-pass filters

If, however, ω_0 and ω_1 are relatively close (e.g. $\omega_0 = 800$ and $\omega_1 = 1200$), the magnitude response exhibits undesirable characteristics: relatively bad attenuation in the stop band and unwanted attenuation in the pass band.



In this situation, i.e. if a tight stop band is required, frequency-domain transformations, similar to those used for band-pass design, can be used.

Outline

Notch Filter Design

Notch filters are band-stop filters with a very narrow stop band.

An ideal filter to reject frequencies close to ω_c has a frequency response given by:

$$H_{\mathsf{NOI}}(\omega) = \begin{cases} 1 & 0 \leq \omega < \omega_c - \varepsilon \\ 0 & \omega_c - \varepsilon \leq \omega \leq \omega_c + \varepsilon \\ 1 & \omega_c + \varepsilon < \omega \end{cases}$$

where ε is small and determines the width of the stop band.

Example: second-order notch filter

A second-order, Butterworth notch filter can be described by

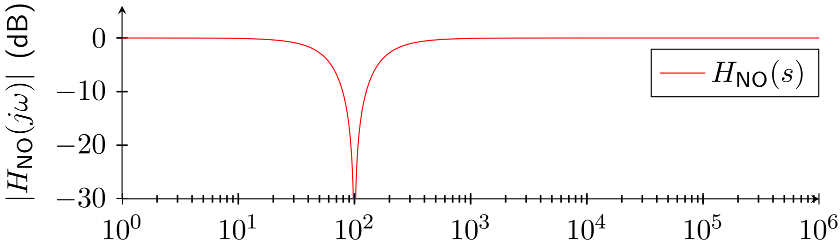
$$H_{\text{NO}}(s) = \frac{s^2 + \omega_c^2}{s^2 + \sqrt{2}s\omega_c + \omega_c^2}.$$

This structure is motivated by the following requirements:

1. $|H_{\text{NO}}(\pm j\omega_c)| = 0$: we therefore require the terms $(s + j\omega_c)(s - j\omega_c) = s^2 + \omega_c^2$ in the numerator;
2. $|H_{\text{NO}}(0)| = 1$ and $|H_{\text{NO}}(\pm j\infty)| = 1$: thus motivating the denominator terms ω_c^2 and s^2 respectively;
3. Stability: we therefore damp the filter's poles through the introduction of $+\sqrt{2}\omega_c s$ in the denominator to give the same poles as a Butterworth filter.

Example: second-order notch filter

The magnitude response of the filter, for $\omega_c = 100$, is given by:



Another observation about this filter's structure can be made:

Let $H_{\mathsf{BS}}(s)$ be the transfer function of a band-stop filter, given by the addition of a low-pass and high-pass filter with corner frequencies $\omega_c - \varepsilon$ and $\omega_c + \varepsilon$ respectively:

$$H_{\mathsf{BS}}(s) = \frac{(\omega_c - \varepsilon)^2}{s^2 + \sqrt{2}s(\omega_c - \varepsilon) + (\omega_c - \varepsilon)^2} + \frac{s^2}{s^2 + \sqrt{2}s(\omega_c + \varepsilon) + (\omega_c + \varepsilon)^2}.$$

Now let $\varepsilon = 0$ and observe that

$$H_{\mathsf{BS}}(s) = H_{\mathsf{NO}}(s) = \frac{s^2 + \omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}.$$

Conclusion :

- A second-order Butterworth notch filter can therefore be seen as a band-stop filter of zero width.
- In practice, the width of the notch can be tuned by adjusting the damping factor.

Outline

Frequency Warping and the Bilinear Transform

Context :

Let T_s be the sampling period for a DT process.

- As we saw in Lecture 9, we can convert a CT transfer function to a DT transfer function via the bilinear transform by substituting:

$$s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right).$$

- In particular, frequencies in CT are mapped to the following DT frequencies:

$$\Omega = 2\arctan\left(\omega\frac{T_s}{2}\right), \quad -\pi < \Omega \leq \pi.$$

However, how to understand what we saw in Lecture 3:

a sinusoid of DT frequency Ω corresponds to a sinusoid at CT frequency Ω/T_s.

Discussion

For small frequencies, this is not an issue, in fact:

$$2\arctan\left(\omega\frac{T_s}{2}\right)\frac{1}{T_s} \to \omega \text{ as } \omega \to 0.$$

For larger frequencies, this can be a problem. See example below.

Example

- Let $T_s = 10^{-3}$ seconds
- and let $H_{\mathsf{LP}}(s)$ be a second-order low-pass filter with corner frequency at $\omega_c = 2000$ rad/s, or $f_c \approx 318$ Hz.
- Applying the bilinear transform results in a low-pass filter $H_{\mathsf{LP}}(z)$ with DT corner frequency of $\Omega_c = 2\arctan\left(\omega_c\frac{T_s}{2}\right) = \pi/2$ rad.
- When actually implemented, the resulting corner frequency in CT is $\frac{\pi}{2}\frac{1}{T_s} = 1570$ rad/s, instead of the desired 2000 rad/s.

Frequency pre-warping : **technique**

Pre-warp the filter corner frequency before doing CT filter design.

1. Let ω_c be the desired CT corner frequency of a DT filter (eg. the corner frequency for a low-pass or high-pass filter, or one of the corner frequencies for a band-pass, etc.)
2. Let T_s be the underlying sampling period, the prewarped CT frequency is

$$\bar{\omega}_c = \frac{2}{T_s} \tan\left(\frac{\omega_c T_s}{2}\right).$$

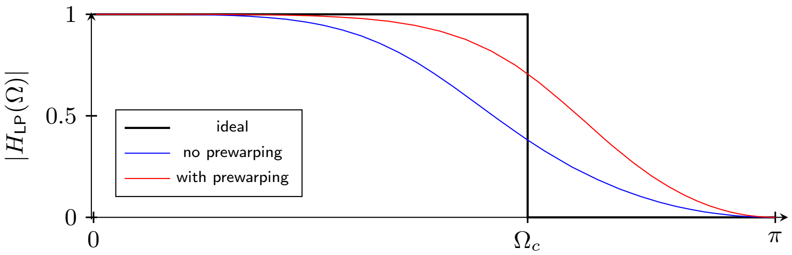
Note: if $\Omega_c = 2\arctan\left(\bar{\omega}_c \frac{T_s}{2}\right)$, then $\frac{\Omega_c}{T_s} = 2\arctan\left(\tan\left(\frac{\omega_c T_s}{2}\right)\right)\frac{1}{T_s} = \omega_c.$

3. Design CT filter with frequency $\bar{\omega}_c$ and obtain $H(s)$.
4. Apply the bilinear transform $s = \frac{2}{T_s}\left(\frac{z-1}{z+1}\right)$ to obtain the DT filter $H(z)$.

Frequency pre-warping technique

- Let $T_s = 10^{-3}$ seconds
- and let $H_{\text{LP}}(s)$ be a second-order low-pass filter with desired corner frequency at $\omega_c = 2000$ rad/s.
- This corresponds to a desired DT corner frequency of $\Omega_c = 2$ rad.
- Following the above procedure we have that $\bar{\omega}_c = 3115$ rad/s.

Below we show the magnitude response of the resulting DT filter $H_{\text{LP}}(z)$ with and without frequency prewarping.



Outline

IIR vs FIR Filter Design

This is a *huge* topic well beyond the scope of this class. CAD tools are well established, for example the *Filter Design Toolbox* in MATLAB. There are two major choices:

Property	FIR	IIR
Stability	Always stable	Can be unstable
Numerical	Errors do not accumulate	Errors can accumulate
Order	Usually needs to be high order	Can be effective at low order
Sensitivity	Can be very sensitive to small filter parameter deviations at higher orders	Can be very sensitive to small filter parameter deviations at higher orders
Shape	Very flexible, arbitrary shapes	Not as flexible
Causality	Does not have to be causal	Almost always causal

IIR vs FIR Filter Design

The choice of filter structure depends on the problem at hand.

Some good advices

- is to start simple
- the first-order IIR low-pass filter is a very good starting point and surprisingly effective.
- Lastly, if you are going to go through the trouble of using a high-order FIR filter, do not use a moving average filter. Significantly better designs exist.