1 Notation

We use two reference frames

NED-frame x-axis points north, y-axis points east and z-axis points down, assumed to be inertial.

Body-frame Non-inertial reference frame attached to the the rigid-body.

In addition we assume that the sensor frames are all aligned with the body frame. Bold lowercase letters are used to indicate a vector and bold uppercase a matrix. Superscript is used to indicate the reference frame the vector is expressed in, for instance the vector $\boldsymbol{a}^b$ is expressed in the body frame. The transpose is denoted by $^\top$

2 Cross axis sensitivity on the BMI270

The BMI270 gyro according to the datasheet has a cross axis sensitivity (cas) correction step give as

$$
\begin{aligned}
\mathrm{Rate}_x &= \mathrm{DATA\_15} \ll 8 + \mathrm{DATA\_14} \\
&\quad - \mathrm{GYR\_CAS.factor\_zx} \cdot (\mathrm{DATA\_19} \ll 8 + \mathrm{DATA\_18})/2^9 \\
\mathrm{Rate}_y &= \mathrm{DATA\_17} \ll 8 + \mathrm{DATA\_16} \\
\mathrm{Rate}_z &= \mathrm{DATA\_19} \ll 8 + \mathrm{DATA\_18}
\end{aligned}
$$

where GYR_CAS.factor_zx is an internal parameter of the sensor that is currently unavailable. The cas correction step can be summarized as

$$\bar{\omega}_x = \omega_x + \alpha \cdot \omega_z \tag{1}$$

$$\bar{\omega}_y = \omega_y \tag{2}$$

$$\bar{\omega}_z = \omega_z \tag{3}$$

where $\alpha = -\mathrm{GYR\_CAS.factor\_zx}/2^9$. Since we cannot know the parameter we will use an extended Kalman filter to estimate this unknown constant α.

3 Sensor model

In the development of the EKF we will use the following gyro sensor model

$$\boldsymbol{\omega}_m^b = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\omega}_t^b + \boldsymbol{\beta}_\omega + \boldsymbol{\eta}_\omega \tag{4}$$

where $\boldsymbol{\omega}_m^b = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^\top$ is the measured output of the sensor, $\boldsymbol{\omega}_t^b$ is the true angular velocity, $\boldsymbol{\beta}_\omega = \begin{bmatrix} \beta_x & \beta_y & \beta_z \end{bmatrix}$ is the sensor bias and $\boldsymbol{\eta}_\omega$ is white gaussian noise with variance σ_ω^2. The sensor model is just (1)-(3) with added bias and noise terms.

4 Continuous-Discrete Extended Kalman-Filter

4.1 Attitude Kinematics

The attitude kinematics of a rigid-body can be expressed using Euler angles as

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\cos^{-1}\theta & \cos\phi\cos^{-1}\theta \end{bmatrix} \boldsymbol{\omega}_t^b \tag{5}$$

where ϕ, θ and ψ are the roll, pitch and yaw angles.

4.2 Prediction

Using (5) we can express the nominal attitude kinematics $\boldsymbol{\eta}_\omega = 0$) with respect to the measured angular velocity using (4)

$$\boldsymbol{\omega}_t^b = \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\boldsymbol{\omega}_m^b - \boldsymbol{\beta}_\omega\right) \tag{6}$$

and by inserting this into (5) we get

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\cos^{-1}\theta & \cos\phi\cos^{-1}\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\boldsymbol{\omega}_m^b - \boldsymbol{\beta}_\omega\right) \tag{7}$$

We can now augment these equations with the dynamics of the bias and cas parameter under the assumption that they are constant

$$\dot{\boldsymbol{\beta}}_\omega = 0 \tag{8}$$

$$\dot{\alpha} = 0. \tag{9}$$

Now let

$$\boldsymbol{x} = \begin{bmatrix} \phi \\ \theta \\ \psi \\ \beta_x \\ \beta_y \\ \beta_z \\ \alpha \end{bmatrix}, \qquad \boldsymbol{u} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{10}$$

be the state and input for the EKF, then

$$\boldsymbol{x}_k^- = \boldsymbol{x}_{k-1} + \int \boldsymbol{f}(\boldsymbol{x}_{k-1}, \boldsymbol{u}_{k-1})\,\mathrm{d}t \tag{11}$$

$$\boldsymbol{P}_k^- = \boldsymbol{P}_{k-1} + \int \boldsymbol{P}_{k-1}\boldsymbol{F}_{k-1}^\top + \boldsymbol{F}_{k-1}\boldsymbol{P}_{k-1} + \boldsymbol{Q}\,\mathrm{d}t \tag{12}$$

where $\boldsymbol{f}(\boldsymbol{x}_{k-1}, \boldsymbol{u}_k)$ is the right side of (7)-(9), $\boldsymbol{Q}$ is the system noise covariance, $\boldsymbol{P}$ is the error covariance and

$$\boldsymbol{F}_{k-1} = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_{k-1}, \boldsymbol{u}=\boldsymbol{u}_{k-1}} = \begin{bmatrix} a_{11} & a_{12} & 0 & -1 & a_{15} & a_{16} & a_{17} \\ a_{21} & 0 & 0 & 0 & a_{25} & a_{26} & 0 \\ a_{31} & a_{32} & 0 & 0 & a_{35} & a_{36} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$a_{11} = \cos\phi \tan\theta(\omega_y - \beta_y) - \sin\phi \tan\theta(\omega_z - \beta_z) \tag{14}$$
$$a_{12} = \sin\phi(1 + \tan^2\theta)(\omega_y - \beta_y) + \cos\phi(1 + \tan^2\theta)(\omega_z - \beta_z) \tag{15}$$
$$a_{15} = -\sin\phi \tan\theta \tag{16}$$
$$a_{16} = \alpha - \cos\phi \tan\theta \tag{17}$$
$$a_{17} = -(\omega_z - \beta_z) \tag{18}$$
$$a_{21} = -\sin\phi(\omega_y - \beta_y) - \cos\phi(\omega_z - \beta_z) \tag{19}$$
$$a_{25} = -\cos\phi \tag{20}$$
$$a_{26} = \sin\phi \tag{21}$$
$$a_{31} = \cos\phi \cos^{-1}\theta(\omega_y - \beta_y) - \sin\phi \cos^{-1}\theta(\omega_z - \beta_z) \tag{22}$$
$$a_{32} = \sin\phi \cos^{-2}\theta \sin\theta(\omega_y - \beta_y) + \cos\phi \cos^{-2}\theta \sin\theta(\omega_z - \beta_z) \tag{23}$$
$$a_{35} = -\sin\phi \cos^{-1}\theta \tag{24}$$
$$a_{36} = -\cos\phi \cos^{-1}\theta. \tag{25}$$

4.3 Correction

Let

$$\boldsymbol{z} = \begin{bmatrix} \hat{\boldsymbol{a}}^b \\ \hat{\boldsymbol{m}}^b \end{bmatrix} + \boldsymbol{\eta}_z \tag{26}$$

with $\hat{\boldsymbol{a}}^b = \boldsymbol{a}^b/||\boldsymbol{a}^b||$ and $\hat{\boldsymbol{m}}^b = \boldsymbol{m}^b/||\boldsymbol{m}^b||$ being the normalized accelerometer and magnetometer measurements. The measurement equation is

$$\boldsymbol{h}(\boldsymbol{x}) = \begin{bmatrix} -\boldsymbol{R}^\top \hat{\boldsymbol{g}}^n \\ \boldsymbol{R}^\top \hat{\boldsymbol{m}}^n \end{bmatrix} \tag{27}$$

where $\boldsymbol{R}$ is the current estimated attitude constructed from the Euler angles, $\hat{g}^n = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^\top$ and $\hat{m}^n$ is the normalized local magnetic field vector. The correction step is then

$$\boldsymbol{K}_k = \boldsymbol{P}_k^- \boldsymbol{H}_k^\top \left(\boldsymbol{H}_k \boldsymbol{P}_k^- \boldsymbol{H}_k^\top + \boldsymbol{L}\right)^{-1} \tag{28}$$
$$\boldsymbol{x}_k^+ = \boldsymbol{x}_k^- + \boldsymbol{K}_k \left(\boldsymbol{z}_k - \boldsymbol{h}(\boldsymbol{x}_{\boldsymbol{k}}^-)\right) \tag{29}$$
$$\boldsymbol{P}_k^+ = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}_k)\, \boldsymbol{P}_k^- \tag{30}$$

where L is the measurement noise covariance and

$$\boldsymbol{H}_k = \left.\frac{\partial \boldsymbol{h}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_k^-} = \begin{bmatrix} 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & 0 & 0 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

$$\begin{aligned}
a_{12} &= \cos\theta \\
a_{21} &= -\cos\phi\cos\theta \\
a_{22} &= \sin\phi\sin\theta \\
a_{31} &= \cos\theta\sin\phi \\
a_{32} &= \cos\phi\sin\theta \\
a_{42} &= -\hat{m}_z\cos\theta - \hat{m}_x\cos\psi\sin\theta - \hat{m}_y\sin\psi\sin\theta \\
a_{43} &= \hat{m}_y\cos\psi\cos\theta - \hat{m}_x\cos\theta\sin\psi \\
a_{51} &= \hat{m}_x\left(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta\right) - \hat{m}_y\left(\cos\psi\sin\phi - \cos\phi\sin\psi\sin\theta\right) \\
&\quad + \hat{m}_z\cos\phi\cos\theta \\
a_{52} &= \hat{m}_x\cos\psi\cos\theta\sin\phi - \hat{m}_z\sin\phi\sin\theta + \hat{m}_y\cos\theta\sin\phi\sin\psi \\
a_{53} &= -\hat{m}_x\left(\cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta\right) - \hat{m}_y\left(\cos\phi\sin\psi - \cos\psi\sin\phi\sin\theta\right) \\
a_{61} &= \hat{m}_x\left(\cos\phi\sin\psi - \cos\psi\sin\phi\sin\theta\right) - \hat{m}_y\left(\cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta\right) \\
&\quad - \hat{m}_z\cos\theta\sin\phi \\
a_{62} &= \hat{m}_x\cos\phi\cos\psi\cos\theta - \hat{m}_z\cos\phi\sin\theta + \hat{m}_y\cos\phi\cos\theta\sin\psi \\
a_{63} &= \hat{m}_x\left(\cos\psi\sin\phi - \cos\phi\sin\psi\sin\theta\right) + \hat{m}_y\left(\sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta\right).
\end{aligned}$$

5 Conclusion

The EKF will give a state estimate which can be used to correct the gyro measurement before using it for feedback. In other words the gyro corrected measurement can be calculate using the EKF as

$$\boldsymbol{\omega}^b = \begin{bmatrix} 1 & 0 & -\alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\boldsymbol{\omega}_m^b - \boldsymbol{\beta}_\omega\right). \tag{32}$$