Limitations and Improvements of the Intelligent Driver Model (IDM)

Saleh Albeaik, Alexandre Bayen, Maria Teresa Chiri, Xiaoqian Gong, Amaury Hayat, Nicolas Kardous, Alexander Keimer, Sean T. McQuade, Benedetto Piccoli, Yiling You

Abstract

This contribution analyzes the widely used and well-known "intelligent driver model" (briefly IDM), which is a second order car-following model governed by a system of ordinary differential equations. Although this model was intensively studied in recent years for properly capturing traffic phenomena and driver braking behavior, a rigorous study of the well-posedness of solutions has, to our knowledge, never been performed. First it is shown that, for a specific class of initial data, the vehicles' velocities become negative or even diverge to $-\infty$ in finite time, both undesirable properties for a car-following model. Various modifications of the IDM are then proposed in order to avoid such ill-posedness. The theoretical remediation of the model, rather than *post facto* by ad-hoc modification of code implementations, allows a more sound numerical implementation and preservation of the model features. Indeed, to avoid inconsistencies and ensure dynamics close to the one of the original model, one may need to inspect and clean large input data, which may result practically impossible for large-scale simulations. Although well-posedness issues occur only for specific initial data, this may happen frequently when different traffic scenarios are analyzed, and especially in presence of lane-changing, on ramps and other network components as it is the case for most commonly used micro-simulators. On the other side, it is shown that well-posedness can be guaranteed by straightforward improvements, such as those obtained by slightly changing the acceleration to prevent the velocity from becoming negative.

Keywords: IDM, Intelligent driver model, system of ODEs, discontinuous ODEs, traffic modelling, microscopic traffic modelling, car following model, well-posedness of ODEs, existence and uniqueness of solutions of ODE;
2010 MSC: 34A12, 34A38, 65L05, 65L08

1. Introduction

The field of car following modeling historically goes back to the early 1950' (and probably before) [7, 11]. Most of the early work in this field focused on establishing the model equations, without paying much attention to the mathematical framework required to characterize solutions to the resulting ordinary differential equations (ODEs) describing the motion of the vehicles. The models are mainly classified into *acceleration models* for longitudinal movement, *lane-changing models* for lateral movement and *decisional models* for discrete-choice situations. Among all the car-following models introduced so far ([3, 5, 6, 22, 23] to just name a few) it is worth mentioning the Gazis-Herman-Rothery (GHR) model [7] which determines the relative velocity between two-lanes based vehicles, the Safe Distance Model [12], the Optimal Velocity Model [4] in which the acceleration of the single vehicle is controlled according to the velocity of the leading vehicle, and the Intelligent Driver Model (IDM) which is subject of analysis in the present work. For a comprehensive overview of the main car-following models we refer to [28, 20, 15].

The IDM has been introduced in [27] and is a deterministic time-continuous model describing the dynamics of the positions and velocities of every vehicle. Similarly to any car-following model, the idea behind it is that drivers control their vehicles to react to the stimulus from preceding vehicles. It aims to balance two different aspects, the necessity to keep safe separation with the vehicle in front and the desire to achieve "free flow" speed. This model presents some peculiarities which made it subject of intense research in the last two decades. Indeed, it is constructed to be *collision-free*, all the parameters can be interpreted and

Preprint submitted to Elsevier *April 7, 2021*

empirically measured, the stability of the model can be calibrated to empirical data, and there exists an equivalent macroscopic counterpart [13]. In literature we find many extensions of the original IDM, each of which seeks to incorporate new realistic features. The Enhanced IDM [16] presents an improved heuristic of the IDM useful for multi-lane simulations, which prevents the model from "over-reactions" even when the driver of the leading vehicle suddenly brakes with the maximum possible deceleration.

The Foresighted Driver Model (FDM) starts from the IDM and assumes that a driver acts in a way that balances predictive risk (due to possible collisions along his route) with utility (time required to travel, smoothness of the ride) [10].

Other extensions of the IDM aim to improve the driver safety and to respect the vehicle capability [24], to strengthen the power of each vehicle in proportion to the immediately preceding vehicle [17], to incorporate the spatially varying velocity profile to account the variation in different types of maneuvers through intersection [18]. Another natural extension is given by Multi-anticipative IDM [30] which models the reaction of driver to several vehicles ahead just by summing up the corresponding vehicle-vehicle pair interactions with the same weight coefficients.

More recently, also stochastic versions of the IDM have been introduced: to describe a probabilistic motion prediction applicable for long term trajectory planning [14], to study mechanisms behind traffic flow instabilities, indifference regions of finite human perception thresholds and external noise [29], to incorporate context-dependent upper and lower bounds on acceleration [26, 25].

Throughout the decades, most of the engineering community worked on improving the ability of the models to capture specific behavioral phenomena, at the expense of the characterization of the solutions. Thus, to this day, only few articles use models, and corresponding solutions, that are well characterized in terms of existence, uniqueness, and regularity. An example of this practice is provided by the double integrator $\ddot{x}(t) = u \in U$, where U is the input set, used abundantly as a canonical example in numerous control articles. On the other hand, when models have inherent flaws leading to unbounded or undefined solutions, as in the case of unbounded acceleration, ad hoc methods have been traditionally applied *post facto* by engineering the numerical implementations. For instance, in commonly used microsimulation tools, such as SUMO [19], Aimsun [2] and others, unbounded quantities are clipped, leading to "acceptable" numerical solutions. However, in the process, the fidelity to the original model is compromised, and the numerical simulations may not represent any instantiation of the model. Consequently, the properties of the considered continuous model might be lost as well. Finally, the process of clipping can introduce additional issues, not necessarily present in the originally model, and prevent the definition of any theoretical models corresponding to the obtained numerical simulations. The present article thus attempts to provide a full pipeline in which the model is first mathematically well defined (including existence, uniqueness and regularity characterization of the solutions), and then numerically implemented using appropriate numerical differentiation schemes. The final achievement is a thorough correspondence between theory and implementation.

1.1. The aim of this contribution

The introduced IDM has two mathematical and modelling drawbacks:

- The velocities of specific vehicles might become negative at specific times, which might not be desirable from a modelling point of view.
- The velocities of specific vehicles might diverge to $-\infty$ in finite time, so that the solution of the system of ODE's ceases to exist.

We will discuss these drawbacks and determine under which conditions on the initial datum and parameters they can happen. Additionally, we will establish the well-posedness of the IDM for certain parameters. At last, we will present several improvements so that the solutions exist on every finite time horizon.

1.2. Structure of this article

The paper is organized in the following way. In Section 2 we review the classical Intelligent Driver Model (IDM) and describe briefly the physical meaning of the parameters involved. Well-posedness of the IDM for small time horizon is stated in Theorem 1. In Section 3 we analyze peculiar and possibly pathological behaviors of the model. Specifically, we provide explicit settings in which it produces negative velocities (Example 1), negative velocities and blow-up of the solution in finite time (Example 2), negative velocities and blow-up in finite time (Example 3). Section 4 collects the main results of this work. Existence and uniqueness of a solution for all times and "safe" initial configurations of the drivers are stated and proved Theorem 2. Moreover, these solutions show to have finite speed as long as they exist (Lemma 4) and to be "collision free" in the sense that the minimum distance between vehicles remains bounded, as proved in Theorem 3.

Section 5 is devoted to the exploration, analysis and comparison of adjustments to the classic IDM in order to avoid the problems mentioned in Section 3 for general initial data. To this end, we introduce four slightly modified versions of the IDM for which well-posedness is proved: the *projected IDM*, the *acceleration projected IDM*, the *vra- IDM* (velocity regularized acceleration) and the *dra- IDM* (distance regularized acceleration) defined respectively in Definition 4, Definition 5, Definition 6 and Definition 7. A further and more drastic adjustment to the classic model is proposed in Definition 8 which involves a discontinuous acceleration, and therefore denoted as *discontinuous improvement*.

Finally, in Section 6 we draw conclusions from our work and mention possible research directions opened by this contribution.

2. The intelligent driver model (IDM): Definitions and basic results

In this section we introduce the intelligent driver model (IDM) as the following system of ordinary differential equations. To this end, we require to define the acceleration function as follows:

Definition 1 (The IDM acceleration). *Let $T \in \mathbb{R}_{>0}$ be fixed. For a parameter set $(a, b, v_{free}, \tau, s_0, l, \delta) \in \mathbb{R}^3_{>0} \times (0, T) \times \mathbb{R}^2_{>0} \times \mathbb{R}_{>1}$ we define the following IDM car-following acceleration on the set*

$$\mathcal{A} := \{(x, v, x_l, v_l) \in \mathbb{R}^4 : \ x_l - x - l > 0\},$$
$$\mathrm{Acc} : \begin{cases} \mathcal{A} & \to \mathbb{R} \\ (x, v, x_l, v_l) & \mapsto a\bigg(1 - \big(\tfrac{|v|}{v_{free}}\big)^{\delta} - \Big(\tfrac{2\sqrt{ab}(s_0+v\tau)+v(v-v_l)}{2\sqrt{ab}(x_l-x-l)}\Big)^2\bigg). \end{cases}$$

Remark 1 (Absolute values in the IDM Acceleration). *It is worth mentioning that in most literature the parameter δ is not precisely specified except that it is assumed to be positive. However, as we will show, velocities can become negative and this is why we assumed that the acceleration term in Definition 1 involves the absolute value of the velocity so that the term*

$$\big(\tfrac{|v|}{v_{free}}\big)^{\delta} \tag{1}$$

is well defined for all $v \in \mathbb{R}$ and $\delta \in \mathbb{R}_{>0}$. Obviously, this is only one choice and it might be more reasonable to replace this by

$$\mathrm{sgn}(v)\big(\tfrac{|v|}{v_{free}}\big)^{\delta} \tag{2}$$

so that this term contributes positive to the acceleration for negative velocities and indeed counteracting a negative velocity. For $\delta \in 2\mathbb{N}_{\geq 1} + 1$ Eq. (2) can actually be replaced by the version without the absolute value and the same is true for $\delta \in 2\mathbb{N}_{\geq 1}$ with the drawback that this part of the acceleration will always remain negative as we also assume for now in Definition 1.

An analysis similar as the one in this paper can then be carried out with mentioning that although in this case the solution's velocity can diverge to $-\infty$. However, in the case that $\delta \in \mathbb{R}_{>4}$ it can be shown that the solution always exists and the velocity cannot diverge. We do not go into details.

As we will require for the leader a specific acceleration, the "free-flow acceleration", we define as follows

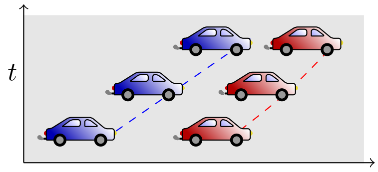

Figure 1: For $t \in [0,T]$ the leader $x_l(t)$ with its dynamics determined by the acceleration $u_{\text{lead}}(t)$ and the follower $x(t)$ with its dynamics governed by the classical IDM (acceleration Acc as in Definition 1). The overall dynamics is stated in Definition 3. The follower approaches the leader. Will they collide?

Definition 2 (Free flow acceleration)**.** *For $(a, v_{free}, \delta) \in \mathbb{R}^2_{>0} \times \mathbb{R}_{>1}$ the* ***free flow*** *acceleration is defined by*

$$\mathrm{Acc}_{\text{front}} : \begin{cases} \mathbb{R}^2 & \to \mathbb{R} \\ (x, v) & \mapsto a\Big(1 - \big(\frac{|v|}{v_0}\big)^{\delta}\Big). \end{cases}$$

Having defined the acceleration function, we are ready to present Definition 3:

Definition 3 (The IDM)**.** *Given Definition 1, we call the following system of ordinary differential equations in position $\boldsymbol{x} = (x_l, x) : [0,T] \to \mathbb{R}^2$ and velocity $\boldsymbol{v} = (v_l, v) : [0,T] \to \mathbb{R}^2$*

$$\begin{aligned} \dot{x}_l(t) &= v_l(t), && t \in [0,T], \\ \dot{v}_l(t) &= u_{lead}(t), && t \in [0,T], \\ \dot{x}(t) &= v(t), && t \in [0,T], \\ \dot{v}(t) &= \mathrm{Acc}(x(t), v(t), x_l(t), v_l(t)), && t \in [0,T], \\ (x_l(0), x(0)) &= (x_{l0}, x_0), \\ (v_l(0), v(0)) &= (v_{l0}, v_0). \end{aligned} \tag{3}$$

with leading dynamics $u_{lead} : [0,T] \to \mathbb{R}$ the car-following ***IDM****. $(x_0, x_{l0}, v_0, v_{l0}) \in \mathbb{R}^2 \times \mathbb{R}^2_{\geq 0}$ are initial positions and velocities.*

This is schematically illustrated in Fig. 1 We give a short overview of some of the meanings of the parameters in the IDM.

Remark 2 (Meaning of the previously introduced parameters)**.** *The parameters a, b, v_{free}, τ, s_0, l and δ, introduced in Definition 1, are model parameters which have – according to [27] – the following meaning:*

acceleration a: *the maximum vehicle acceleration;*

comfortable braking deceleration b: *a positive number;*

desired velocity v_{free}: *the velocity the vehicle would drive at in free traffic;*

desired time headway τ: *the minimum possible time to the vehicle in front;*

minimum spacing s_0: *a minimum desired net distance;*

the length of the vehicle l

the acceleration exponent δ: *Specifying how the acceleration decreases when approaching the desired velocity v_{free}.*

Table 1 shows some suggested values for the parameters already identified in [27].

For the system to be physically reasonable we require some additional assumptions on the order of the initial position and other parameters for the acceleration functions. This is made precise in the following Assumption 1.

Table 1: According to [27] typical and physical meaningful variables for the IDM

Parameters	Variable	Suggested value
Maximum acceleration	a	$0.73\ m/s^2$
Desired deceleration	b	$1.67\ m/s^2$
Desired velocity	v_{free}	$120\ km/h$
Desired time headway	τ	$1.6\ s$
Minimum spacing	s_0	$2\ m$
Length of the vehicle	l	$5\ m$
Acceleration exponent	δ	4

Assumption 1 (Assumptions on input datum and more)**.** *We assume that*

Leading velocity: $u_{lead} \in \mathcal{U}_{lead} := \Big\{u \in L^\infty((0,T)) : v_{l0} + \int_0^t u(s)\,\mathrm{d}s \geq 0\ \forall t \in [0,T]\Big\}.$

Input parameters for Acc**:** $(a, b, v_{free}, \tau, s_0, l, \delta) \in \mathbb{R}^3_{>0} \times (0,T) \times \mathbb{R}^2_{>0} \times \mathbb{R}_{>1}.$

Physical relevant initial datum: $(x_0, x_{l0}, v_0, v_{l0}) \in \mathbb{R}^2 \times \mathbb{R}^2_{\geq 0}:\ x_0 < x_{l0} - l.$

The previous assumption on the involved datum enables it to prove the well-posedness of solutions on a significantly small time horizon, i.e., that there exists a solution on the time horizon and that this solution is unique:

Theorem 1 (Well-posedness of sufficiently small time horizon)**.** *Given $N \in \mathbb{N}_{\geq 0}$ and Assumption 1, there exists a small enough time $T^* \in \mathbb{R}_{>0}$ so that the IDM in Definition 3 admits a unique solution $(x, x_l) \in W^{2,\infty}([0,T^*])^2$.*

Proof. The right hand side of Definition 3 is around the initial datum in Assumption 1 locally Lipschitz-continuous. The existence and uniqueness on a small time horizon then follows by the Picard-Lindelöf Theorem ([8, Chapter 4] or [9, Thm 1.3]). □

3. Counterexamples

Given Theorem 1, the next natural questions consist of whether the solution on the small time horizon can be extended to any finite time horizon and whether the model remains reasonable. As it turns out, neither points hold if we do not restrict our initial datum beyond Assumption 1. We present the shortcomings in the following.

3.1. Negative velocity

In this subsection, we show that the IDM can develop negative velocities for the following vehicle, although the leading vehicle might drive with positive speed. The reason for this is that if the following vehicle is too close to the leading vehicle, it needs to slow down. Assume now that it actually has already zero velocity, it will need to move backwards to make it to the "safety" distance s_0 it aims for.

Example 1 (Negative velocity)**.** *Assume that $x_0 = x_{l0} - l - \varepsilon$ for $\varepsilon \in \mathbb{R}_{>0}$ yet to be determined and $v_0 = 0$. Then, we compute the change of velocity for the following vehicle and have for $t \in [0,T]$ according to Definition 3*

$$\dot{v}(t) = a\bigg(1 - \big(\tfrac{|v(t)|}{v_{free}}\big)^\delta - \Big(\tfrac{2\sqrt{ab}(s_0 + v(t)\tau) + v(t)(v(t) - v_l(t))}{2\sqrt{ab}(x_l(t) - x(t) - l)}\Big)^2\bigg).$$

Plugging in $t = 0$ leads to

$$\dot{v}(0) = a\bigg(1 - \Big(\tfrac{2\sqrt{ab}s_0}{2\sqrt{ab}\varepsilon}\Big)^2\bigg) = a\Big(1 - \big(\tfrac{s_0}{\varepsilon}\big)^2\Big).$$

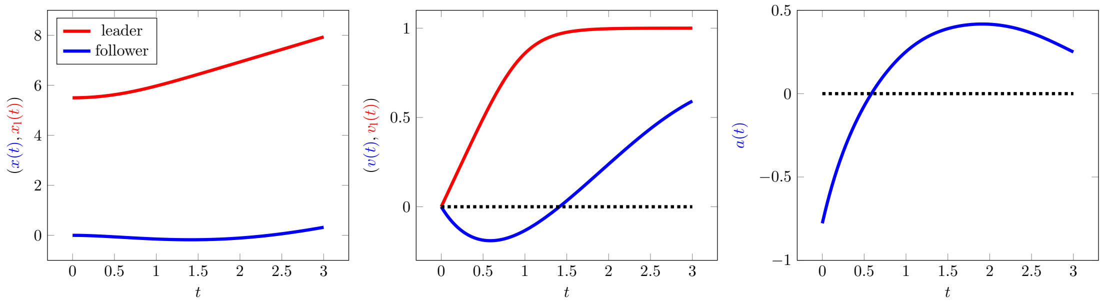

Figure 2: The IDM with parameters $a = 1,\ b = 2,\ v_{\text{free}} = 1,\ \tau = 1.6,\ l = 4,\ s_0 = 2,\ d = 4$ and datum $x_0 = 0,\ x_{l0} = l + 1.5 < l + s_0,\ v_0 = 0,\ v_{l0} = 0$. **Left** represents vehicles' positions, **middle** vehicles' velocities and **right:** followers acceleration. The leader follows the free flow acceleration profile as in Definition 2. As the initial distance between the two vehicles is smaller than s_0, the following vehicle moves backwards to increase the space.

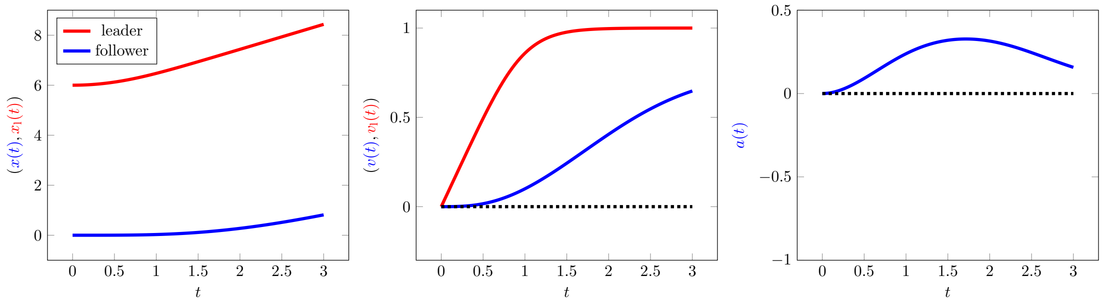

Figure 3: Continuation of Fig. 2: The IDM with parameters $a = 1,\ b = 2,\ v_{\text{free}} = 1,\ \tau = 1.6,\ l = 4,\ s_0 = 2,\ d = 4$ and datum $x_0 = 0,\ x_{l0} = l + 2 = l + s_0,\ v_0 = 0,\ v_{l0} = 0$. The leader follows the free flow acceleration as in Definition 2. The initial distance between the two vehicles is large enough so that the following vehicle does not attain negative velocity.

Thus, whenever $\varepsilon < s_0$, the following vehicle has – at least for small time horizon – a negative speed, although the leading vehicle drives with arbitrary speed $v_l \in \mathbb{R}$. This is also detailed in the following Fig. 2 and in Fig. 3 demonstrated that for larger spacing this does not occur:

A more reasonable approach for avoiding this type of behavior is that the following car just waits until the leading car has moved farther away. This can be achieved by adjusting the model accordingly as done in Section 5.

3.2. Velocity exploding in finite time

In this subsection, we show that the solution can cease to exist in finite time. We first present an example, with fixed parameters, to explain the reasons behind this phenomenon. Then, we generalize the example and illustrate how this phenomenon may occur for parameters in a whole region of the space.

Example 2 (Negative velocity and a blow-up of the solution in finite time)**.** *Assume the constants and initial data are as in Example 1, with $0 < \varepsilon < 1$ and positive initial velocity of the leading vehicle $v_{l0} \geq 0$. The leading vehicle's position is given by*

$$x_l(t) = x_{l0} + v_{l0}t + \tfrac{u}{2}t^2,$$

and the leading vehicle's velocity by

$$v_l(t) = v_{l0} + ut.$$

Plugging this into the equations for the follower, we obtain the following system of ODEs

$$\dot{x}(t) = v(t)$$

$$\dot{v}(t) = a\Bigg(1 - \big(\tfrac{|v(t)|}{v_{free}}\big)^{\delta} - \Big(\tfrac{2\sqrt{ab}(s_0+v(t)\tau)+v(t)(v(t)-v_l(t))}{2\sqrt{ab}\cdot(x_l(t)-x_{l0}+\varepsilon-\int_0^t v(s)\,\mathrm{d}s)}\Big)^2\Bigg)$$
$$x(0) = x_{l0} - l - \varepsilon$$
$$v(0) = 0.$$

Now we fix $v_{free} = 1 = a$*;* $s_0 = 16$*;* $b = \frac{1}{4a}$*;* $\tau = 8$*;* $v_{l0} = 0$*;* $u = 0$*;* $\delta = 4$*;* $x_{l0} = 0$*, which gives*

$$\dot{x}(t) = v(t)$$
$$\dot{v}(t) = 1 - \big(v(t)\big)^4 - \Big(\tfrac{(4+v(t))^2}{\varepsilon-\int_0^t v(s)\,\mathrm{d}s}\Big)^2.$$

We now show that there exists $\bar{t}$ *such that* $v(\bar{t}) < -1$*. Assume, by contradiction, that* $v(t) \geq -1$ *on* $t \in [0,1]$*, then since* $\varepsilon < 1$ *for* $t \in [0,1]$ *we get*

$$\dot{v}(t) \leq 1 - \Big(\tfrac{(4+v(t))^2}{\varepsilon-\int_0^t v(s)\,\mathrm{d}s}\Big)^2 \leq 1 - \big(\tfrac{4}{\varepsilon+1}\big)^2 \leq 1 - \big(2\big)^2 = -3, \tag{4}$$

therefore $v(1) \leq -3 < -1$ *reaching a contradiction.*

Blow-up of the solution in finite time. Let $t^* \in (0,T]$ *denote the first time such that* $v(t^*) < -1$*. Then, recalling the semi-group property of ODEs, we can consider the initial value problem in* $\tilde{v} : [t^*,T] \to \mathbb{R}$

$$\tilde{v}(t^*) = v(t^*) < -1$$
$$\dot{\tilde{v}}(t) = 1 - \big(\tilde{v}(t)\big)^4 - \Big(\tfrac{(4+\tilde{v}(t))^2}{\varepsilon-\int_0^t \tilde{v}(s)\,\mathrm{d}s}\Big)^2 \qquad t \in [t^*,T].$$

Estimating $\dot{\tilde{v}}$*, we have*

$$\dot{\tilde{v}}(t) \leq 1 - \big(\tilde{v}(t)\big)^4 \quad \forall t \in [t^*,T].$$

As the initial value $\tilde{v}(t^*) < -1$ *we obtain that* $\tilde{v}$ *is monotonically decreasing and we can thus estimate*

$$\dot{\tilde{v}}(t) \leq 1 - \big(\tilde{v}(t)\big)^2 \quad \forall t \in [t^*,T]$$

which can be solved explicitly to obtain

$$\tilde{v}(t) \leq \tfrac{v(t^*)\exp(2t)+v(t^*)+\exp(2t)-1}{\exp(2t)+1+v(t^*)(\exp(2t)-1)}.$$

However, the right hand side goes to $-\infty$ *when* $t \to \frac{1}{2}\ln\Big(\frac{-1+v(t^*)}{1+v(t^*)}\Big)$ *and thus, also the solution for* v *ceases to exist for* $t \geq t^{**} + \frac{1}{2}\ln\Big(\frac{-1-v(t^*)}{1+v(t^*)}\Big)$*. This is illustrated in Fig. 4.*

In this example, we chose very specific parameters for convenience. However, the same behavior and divergence of the speed to $-\infty$ in finite time can be shown for a rather general range of parameters. This is presented in Example 3. To prove this, we start by showing the following Lemma.

Lemma 1 (Sufficiently negative velocity in small time)**.** *Assume that the initial velocity of the following vehicle* $v_0 = 0$*, the initial velocity of the leading vehicle* $v_{l0} > 0$*, and the initial positions of the two vehicles are separated by* $l + \varepsilon$ *for some* $\varepsilon \in (0, s_0)$*, i.e.,* $x_{l0} - x_0 - l = \varepsilon < s_0$*. Choose parameters* s_0*,* τ *and* v_{free} *such that* $-\frac{s_0}{1.01\tau} < -v_{free}$*, let* $\delta > 1$ *and let* $v_{\max} > 0$ *be an upper bound for the vehicles' velocities. Then there exists* $t^{**} \in [0,T]$*, such that* $v(t^{**}) < -v_{free}$*.*

Proof. We consider two different cases:

If there exists $t^* \in [0,T]$, such that $v(t^*) \leq -\frac{s_0}{1.01\tau} < -v_{\mathrm{free}}$, then we chose $t^{**} = t^*$.

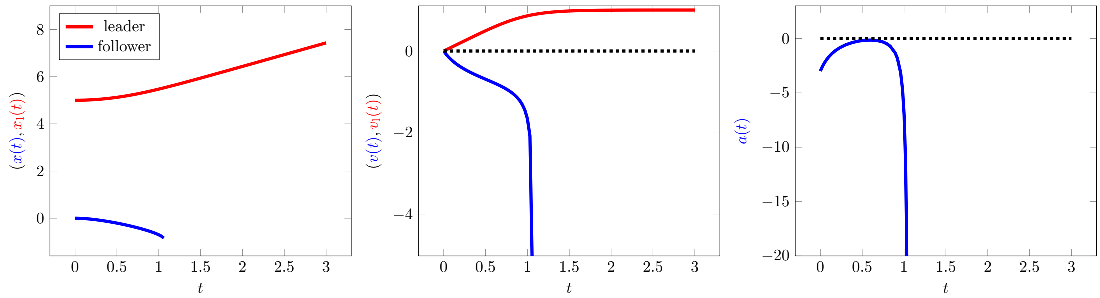

Figure 4: Same parameters as in Fig. 2, but $x_{l0} = l + 1 < l + s_0$. **Left:** Positions of the vehicles; **Middle:** Velocities of the vehicles, **Right:** Acceleration of the follower. Velocity of the follower diverging to $-\infty$ around $t \approx 1.06$. The solution ceases to exist for larger t.

Otherwise, we assume that for every $t \in [0, T]$, $v(t) > -\frac{s_0}{1.01\tau}$, i.e.,

$$s_0 + v(t)\tau > \tfrac{s_0}{101} > 0 \tag{5}$$

and $v(t) \geq -v_{\text{free}}$. By the definition of $v_{\max}$ and v_{free}, we have for every $t \in [0, T]$, $v_l(t) - v(t) \leq v_{\max} + v_{\text{free}}$. Furthermore, since $v_l(t) > 0$ and $v(t) < 0$, we have $v(t)(v(t) - v_l(t)) > 0$. Then one can find an upper bound for the distance between the two vehicles. That is, for every $t \in [0, T]$,

$$x_l(t) - x(t) - l \leq \varepsilon + (v_{\max} + v_{\text{free}})t. \tag{6}$$

Therefore, for every $t \in [0, T]$,

$$\begin{aligned}
\dot{v}(t) &= a\left(1 - \left(\tfrac{|v(t)|}{v_{\text{free}}}\right)^{\delta} - \left(\tfrac{2\sqrt{ab}(s_0+v(t)\tau)+v(t)(v(t)-v_l(t))}{2\sqrt{ab}(x_l(t)-x(t)-l)}\right)^2\right) &(7)\\
&\leq a\left(1 - \left(\tfrac{2\sqrt{ab}(s_0+v(t)\tau)+v(t)(v(t)-v_l(t))}{2\sqrt{ab}(x_l(t)-x(t)-l)}\right)^2\right) &(8)\\
&\leq a\left(1 - \left(\tfrac{s_0+v(t)\tau}{x_l(t)-x(t)-l}\right)^2\right) &(9)\\
&\leq a\left(1 - \left(\tfrac{s_0+v(t)\tau}{\varepsilon+(v_{\max}+v_{\text{free}})t}\right)^2\right) &(10)\\
&\leq a\left(1 - \left(\tfrac{\frac{s_0}{101}}{\varepsilon+(v_{\max}+v_{\text{free}})t}\right)^2\right). &(11)
\end{aligned}$$

Note that inequality (8) is true since $\frac{|v(t)|}{v_{\text{free}}}$ is non-negative, inequality (9) is due to the fact that for every $t \in [0, T]$, $v(t)(v(t) - v_l(t)) > 0$, inequality (10) is because of the upper bounded for the distance between the two vehicles given by inequality (6), and inequality (11) is due to inequality (5).

Hence, for every $t \in [0, T]$,

$$\begin{aligned}
v(t) &\leq a\int_0^t \left(1 - \left(\tfrac{\frac{s_0}{101}}{\varepsilon+(v_{\max}+v_{\text{free}})s}\right)^2\right) \mathrm{d}s\\
&= a\left(t + \tfrac{s_0^2}{101^2(v_{\max}+v_{\text{free}})}\left(\tfrac{1}{\varepsilon+(v_{\max}+v_{\text{free}})t} - \tfrac{1}{\varepsilon}\right)\right) = at\left(1 - \tfrac{s_0^2}{101^2}\tfrac{1}{\varepsilon(\varepsilon+(v_{\max}+v_{\text{free}})t)}\right).
\end{aligned}$$

Setting $t = \varepsilon$, we have,

$$v(\varepsilon) \leq a\varepsilon\left(1 - \tfrac{s_0^2}{101^2}\tfrac{1}{\varepsilon(\varepsilon+(v_{\max}+v_{\text{free}})\varepsilon)}\right) = a\tfrac{101^2\varepsilon(\varepsilon+(v_{\max}+v_{\text{free}})\varepsilon)-s_0^2}{101^2(\varepsilon+(v_{\max}+v_{\text{free}})\varepsilon)} \to -\infty \text{ as } \varepsilon \to 0.$$

Note that the upper bound for the velocity at time $t = \varepsilon$ goes to negative infinity as ε goes to zero. Therefore, for $\varepsilon > 0$ small enough, we have,

$$v(\varepsilon) \leq -v_{\text{free}}.$$

Hence, there exists $t^{**} = \varepsilon \in [0, T]$, such that $v(t^{**}) < -v_{\text{free}}$. $\square$

The previous Lemma guarantees that for properly chosen initial datum and velocity the follower's velocity can become more negative than $-v_0$, the negative free-flow velocity. This enables us to prove that the solution ceases to exist in finite time. This is related to the famous example of a blowup of ODEs in finite time:

Example 3 (Negative velocity and a blow-down of the velocity in finite time)**.** *Again we assume the parameters and initial data as in Lemma 1. Then by Lemma 1, there exists $t^{**} \in [0, T]$, such that $v(t^{**}) < -v_0$. Recall that $\delta > 1$. For any time t such that $v(t) < -v_0$,*

$$\dot{v}(t) \leq a\Big(1 - \big(\tfrac{|v(t)|}{v_0}\big)^{\delta}\Big) < 0$$

Thus v is strictly decreasing on the time interval $[t^{**}, T]$. In addition, we are going to show that v cease to exist in finite time and that there exists $t_1 > 0$ such that $\lim\limits_{t \to t_1} v(t) = -\infty$. Assume by contradiction that this is not the case. Then as $v(T) < -v_0$ the solution remains strictly decreasing as long as it exists and as does not reach $-\infty$ in finite time the solution can be extended on $[0, +\infty)$ and is strictly decreasing on $[T, +\infty)$. Therefore we have the following

$$\int_T^t \frac{\dot{v}(s)}{1 - \Big(\frac{|v(s)|}{v_{free}}\Big)^{\delta}} ds \geq a(t - T), \ \forall t \in [T, +\infty). \tag{12}$$

As v is strictly decreasing we can perform a change of variable in the integral by setting $y = -v(s)$ to get

$$\int_{-v(T)}^{-v(t)} \frac{1}{\Big(\frac{y}{v_{free}}\Big)^{\delta} - 1} dy \geq a(t - T), \ \forall t \in [T, \infty). \tag{13}$$

Note that $-v(T) > v_0$. We denote $\eta := -v(T) - v_{free} > 0$, Eq. (13) *implies*

$$\int_{v_{free}+\eta}^{+\infty} \frac{1}{\Big(\frac{y}{v_{free}}\Big)^{\delta} - 1} dy \geq a(t - T), \ \forall t \in [T, \infty). \tag{14}$$

Letting $t \to +\infty$ this implies that

$$\int_{v_{free}+\eta}^{+\infty} \frac{1}{\Big(\frac{y}{v_{free}}\Big)^{\delta} - 1} dy = \infty, \tag{15}$$

but because $\delta > 1$ we have $\int_{v_{free}+\eta}^{+\infty} \frac{1}{\Big(\frac{y}{v_{free}}\Big)^{\delta} - 1} dy \in \mathbb{R}$, which gives a contradiction. Therefore, v ceases to exist and converges to $-\infty$ in finite time.

As we have seen from the previous Example 3 the velocity can blow up in finite time. However, what is not clear is whether the position of the car can consequently also explode. Thanks to the relation between position and velocity, i.e., $x'(t) = v(t),\ t \in [0, T]$ this is a matter of whether $v \in L^1((0, t^*))$ if t^* is the time where the velocity goes to $-\infty$. And indeed, it can be shown that this holds true and the position remains bounded:

Corollary 1 (Boundedness of the position in the case of a blow up of velocity)**.** *Let Assumption 1, $\delta \in \mathbb{R}_{>2}$ and assume that – as investigated in Example 3 – there exists a time horizon $t^* \in \mathbb{R}_{>0}$ so that*

$$\lim_{t \nearrow t^*} v(t) = \lim_{t \nearrow t^*} \dot{x}(t) = -\infty.$$

Then, the position at the time of the blow up remains finite, i.e.

$$\exists c \in \mathbb{R} : \lim_{t \nearrow t^*} x(t) = c$$

or equivalently stated

$$v \in L^1((0, t^*)).$$

Proof. The proof consists of showing that the L^1 mass of the velocity remains bounded. To this end, we estimate the acceleration from above. Choose $t_1 \in [0, t^*]$ so that

$$\Big((v(s)^2 - v(s)v_l(s)) + 2\sqrt{ab}(s_0 + v(s)\tau)\Big)^2 > 0 \wedge v(s) \leq -\max(2v_{\text{free}}, 1) \qquad \forall s \in [t_1, t^*).$$

Such a t_1 always exists as v diverges to $-\infty$ so that for s close enough to t^* the quadratic term in the previous estimate will always outnumber the affine linear term and v_l is essentially bounded. Then, recalling the definition of the IDM in Definition 1 we have for $s \in [t_1, t^*)$

$$\dot{v}(t) \leq -\frac{a}{2}\left(\tfrac{|v(t)|}{v_{\text{free}}}\right)^\delta. \tag{16}$$

Assuming $\delta \in \mathbb{R}_{>2}$ divide by $|v(t)|^{\delta-1}$ (this is possible because v is strictly decreasing and $v(t_1) < -1$), and integrating between t_1 and $t \in [t_1, t^*)$, one has

$$\begin{aligned}\int_{t_1}^{t} \tfrac{\dot{v}}{|v(\tau)|^{\delta-1}} d\tau &\leq -\tfrac{a}{2v_{\text{free}}^\delta}\int_{t_1}^{t} |v(\tau)| d\tau,\\ \tfrac{(-v)^{2-\delta}(t)}{\delta-2} - \tfrac{(-v)^{2-\delta}(t_1)}{\delta-2} &\leq -\tfrac{a}{2v_{\text{free}}^\delta}\int_{t_1}^{t} |v(\tau)| d\tau.\end{aligned} \tag{17}$$

Dividing by $-a/2v_{\text{free}}^\delta < 0$ and letting $t \to t^*$, this gives

$$\left(\tfrac{a}{2v_{\text{free}}^\delta}\right)^{-1} \tfrac{(-v)^{2-\delta}(t_1)}{\delta-2} \geq \|v\|_{L^1(t_1,t^*)}. \tag{18}$$

Hence, $\|v\|_{L^1(t_1,t^*)} < +\infty$.

□

This is particularly interesting as it also illustrates that the model behaves still reasonable (even in the case of a diverge of the velocity to $-\infty$) and underlines the fact that a change in the acceleration to prevent the velocity to diverge might be enough to "improve" the model (compare Section 5).

4. Well-posedness for specific initial datum and all times

In this section we state results guaranteeing existence and uniqueness of solutions on every finite time horizon under (suitable) conditions on the initial datum and the parameters.

Lemma 2 (Follower's velocity reaching leader's velocity in specific situations)**.** *Assume that for the dynamics in Definition 3 with parameters as in Assumption 1, the leading vehicle follows the free flow velocity, that is,*

$$\dot{v}_l(t) = a\left(1 - \left(\tfrac{v_l(t)}{v_{free}}\right)^\delta\right), \qquad t \in [0, T] \tag{19}$$

with $\delta > 0$ being even and in addition the initial positions and velocities of the vehicles satisfy

$$v_0 > v_{l0} > 0 \ \wedge x_{l0} - x_0 - l = s_0. \tag{20}$$

Then, there exists a certain time $t_1 \in [0, T]$ such that $v(t_1) = v_l(t_1)$.

Proof. Assume that such a $t_1 \in [0,T]$ would not exist. Then, by the continuity of the velocity and the initial velocity, for every $t \in [0,T]$, we have $v(t) > v_l(t)$. Furthermore, for every $t \in [0,T]$, $x_l(t) - x(t) - l < s_0$. Then for $t = \frac{v_0 - v_{l0} + 1}{a} > 0$ we obtain, using the acceleration profile in Definition 1

$$\begin{aligned} v(t) - v_l(t) &= v_0 - v_{l0} \\ &\quad + \int_0^t a \left(\tfrac{v_l(s)^\delta - v(s)^\delta}{v_{\text{free}}^\delta} - \left(\tfrac{2\sqrt{ab}(s_0 + v(s)\tau) + v(s)(v(s) - v_l(s))}{2\sqrt{ab}(x_l(s) - x(s) - l)} \right)^2 \right) \mathrm{d}s \\ &\le v_0 - v_{l0} - \int_0^t a \left(\left(\tfrac{2\sqrt{ab}(s_0 + v(s)\tau) + v(s)(v(s) - v_l(s))}{2\sqrt{ab}(x_l(s) - x(s) - l)} \right)^2 \right) \mathrm{d}s \\ &\le v_0 - v_{l0} - \int_0^t a \left(\tfrac{s_0}{x_l(s) - x(s) - l} \right)^2 \mathrm{d}s \\ &\le v_0 - v_{l0} - at \\ &= -1 \end{aligned}$$

which is a contradiction. Therefore, there exists $t_1 \in [0,T]$, such that $v(t) > v_l(t)$ for every $t \in [0,t_1)$ and $v(t_1) - v_l(t_1) = 0$. ☐

Lemma 3 (No potential collision before equilibrium of velocities)**.** *Let the assumptions of Lemma 2 hold, and in addition that the model parameters and initial datum satisfy*

$$s_0 - \tfrac{(v_0 - v_{l0})^2}{2a} > 0. \tag{21}$$

Then, the vehicles do not collide before they reach to the same velocity.

Proof. Recall from the proof of Lemma 2 that t_1 is the first time when the two vehicles reach to the same velocity. That is, for every $t \in [0,t_1)$, $v(t) > v_l(t)$ and $v(t_1) = v_l(t_1)$. Furthermore, for every $t \in [0,t_1]$,

$$\begin{aligned} x_l(t) - x(t) &= x_{l0} - x_0 + \int_0^t v_l(s) - v(s)\,\mathrm{d}s \\ &= x_{l0} - x_0 + (v_{l0} - v_0)t \\ &\quad + \int_0^t \left(\int_0^s a \left(\tfrac{v(\tilde t)^\delta - v_l(\tilde t)^\delta}{v_{\text{free}}^\delta} + \left(\tfrac{2\sqrt{ab}(s_0 + v(\tilde t)\tau) + v(\tilde t)(v(\tilde t) - v_l(\tilde t))}{2\sqrt{ab}(x_l(\tilde t) - x(\tilde t) - l)} \right)^2 \right) \mathrm{d}\tilde t \right) \mathrm{d}s \\ &\ge x_{l0} - x_0 + (v_{l0} - v_0)t + \int_0^t \left(\int_0^s a \left(\tfrac{2\sqrt{ab}(s_0 + v(\tilde t)\tau) + v(\tilde t)(v(\tilde t) - v_l(\tilde t))}{2\sqrt{ab}(x_l(\tilde t) - x(\tilde t) - l)} \right)^2 \mathrm{d}\tilde t \right) \mathrm{d}s \\ &\ge x_{l0} - x_0 + (v_{l0} - v_0)t + a \int_0^t \left(\int_0^s \left(\tfrac{s_0}{(x_l(\tilde t) - x(\tilde t) - l)} \right)^2 \mathrm{d}\tilde t \right) \mathrm{d}s \\ &\ge x_{l0} - x_0 + (v_{l0} - v_0)t + a \int_0^t \left(\int_0^s 1\,\mathrm{d}\tilde t \right) \mathrm{d}s \\ &= x_{l0} - x_0 + (v_{l0} - v_0)t + \tfrac{t^2}{2} a \\ &= l + s_0 + (v_{l0} - v_0)t + \tfrac{t^2}{2} a \\ &\ge l + s_0 - \tfrac{(v_0 - v_{l0})^2}{2a} > l. \end{aligned} \tag{22}$$

Thereby, we applied to condition on the initial velocities in relation to s_0 and the acceleration a in Eq. (21). ☐

Theorem 2 (Well-posedness for specific input datum)**.** *Assume that for the dynamics in Definition 3 with parameters as in Assumption 1, the leading vehicle follows the free flow velocity as defined in* (19)*, and the model parameters and initial datum satisfy* (20) *and* (21)*, then there exists* $\varepsilon_0 \in (0, s_0)$ *such that*

$$x_l(t) - x(t) - l \ge \varepsilon_0 \quad \forall t \in [0,T].$$

Proof. We again consider t_1 as defined in the proof of Lemma 2. That is, for every $t \in [0, t_1)$, $v(t) > v_l(t)$ and $v(t_1) - v_l(t_1) = 0$. Note that

$$\dot{v}(t_1) - \dot{v}_l(t_1) = -a\left(\frac{s_0+v(t_1)\tau}{x_l(t_1)-x(t)-l}\right)^2 < 0.$$

Therefore, there exist $t_2 > 0$, such that $v(t) < v_l(t)$ for every $t \in (t_1, t_2)$.

Hence, it is enough to show that $x_l(t_1) - x(t_1) - l \geq \varepsilon_0$ for $0 < \varepsilon_0 = s_0 + (v_{l0} - v_0)t_1 + \frac{t_1^2}{2}a < s_0$. By the previous Lemma 3 and in particular by (22), it holds that

$$\begin{aligned} x_l(t_1) - x(t_1) &\geq x_{l0} - x_0 + (v_{l0} - v_0)t_1 + \tfrac{t_1^2}{2}a, \\ &= s_0 + l + (v_{l0} - v_0)t_1 + \tfrac{t_1^2}{2}a. \end{aligned}$$

Thus, the only thing left to show is that $0 < s_0 + (v_{l0} - v_0)t_1 + \frac{t_1^2}{2}a < s_0$. To this end, recall that $v_l(t_1) - v(t_1) = 0$ implies

$$v_{l0} - v_0 = \int_0^{t_1} a\left(\frac{v_l(t)^\delta - v(t)^\delta}{v_{\text{free}}^\delta} - \left(\frac{2\sqrt{ab}(s_0+v(t)\tau)+v(t)(v(t)-v(t))}{12\sqrt{ab}(x_l(t)-x(t)-l)}\right)^2\right) \mathrm{d}t \leq -at_1.$$

Therefore, we obtain

$$(v_{l0} - v_0)t_1 + \tfrac{t_1^2}{2}a = t_1\left((v_{l0} - v_0) + \tfrac{t_1}{2}a\right) \leq t_1\left(-at_1 + \tfrac{t_1}{2}a\right) < 0$$

and thus, $\varepsilon_0 = s_0 + (v_{l0} - v_0)t_1 + \frac{t_1^2}{2}a < s_0$.

Furthermore, we claim that $\varepsilon_0 = s_0 + (v_{l0} - v_0)t_1 + \frac{t_1^2}{2}a > 0$. Indeed, for every $t \in [0, t_1]$, then

$$\dot{v}_l(t) - \dot{v}(t) = a\left(\frac{v(t)^\delta - v_l(t)^\delta}{v_{\text{free}}^\delta} + \left(\frac{2\sqrt{ab}(s_0+v(t)\tau)+v(t)(v(t)-v_l(t))}{2\sqrt{ab}(x_l(t)-x(t)-l)}\right)^2\right) > a.$$

Additionally, by Lemma 3, the vehicles do not collide before time t_1. Thus $(v_{l0} - v_0)t_1 + \frac{t_1^2}{2}a < -s_0$. Therefore, $\varepsilon_0 = s_0 + (v_{l0} - v_0)t_1 + \frac{t_1^2}{2}a > 0$. □

The next lemma shows that the drivers velocity remains bounded from above as long as a solution of the IDM exists.

Lemma 4 (Boundedness of the velocity). *As long as the solution to the IDM in Definition 3 exists, the velocity of the following vehicle satisfies*

$$v(t) \leq \max\{v_0, v_{free}\} \quad \forall t \in [0, T]. \tag{23}$$

Proof. We prove this result in two steps:

Velocity above v_{free}: Assume that the velocity of the follower satisfies at any given time $t \in [0, T]$

$$v(t) \geq v_{\text{free}}.$$

Then, plugging this into the acceleration function of the IDM gives

$$\dot{v}(t) = a\left(1 - \left(\frac{|v(t)|}{v_{\text{free}}}\right)^\delta - \left(\dots\right)^2\right) < 0$$

so that the velocity is monotonically decreasing and Eq. (23) is satisfied.

Velocity below v_{free}: Then, obviously Eq. (23) is satisfied and the proof is complete.

□

The following Theorem 3 states that whenever a solution to the IDM for a general acceleration profile u_{lead} for the follower exists and the initial datum is chosen according to Assumption 1 that a minimal distance between two cars is ensured. Theorem 3 is thus analogue to Theorem 2, however the minimal distance ε_0 is replaced by another smaller lower bound ε_0^*.

Theorem 3 (Lower bound on the vehicle's distance)**.** *Recall Definition 3 and the corresponding parameters in Assumption 1. Let $v_{max} \in \mathbb{R}_{>0}$ as instantiated in Lemma 4 be an upper bound for the velocity. Then, there exists $\varepsilon_0^* \in \mathbb{R}_{>0}$ such that if the initial position satisfies*

$$x_{l0} - x_0 - l > \varepsilon_0^*,$$

then the distance of the vehicles remains bounded from below over all times, i.e.

$$x_l(t) - x(t) - l \geq \varepsilon_0^* \qquad \forall t \in [0, T]$$

as long as the solution $\boldsymbol{x}$ exists. In particular, one may choose

$$\varepsilon_0^* = \tfrac{a s_0^2 \delta^*}{8(a\delta^* + 2v_{max}^2)} \quad \textit{with} \quad \delta^* = \tfrac{-3v_{max}^2 + \sqrt{9v_{max}^4 + 4a^2 s_0^2}}{2a}.$$

Note that ε_0^* is independent on the vehicles' initial velocities.

Proof. Assume that we have $x_{l0} - x_0 - l > \varepsilon_0$, for some $\varepsilon_0 > 0$. Let $t^* \in [0, T]$ be the first time such that

$$x_l(t^*) - x(t^*) - l = \varepsilon_0.$$

This means that for any $s \in [0, t^*]$: $x_l(s) - x(s) - l > \varepsilon_0$. It thus suffices to show that $v_l(t^*) - v(t^*) \geq 0$.

We prove by contradiction and assume that $v_l(t^*) - v(t^*) < 0$. Then, by the continuity of the velocity, there exists $t_0 < t^*$, such that for every $s \in [t_0, t^*)$, $v_l(s) - v(s) < 0$. In addition, the position difference of the two vehicles, $x_l - x$, decreases over the time interval $[t_0, t^*)$.

Assume that $x_l(t_0) - x(t_0) - l = \varepsilon_0 + \delta_0$ for some $\delta_0 > 0$. Note that $t^* - t_0 \geq \frac{\delta_0}{2v_{\max}}$. By the continuity of the position, $\delta_0 \to 0$ as $t_0 \to t^{*-}$. Thus, one can choose $t_0 > 0$ such that $\delta_0 < \min\left\{\varepsilon_0, \frac{-3v_{\max}^2 + \sqrt{9v_{\max}^4 + 4a^2 s_0^2}}{2a}\right\}$. Furthermore, we have,

$$\begin{aligned}
&v_l(t^*) - v(t^*) \\
&= v_l(t_0) - v(t_0) + \int_{t_0}^{t^*} u(s)\,\mathrm{d}s - \int_{t_0}^{t^*} a\left(1 - \left(\tfrac{|v(s)|}{v_{\text{free}}}\right)^\delta - \left(\tfrac{2\sqrt{ab}(s_0 + v(s)\tau) + v(s)(v(s) - v_l(s))}{2\sqrt{ab}(x_l(s) - x(s) - l)}\right)^2\right)\mathrm{d}s \\
&\geq v_l(t_0) - v(t_0) + \int_{t_0}^{t^*} u(s)\,\mathrm{d}s - \int_{t_0}^{t^*} a\,\mathrm{d}s + a\int_{t_0}^{t^*} \left(\tfrac{2\sqrt{ab}(s_0 + v(s)\tau) + v(s)(v(s) - v_l(s))}{2\sqrt{ab}(x_l(s) - x(s) - l)}\right)^2 \mathrm{d}s && (24)\\
&\geq -v(t_0) - \int_{t_0}^{t^*} a\,\mathrm{d}s + a\int_{t_0}^{t^*} \tfrac{s_0^2}{(x_l(s) - x(s) - l)^2}\,\mathrm{d}s && (25)\\
&\geq -v(t_0) - a(t^* - t_0) + a s_0^2 \int_{t_0}^{t^*} \tfrac{1}{(x_l(s) - x(s) - l)^2}\,\mathrm{d}s \\
&\geq -v(t_0) - a(t^* - t_0) + a s_0^2 \int_{t_0}^{t^*} \tfrac{1}{(\varepsilon_0 + \delta_0)^2}\,\mathrm{d}s && (26)\\
&\geq -v(t_0) - a(t^* - t_0) + \tfrac{a s_0^2}{(\varepsilon_0 + \delta_0)^2}(t^* - t_0) \\
&\geq -v(t_0) + \left(\tfrac{a s_0^2}{(\varepsilon_0 + \delta_0)^2} - a\right)(t^* - t_0) \\
&\geq -v(t_0) + \left(\tfrac{a s_0^2}{(\varepsilon_0 + \delta_0)^2} - a\right)\tfrac{\delta_0}{2v_{\max}}. && (27)
\end{aligned}$$

The above inequality (24) is due to the positivity of $\left(\frac{|v(s)|}{v_0}\right)^{\delta}$, inequality (25) is because of that the velocity of the leading vehicle, $v_l(t) > 0$ for every $t \in [0, T]$, inequality (26) is true since for every $s \in [t_0, t^*)$, $x_l(s) - x(s) - l < \varepsilon_0 + \delta_0$, and the last inequality (27) is derived by the fact that $t^* - t_0 > \frac{\delta_0}{2v_{\max}}$.

Note that since $\delta_0 \in \left(0, \frac{-3v_{\max}^2+\sqrt{9v_{\max}^4+4a^2s_0^2}}{2a}\right)$,

$$\lim_{\varepsilon_0\to 0}\left(\frac{as_0^2}{(\varepsilon_0+\delta_0)^2} - a\right)\delta_0 = a\left(\frac{s_0^2}{\delta_0^2} - 1\right)\delta_0 > 3v_{\max}^2.$$

For convenience, we set $\delta^* = \frac{-3v_{\max}^2+\sqrt{9v_{\max}^4+4a^2s_0^2}}{2a}$. Then $\delta_0 \in (0, \min\{\varepsilon_0, \delta^*\})$, and

$$\left(\frac{as_0^2}{(\varepsilon_0+\delta_0)^2} - a\right)\delta_0 > \frac{as_0^2\delta_0}{4\varepsilon_0^2} - a\delta^*.$$

Let $\varepsilon_0^* = \sqrt{\frac{as_0^2\delta^*}{4(a\delta^*+2v_{\max}^2)}}$. In particular, one can choose $\varepsilon_0 = \varepsilon_0^*$, then $\left(\frac{as_0^2}{(\varepsilon_0^*+\delta_0)^2} - a\right)\delta_0 > 2v_{\max}^2$. Therefore, using this in (27), $v_1(t^*) - v(t^*) \ > 0$ which is a contradiction.

□

Corollary 2 (Global well-posedness for a specific choice of parameters)**.** *Let the assumption of Theorem 3 hold and in addition for the parameters that*

$$s_0 \le \frac{as_0^2\delta^*}{8(a\delta^*+2v_{max}^2)} \quad \textit{with} \quad \delta^* = \frac{-3v_{max}^2+\sqrt{9v_{max}^4+4a^2s_0^2}}{2a}. \tag{28}$$

Then, we obtain

$$\dot{x}(t) = v(t) \ge 0 \quad \forall t \in [0, T],$$

and in particular, the well-posedness of the solution on every finite time horizon.

Proof. Let $t^0 \in [0, T]$ be a time at which $v(t^0) = 0$. It is enough to show that $\dot{v}(t^0) \ge 0$. Recalling the change of velocity according to Definition 1 and inserting $v(t^0) = 0$ we have

$$\dot{v}(t^0) = a\left(1 - \left(\frac{s_0}{(x_1(t^0)-x(t^0)-l)}\right)^2\right).$$

However, following Theorem 3, we know that $x_1(t) - x(t) - l \ge \varepsilon_0^*$ and as the parameters are chosen according to Eq. (28) we have

$$x_1(t^0) - x(t^0) - l \ge s_0,$$

and thus

$$\dot{v}(t^0) = a\left(1 - \left(\frac{s_0}{(x_1(t^0)-x(t^0)-l)}\right)^2\right) \ge 0.$$

This means that the velocity cannot drop below zero. As $t^0 \in [0, T]$ was arbitrarily chosen to satisfy $v(t^0) = 0$, we obtain the claim. □

5. Several improvements for the IDM

In this section, we present several improvements of the IDM to fix the problems illustrated in Section 3 for general initial datum. Before doing this, however, we present some other numerics on how the classical IDM behaves for specific data. This will serve as a comparison to the proposed improvements later:

Example 4 (Some additional numerical results for the classical IDM)**.** *All examples – except those who are physically unreasonable (compare Section 5.2) – will be tested on three different scenarios:*

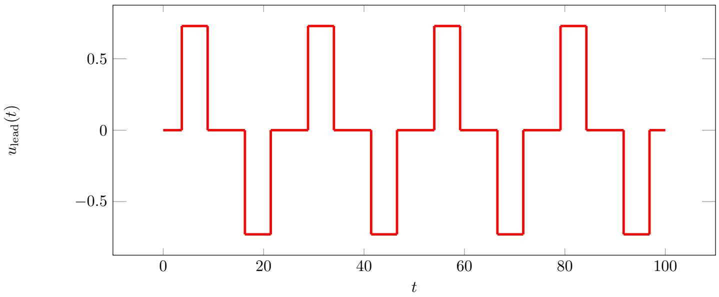

Figure 5: The acceleration profile u_{lead} of the leader. This is intended to specify a leader that repeats the pattern "accelerate, constant velocity, decelerate, constant velocity".

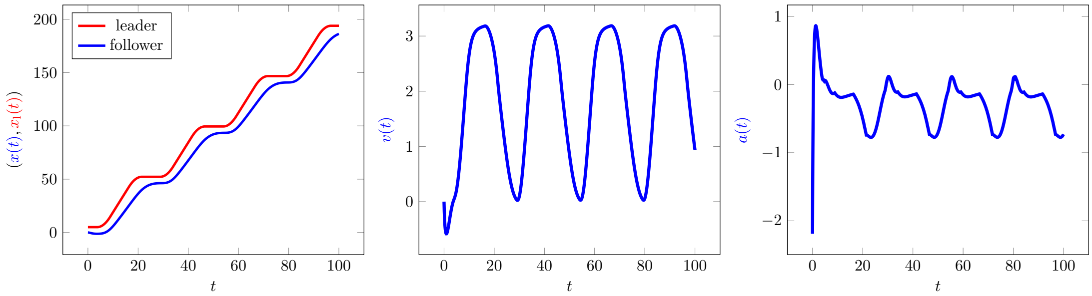

Figure 6: The original IDM, with parameters $x_0 = 0$, $x_{l0} = l + 1 < l + s_0$, $v_0 = 0 = v_{l0} = 0$. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in Fig. 5. **Left** is the position of both leader and follower, **middle** the velocity of the follower and **right** is the acceleration of the follower. We leave the velocity and acceleration profile of the leader out as it is fully determined by the given u_{lead}.

1. *As the set of parameters where one can observe a negative velocity of the follower in the original IDM – see Example 1 and Fig. 2.*

2. *As the set of parameters where one can observe that the velocity of the follower diverges to $-\infty$ in finite time in the original IDM – see Example 2 and Fig. 4.*

3. *A heavy stop and go wave traffic situation with the leader's acceleration satisfying $u_{lead} \equiv a \cdot \mathbb{1}_{\{t\in[0,T]:\ \sin(t/4)\geq 0.8\}} - a \cdot \mathbb{1}_{\{t\in[0,T]:\ \sin(t/4)\leq -0.8\}}$, $a = 0.73,\ b = 1.67,\ v_{free} = \frac{120}{36},\ \tau = 1.6,\ l = 4,\ s_0 = 2,\ d = 4$ illustrated in Fig. 5.*

5.1. Projection on nonnegative velocities and restricting the maximal deceleration

A straight forward improvement consists of projecting the velocity to nonnegative values. This is detailed in the following Definition 4:

Definition 4 (IDM with projection to nonnegative velocities)**.** *Given Assumption 1, we replace the acceleration in Definition 1 and velocity for the IDM model in Definition 3 by*

$$\dot{x}(t) = \max\{v(t), 0\}, \qquad t \in [0, T] \tag{29}$$

$$\dot{v}(t) = \operatorname{Acc}\big(x(t), \max\{v(t), 0\}, x_l(t), v_l(t)\big), \qquad t \in [0, T] \tag{30}$$

*and call the model the **velocity projected IDM**.*

Theorem 4 (Existence and uniqueness of solutions for small times)**.** *Given Assumption 1 the velocity projected IDM in Definition 4 admits on a sufficiently small time horizon $T^* \in \mathbb{R}_{>0}$ a unique solution $(x, v) \in W^{1,\infty}((0, T^*))^2$.*

Proof. The proof is almost identical to the proof of Theorem 1 when recalling that the right hand side is still locally Lipschitz-continuous. We do not go into details. ☐

However, as we will see the model's solution will not exist for general initial datum:

Example 5 (The following vehicle waits for too long to start driving)**.** *Assume that the constants and initial data as in Example 1 with $\varepsilon < s_0$. Then, the leading vehicle's trajectory can be computed as*

$$x_l(t) = x_{l0} + v_{l0}t + \tfrac{u}{2}t^2, t \in [0, T],$$

and the leading vehicle's velocity can be computed as

$$v_l(t) = v_{l0} + ut, t \in [0, T].$$

Plugging this into the change of the vehicle's velocity we obtain the following system of ODEs

$$\begin{aligned}
\dot{x}(t) &= \max\{v(t), 0\}\\
\dot{v}(t) &= a\Big(1 - \big(\tfrac{\max\{v(t),0\}}{v_0}\big)^{\delta} - \Big(\tfrac{2\sqrt{ab}(s_0+\max\{v(t),0\}\tau)+\max\{v(t),0\}(\max\{v(t),0\}-v_l(t))}{2\sqrt{ab}(x_l(t)-x_{l0}+\varepsilon-\int_0^t \max\{v(s),0\}\,\mathrm{d}s)}\Big)^2\Big)\\
x(0) &= x_{l0} - l - \varepsilon\\
v(0) &= 0.
\end{aligned}$$

Note that

$$\dot{v}(0) = a\left(1 - \left(\tfrac{s_0}{\varepsilon}\right)^2\right) < 0,$$

therefore, there exists some small time interval $[0, t_1]$ such that for every $t \in [0, t_1]$, $v(t) < 0$. During the time interval $[0, t_1]$, the distance between the two vehicles is

$$l + \varepsilon + x_l(t) - x_{l0} = l + \varepsilon + v_{l0}t + \frac{u}{2}t^2, t \in [0, t_1].$$

Thus, for every $t \in [0, t_1]$,

$$\dot{v}(t) = a\bigg(1 - \Big(\tfrac{s_0}{\varepsilon+v_{l0}t+\frac{u}{2}t^2}\Big)^2\bigg).$$

Note that $\dot{v}\colon [0, t_1] \mapsto \mathbb{R}$ is strictly increasing. Without loss of generality, we assume that the initial velocity of the leading vehicle is $v_{l0} = 0$ and the acceleration of the leading vehicle is $u = 2$. Then for every $t \in [0, t_1]$,

$$v(t) = \int_0^t a\Big(1 - \Big(\tfrac{s_0}{\varepsilon+s^2}\Big)^2\Big)\,\mathrm{d}s = at - 4as_0^2\int_0^t \left(\tfrac{1}{s^2+\varepsilon}\right)^2\,\mathrm{d}s = at - 4as_0^2\left(\frac{\frac{\sqrt{\varepsilon}t}{\varepsilon+t^2}+\arctan\left(\frac{t}{\sqrt{\varepsilon}}\right)}{2\varepsilon^{\frac{3}{2}}}\right).$$

In particular, as illustrated in Fig. 7, we have, t_1 increases as ε decreases. That is, the smaller the initial distance between the two vehicles, the longer it takes the following vehicle to recover its positive velocity. Another example illustrates the projected velocity model numerically with regard to other scenarios:

Example 6 (Velocity projected IDM)**.** *As can be observed the actual velocity in all the three different scenarios is bounded from below by zero and the solution exists on the entire time horizon considered. However, the projection operator leads to the problem that the follower waits too long until they speed up. This can be observed in particular in Figs. 8 and 9 where the distance of the two vehicles after both have started speeding up ($t \approx 5$) is approximately around 8.5 which is quite far from the comfortable vehicle distance s_0 and thus leading to a too large distance. Same can be observed in Fig. 10 for smaller time.*

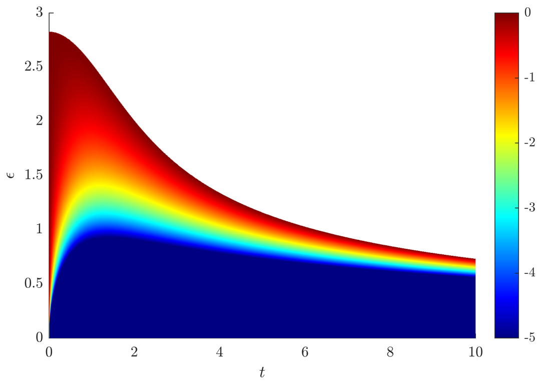

Figure 7: Velocity dependent on epsilon and time. The dark blue indicates values less or equal -5 and the white area positive function values, so that the red curve separating the white and colored region can be seen as the values where the velocity is actually zero. In particular, as the initial distance between two vehicles ε increases, the time when the following vehicle recovers its positive velocity t_1 decreases.

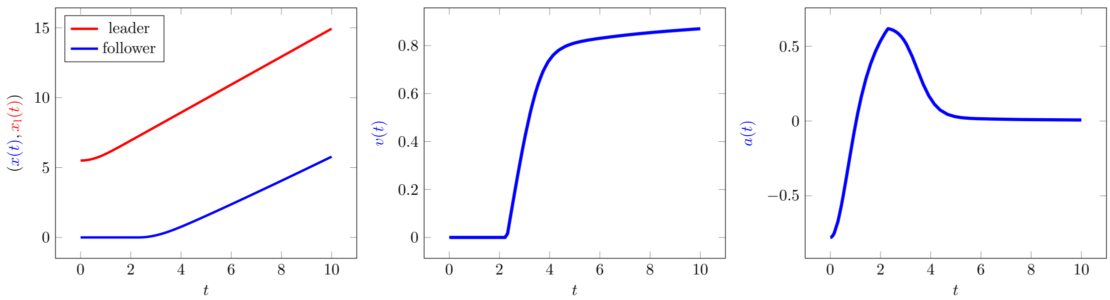

Figure 8: The IDM with projection as in Definition 4 and same parameters as in Fig. 2 with $x_0 = 0,\ x_{10} = l + 1.5 < l + s_0$. **Left** positions of the vehicles, **middle** the velocity of the follower, and **right** the acceleration of the follower. The leader follows the free-flow acceleration as in Eq. (19). The follower stays still and waits until there is a safe space to speed up.

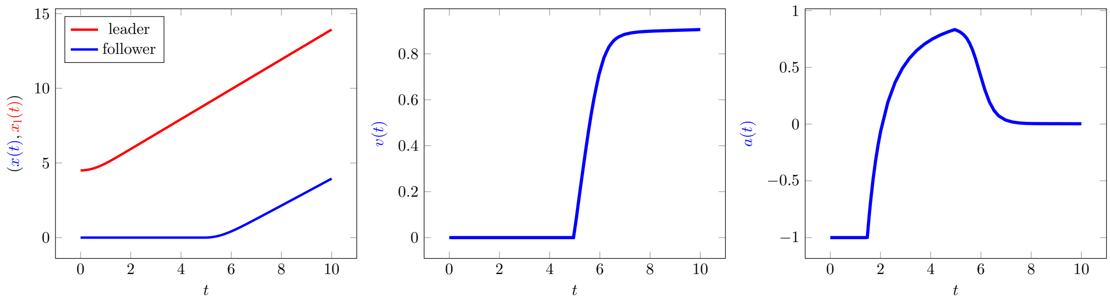

Figure 9: The IDM with projection, with the same parameters as in Fig. 2 and $x_0 = 0,\ x_{10} = l + 0.5 < l + s_0,\ v_0 = 0 = v_{10} = 0$. **Left** vehicles' positions, **middle** the follower's velocity and **right** the follower's acceleration. The leader follows the free flow acceleration as in Eq. (19). The follower stays still and waits until there is a safe space to speed up.

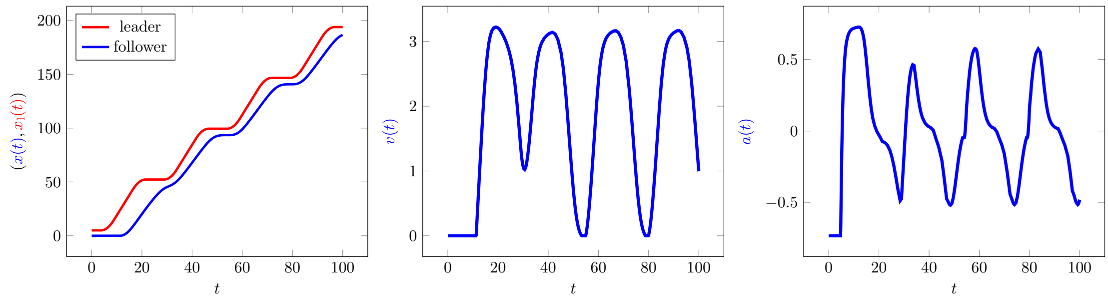

Figure 10: The IDM with projection as in Definition 4 and initial data $x_0 = 0,\ x_{l0} = l + 1 < l + s_0,\ v_0 = v_{l0} = 0$. **Left** position of the vehicles, **middle** the velocity of the follower and **right** the acceleration of the follower. The leader follows the acceleration profile in Fig. 5. The velocities remain positive, however the waiting time for the follower until speeding up is large.

5.2. Projection to nonnegative velocities with bounded deceleration

Another improvement for this is projecting the acceleration to prevent it from becoming too negative. Then, the corresponding model reads as

Definition 5 (IDM with projection to nonnegative velocities and bounded deceleration)**.** *Given Assumption 1, we replace the acceleration in Definition 1 and velocity for the IDM model in Definition 3 by*

$$\begin{aligned} \dot{x}(t) &= \max\{v(t), 0\}, && t \in [0, T] \\ \dot{v}(t) &= \max\{\mathrm{Acc}(x(t), \max\{v(t), 0\}, x_l(t), v_l(t)), -a_{\min}\}, && t \in [0, T] \end{aligned}$$

with a parameter $a_{\min} \in \mathbb{R}_{>0}$ be given and call the model the ***acceleration projected IDM****.*

Theorem 5 (Global existence and uniqueness of solutions)**.** *Given Assumption 1 the acceleration projected IDM in Definition 5 admits for every $T \in \mathbb{R}_{>0}$ a unique solution $(x, v) \in W^{1,\infty}((0, T))$.*

Proof. The proof of existence and uniqueness for small time is almost identical to the proof of Theorem 1 when recalling that the right hand side is still locally Lipschitz-continuous. We do not go into details.

So it remains to show that we can find uniform estimates for $(x(t), v(t)),\ t \in [0, T]$. Obviously,

$$v(t) \geq v_{\text{free}} - a_{\min} t \quad \forall t \in [0, T].$$

Thanks to the structure of Acc (see Definition 1) we also obtain as a bound from above

$$v(t) \leq v_{\text{free}} + at \quad \forall t \in [0, T].$$

As v is uniformly bounded on every finite time horizon, so is x and we are done. □

However, although the previous change of the acceleration profile in Definition 5 looks promising as according to Theorem 5 a solution exists on every finite time horizon, the physical representation, the model itself is unreasonable as the car behind can overtake the leading car – or differently put, the car behind can bump into the leading car without the model noticing it. This is detailed in the following Example 7.

Example 7 (Physical unreasonability)**.** *As the deceleration of the following vehicle is bounded from below by $-a_{\min}$, we can always chose an initial velocity of the follower which leads to the fact that $x_l(t) - x(t) - l \to 0$ in finite time.*

In formulae, assume for simplicity that the leading vehicle has the following trajectory

$$x_l(t) = x_{l0} + v_{l0} t + \tfrac{1}{2} u_{lead} t^2, \qquad t \in [0, T].$$

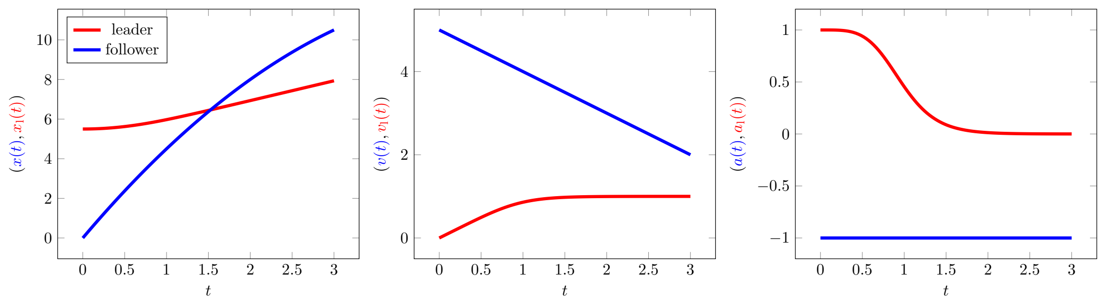

Figure 11: Illustration of the improvement in Definition 5 and its physical unreasonability as shown in Example 7. The parameters are $a = a_{\min} = 1,\ b = 2,\ v_{\text{free}} = 1,\ \tau = 1.6,\ l = 4,\ s_0 = 2,\ d = 4$ and datum $x_0 = 0,\ x_{l0} = l + 1.5,\ v_0 = 5,\ v_{l0} = 0$. **Left** Vehicles' positions, **middle** vehicles' velocities, **right** vehicles' acceleration. The follower overtakes the leader in finite time, although the follower breaks constantly.

with $u_{lead} \in \mathbb{R}_{\geq 0}$ and $(x_{l0}, v_{l0}) \in \mathbb{R} \times \mathbb{R}_{>0}$. Then, we take the difference of the vehicles position with car length $l \in \mathbb{R}_{>0}$ and have for $t \in [0, T]$

$$
\begin{aligned}
&x_l(t) - x(t) - l \\
&\leq x_{l0} + v_{l0}t - l + \tfrac{1}{2}u_{lead}t^2 - x_0 - \int_0^t \max\{v(s), 0\}\,\mathrm{d}s \\
&\leq x_{l0} + v_{l0}t - l + \tfrac{1}{2}u_{lead}t^2 - x_0 - \int_0^t v(s)\,\mathrm{d}s \\
&\leq x_{l0} + v_{l0}t - l + \tfrac{1}{2}u_{lead}t^2 - x_0 \\
&\qquad - \int_0^t v_{free} + \int_0^s \max\{\mathrm{Acc}(x(\tau), \max\{0, v(\tau)\}, x_l(\tau), v_l(\tau)), -a_{\min}\}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\leq x_{l0} + v_{l0}t - l + \tfrac{1}{2}u_{lead}t^2 - x_0 - tv_{free} + \int_0^t \int_0^s a_{\min}\,\mathrm{d}\tau\,\mathrm{d}s \\
&\leq x_{l0} - x_0 - l + (v_{l0} - v_{free})t + \tfrac{1}{2}(u_{lead} + a_{\min})t^2.
\end{aligned}
$$

Obviously, for v_0 sufficiently large, we obtain for small time that $x_l(t) - x(t) < l$. An extreme case for this is illustrated in Fig. 11 where not only the vehicles get closer than l but the follower (in blue) entirely overtakes the leader (in red).

5.3. Velocity regularized acceleration

Another improvement of the IDM is to add a regularization term which will make the third term in the acceleration function Definition 1 of the IDM in Definition 3 become zero if the corresponding velocity approaches zero.

Definition 6 (IDM with velocity regularized acceleration)**.** *Given Assumption 1, we replace acceleration in Definition 1 in Definition 3 by*

$$
\mathrm{Acc}_{vra} : \begin{cases} \mathcal{A} & \to \mathbb{R} \\ (x, v, x_l, v_l) & \mapsto a\left(1 - \left(\frac{|v|}{v_{free}}\right)^{\delta} - h(v)\left(\frac{2\sqrt{ab}(s_0 + v\tau) + v(v - v_l)}{2\sqrt{ab}(x_l - x - l)}\right)^2\right) \end{cases}
$$

*with $\mathcal{A}$ as in Definition 1 and regularization $h \in W^{1,\infty}(\mathbb{R}; \mathbb{R}_{\geq 0})$ be a monotonically increasing function satisfying $h(0) = 0$ and – for a given $\varepsilon \in \mathbb{R}_{>0}$ – $h(v) = 1\ \forall v \in \mathbb{R}_{\geq \varepsilon}$. We call this the **velocity regularized acceleration IDM**.*

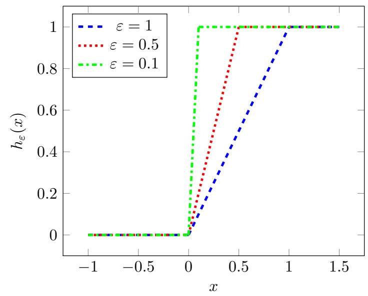

Figure 12: Different choices of the "saturation" function h_ε for $\varepsilon \in \{1, 0.5, 0.1\}$ as suggested in Eq. (31).

Theorem 6 (Existence and Uniqueness of solutions on arbitrary time horizon)**.** *Given Assumption 1 the velocity regularized IDM in Definition 6 admits for every finite time horizon $T \in \mathbb{R}_{\geq 0}$ a unique solution satisfying $x \in W^{2,\infty}((0,T))$ and additionally*

$$\dot{x} \equiv v \geqq 0 \quad \textit{on } [0,T].$$

Proof. To show the well-posedness it suffices to make sure that

- $\exists C \in \mathbb{R}_{>0} : x_{10} - x_0 - l \geq C$ implies that $x_1(t) - x(t) - l \geq C\ \forall t \in [0,T]$. However, this is a direct consequence of Theorem 3 as the regularization h is bounded from above by 1 and for any $t \in [0,T]$,

$$\mathrm{Acc}_{\mathrm{vra}}(x(t), v(t), v_1(t), v_1(t)) \leq \mathrm{Acc}(x(t), v(t), v_1(t), v_1(t)).$$

- $\dot{x}(t) \geq 0\ \forall t \in [0,T]$. However, we know by the Picard-Lindelöf theorem that there exists a solution on a significantly small time horizon $[0,T^*]$ with $T^* \in \mathbb{R}_{>0}$. Assume that the velocity of a given vehicle $\dot{x}$ becomes zero at a given time $T^{**} \in \mathbb{R}_{>0}$. Then, by continuity of the velocity and due to the nonnegativity of the initial velocity there exists a first time $t \in [0,T^{**})$ so that $v(t) = \dot{x}(t) = 0$. Plugging this into the corresponding acceleration we obtain at that time

$$\ddot{x}(t) = \dot{v}(t) = \mathrm{Acc}_{\mathrm{vra}}(x(t), \dot{x}(t), x_1(t), \dot{x}_1(t)) = \mathrm{Acc}_{\mathrm{vra}}(x(t), 0, x_1(t), \dot{x}_1(t)) = a > 0,$$

  thanks to the assumption on h in Definition 6, namely $h(0) = 0$. This means that whenever the velocity becomes zero, the derivative is strictly positive so that the velocity can never become negative. This gives the claim.

□

Remark 3 (Proper choice of the regularization h)**.** *A proper choice for h consists for $\varepsilon \in \mathbb{R}_{>0}$ of*

$$h_\varepsilon : \begin{cases} \mathbb{R} & \to [0,1] \\ v & \mapsto \frac{v}{\varepsilon}\mathbb{1}_{\mathbb{R}_{\geq 0}}(v) \cdot \mathbb{1}_{\mathbb{R}_{\leq \varepsilon}}(v). \end{cases} \tag{31}$$

This is illustrated in the following Fig. 12. As can be seen this "saturation" function is only continuous, resulting in an acceleration function which is not differentiable. Obviously, this could be changed by smoothing h_ε. We do not go into details.

We illustrate the model in the following

Example 8 (Velocity regularized acceleration)**.** *As can be seen in Figs. 13 to 15, with this fix the spacing between the two vehicles is not getting large as had been observed in Figs. 8 to 10 but the follower speeds up immediately when there is enough safe distance to do so. The clipping due to the function h_ε can be observed in particular in the acceleration profile which is nonsmooth. In all Figs. 13 to 15 we choose $\varepsilon = 0.1$.*

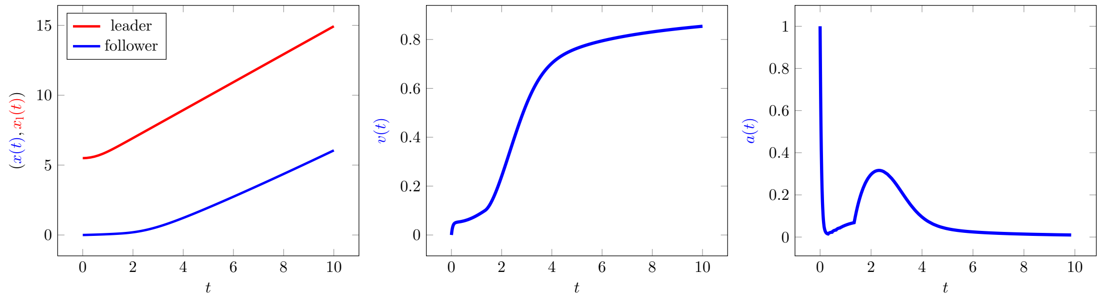

Figure 13: The IDM with velocity regularized acceleration as in Definition 6 and parameters as in Fig. 2. Initial data are $x_0 = 0,\ x_{10} = l + 1.5 < l + s_0,\ v_0 = v_{10} = 0$. **Left** the vehicles' position, **middle** the vehicles' velicities, and **right** the vehicles acceleration. The leader follows the free flow acceleration. The follower stays still and waits until there is a safe space to speed up.

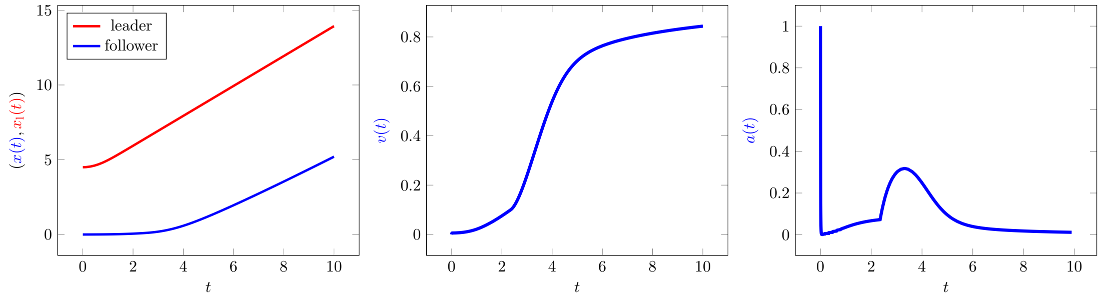

Figure 14: The IDM with velocity regularized acceleration, with initial gap equal 0.5 and same parameters as in Fig. 2. Both vehicles start with 0 velocity, and the leader follows the free flow IDM dynamics. **Left:** $x_0 = 0,\ x_{10} = l + 0.5 < l + s_0,\ v_0 = 0 = v_{10} = 0$. The follower stays still and waits until there is a safe space to speed up. **Right:** the follower remains still in the initial phase.

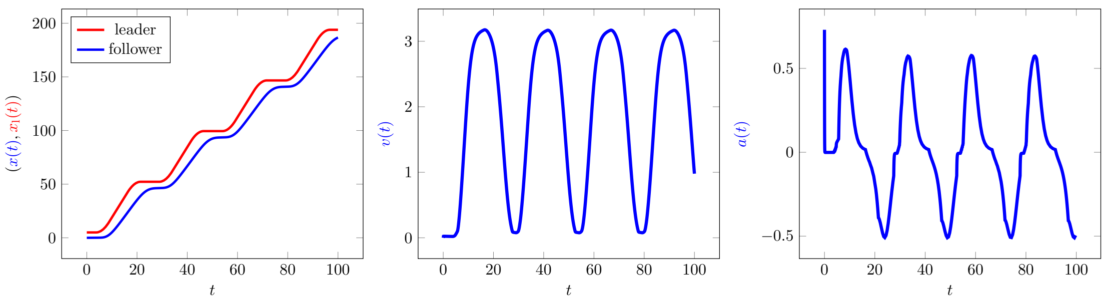

Figure 15: The IDM with velocity regularized acceleration, with initial gap equal 1. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 1 < l + s_0,\ v_0 = 0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

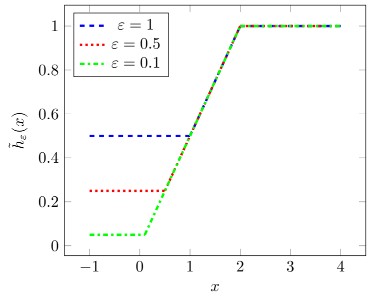

Figure 16: Different choices of the "saturation" for $s_0 = 2$ function h_ε for $\varepsilon \in \{1, 0.5, 0.1\}$ as suggested in Eq. (33).

5.4. IDM with distance regularized acceleration

Similar to the improvement in Section 5.3 we can also regularize the third term in the acceleration function in Definition 1 in space which then reads as

Definition 7 (IDM with distance regularized acceleration)**.** *Given Assumption 1, we replace acceleration in Definition 1 of the IDM in Definition 3 by*

$$\mathrm{Acc}_{dra} : \begin{cases} \mathcal{A} & \to \mathbb{R} \\ (x, v, x_l, v_l) & \mapsto a\Big(1 - \big(\tfrac{|v|}{v_0}\big)^\delta - \tilde{h}(x_l - x - l)\Big(\tfrac{2\sqrt{ab}(s_0 + v\tau) + v(v - v_l)}{2\sqrt{ab}\cdot(x_l - x - l)}\Big)^2\Big) \end{cases} \tag{32}$$

with $\mathcal{A}$ as in Definition 1 and regularization $\tilde{h} \in W^{1,\infty}(\mathbb{R}; \mathbb{R}_{\geq 0})$ be a monotonically increasing function satisfying – for a given $\varepsilon \in (0, s_0)$ – $\tilde{h}(x) = \frac{\varepsilon}{s_0}\ \forall x \in \mathbb{R}_{\leq \varepsilon}$ and $\tilde{h}(x) = 1\ \forall x \in \mathbb{R}_{>s_0}$. We call this the ***distance regularized acceleration IDM****.*

Remark 4 (The choice of $\tilde{h}$ as in Eq. (32))**.** *A possible choice for $\tilde{h}$ and $s_0 \in \mathbb{R}_{>0}$ as in Assumption 1 in Eq. (32) is for $\varepsilon \in (0, s_0)$*

$$\tilde{h}_\varepsilon : \begin{cases} \mathbb{R} & \to [\frac{\varepsilon}{s_0}, 1] \\ x & \mapsto \frac{\varepsilon}{s_0} + (x - \varepsilon)\,\frac{1 - \frac{\varepsilon}{s_0}}{s_0 - \varepsilon}\,\mathbb{1}_{(\varepsilon, s_0)}(x) + \Big(1 - \frac{\varepsilon}{s_0}\Big)\,\mathbb{1}_{[s_0, \infty)}(x). \end{cases} \tag{33}$$

This function is illustrated in Fig. 16:

Theorem 7 (Existence and Uniqueness of solutions on arbitrary finite time horizon)**.** *Given Assumption 1 the distance regularized IDM in Definition 7 admits for every finite time horizon $T \in \mathbb{R}_{>0}$ a unique solution satisfying $x \in W^{2,\infty}((0, T))$ and additionally*

$$\dot{x} \geqq 0.$$

Proof. The proof is similar to the proof to Theorem 6. We decide to omit it. □

Some numerical results for the distance regularized acceleration are illustrated in the following

Example 9 (Distance regularized acceleration)**.** *Very similar to the velocity regularized accelerations are the results for the distance regularized acceleration in Figs. 17 to 19 at least with regard to the position of the vehicles. In Figs. 17 to 19 we choose $\varepsilon = 0.5$.*

5.5. A discontinuous improvement to prevent negative velocity

Our last potential improvement for the IDM which had also been suggested a lot (see for instance [1]) is a improvement which will become active only if the velocity becomes zero and the corresponding acceleration at the time where the velocity is zero is negative. In this way, the following improvement is the most natural one. We state it in the following Definition 8.

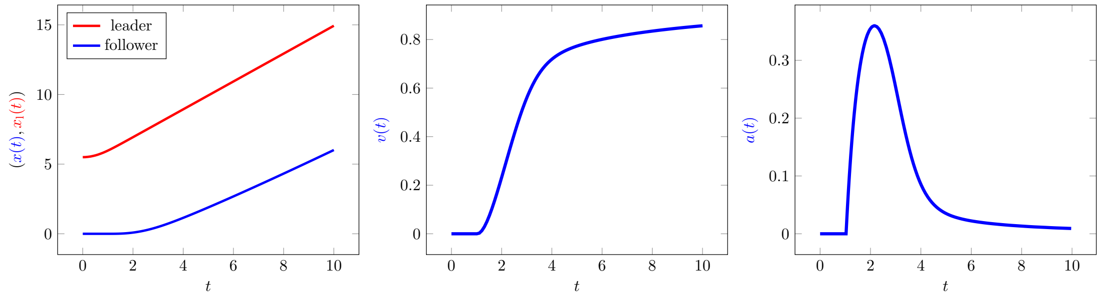

Figure 17: The IDM with distance regularized acceleration, with initial gap equal 1.5 and same parameters as in Fig. 2. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 1.5 < l + s_0,\ v_0 = 0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

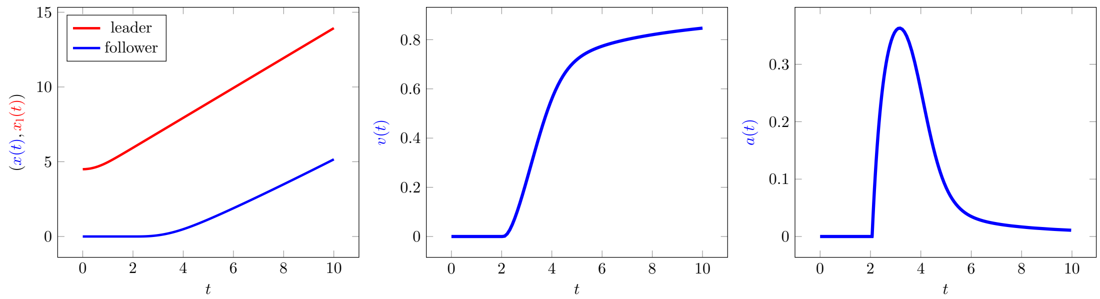

Figure 18: The IDM with distance regularized acceleration, with initial gap equal 0.5 and same parameters as in Fig. 2. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 0.5 < l + s_0,\ v_0 = 0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

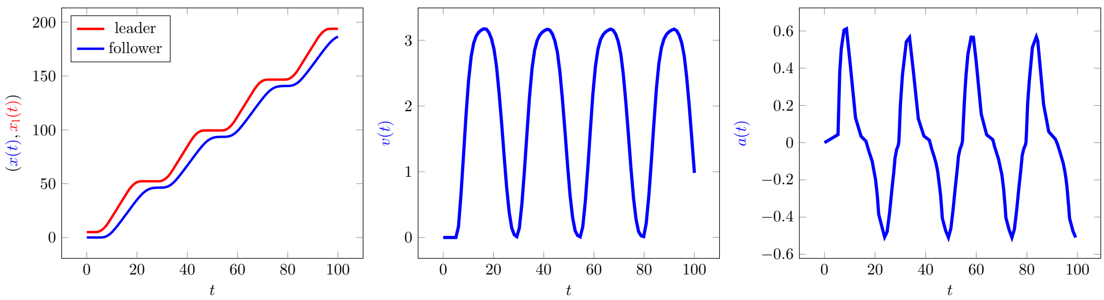

Figure 19: The IDM with distance regularized acceleration, with initial gap equal 1. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 1 < l + s_0,\ v_0 = 0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

Definition 8 (Preventing the velocity from becoming negative by a discontinuous right hand side)**.** *Let Assumption 1 and* Acc *as in Definition 1 be given. Then, replacing in Definition 3 the acceleration of the follower in the following discontinuous way as*

$$\dot{v}(t) = \begin{cases} \mathrm{Acc}(x(t), v(t), x_l(t), v_l(t)) & \text{if } v(t) > 0 \\ \mathrm{Acc}(x(t), v(t), x_l(t), v_l(t)) & \text{if } v(t) = 0 \ \wedge \ x_l(t) - x(t) - l \geq s_0 \\ 0 & \text{if } v(t) = 0 \ \wedge \ x_l(t) - x(t) - l < s_0 \end{cases}, \tag{34}$$

we call this the ***discontinuous IDM****.*

Although one might expect that the introduction of the discontinuity in Eq. (34) might prohibit a solution to exist for all times or also might destroy uniqueness, it actually does not as the following Theorem 8 states:

Theorem 8 (Well-posedness of the discontinuously fixed IDM in Definition 8)**.** *Given Assumption 1 and assuming that the velocity of the leader is only zero at finitely many intervals, the discontinuous improvement of the IDM as in Definition 8 admits a unique solution on every finite time horizon $T \in \mathbb{R}$ and satisfies*

$$x \in W^{2,\infty}((0,T)) : \ \dot{x} \geqq 0$$

Proof. $B = \{(x_l - x, v) : v = 0 \wedge x_l - x - l < s_0\}$.

We consider several different cases:

- Assume that at $t = 0$ we have $v(0) = c > 0$. Obviously, for small time the solution of the system is unique as the right hand side is locally Lipschitz-continuous then. Either, $v(t) > 0 \ \forall t \in [0,T]$. Then, there is nothing more to do as we never run into the discontinuity or $\exists t^* \in (0,T] : v(t^*) = 0$. We distinguish two cases:

  $x_l(t^*) - x(t^*) - l \geq s_0$: However, in this case the right hand side has not changed so that the solution still exists and is unique. As at that time, the velocity is zero and as the leading car never moves backwards, we can never go into the third case where it would hold $x_l(t) - x(t) - l < s_0$ but only back into the first case with strictly positive velocity. In this case, the solution exists and is unique and there is no more to do.

  $x_l(t^*) - x(t^*) - l < s_0$: Then, the velocity of the follower is zero and the solution then can only go into the second case when the leader's position x_l increases, we are automatically left with the case that either we stay in the third case or that we move back to the second case. The second case, however, was already treated previously.

- Assume that we have $v(0) = 0$. Then, we are either in item one or two of the previous case and are done.

As all of these changes only depend on the leader's trajectory x_l which is given independent on the status of the follower, we can conclude the existence and uniqueness of a solution. □

Example 10 (The discontinuous improvement)**.** *In Figs. 20 to 22 we illustrate the dynamics of leader and follower for the discontinuous improvement proposed in Theorem 8. Here, the velocity can become zero and stay zero for an amount of time dependent on the leaders velocity/position, but remains nonnegative. If the spacing between the two cars is large enough, the follower immediately speeds up and does not let the gap increase too much.*

5.6. Comparisons among modifications

We herein summarize the strength and weakness of each modification.

- The acceleration projected IDM in Definition 5 is the most straightforward modification to fix the negative velocities. However as shown in Figs. 8 and 9, it suffers from the fact that the follower waits too long to start driving. It may also lead to physical unreasonability issue with certain initial datum as we point out in Example 7.

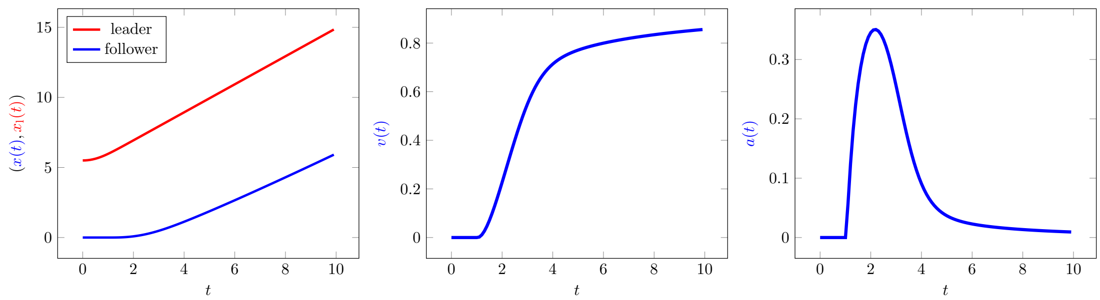

Figure 20: The discontinuous improvement of the IDM, with initial gap equal 1.5 and same parameters as in Fig. 2. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 1.5 < l + s_0,\ v_0 = 0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

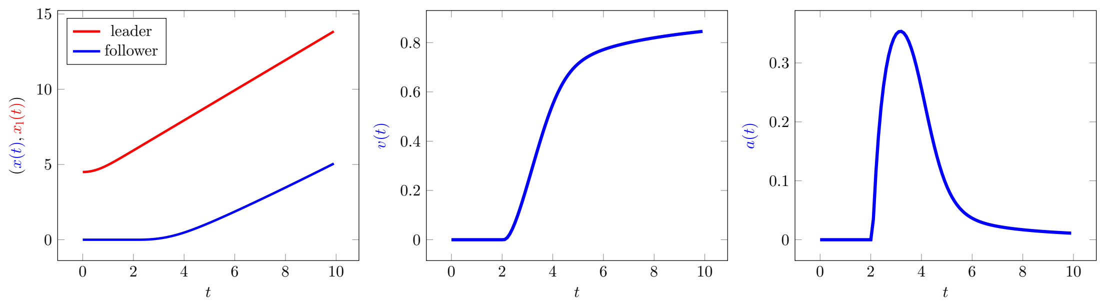

Figure 21: The discontinuous improvement of the IDM, with initial gap equal 0.5 and same parameters as in Fig. 2. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 0.5 < l + s_0,\ v_0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

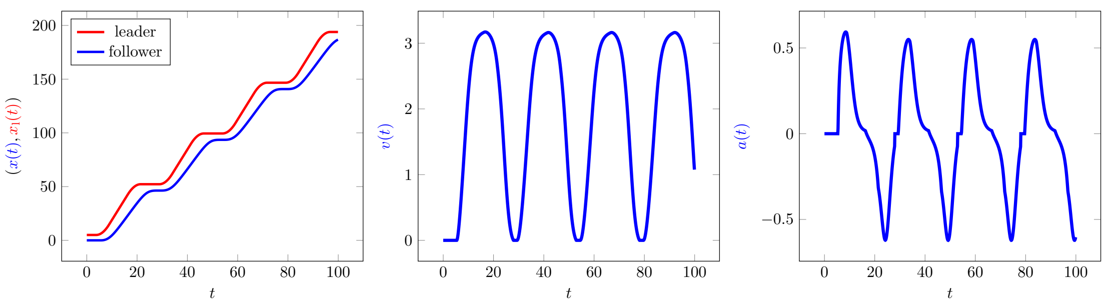

Figure 22: The discontinuous improvement of the IDM, with initial gap equal 1. Both vehicles start with 0 velocity, and the leader follows the acceleration profile in figure Fig. 5. **Left:** $x_0 = 0,\ x_{10} = l + 1 < l + s_0,\ v_0 = v_{10} = 0$. **Right:** again, the velocity always stays nonnegative.

Table 2: The average distance between the leader and follower (and the variance reported in the parenthesis)

Average distance (variance)	Item 1	Item 2	Item 3
acceleration projected IDM	7.99 (1.19)	8.09 (3.55)	14.94 (54.99)
velocity regularized IDM	7.76 (1.00)	7.24 (1.70)	12.81 (25.51)
distance regularized IDM	7.73 (1.02)	7.29 (1.77)	12.79 (25.87)
discontinuous improvement of the IDM	7.75 (1.01)	7.31 (1.75)	12.39 (25.18)

- The velocity regularized IDM in Definition 6 and distance regularized IDM in Definition 7 are both capable of preventing negative velocity, and the follower speeds up immediately when it observes safe distance. However in both modifications one needs to specify the regularization functions h or $\tilde{h}$, which might require extra parameter tuning.
- The discontinuous improvement of the IDM as in Definition 8 prohibits negative velocities, without the need to introduce extra saturation functions.

In order to quantitatively compare the modifications, we compute the average distance (as well as the variance) between the leader and follower over time in Table 2 on three different scenarios we tested.

5.7. Generalization to many cars

In the proposed framework, we have only studied the case where we have one leader and one follower and the leader (in most cases) satisfies an acceleration profile where their velocity is nonnegative. However, as this is somewhat arbitrary, all proposed results and all "improvements" remain valid as long as the velocity of the follower remains non-negative when we generalize this to more than two cars. We make this precise in the following Definition 9 but first introduce the number of cars as well as some physical reasonable assumption on the input datum:

Assumption 2 (Input datum for multiple cars)**.** *Let $N \in \mathbb{N}_{\geq 1}$ be given. We assume that the parameters of the* Acc *satisfy what we have assumed in Assumption 1 as well as the leaders acceleration u_{lead}. Additionally, we assume for the initial datum (position and velocity)*

$$\boldsymbol{x}_0 \in \mathbb{R}^N : \boldsymbol{x}_{0,i} - \boldsymbol{x}_{0,i+1} > l \ \forall i \in \{1, \dots, N-1\} \ \wedge \ \boldsymbol{v}_0 \in \mathbb{R}^N_{\geq 0}.$$

Definition 9 (The car-following model for many cars)**.** *Given Assumption 2 and Definition 1. Then, the dynamics for the IDM with many vehicles $N \in \mathbb{N}_{\geq 1}$ read in $(\boldsymbol{x}, \boldsymbol{v}) : [0,T] \to \mathbb{R}^N \times \mathbb{R}^N$ as*

$$\begin{aligned}
\dot{\boldsymbol{x}}_1(t) &= \boldsymbol{v}_1(t), && && t \in [0,T] \\
\dot{\boldsymbol{v}}_1(t) &= u_{lead}(t), && && t \in [0,T] \\
\dot{\boldsymbol{x}}_i(t) &= \boldsymbol{v}_i(t), && i \in \{2, \dots, N\} && t \in [0,T] \\
\dot{\boldsymbol{v}}_i(t) &= \mathrm{Acc}(\boldsymbol{x}_i(t), \boldsymbol{v}_i(t), \boldsymbol{x}_{i-1}(t), \boldsymbol{v}_{i-1}(t)), && i \in \{2, \dots, N\} && t \in [0,T] \\
\boldsymbol{x}(0) &= \boldsymbol{x}_0, \\
\boldsymbol{v}(0) &= \boldsymbol{v}_0.
\end{aligned} \tag{35}$$

The system for many cars is illustrated in Fig. 23. Having the definition, we obtain the following general result on the well-posedness when changing the right hand side of the ODEs as suggested in the corresponding improvements:

Theorem 9 (Well-posedness of some of the previously discussed models)**.** *Let Assumption 2 hold and Consider as "improvement" of the IDM either the*

- *the velocity regularized acceleration in Definition 6*
- *the distance regularized acceleration in Definition 7*

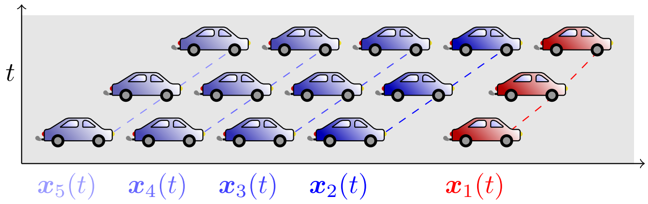

Figure 23: The leader $\boldsymbol{x}_1(t)$ with its dynamics determined by the acceleration $u_{\text{lead}}(t)$ and the following cars $\boldsymbol{x}_2(t)$, $\boldsymbol{x}_3(t)$, $\boldsymbol{x}_4(t)$, $\boldsymbol{x}_5(t)$ with its dynamics governed by the classical IDM Definition 3. The overall dynamics is stated in Definition 9.

- *the discontinuous improvement in Definition 8.*

Then, the system of $2N$ *coupled initial value problems (*$N \in \mathbb{N}_{\geq 1}$*) admits a unique solution on every given time horizon* $T \in \mathbb{R}_{>0}$ *and the solutions satisfy*

$$\boldsymbol{x} \in W^{2,\infty}((0,T);\mathbb{R}^N): \ \dot{\boldsymbol{x}} :\equiv \boldsymbol{v} \geqq \boldsymbol{0} \ \text{on } [0,T].$$

Proof. The proof consists of recalling that the proofs of the corresponding improvements all worked for any leaders acceleration proposed in Assumption 2 as long as the corresponding velocity would remain nonnegative. As the dynamics are only "one-directionally" coupled, i.e. the dynamics of the follower depend on the leader but not vice versa, we can use an inductive argument and first look on the dynamics for $\boldsymbol{x}_1, \boldsymbol{x}_2$. According to Theorem 6, Theorem 7, Theorem 8 we obtain the well-posedness. However, additionally also that the velocity $\boldsymbol{v}_2$ is nonnegative. As a next step we can thus consider the dynamics of $\boldsymbol{x}_2, \boldsymbol{x}_3$. As $\boldsymbol{x}_2$ satisfies the non-negativity of velocities we can again apply the stated well-posedness results for two cars. This can be iterated until we have reached $\boldsymbol{x}_N$ and we conclude with the existence and uniqueness of solutions and the nonnegativity of the velocities. □

6. Conclusions and future work

In this contribution, we have demonstrated the ill-posedness of the rather often used **intelligent driver model** (IDM) for specific initial datum and have presented some improvements to avoid these problems. The proposed work builds a solid foundation for future work on **1)** Multi-lane traffic with lane-changing. The lane-changing requires to know under which condition a car can change lane and how the well-posedness of solutions is affected by the lane change. With the proposed improvements, the well-posedness can be guaranteed easily. For optimizing lane-changing to smooth traffic and avoid stop and go waves (thus saving energy) we will next consider a suitable hybrid optimal control problem based on the proposed dynamics. **2)** Signalized junctions/intersection modelled with a form of the IDM also requires the proposed well-posedness in particular as a red traffic light necessitates the vehicles in front of it to stop (velocity becomes zero). **3)** Stability of solutions with regard to the model parameters and comparison of stability for the different suggested improvements. **4)** Implementations of the models presented in this article. While the numerical work presented above results from `matlab` [21] implementations and the use of `ODE45` and similar routines, it would be interesting to also theoretically study the discretization of these equations with standard finite difference schemes to see what guarantees can be provided for the numerical solutions (for example order of the numerical schemes, error bounds on the numerical solution etc.). **5)** Finally, it would also be of great interest for the improvements of the IDM model presented here to validate them against field data, or to see if the resulting microsimulation implementations (SUMO [19], Aimsun [2], etc.) match experimental data for specific choices of numerical parameters. In particular, it would be interesting to measure discrepancies with the prior IDM improvements in their implementation, and compare their respective performances.

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant Numbers CNS-1837244 (A. Bayen), CNS-1837481 (B. Piccoli). The research is based upon work supported by the

U.S. Department of Energy's Office of Energy Efficiency and Renewable Energy (EERE) under the Vehicle Technologies Office award number CID DE-EE0008872. The views expressed herein do not necessarily represent the views of the U.S. Department of Energy or the United States Government.

References

[1] Longitudinal Traffic model: The IDM. `https://mtreiber.de/MicroApplet/IDM.html#IDMparameters`. Accessed: 2021-01-21.

[2] AIMSUN. aimsun.

[3] M. Bando, K. Hasebe, K. Nakanishi, A. Nakayama, A. Shibata, and Y. Sugiyama. Phenomenological study of dynamical model of traffic flow. Journal de Physique I, 5(11):1389–1399, 1995.

[4] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama. Dynamical model of traffic congestion and numerical simulation. Phys. Rev. E, 51:1035–1042, Feb 1995.

[5] S. Bexelius. An extended model for car-following. Transportation Research, 2(1):13–21, 1968.

[6] M. Brackstone and M. McDonald. Car-following: a historical review. Transportation Research Part F: Traffic Psychology and Behaviour, 2(4):181–196, 1999.

[7] R. E. Chandler, R. Herman, and E. W. Montroll. Traffic dynamics: Studies in car following. Operations Research, 6(2):165–184, 1958.

[8] R. Cooke and V.I. Arnold. Ordinary Differential Equations. Springer Textbook. Springer Berlin Heidelberg, 1992.

[9] N. Levinson Earl A. Coddington. Theory of ordinary differential equations. R.E. Krieger, 1984.

[10] J. Eggert, F. Damerow, and S. Klingelschmitt. The foresighted driver model. In 2015 IEEE Intelligent Vehicles Symposium (IV), pages 322–329, 2015.

[11] D. C. Gazis, R. Herman, and R. W. Rothery. Nonlinear follow-the-leader models of traffic flow. Operations research, 9(4):545–567, 1961.

[12] P.G. Gipps. A behavioural car-following model for computer simulation. Transportation Research Part B: Methodological, 15(2):105 – 111, 1981.

[13] D. Helbing, A. Hennecke, V. Shvetsov, and M. Treiber. Micro- and macro-simulation of freeway traffic. Mathematical and Computer Modelling, 35(5):517 – 547, 2002.

[14] S. Hoermann, D. Stumper, and K. Dietmayer. Probabilistic long-term prediction for autonomous vehicles. In 2017 IEEE Intelligent Vehicles Symposium (IV), pages 237–243, 2017.

[15] S.P. Hoogendoorn and P.HL. Bovy. State-of-the-art of vehicular traffic flow modelling. Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering, 215(4):283–303, 2001.

[16] A. Kesting, M. Treiber, and D. Helbing. Enhanced intelligent driver model to access the impact of driving strategies on traffic capacity. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 368(1928):4585–4605, Oct 2010.

[17] Z. Li, W. Li, S. Xu, and Y. Qian. Stability analysis of an extended intelligent driver model and its simulations under open boundary condition. Physica A: Statistical Mechanics and its Applications, 419:526 – 536, 2015.

[18] M. Liebner, M. Baumann, F. Klanner, and C. Stiller. Driver intent inference at urban intersections using the intelligent driver model. In 2012 IEEE Intelligent Vehicles Symposium, pages 1162–1167, 2012.

[19] P. A. Lopez, M. Behrisch, L. Bieker-Walz, J. Erdmann, Y.-P. Flötteröd, R. Hilbrich, L. Lücken, J. Rummel, P. Wagner, and E. Wießner. Microscopic traffic simulation using SUMO. In The 21st IEEE International Conference on Intelligent Transportation Systems. IEEE, 2018.

[20] B. N. Matcha, S. Namasivayam, M. H. Fouladi, K. C. Ng, S. Sivanesan, and S. Y. E. Noum. Simulation strategies for mixed traffic conditions: A review of car-following models and simulation frameworks. The Journal of Engineering, 2020:8231930, 2020.

[21] MATLAB. 9.8.0.1359463 (R2020a). The MathWorks Inc., Natick, Massachusetts, 2021.

[22] G. Orosz, B. Krauskopf, and R. E. Wilson. Bifurcations and multiple traffic jams in a car-following model with reaction-time delay. Physica D: Nonlinear Phenomena, 211(3-4):277–293, 2005.

[23] G. Orosz and G. Stépán. Subcritical hopf bifurcations in a car-following model with reaction-time delay. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 462(2073):2643–2670, 2006.

[24] Oussama, T. Peter, H. Zebiri, B. Mourllion, and M. Basset. Modified intelligent driver model for driver safety and traffic stability improvement. IFAC Proceedings Volumes, 46(21):744 – 749, 2013. 7th IFAC Symposium on Advances in Automotive Control.

[25] J. Schulz, C. Hubmann, J. Löchner, and D. Burschka. Interaction-aware probabilistic behavior prediction in urban environments. 2018 IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS), pages 3999–4006, 2018.

[26] J. Schulz, C. Hubmann, J. Löchner, and D. Burschka. Multiple model unscented kalman filtering in dynamic bayesian networks for intention estimation and trajectory prediction. 2018 21st International Conference on Intelligent Transportation Systems (ITSC), pages 1467–1474, 2018.

[27] M. Treiber, A. Hennecke, and D. Helbing. Congested traffic states in empirical observations and microscopic simulations. Physical review E, 62(2):1805, 2000.

[28] M. Treiber and A. Kesting. Traffic flow dynamics: Data, models and simulation. 2012.

[29] M. Treiber and A. Kesting. The intelligent driver model with stochasticity -new insights into traffic flow oscillations.

Transportation Research Procedia, 23:174 – 187, 2017. Papers Selected for the 22nd International Symposium on Transportation and Traffic Theory Chicago, Illinois, USA, 24-26 July, 2017.

[30] M. Treiber, A. Kesting, and D. Helbing. Delays, inaccuracies and anticipation in microscopic traffic models. Physica A: Statistical Mechanics and its Applications, 360(1):71 – 88, 2006.