3F. Differential equations: Separation of variables

1c)

$$\frac{dy}{dx} = \frac{3}{\sqrt{y}}$$

$$\int \sqrt{y} \, dy = \int 3 \, dx$$

$$\frac{2}{3} y^{3/2} = 3x + C$$

$$\boxed{y = \left(\frac{9}{2}x + \frac{3}{2}C\right)^{2/3}}$$

1d)

$$\frac{dy}{dx} = xy^2$$

$$\int \frac{1}{y^2} \, dy = \int x \, dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 + C$$

$$\boxed{y = -\frac{1}{\frac{1}{2}x^2 + C}}$$

2a)

$$\frac{dy}{dx} = 4xy$$

$$\int \frac{1}{y} \, dy = \int 4x \, dx$$

$$\ln|y| = 2x^2 + C$$

$$|y| = e^{2x^2 + C}$$

$$y = \pm e^{2x^2 + C}$$

$$y(1) = 3 \Rightarrow 3 = e^{2+C} \Rightarrow e^C = \frac{3}{e^2} \Rightarrow C = \ln\left(\frac{3}{e^2}\right) = \ln 3 - 2 \Rightarrow \boxed{y(3) = 3e^{16}}$$

2e)

$$\frac{dy}{dx} = e^y$$

$$\int e^{-y} \, dy = \int dx$$

$$-e^{-y} = x + C$$

$$e^{-y} = -x - C$$

$$-y = \ln(-x - C)$$

$$y(3) = 0 \Rightarrow 0 = \ln(-3 - C) \Rightarrow e^0 = -3 - C \Rightarrow C = -4 \Rightarrow \boxed{y(0) = -\ln 4}$$

4b)

$$\int \frac{dT}{T_e - T} = \int k \, dt$$

$$-\ln|T_e - T| = kt + C$$

$$T_e - T = e^{-kt - C}$$

$$T_e - T = Ae^{-kt} \, ; \, A = e^{-C}$$

$$T = T_e - Ae^{-kt}$$

$$T_0 = T_e - A$$

$$A = T_0 - T_e$$

$$\boxed{T = T_e - (T_0 - T_e)e^{-kt}}$$

4c)

$$T = T_e - \frac{1}{e^{kt}}(T_0 - T_e)$$

$$t \to \infty \, \Rightarrow \, \frac{1}{e^{kt}} \to 0 \, \Rightarrow \, \boxed{T \to T_e} \, \checkmark$$

4d)

$$-kt = \ln\left(\frac{T_e - T}{T_0 - T_e}\right)$$

$$t = -\frac{\ln\left(\frac{T_e - T}{T_0 - T_e}\right)}{k}$$

$$8 = -\frac{\ln\left(\frac{40 - 200}{680 - 40}\right)}{k} \, \Rightarrow \, k = \frac{\ln\frac{1}{4} + i\pi}{8} \, \Rightarrow \, t = -\frac{8\ln\left(\frac{T_e - T}{T_0 - T_e}\right)}{\ln\frac{1}{4} + i\pi}$$

something isn't adding up...