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Higher Algebraic K-Theory

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Introduction

Algebraic K-theory studies a family of functors that associate rings and algebraic varieties (or more generally exact categories) to abelian groups, known as algebraic K-groups and denoted K_n. The classical theory originated in the 1950s and 1960s through the foundational work of Grothendieck on K_0, followed by Bass and Whitehead on K_1, and Milnor on K_2, establishing the lower K-groups.

A major advancement came in the 1970s when Quillen defined higher K-groups K_n for all natural n, unifying and vastly extending the scope of algebraic K-theory. These groups capture deep arithmetic and geometric information about rings and, more generally, algebraic varieties.

Over time, the theory has revealed deep connections across diverse mathematical disciplines, particularly serving as conceptual bridges amongst algebraic number theory, algebraic geometry, arithmetic geometry, and algebraic topology. This poster outlines the definition of K-groups, their properties, some explicit computations, and their significant connections to number theory and arithmetic geometry.

In this poster, R is a commutative ring with unity and all modules are finitely generated, although the former is not usually necessary.

K_0 (Grothendieck group)

A **projective module** is a summand of a free-module. Projective modules form a monoid $\mathbf{P}$ via $P + P' \coloneqq P \oplus P'$. Over PIDs or local rings, projective modules are free (i.e. have a basis). The group K_0 is defined to be formal group closure of $\mathbf{P}$.

Example. Consider $R = \mathbb{Z}$. Because $\mathbb{Z}$ is a PID, finitely generated projective modules over R are free. Thus, $\mathbf{P} \cong \{\mathbb{Z}^n | n \in \mathbb{N}\} \cong \mathbb{N}$, where the isomorphisms are as monoids. The group closure is thus $\mathbb{Z}$, and thus $K_0(R) \cong \mathbb{Z}$.

Likewise, the K_0 of any PID or local ring is $\mathbb{Z}$. Additionally, the tensor product $\otimes$ distributes over $\oplus$, and thus gives K_0 the structure of an commutative ring with unity $1 \coloneqq R$. By examining the above definition, K_0 can be seen as a functor from commutative rings with unity to abelian groups.

Proposition. The functor K_0 respects binary products. That is, $K_0(R \times R') \cong K_0(R) \times K_0(R')$.

Example. We have $K_0(\mathbb{Z}[\zeta_p]) \cong K_0(\mathbb{Z}[\mu_p])$, where the right side denotes the group ring, and μ_p is the pth roots of unity.

K_0 of a Dedekind domain

A **Dedekind domain** R is a domain such that for any $I \subset J$ ideals there is some ideal $J' \subseteq R$ with $I = JJ'$. In a Dedekind domain, any non-zero fractional ideal (R modules $I \subset \mathsf{Frac}(R) = K$ with $rI \subseteq R$ for some nonzero $r \in R$) is invertible; that is, there is some J with $IJ = R$. Let $\mathcal{I}_K$ be the set of non-zero fractional ideals. It can be given an abelian group structure by $I +_G J \coloneqq IJ$ and contains as a subgroup the principal ideals $\mathcal{P}_K$. The quotient $\mathsf{Cl}(K) \coloneqq \mathcal{I}_K/\mathcal{P}_K$ is the **class group**, which measures lack of unique factorization. The size of the class group is called the **class number**.

Theorem. R is a PID if and only if R is a UFD if and only if $\mathsf{Cl}(K) \cong 1$.

Example. If K is a number field (i.e. $[K : \mathbb{Q}] < \infty$) then $\mathcal{O}_K$ (the elements which are roots of monic polynomials over $\mathbb{Z}$) is a Dedekind domain. We get an exact sequence

$$0 \to \mathcal{O}_K^\times \to K^\times \to \mathcal{I}_K \to \mathsf{Cl}(K) \to 0.$$

Lemma. A finitely generated R-module M is projective if and only if it is isomorphic to a direct sum of ideals.

Theorem. (Steinitz) The map $I_1 \oplus \cdots \oplus I_k \to (k, I_1 \cdots I_k) \in \mathbb{N} \times \mathsf{Cl}(K)$ induces an isomorphism $K_0(R) \cong \mathbb{Z} \oplus \mathsf{Cl}(K)$.

Example. We have $K_0(\mathbb{Z}[(1+\sqrt{-5})/2]) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, $K_0(\mathbb{Z}[\zeta_{37}]) \cong \mathbb{Z} \oplus \mathbb{Z}/37\mathbb{Z}$, and $K_0(\mathbb{Z}[\zeta_{29}]) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\otimes 3}$.

K_1 (Whitehead Group)

Consider the map $\mathsf{GL}_n(R) \hookrightarrow \mathsf{GL}_{n+1}(R)$ via $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. This is a directed system with limit $\mathsf{GL}(R)$, which inherits a group structure. Informally, this is the set of infinite matrices with only finitely many differences from the identity. Let $E_n(R)$ denote the subgroup of $\mathsf{GL}_n(R)$ generated by elementary matrices $e_{ij}(r)$ (i.e. identity with the ij entry changed to be r for $i \neq j$). We likewise have $E_n(R) \hookrightarrow E_{n+1}(R)$ and thus some limit $E(R) \subseteq \mathsf{GL}(R)$.

Definition. The K_1 of a ring R is defined to be $\mathsf{GL}(R)/E(R)$.

Proposition. We have $E(R) = [\mathsf{GL}(R), \mathsf{GL}(R)]$, the commutator subgroup of $\mathsf{GL}(R)$.

Corollary. We have $K_1(R) \cong H_1(\mathsf{GL}(R), \mathbb{Z})$, where H_i denotes group homology.

Observation. The determinant of every element of $E_n(R)$ is 1, and thus the unit group $R^\times$ of R will be a direct summand of K_1. Consequently, we have $K_1(R) \cong R^\times \oplus (\mathsf{SL}(R)/E(R))$, where the second summand is denoted SK_1.

Theorem. Let R be a local ring or a Euclidean domain. Then $K_1(R) \cong R^\times$.

Example. Let $R = \mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. Then $K_1(R) \cong (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. We also have $K_1(\mathbb{Z}) \cong \{\pm 1\}$ and $K_1(\mathbb{R}) \cong \mathbb{R}^\times \cong \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$.

Connection to Algebraic Topology. (Mayer-Vietoris) For $I \subseteq R$ an ideal mapped isomorphically via $f : R \to S$ such that quotienting and mapping to S commute, we have the following exact sequence

$$K_1(R) \to K_1(S) \oplus K_1(R/I) \xrightarrow{\pm} K_1(S/I) \xrightarrow{\partial} K_0(R) \to K_0(S) \oplus K_0(R/I) \xrightarrow{\pm} K_0(S/I).$$

K_2 (Milnor Group)

For $n \geq 3$, consider the group $\mathsf{St}_n(R)$ defined by generators $x_{ij}(r)$ for $r \in R$ with the relations

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s), \quad [x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & i \neq l, j \neq k \\ x_{kj}(-rs) & i = l, j \neq k \\ x_{il}(rs) & i \neq l, j = k. \end{cases}$$

There is a natural homomorphism ϕ_n via $x_{ij}(r) \mapsto e_{ij}(r) \subseteq E_n(R)$. Now put $\mathsf{St}(R)$ as the direct limit of $\mathsf{St}_n(R)$. The ϕ define a surjective map $\mathsf{St}(R) \to E(R)$.

Definition. $K_2(R) \coloneqq \mathrm{Ker}(\phi)$. Consequently, the following is exact

$$0 \to K_2(R) \to \mathsf{St}(R) \xrightarrow{\phi} \mathsf{GL}(R) \to K_1(R) \to 0.$$

Theorem. (Steinberg) The center of $\mathsf{St}(R)$ is precisely $K_2(R)$.

Corollary. We have $K_2(R) \cong H_2(E(R), \mathbb{Z})$.

For $R = \mathbb{Z}$, we can compute $K_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, but this is a unwieldy definition in general. However, for $R = F$, a field, it reduces by the following theorem.

Theorem. (Matsumoto) The group $K_2(F)$ is isomorphic to the group generated by $\{x, y\}$ with $x, y \in F^\times$ that is linear on each coordinate and satisfies $\{x, 1-x\} = 1$ for $x \neq 0, 1$.

This can be naturally seen via $\{r, s\} = [x_{12}(r), x_{13}(s)]$ and checking the identities.

Computation. We have $-r = (1-r)/(1-r^{-1})$, so

$$\{r, -r\} = \{r, 1-r\}\{r, 1-r^{-1}\}^{-1} = \{r^{-1}, 1-r^{-1}\} = 1.$$
$$\{r, s\}\{s, r\} = \{r, (-r)s\}\{s, (-s)r\} = \{rs, -rs\} = 1.$$

Corollary. We have $K_2(\mathbb{F}_q) = 1$, where $\mathbb{F}_q$ denotes the finite field on q elements.

Proof. Consider some generator x of $\mathbb{F}_q^\times$. Then $\{x, x\}$ generates $K_2(\mathbb{F}_q)$. For q even, we have $\{x, x\} = \{x, -x\} = 1$. For q odd, find some u non-square such that $1-u$ is not a square by the pigeon hole principle. Then for some odd numbers n, m, we have

$$1 = \{u, 1-u\} = \{x^n, x^m\} = \{x, x\}^{nm} = \{x, x\},$$

where the last equality is since $\{x, x\}^2 = 1$. □

Higher K-groups

For any group G, we write BG to denote the classifying space of G.

Quillen + construction. Given a topological space X and perfect $H \trianglelefteq G = \pi_1(X)$, we say that $f : X \to X^+$ is the $+$ construction if H is the kernel of $f_* : \pi_1(X) \to \pi_1(X^+)$ and induces isomorphisms on all homology groups (this is possible since $H_1 = \pi_1^{\mathrm{ab}}$).

Definition. Consider the perfect normal subgroup $E(R) \subseteq \mathsf{GL}(R)$. Then for $n > 0$,

$$K_n(R) = \pi_n(B\mathsf{GL}(R)^+).$$

Proposition. The functor K_n from rings to abelian groups respects biproducts.

Theorem. We have $K_3(R) = H_3(\mathsf{St}(R), \mathbb{Z})$.

Proof. $K_3(R) \coloneqq \pi_3(B\mathsf{GL}(R)^+) \cong \pi_3(B\mathsf{St}(R)^+) \overset{\star}{\cong} H_3(B\mathsf{St}(R)^+) \cong H_3(\mathsf{St}(R))$, where the $\star$ isomorphism is due to lower homology groups vanishing.

Computation. $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\mathbb{Z}$, and $K_n(\mathbb{Z})$ are related to the **Bernoulli numbers** for $n \equiv 2, 3 \pmod 4$.

Milnor K-Theory

When trying to generalize K-theory, Milnor observed that for fields $R = F$, we have $K_1(F) = F^\times$. The following ad-hoc definition for $n > 0$ turned out to be surprisingly close to K_n, and is now known as Milnor's K-theory.

$$K_n^M(F) = \frac{F^\times \otimes F^\times \otimes \cdots \otimes F^\times}{\langle a_1 \otimes a_2 \cdots \otimes a_n : a_i + a_{i+1} = 1 \rangle}.$$

There exists a natural map $K_n^M(F) \to K_n(F)$ that is an isomorphism for $n \leq 2$. For $n > 2$ this map is no longer necessarily an isomorphism. For example, for $n \geq 2$ we have $K_{2n-1}^M(\mathbb{F}_q) \cong 1$, but $K_{2n-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^n - 1)\mathbb{Z}$.

Example. One can compute

$$K_2^M(\mathbb{Q}) \cong K_2(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_p \mathbb{Z}/(p-1)\mathbb{Z}.$$

Milnor K-theory ($K_n^M(F)$) is deeply related to arithmetic objects such as the Brauer group $\mathsf{Br}(F)$ via the **Merkurjev-Suslin** Theorem ($n = 2$), the abelianized Galois group $\mathbf{G}_F^{\mathrm{ab}}$ (via **Kato's higher dimensional class field theory**), and more generally ètale cohomology via the **Bloch-Kato conjecture** (n arbitrary), proved by Voevodsky, which says that

$$K_n^M(F)/\ell \cong H_{et}^n(F, \mu_\ell^{\otimes n})$$

for some ℓ invertible in F.

K-theory and Number Theory

Computing K-groups in general is very difficult. For example, we can currently classify $K_n(\mathbb{Z})$ for $n = 0, 4$ or $n \not\equiv 0 \pmod 4$. It is conjectured to be trivial for $4n > 0$.

In 1992, Kurihara showed that this statement is equivalent to p not dividing the class number of the maximal real subfield of $\mathbb{Q}(\zeta_p)$, which is equal to $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$.

In this way, K-theory provides a way to relate purely algebraic results to number theoretic results. Indeed, the K-theory of rings of integers of number fields is always a finitely generated abelian group, and is related to special values of L-functions.

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