# Classification of Finitely Often Self-Intersecting Loops in $\mathbb{R}^2$ under equivalence given by isomorphism of corresponding Region Graphs.

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**Abstract**

Abstract placeholder

## 1 Introduction

This work stems from an intuitive observation that for any loop one can draw on a sheet of paper there seems to exist a colouring, which assigns one of two possible colours to any region enclosed by the loop, such that no two neighbouring regions share the same colour. Figure 1 gives a graphical representation of this intuition. We can go further and view coloured regions as vertices in a graph, with vertices representing any two neighbouring regions connected with an edge. Figure 2 shows such a graph derived from the loop in Figure 1. It is natural to consider all possible graphs that can be constructed in such way. Can we describe this collection precisely using familiar graph theoretic adjectives (bipartite, planar, connected, etc.)?

In this work we make rigorous and prove the 2-colourability intution suggested by Figure 1 for a certain class of loops. We also identify the class of graphs corresponding to this class of loops, by constructing an appropriate surjective function between the two classes.

**Definition 1** (Almost Injectivity)**.** Let $f : X \mapsto Y$ be a function. Let us say that $f$ is almost injective, if $f(x_1) = f(x_2) \implies x_1 = x_2$ holds for all but finitely many $x_i \in X$.

**Example 1** (Almost Injective Circle)**.** $f : [0, 1] \mapsto \mathbb{R}^2$ defined by

$$f(x) = (\sin(2\pi x), \cos(2\pi x))$$

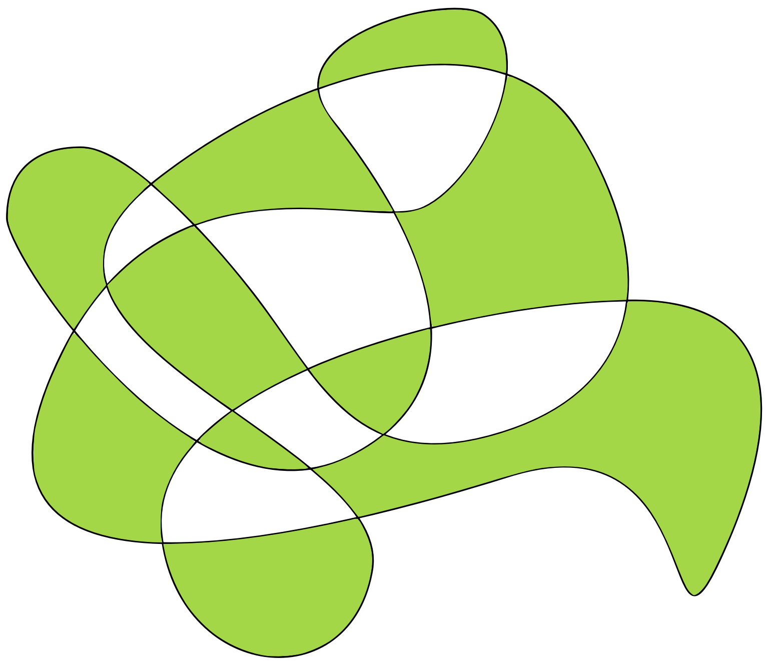

Figure 1: 2-colouring of a closed loop drawn on a sheet of paper.

is an almost injective function, which is injective on $(0,1)$ and fails to be injective on $\{0,1\}$. Figure 1 is a plot of another almost-injective function.

**Definition 2** (Crossing Set)**.** Let $f : X \mapsto Y$ be an almost injective function. Let the crossing set $C_f$ of $f$ be defined by

$$C_f = \{x_1, x_2 \mid f(x_1) = f(x_2) \wedge x_1 \neq x_2\}$$

**Example 2.** The set $\{0,1\}$ is a crossing set of the almost injective circle from Example 1.

**Definition 3** (Crossing Equivalence)**.** Let $C_f$ be the crossing set of of an almost injective function $f$. Let crossing equivalence $\sim_C$ be a relation on $C_f$ defined by $c_1 \sim_C c_2 \iff f(c_1) = f(c_2)$ for any $c_1, c_2 \in C_f$.

**Proposition 1** (Crossing Equivalence Relation)**.** *Crossing equivalence $\sim_C$ is an equivalence relation on the crossing set $C_f$ of an almost injective function $f$.*

*Proof.* Reflexivity, Symmetry and Transitivity follow easily from properties of set equality $=$, and that $f$ is a function. $\square$

**Definition 4** (FOSIL)**.** Let $\gamma : [0,1] \mapsto \mathbb{R}^2$ be a continuous, almost injective function with $\gamma(0) = \gamma(1)$. We will say that $\gamma$ is a finitely often self-intersecting loop in $\mathbb{R}^2$, or FOSIL in short.

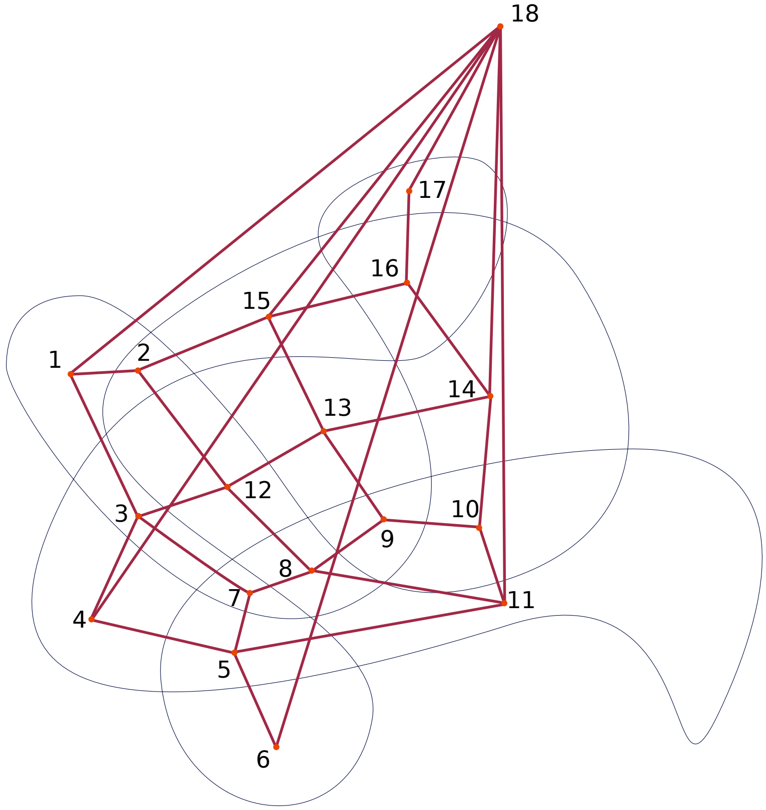


Figure 2: Region graph of the closed loop from Figure 1. Vertices corresponding to regions enclosed by the loop are numbered.

**Definition 5** (Region Number)**.** Let $\gamma$ be a FOSIL. Let $\sim_C$ be a crossing equivalence on $C_\gamma$, the crossing set of $\gamma$. Define region number of $\gamma$, $\rho_\gamma$ as

$$\rho_\gamma = 1 + \sum_{[c] \in C_\gamma / \sim_C} (|[c]| - 1)$$

.

**Definition 6** (Regions)**.** Let $\gamma$ be a FOSIL. Let $R^\gamma$ denote a collection of open, connected, disjoint subsets of $\mathbb{R}^2$, whose union gives the complement of $\gamma$, that is $\mathbb{R}^2 \setminus \gamma([0,1])$. Then $R^\gamma$ is called a collection of regions of $\gamma$, and $R_i \in R^\gamma$ is called a region.

**Theorem 1** (Jordan Curve Theorem for FOSILs)**.** *Let $\gamma$ be a FOSIL, $R^\gamma$ be a collection of regions of $\gamma$, and $\rho_\gamma$ be the region number of $\gamma$. Then the collection of regions is unique, and*

$$|R^\gamma| = \rho_\gamma$$

This auxiliary result has a rather technical proof, to which we devote a separate chapter.

**Definition 7** (Edge-Connectedness)**.** Let $\gamma$ be a FOSIL. Let $R_1, R_2 \in R^\gamma$ be regions of $\gamma$, and $C_\gamma$ be the crossing set of $\gamma$. We will say that $R_1$ and $R_2$ are edge-connected if $R_1 \neq R_2$ and there exists an open, connected subset of $[0,1]$, say $S$, such that $S \cap C_f = \emptyset$, and such that $R_1 \cup \gamma(S) \cup R_2$ is connected.

**Proposition 2** (Edge-Connectedness Reflexivity)**.** *Edge connectedness on the collection of regions of $\gamma$ is a reflexive relation.*

*Proof.* Reflexivity follows from commutativity of set union $\cup$. □

**Definition 8** (Graph)**.** Let $V$ be a finite set. Let $E$ be a subset of $V \times V$. Then $G = (V, E)$ is called a graph if for any $(v_1, v_2) \in E$ we have that $(v_2, v_1) \in E$. $v \in V$ is called a vertex in G, and $(v_1, v_2) \in E$ is called an edge in G.

**Definition 9** (Region Graph)**.** Let $R^\gamma$ be the collection of regions of a FOSIL $\gamma$. Let $E = \{(R_1, R_2) \in R^\gamma \times R^\gamma \mid R_1 \,\text{edge-connected}\, R_2\}$. Then $G_\gamma = (R^\gamma, E)$ is called a region graph of a FOSIL $\gamma$.

**Proposition 3** (Region Graph is a Graph)**.** *The region graph $G_\gamma = (V, E)$ of a FOSIL $\gamma$ is a graph.*

*Proof.* By Jordan Curve Theorem for FOSILs, $V$ is uniquely defined and finite. Let $(R_1, R_2) \in E$. Then $(R_2, R_1) \in E$ by reflexivity of edge-connectedness. □

**Definition 10** (Graph Isomorphism)**.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $|V_1| = |V_2|$ and $|E_1| = |E_2|$. We will say that $G_1$ is graph isomorphic, $\sim_G$, to $G_2$ if there exists an injective function $\psi : V_1 \mapsto V_2$ such that for each $(v_1, v_2) \in E_1$, we have that $(\psi(v_1), \psi(v_2)) \in E_2$.

**Proposition 4** (Graph Isomorphism Equivalence Relation)**.** *Let $\mathfrak{G}$ be a collection of graphs. Then $\sim_G$ is an equivalence relation on $\mathfrak{G}$.*

*Proof.* TODO Symmetry, transitivity, reflexivity. $\square$

**Definition 11** (Path in Graph)**.** Let $G = (V, E)$ be a graph. Let $P = \{v_k\}$ be a sequence of vertices in G, such that for any $v_i, v_{i+1} \in P$, we have that $(v_i, v_{i+1}) \in E$. Then $P$ is called a path in graph G.

**Definition 12** (Connected Graph)**.** Let $G = (V, E)$ be a graph. Let us say that $G$ is connected if there exists a path $P$ in $G$, such that for any $v \in V$ we have that $v \in P$.

**Definition 13** (Bipartite Graph)**.** Let $G = (V, E)$ be a graph. If there exist disjoint $S_1, S_2 \subseteq V$ with $S_1 \cup S_2 = V$ such that for any $v_i, w_i \in S_i$ we have $(v_i, w_i) \notin E$, then G is called a bipartite graph.

**Definition 14** (Planar Graph)**.** TODO define planar graphs.

**Theorem 2** (FOSIL Classification Theorem)**.** *Let $\mathfrak{F}$ be the collection of all FOSILs. Let $\mathfrak{G} = \{G_\gamma | \gamma \in \mathfrak{F}\}$ be the collection of corresponding region graphs. Let $\mathfrak{B}$ be the collection of all connected, bipartite, planar graphs, containing more than 1 vertex. By abuse of notation, let $\sim_G$ denote the equivalence relation given by graph isomorphism on both $\mathfrak{G}$ and $\mathfrak{B}$. The function*

$$\Phi : \mathfrak{G}/\sim_G \mapsto \mathfrak{F}/\sim_G$$

*given by*

$$\Phi([G]_{\mathfrak{G}}) = [G]_{\mathfrak{B}}$$

*is a well-defined bijection.*

*In other words, $\mathfrak{G} \subseteq \mathfrak{B}$ and for any $B \in \mathfrak{B}$ there is a $G \in \mathfrak{G}$ with $G \sim_G B$.*

We devote the next chapter to the proof of this theorem.

## 2 FOSIL Classification Theorem

The inclusion bit should follow from Hopf's degree theorem. Surjectivity is much trickier, but there seems to be an algorithm to go from any bipartite graph to a FOSIL, but I don't quite have the details yet.

## 3 Jordan Curve Theorem for FOSILs

The starting point should be the proof of Jordan-Holder. Two proofs in particular seem of interest [1], [2].

## References

[1] T. Hales. Jordan's proof of the Jordan Curve Theorem. *Studies in Logic, Grammar and Rhetoric*, 10(23):116–131, 2007.

[2] C. Thomassen. The Jordan-Schonflies theorem and the classification of surfaces. *The American Mathematical Monthly*, 99(2):116–131, 1992.