Classification of Finitely Often Self-Intersecting Loops in $\mathbb{R}^2$ under equivalence given by isomorphism of corresponding Region Graphs.

Adam Kurkiewicz
School of Computer Science
University of Glasgow
United Kingdom

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Abstract

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1 Introduction

Definition 1 (Almost Injectivity)**.** Let $f : X \mapsto Y$ be a function. Let us say that f is almost injective, if $f(x_1) = f(x_2) \implies x_1 = x_2$ holds for all but finitely many $x_i \in X$.

Definition 2 (Crossing Set)**.** Let $f : X \mapsto Y$ be an almost injective function. Let the crossing set C_f of f be defined by

$$C_f = \{x_1, x_2 \mid f(x_1) = f(x_2) \wedge x_1 \neq x_2\}$$

Definition 3 (Crossing Equivalence)**.** Let C_f be the crossing set of of an almost injective function f. Let crossing equivalence $\sim_C$ be a relation on C_f defined by $c_1 \sim_C c_2 \iff f(c_1) = f(c_2)$ for any $c_1, c_2 \in C_f$.

Proposition 1 (Crossing Equivalence Relation)**.** *Crossing equivalence $\sim_C$ is an equivalence relation on the crossing set C_f of an almost injective function f.*

Proof. Reflexivity, Symmetry and Transitivity follow easily from properties of set equality =, and that f is a function. $\square$

Definition 4 (FOSIL)**.** Let $\gamma : [0, 1] \mapsto \mathbb{R}^2$ be a continuous, almost injective function with $\gamma(0) = \gamma(1)$. We will say that γ is a finitely often self-intersecting loop in $\mathbb{R}^2$, or FOSIL in short.

Definition 5 (Region Number)**.** Let γ be a FOSIL. Let $\sim_C$ be a crossing equivalence on C_γ, the crossing set of γ. Define region number of γ, ρ_γ as

$$\rho_\gamma = 1 + \sum_{[c] \in C_\gamma / \sim_C} (|[c]| - 1)$$

.

Definition 6 (Regions)**.** Let γ be a FOSIL. Let R^γ denote a collection of open, connected, disjoint subsets of $\mathbb{R}^2$, whose union gives the complement of γ, that is $\mathbb{R}^2 \setminus \gamma([0,1])$. Then R^γ is called a collection of regions of γ, and $R_i \in R^\gamma$ is called a region.

Theorem 1 (Jordan Curve Theorem for FOSILs)**.** *Let γ be a FOSIL, R^γ be a collection of regions of γ, and ρ_γ be the region number of γ. Then the collection of regions is unique, and*

$$|R^\gamma| = \rho_\gamma$$

This auxiliary result has a rather technical proof, to which we devote a separate chapter.

Definition 7 (Edge-Connectedness)**.** Let γ be a FOSIL. Let $R_1, R_2 \in R^\gamma$ be regions of γ, and C_γ be the crossing set of γ. We will say that R_1 and R_2 are edge-connected if $R_1 \neq R_2$ and there exists an open, connected subset of $[0,1]$, say S, such that $S \cap C_f = \emptyset$, and such that $R_1 \cup \gamma(S) \cup R_2$ is connected.

Proposition 2 (Edge-Connectedness Reflexivity)**.** *Edge connectedness on the collection of regions of γ is a reflexive relation.*

Proof. Reflexivity follows from commutativity of set union $\cup$. $\square$

Definition 8 (Graph)**.** Let V be a finite set. Let E be a subset of $V \times V$. Then $G = (V, E)$ is called a graph if for any $(v_1, v_2) \in E$ we have that $(v_2, v_1) \in E$. $v \in V$ is called a vertex in G, and $(v_1, v_2) \in E$ is called an edge in G.

Definition 9 (Region Graph)**.** Let R^γ be the collection of regions of a FOSIL γ. Let $E = \{(R_1, R_2) \in R^\gamma \times R^\gamma \mid R_1 \, \text{edge-connected} \, R_2\}$. Then $G_\gamma = (R^\gamma, E)$ is called a region graph of a FOSIL γ.

Proposition 3 (Region Graph is a Graph)**.** *The region graph $G_\gamma = (V, E)$ of a FOSIL γ is a graph.*

Proof. By Jordan Curve Theorem for FOSILs, V is uniquely defined and finite. Let $(R_1, R_2) \in E$. Then $(R_2, R_1) \in E$ by reflexivity of edge-connectedness. $\square$

Definition 10 (Graph Isomorphism)**.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with $|V_1| = |V_2|$ and $|E_1| = |E_2|$. We will say that G_1 is graph isomorphic, $\sim_G$, to G_2 if there exists an injective function $\psi : V_1 \mapsto V_2$ such that for each $(v_1, v_2) \in E_1$, we have that $(\psi(v_1), \psi(v_2)) \in E_2$.

Proposition 4 (Graph Isomorphism Equivalence Relation)**.** *Let $\mathfrak{G}$ be a collection of graphs. Then $\sim_G$ is an equivalence relation on $\mathfrak{G}$.*

Proof. TODO Symmetry, transitivity, reflexivity. $\square$

Definition 11 (Path in Graph)**.** Let $G = (V, E)$ be a graph. Let $P = \{v_k\}$ be a sequence of vertices in G, such that for any $v_i, v_{i+1} \in P$, we have that $(v_i, v_{i+1}) \in E$. Then P is called a path in graph G.

Definition 12 (Connected Graph)**.** Let $G = (V, E)$ be a graph. Let us say that G is connected if there exists a path P in G, such that for any $v \in V$ we have that $v \in P$.

Definition 13 (Bipartite Graph)**.** Let $G = (V, E)$ be a graph. If there exist disjoint $S_1, S_2 \subseteq V$ with $S_1 \cup S_2 = V$ such that for any $v_i, w_i \in S_i$ we have $(v_i, w_i) \notin E$, then G is called a bipartite graph.

Definition 14 (Planar Graph)**.** TODO define planar graphs.

Theorem 2 (FOSIL Classification Theorem)**.** *Let $\mathfrak{F}$ be the collection of all FOSILs. Let $\mathfrak{G} = \{G_\gamma | \gamma \in \mathfrak{F}\}$ be the collection of corresponding region graphs. Let $\mathfrak{B}$ be the collection of all connected, bipartite, planar graphs, containing more than 1 vertex. By abuse of notation, let $\sim_G$ denote the equivalence relation given by graph isomorphism on both $\mathfrak{G}$ and $\mathfrak{B}$. The function*

$$\Phi : \mathfrak{G}/\sim_G \mapsto \mathfrak{F}/\sim_G$$

given by

$$\Phi([G]_{\mathfrak{G}}) = [G]_{\mathfrak{B}}$$

is a well-defined bijection.

In other words, $\mathfrak{G} \subseteq \mathfrak{B}$ and for any $B \in \mathfrak{B}$ there is a $G \in \mathfrak{G}$ with $G \sim_G B$.

We devote the next chapter to the proof of this theorem.

2 FOSIL Classification Theorem

The inclusion bit should follow from Hopf's degree theorem. Surjectivity is much trickier, but there seems to be an algorithm to go from any bipartite graph to a FOSIL, but I don't quite have the details yet.

3 Jordan Curve Theorem for FOSILs

The starting point should be the proof of Jordan-Holder. Two proofs in particular seem of interest [1], [2].

References

[1] T. Hales. Jordan's proof of the Jordan Curve Theorem. *Studies in Logic, Grammar and Rhetoric*, 10(23):116–131, 2007.

[2] C. Thomassen. The Jordan-Schonflies theorem and the classification of surfaces. *The American Mathematical Monthly*, 99(2):116–131, 1992.