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Optimal control of inverted pendulum system using PID controller, LQR and MPC

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Abstract. Inverted pendulum is a highly nonlinear system. Here we propose an optimal control technique for the control of an inverted Pendulum - cart system. The system is modeled, linearized and controlled. Here**,** the control objective is to control the system such that when the cart reaches a desired position the inverted pendulum stabilizes in the upright position. Initially PID controller is used to control the system. Later, Linear Quadratic Regulator (LQR) a well-known optimal control technique which makes use of the states of the dynamical system and control input to frame the optimal control decision is used. Various combinations of both PID and LQR controllers are implemented. To validate the robustness of the controller, the system is simulated with and without disturbance. Finally the system is also controlled using Model Predictive controller (MPC). MPC has well predictive ability to calculate future events and implement necessary control actions. The performance of the system is compared and analyzed.

1. Introduction

The inverted pendulum is a highly unstable system which is very difficult to control, thus making it a significant classical problem for research in control system domain [1]. Being utmost important it is used for analyzing and designing of control laws. The system comprises of a cart on which a pole mounted and it moves horizontally. It posses a stable and unstable equilibrium state. The pendulum orients downwards in the stable equilibrium state and in vertically upward direction in the latter state. Hence in order to align the pendulum in the latter position a force is required [2].

To frame a control law for a system first we have to obtain a dynamical model of that system. The control law is designed based on this dynamic model so that required performance is achieved. Most of the systems that have dynamical model are non-linear [2]. Being a non-linear system designing of control laws for inverted pendulum is very challenging. PID controllers are broadly utilized as a part of process control framework. To obtain best behavior from the system we use optimal control techniques with PID controller [3]. All control problems has a performance index. The Optimal control technique either maximizes or minimizes this performance index. Many optimal methods and algorithms are used to obtain best performance. In this paper we use LQR method to obtain optimal control of inverted pendulum, which is highly non-linear system[4].

In order to compare the performance, the system is controlled using model predictive controller (MPC). MPC optimizes the current parameters and it can predict the future values and thus it takes control action accordingly [9]. Initially system is controlled using PID controller and then controlled using MPC. Later LQR, the optimal control technique is used to control the system. In all

the above cases the system is analyzed with and without disturbances. The system performances are compared.

This paper has five sections. Section one gives a brief idea about the paper. Section two deals with the modeling and linearization of the system. Section three describes the designing of various control strategies. Section four depicts the simulation results and observations. The paper concludes with the results obtained in section five.

2. Modelling of Pendulum -Cart System

Here we consider a pendulum cart system.Figure2.1 represents the free body diagram of the system. Here we assume that the rod of the pendulum is mass-less and the hinge to which the pendulum is fixed is frictionless. The mass . of the . pendulum is concentrated at the center . of gravity . of the pendulum which . is located at pendulum ball's center. The mass of the cart is represented as M_c and the mass of pendulum is represented as m_p. The control force $u(t)$ acts along the x direction of the cart. The rod's length is represented as l.The . angle by which the . pendulum is . tilted represented as θ [5] .
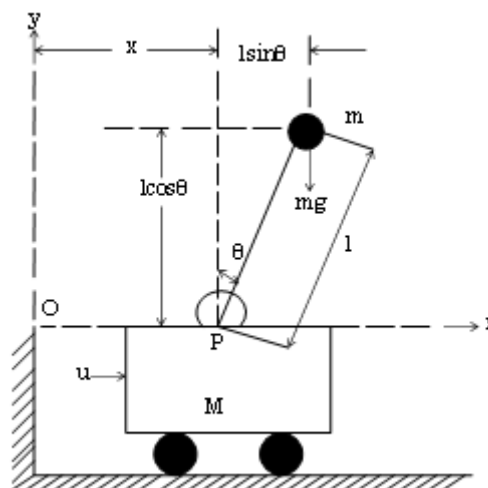
Figure 1. Pendulum-cart system

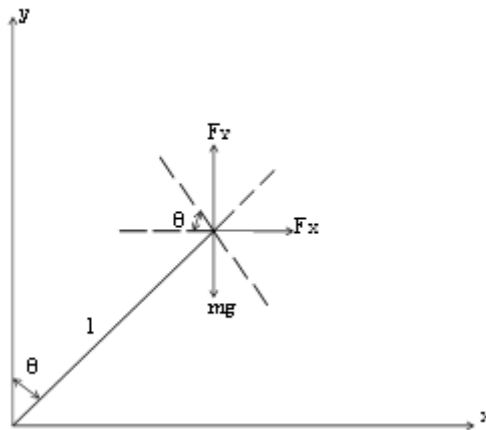The force equation along the x direction is [5]

$$M_C\frac{d^2x}{dt^2}+m_p\frac{d^2x_g}{dt^2} \tag{1}$$

$$x_g = x + l_p sin\theta \; , y_g = l_p cos\theta \tag{2}$$

We have

$$(M_c + m_p)\ddot{x} - m_p l_p sin\theta\dot{\theta}^2 + m_p l_p cos\theta\ddot{\theta} \tag{3}$$

The torque balance of the system is represented in Fig.2

Figure 2. Vector diagram of torque balance

The torque balance equation is:

$$(F_X cos\theta)l_p - (F_Y sin\theta)l_p = (m_c sin\theta)l_p \tag{4}$$

Where $F_X = m_p \frac{d^2 x_g}{dt^2}$ is the force along x direction and $F_Y = m_p \frac{d^2 y_g}{dt^2}$ is the force along y direction.

After simplifying we get

$$m_p \ddot{x} cos\theta + m_p l_p \ddot{\theta} = m_p g sin\theta \tag{5}$$

Thus we have

$$\ddot{x} = \frac{u + m_p l_p (sin\theta)\dot{\theta}^2 - m_p g cos\theta sin\theta}{M_c + m_p - m_p cos^2\theta} \tag{6}$$

$$\ddot{\theta} = \frac{u cos\theta + (M_c + m_p)g sin\theta + m_p l_p (cos\theta sin\theta)\dot{\theta}^2}{m_p l_p cos^2\theta - (M_c + m_p)l_p} \tag{7}$$

The equation (6) and (7) represents the pendulum - cart system's Dynamical model.

2.1. State space model of system

The equations (6) and (7) are to be represented in state space [1].

$$\frac{dx}{dt} = f(x, u, t) \tag{8}$$

The state variables are

$$x_1 = \theta, x_2 = \dot{\theta} = \dot{x}_1, x_3 = x, x_4 = \dot{x} = \dot{x}_3 \tag{9}$$

The state space equations are

$$\frac{dx}{dt} = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ .x_4 \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{10}$$

Where,

$$f_1 = x_2 \tag{11}$$

$$f_2 = \frac{u\cos x_1 - (M_c + m_p)g\sin x_1 + m_p l_p (\cos x_1 \sin x_1) x_2{}^2}{m_p l_p \cos^2 x_1 - (M_c + m_p) l_p} \tag{12}$$

$$f_3 = x_4 \tag{13}$$

$$f_4 = \frac{u + m_p l_p (\sin x_1) x_2{}^2 - m_p g \cos x_1 \sin x_1}{M_c + m_p - m_p \cos^2 x_1} \tag{14}$$

The output equation is represented

$$y = Cx \ or \ y = \begin{bmatrix} \theta \\ x \end{bmatrix} = Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} \tag{15}$$

2.2. Linearisation of the inverted pendulum

The objective is to place the pendulum in erect position that is $\theta = 0$. The system is linearized about this equilibrium point by jacobian method [6]. The system is linearized with the condition $x_0 = 0$ and $u_0 = 0$. The linearized system is in the form

$$\frac{d\delta x}{dt} = J_x(x_0, u_0)\delta x + J_u(x_0, u_0)\delta u \tag{16}$$

After linearization the terms of the jacobian matrix are found systematically. We get

$$J_x(x_0, u_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(M_c + m_p)g}{M_c l_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-m_p g}{M_C} & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

After differentiating with respect to u we get,

$$J_u(x_0, u_0) = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} \\ \dfrac{\partial f_2}{\partial u} \\ \dfrac{\partial f_3}{\partial u} \\ \dfrac{\partial f4}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{\cos x_1}{m_p l_p \cos^2 x_1 - (M_c + m_l)l_p} \\ 0 \\ \dfrac{1}{M_c + m_p - m_p \cos^2 x_1} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{1}{M_c l_c} \\ 0 \\ \dfrac{1}{M_c} \end{bmatrix} \tag{18}$$

After simplifying we get

$$\frac{d\delta x}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(M_c + m_p)g}{M_c l_p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m_p g}{M_c} & 0 & 0 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 0 \\ -\dfrac{1}{M_c l_p} \\ 0 \\ \dfrac{1}{M_c} \end{bmatrix} \delta u \tag{19}$$

This is the linearized form of the system in perturbation form. Equation 19 can be generalized as

$$\frac{d\delta x}{dt} = A\delta x + B\delta u \tag{20}$$

3. Control methods

Different methods of control . are used to bring the system. to stable position.

3.1. PID control

First we use PID controller to bring the inverted pendulum at equilibrium point when the cart reaches at desired position. Here we use 2 PID controllers. One PID controller to regulate the .cart position and other one to control pendulum. The PID controller equation is

$$u_c = K_{pc} . e_x(t) + . K_{ic} . \int e_x . (t)dt + . K_{dc} \frac{de_x(t)}{dt} \tag{21}$$

$$u_p = . K_{pp} e_\theta(t) + K_{ip} . \int e_\theta (t)dt + K_{dp} . \frac{de_\theta(t)}{dt} \tag{22}$$

Here $e_x(t)$. represent the position error, $e_\theta(t)$ represent error angle, u_c represent cart control signal and u_p represents the pendulum angle control signal [2]. The reference position is 0.1m and the reference angle is 0.The controller's tuning has large effect on the control action. The . parameters of the controllers are found through trial and error method. The system using PID controller is simulated in MATLAB[1].

3.2. Optimal control using LQR

Optimal control techniques are used to get best performance from the system with respect to the specified criteria [8]. The main objective of optimal technique to find the best control signals that results the best performance of the . .system by satisfying the specified physical .constraints and by maximizing or minimizing the performance index [1]. Linear . quadratic regulator . (LQR) is a control technique that . is optimal. Linear quadratic regulator (LQR),is a well known optimal control technique, that makes use of dynamical . system's states and to frame the optimal control law. They denote a robust control technique [4].

3.2.1 Framing of optimal law

By taking the initial conditions x_0=[0 0 0 0] .^{T}. The state space . equation is

$$\dot{x} . = Ax + Bu \tag{23}$$

Where $x = \begin{bmatrix} \theta & \dot{\theta} & x & \dot{x} \end{bmatrix}$

The feedback signal is $u = -Kx$. We get

$$\dot{x} = (A - BK)x .\tag{24}$$

The cost function is $J = \int_{t_0}^{t_f}(x^T Q x + u^T R u)dt$ $\qquad(25)$

Where Q is a .positive semi .definite matrix and .R is the positive definite symmetric constant .matrix.

K is obtained in LQR gain vector from

$$K=R^{-1}B^{T}P\tag{26}$$

P is a positive . definite constant matrix . and it is found from the algebraic . Riccati equation . (ARE)

$$A\backslash.^{T}P + P .A - PBR^{-1}B^{T}P + .Q = 0\tag{27}$$

For LQR we give four states as feedback. They are cart position (x), cart velocity ($\dot{x}$), pendulum angle *(θ)* and angular velocity ($\dot{\theta}$) .These are considered as available measurements. The LQR output is added negatively to PID control to obtain optimal control. The system is simulated using different combinations of PID and LQR technique.

3.3. Model Predictive Control

The function of MPC is to compute the value to be used in the input in real time for problem optimization. MPC makes use of the input and output values from the earlier process to predict the following output in a definite size of horizon of time. The optimisation starts after the calculation of time at that desired horizon [10].

MPC calculates the optimal signal corresponding to each input by considering the constraints. It predicts the future by making use of the input output values of the preceding state. After that the time is shifted and the calculation of input is done again to obtain an optimal solution which can be used for the further calculations [11] . The process repeats in such a manner so as to always produce an optimal control.

4. Simulation Results

The non-linear inverted pendulum is modelled, analysed and controlled using the Matlab-Simulink models for cases . of with and without disturbance . input [1]. The system parameters are . cart's Mass *(M_C)* =2.4 kg; pendulum's mass *(M_p)*=0.23 kg; Cart track's length (L_C =±0.5m); the coefficient of friction between cart and .pole rotation is taken as .negligible. The values of input . disturbance parameters used for the .simulation are seed = 23341, magnitude of band limited white noise power as 0.001 and the sampling values time is taken as 0.01 [5].

Next step is to design LQR for which the system matrices are required and are obtained as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 29.8615 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.9401 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -1.1574 \\ 0 \\ 0.4167 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By choosing $R = 1$ and

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 500 & 0 \\ 0 & 0 & 0 & 250 \end{bmatrix}$$

The gain vector of the LQR is attained as

$$K = \begin{bmatrix} -137.7896 & -25.9783 & -22.3607 & -27.5768 \end{bmatrix}$$

The optimal control of the inverted pendulum is implemented using three control combinations [1]:

1. Control using two PID controllers – i) Cart PID ii) Angle PID
2. Control using two PID controllers (combination of both cart and angle PID) with LQR
3. Control using one PID with LQR

The values of PID parameters which are tuned for these control schemes for the cases of with and without disturbance is tabulated in tables 1 and 2.   .

Table 1. Parameters of PID controller without disturbance

Control Schemes	Angle PID control			Cart PID control		
	K_{pa}	K_{ia}	K_{da}	K_{pca} K_{dca}	K_{ica}	
PID	-40	0	-8	-1	0	-3
LQR + 2 PID	1	1	1	1.5	-7.5	5
LQR + 1 PID	-	-	-	1.5	-7.5	5

Table 2. Parameters of PID controller with disturbance

| Control Schemes	Angle PID control			Cart PID control		
	K_{pa}	K_{ia}	K_{da}	K_{pca}	K_{ica}	K_{dca}
PID	-40	0	-8 .	-1.25	0	-3.6
LQR + 2 PID	1.	1	1	1.5	-7.5	5
LQR + 1 PID	-	-	-	1.5	-7.5	5.

Where K_{pa}, K_{ia} and K_{da} are the propotional , integral and differential gains of the pendulum respectively, and K_{pca}, K_{ica} and K_{dca} are the propotional , integral and differential gains of the cart respectively The first control method is using only PID controllers is simulated in MATLAB Simulink and both conditions of without and with disturbance are depicted in Fig 3 and Fig 5 respectively. The disturbance input added to the system is band limited white noise. The parameters considered for measurement are the position of the cart x and .pendulum's angle θ.
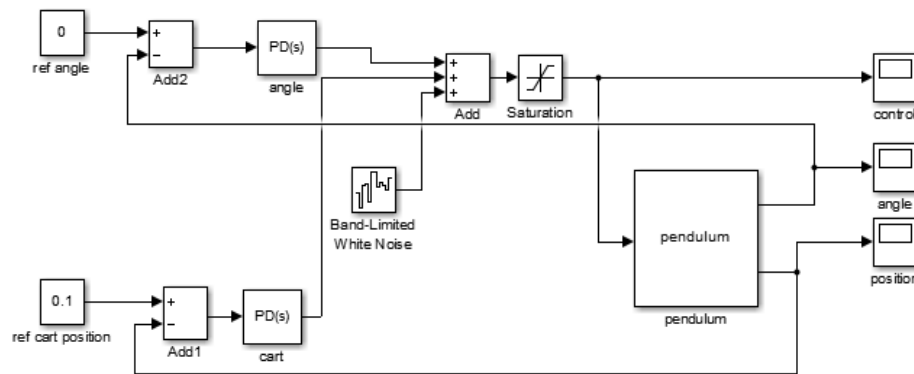
The references for cart position and pendulum angle are 0.1m and 0 rad respectively. The results for simulation are shown in figures 4 and 6 respectively.
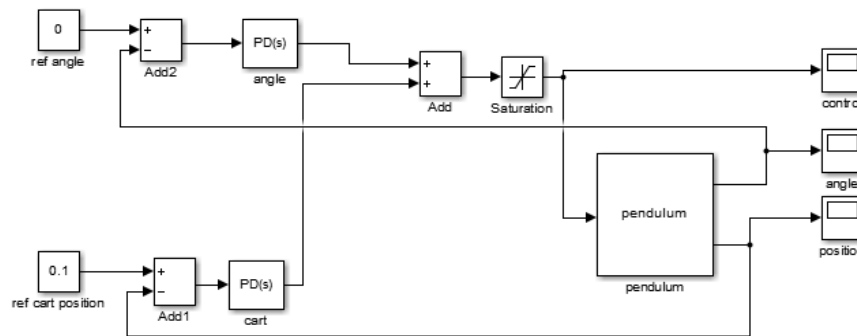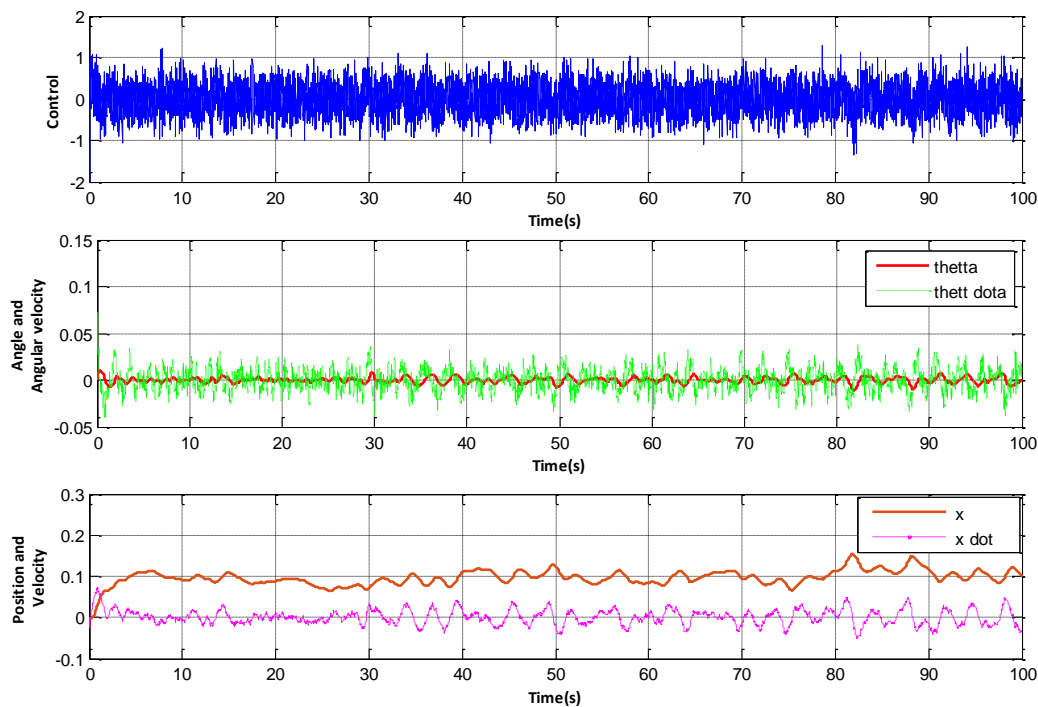

Figure 3. Control of system with two PID controllers without disturbance
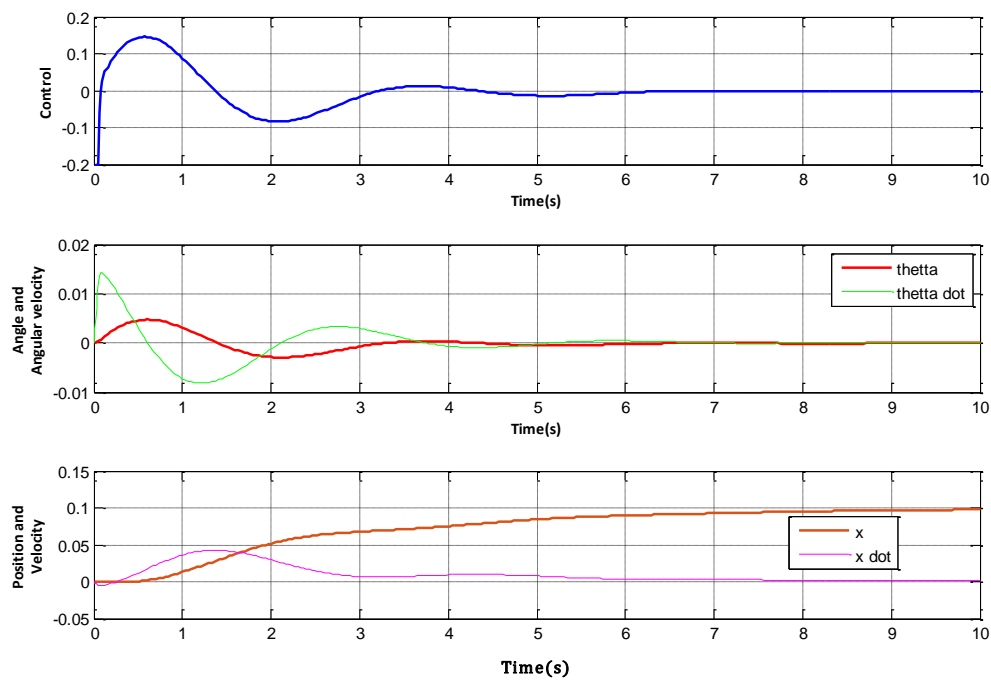
Figure 4. Responses of control force , pendulum angle and cart position without disturbance.

**Figure 5.** Control of system with two PID controllers with disturbance

Figure 6. Responses of control force , pendulum angle and cart position with disturbance.

Figure 4 and figure 6 shows that for the case of with disturbance the pendulum after two small overshoots stabilises in the vertically upward position and in the case of without disturbance stabilises with minor oscillations.

The Simulink models of the nonlinear pendulum cart system for the optimal control utilizing two PID controllers (angle PID and cart PID) with LQR control strategy for both instances of with and without disturbances input are shown in figs . 7 and 9 .respectively [1]. The LQR is fed with the states of the system i.e $x, \dot{x}, \theta, \dot{\theta}$ which is designed using system's linear space .model. The variables of control are chosen as angle θ and cart position x [4]. The   input disturbance added to the system

is band limited white noise. The references taken are as before. Figs. 8 and 10 shows the simulation results. The efficiency and strength of the system is depicted by the simulation results.
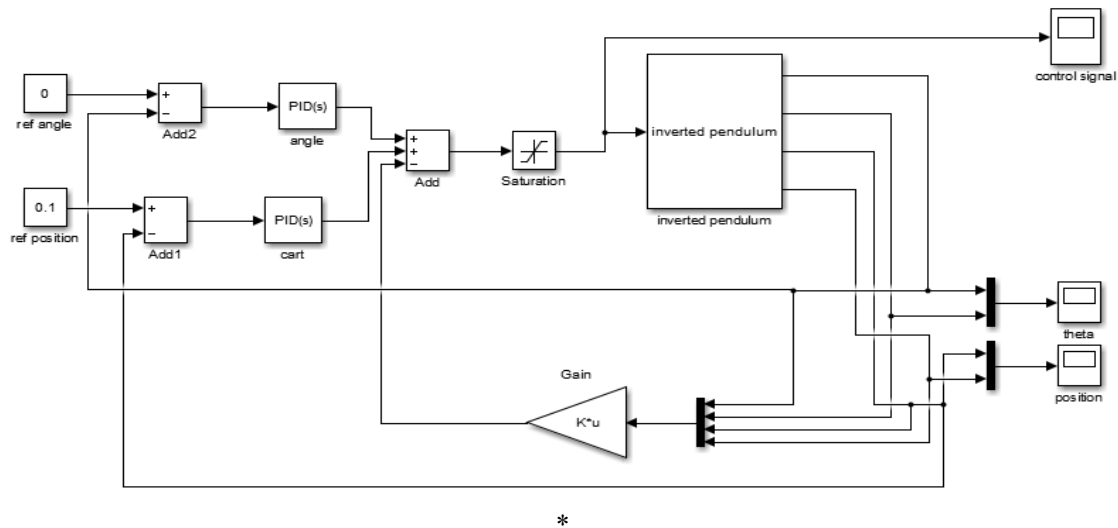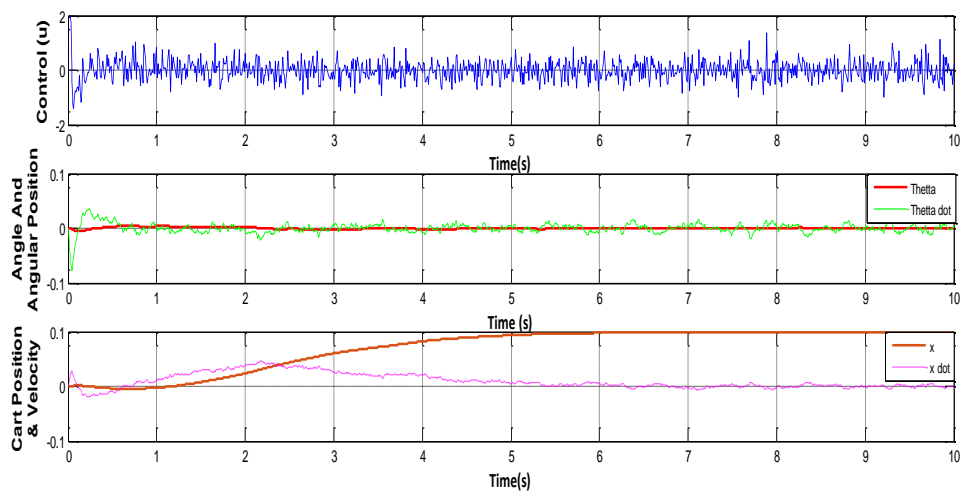

Figure 7. Control using two PID controllers with LQR without disturbance
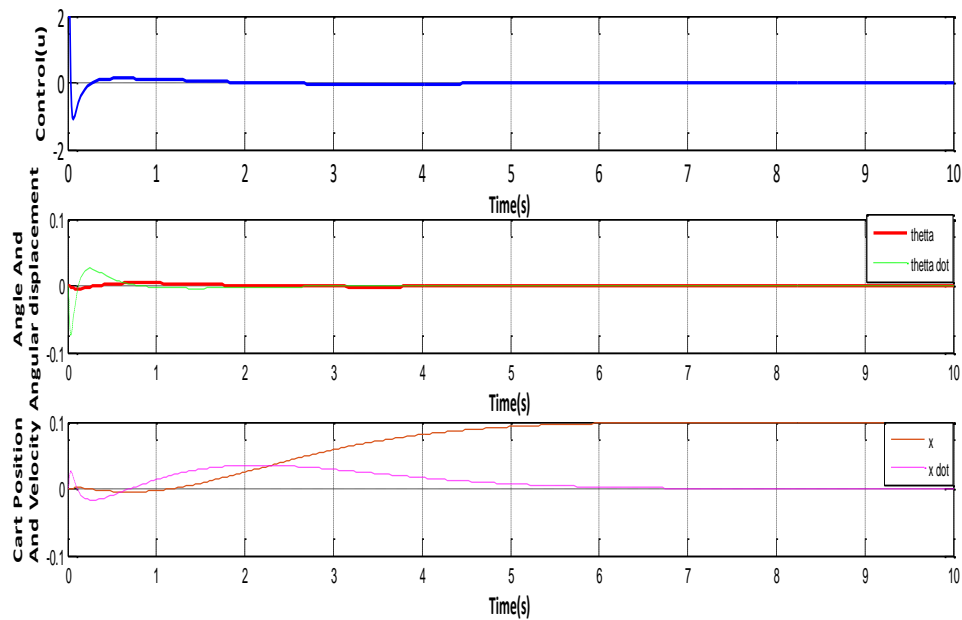
Figure 8. Responses of control signal, pendulum angle, angular displacement, Cart position and cart velocity without disturbance
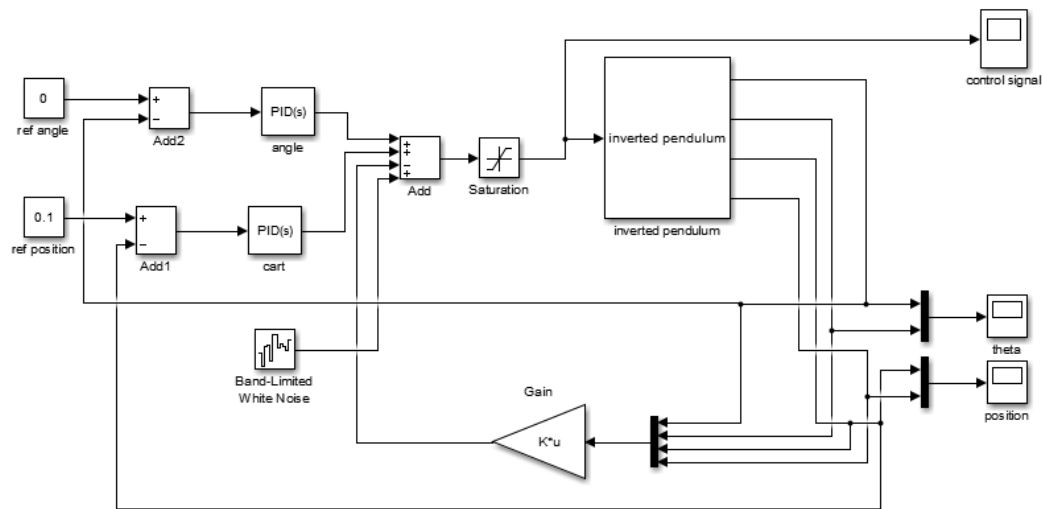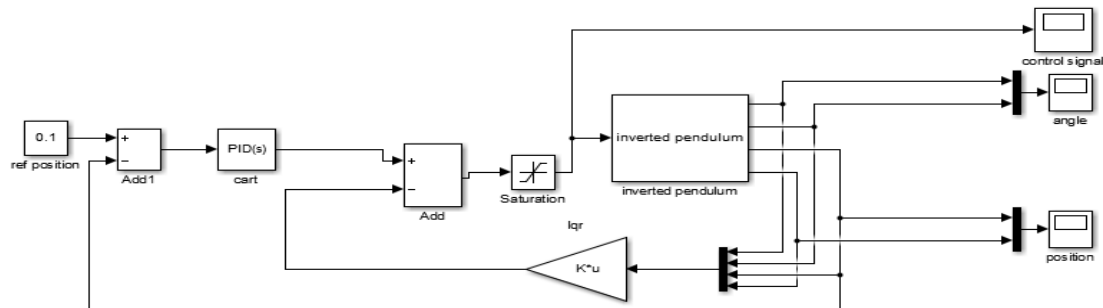
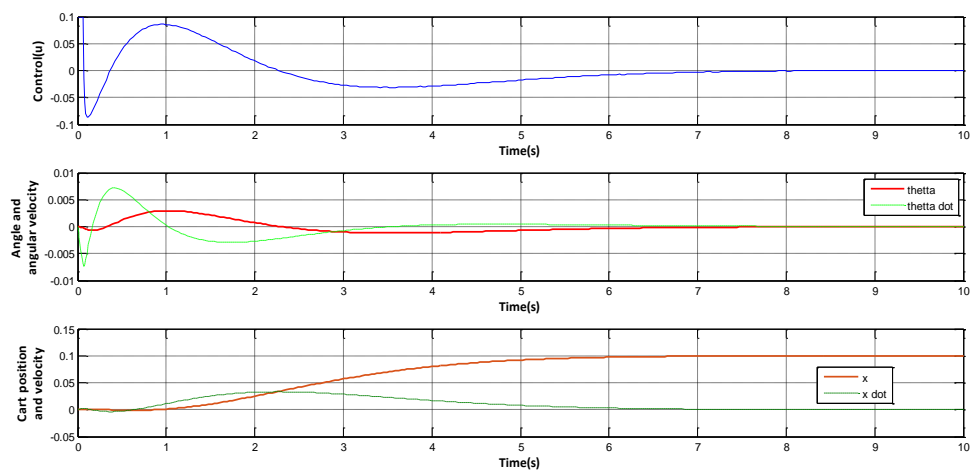**Figure 9.** Control using two PID and LQR with disturbance

Figure 10. Responses of control signal, pendulum angle, angular displacement, Cart position and cart velocity with disturbance

The next control strategy using one PID controller and LQR without and with disturbance is modelled in Simulink Matlab as shown in figs. 11 and 13 PID controller is used here for cart position control only and not for angle control [1]. This method resembles the second one in all aspects apart from the number of controllers used. Cart PID controller is only and it is set with a reference of 0.1m.
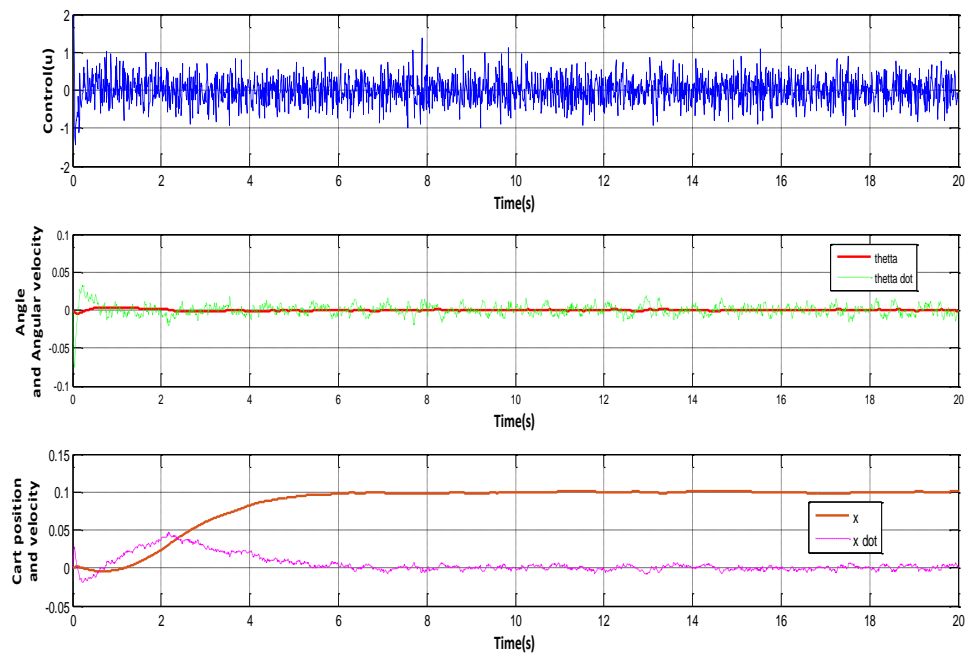
Figure 11. Control of the .system . using cart PID . and LQR without disturbance

Figure 12. Responses of control signal, pendulum angle, angular displacement, Cart position and cart velocity with .disturbance
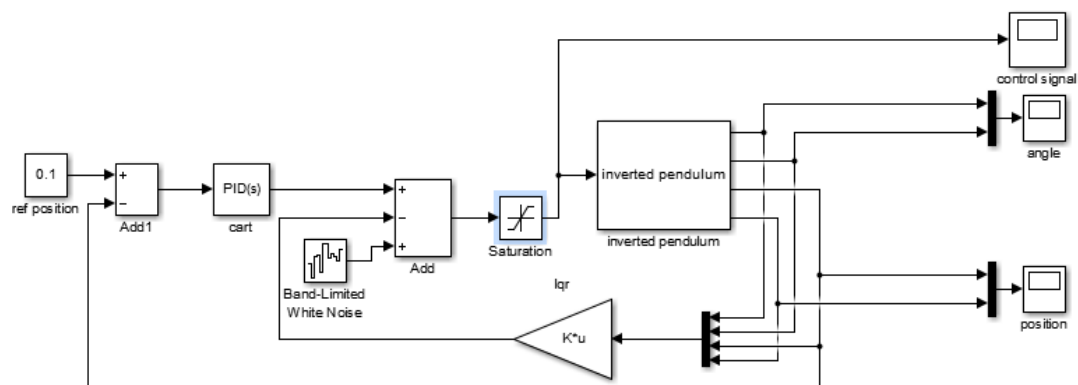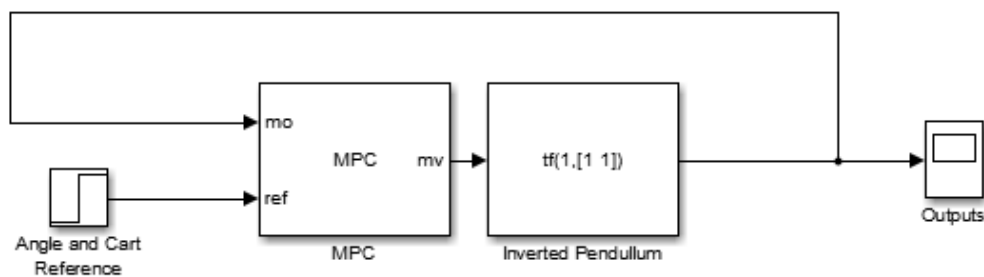

Figure 13. Control of inverted pendulum using cart PID and LQR with disturbance
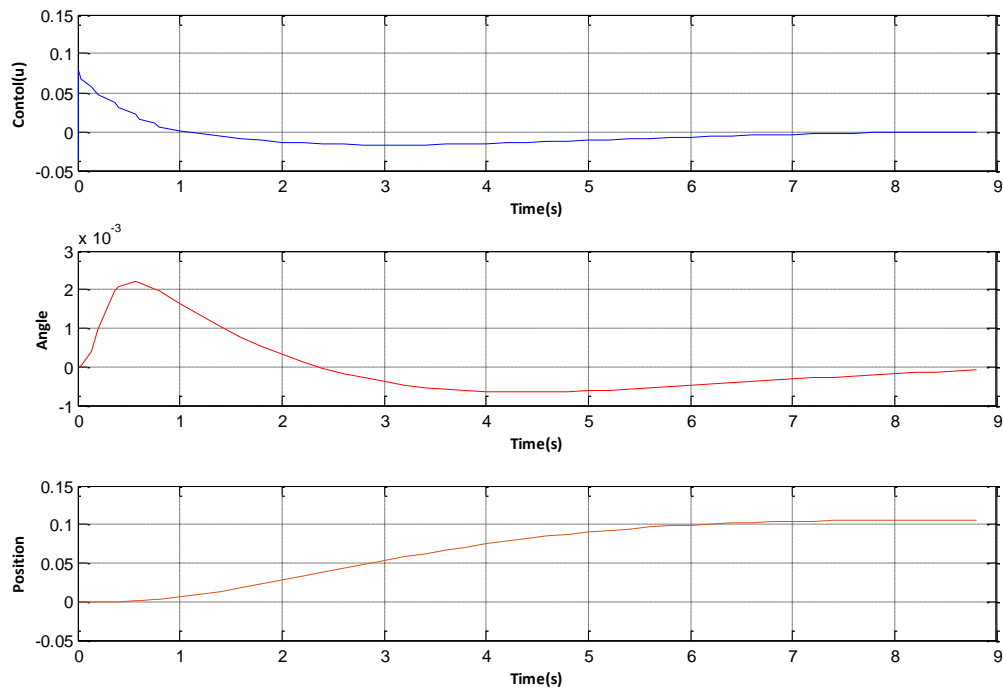
Figure 14. Responses of control signal, pendulum angle, angular displacement, Cart position and cart velocity with disturbance

As the LQR is .designed using the .state space feedback of the . system with .the pendulum in the vertical position the angle control is automatically taken care of [8]. The responses .of angle θ, . angular velocity $\dot{\theta}$, .cart position x, . cart velocity $\dot{x}$ and . control signal u .are plotted as shown in figs 12 and 14. The results of simulation justify the effective robustness of the system.

The system is now controlled using Model predictive Controller as shown in Figure 15. The response obtained is plotted in Figure 16.

Figure 15. MPC Control of Inverted Pendulum

Figure 16. Responses Of Control signal, Pendulum angle and cart position.

The   maximum. absolute values of different states of the system and control signal for cases of with and without disturbances are tabulated in tables 3 and 4 respectively.

Table 3. System states' and control signal's maximum absolute magnitudes in the absence of disturbance

Control Schemes	θ	$\dot{\theta}$	x	$\dot{x}$	u
P.ID	0.0047	-	0.0976	-	0.15
2 PID +LQR	0.0003	0.00087	0.1	0.0003	0.15
1 PID + LQR	0.0003	0.00071	0.1	0.00033	0.15
MPC	0.0006	-	0.106	-	

Table 4. System states and control signal's maximum absolute magnitudes in the presence of disturbance

Control Schemes	θ	$\dot{\theta}$	x	$\dot{x}$	u
PID	0.012	-	0.09	-	1.35
2 PID + LQR	0.003	0.02	0.1	0.046	1.38
1 PID + LQR	0.003	0.021	0.103	0.044	1.39

By relating the responses of the above control schemes it is found that the ones using LQR control technique stands superior to the one using PID control only. Although the scheme using two PID and LQR and single PID and LQR are similar but it is found that the former is a little improved

compared to the latter as it consist of angle controller which in turn contributes to additional degree of freedom. On the other hand the combination of one PID and LQR is simple in structure.

From all these results we observe that these control strategies are proficient and robust. Among the strategies the PID +LQR is selected as the superior one for the optimal control of the system due to its simple structure, robustness and efficiency.

5. Conclusions

To control the non-linear inverted pendulum, PID control and LQR, a control strategy for optimal control has been implemented. It is analysed for cases of using disturbance and without disturbance is executed. The LQR is directly fed with all the states of the system which are obtainable for measurement. The design of LQR is done using the model's state space. LQR's optimal value is added to the PID controller to achieve the resulting optimal control. Matlab is used to develop the models and to analyse the response outputs. The tuning of the controllers to achieve the most optimal control is attained by trial and error method. The results of the simulation justify the relative effectiveness of using LQR control method. The pendulum stabilises in the vertical position and the cart reaches the desired position even in the presence of disturbance input. The study of the control responses depict that the performance of the PID +LQR controls method is superior to the one using PID only. This concludes with the statement that, of all the control schemes performed, LQR + PID is effective, robust and simple. .

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