

1 Finite Difference Methods

General System Formulas

The general system is a BVP set up as: $x'' = F(t, x, x')$ with boundary values $x(a) = \alpha$ and $x(b) = \beta$. Here $t = a + ih$ and $h = \frac{b-a}{n}$ where $(0 \leq i \leq n)$

Central Difference Formulas

For the frist derivative the formula is:

$$x'(t) = \frac{x(t+h) - x(t-h)}{2h}$$

$$\rightarrow x'_i = \frac{x_{i+1} - x_{i-1}}{2h}$$

For the second derivative the formula is:

$$x''(t) = \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}$$

$$\rightarrow x''_i = \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}$$

Matrix Formulas

The formulas also take a matrix form in a linear system $Ax = b$. The pricipal equati-on of the system is found in the form:

$$a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i$$

In $Ax = b$ form this looks like:

$$A = \begin{bmatrix} d_1 & c_1 & & \\ a_2 & d_2 & c_2 & \\ & & \ddots & \\ & & & a_n & d_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 - a_1 \alpha \\ b_2 \\ \vdots \\ b_n - c_n \beta \end{bmatrix}$$

2 Shooting Method Formulas

General System Formulas

Similar to finite methods. The general system is a BVP set up as: $x'' = F(t, x, x')$ with boundary values $x(a) = \alpha$ and $x(b) = \beta$. Here we may also have formulas for hypothesized terminal values at b in the form $x(b) \approx \beta \approx \phi(z)$. z values will be of an IVP form $x'(a) = z_{IVP}$

The error formula for $\phi(z)$ is given by: $\epsilon = |\phi(z) - \beta|$

Iterative z formulas

The general form iterative z formula is as follows:

$$z_{n+1} = z_n + \frac{\beta - \phi(z_n)}{\phi(z_n) - \phi(z_{n-1})} (z_n - z_{n-1})$$

In the simple form the formula is:

$$z_3 = z_2 + \frac{\beta - \phi(z_2)}{\phi(z_n) - \phi(z_1)} (z_2 - z_1)$$

3 Linear Algebra Overview

LU Decomposition Formulas

Here $A = LU$, this would look like the following:

$$A = LU, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

to make this fit on the cheatsheet I have broken A into several vectors, $A = [a_1|a_2|a_3]$, so these parts of A are:

$$a_1 = \begin{bmatrix} l_{11} \\ l_{21} \\ l_{31} \end{bmatrix}, \quad a_2 = \begin{bmatrix} l_{11}u_{12} \\ l_{21}u_{12} + l_{22}u_{22} \\ l_{31}u_{12} + l_{32}u_{22} \end{bmatrix}$$

$$a_3 = \begin{bmatrix} l_{11}u_{13} \\ l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}$$

Ax = b, LU Decomposition Formulas

Note that before $Ax = b$ and we also have $A = LU$, thus $LUx = b$. With a substituti-on we get $Ux = y$, so $Ly = b$.

LU Decomposition Inverse Formula

This is similar to normal LU decompo-sition. The formula takes the form of $AX = I$ where all matrices are n by n, so $X = [x_1|x_2|\dots|x_n]$ with the standard iden-tity $I = [I_1|I_2|\dots|I_n]$.

Gaussian Inverse Formula

The gaussian inverse does the following:

$$[A|I] \rightarrow [I|A^{-1}]$$

Cholesky Factorization Formulas

This is similar to LU decomposition, only $U = L^T$. So: $A = LL^T$ which means that:

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix},$$

$$A = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Then solving for the system and getting a cleaner L at the end gives the formula:

$$L = \begin{bmatrix} \sqrt{a_{11}} & 0 & 0 \\ \frac{a_{21}}{l_{11}} & \sqrt{a_{22} - l_{21}^2} & 0 \\ \frac{a_{31}}{l_{11}} & \frac{a_{32} - l_{31}l_{21}}{l_{22}} & \sqrt{A_{33} - l_{31}^2 - l_{32}^2} \end{bmatrix}$$

But the actual Cholesky factorization is just the matrix L .

Gaussian Elimination Formulas

Gaussian elimination is an $Ax = b$ solving method. It makes an upper triangual matrix with the form: $[A|b]$

4 Generally Useful Maths

Trig Properties

$$\begin{aligned} \sin^2 x + \cos^2 x &= 1 & \sec x &= \frac{1}{\cos x} \\ 2 \sin x &= \sin x \cos x \\ \tan x &= \frac{\sin x}{\cos x} = \frac{1}{\cot x} & \csc x &= \frac{1}{\sin x} \\ \frac{d}{dx} \sin x &= \cos x & \frac{d}{dx} \cos x &= -\sin x \\ \frac{d}{dx} \tan x &= \sec^2 x & \frac{d}{dx} \cot x &= -\csc^2 x \\ \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx} \arccos x &= \frac{-1}{\sqrt{1-x^2}} \\ \frac{d}{dx} \arctan x &= \frac{1}{1+x^2} & \frac{d}{dx} \sec x &= \sec x \tan x \end{aligned}$$

Log & Exp Properties

$$\log x^n = n \log x \quad \log\left(\frac{1}{x}\right) = -\log x$$

$$\log_a x = \frac{\log_b x}{\log_a b} \quad \frac{d}{dx} e^{ax} = ae^{ax}$$

$$x^0 = 1 \quad x^n \cdot x^m = x^{n+m} \quad x^{-n} = \frac{1}{x^n}$$

$$\log_a x^n = n \log_a x \quad \frac{e^{-nx}}{e^x} = e^{-(n+1)x}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \frac{d}{dx} a^{g(x)} = \ln(a) a^{g(x)} g'(x)$$

$$\frac{d}{dx} a^{g(x)} = \ln(a) a^{g(x)} g'(x) \quad \frac{d}{dx} b^x = b^x \ln x$$

$$\frac{d}{dx} e^{g(x)} = g'(x) e^{g(x)} \quad \frac{d}{dx} a^x = a^x \ln a$$

$$\frac{d}{dx} \log_a(g(x)) = \frac{g'(x)}{\ln(a)g(x)}$$

Other Derivative Rules

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

$$\frac{d}{dx} f(x)/g(x) = \frac{(f'(x)g(x) - g'(x)f(x))}{g(x)^2}$$

In Class Terminology

the relative error formula: $\frac{|x-\hat{x}|}{x}$

more generally, with \hat{x}, \hat{y} being rounded

terms we get relative error as:

$$\frac{(x-y) - (\hat{x} - \hat{y})}{(x-y)} = \text{relative error}$$