UGA CSCI 4150/6150
Caleb Ashmore Adams

1 Finite Difference Methods

General System Formulas

The general system is a BVP set up as: $x'' = F(t,x,x')$ with boundary values $x(a) = \alpha$ and $x(b) = \beta$. Here $t = a + ih$ and $h = \frac{b-a}{n}$ where $(0 \le i \le n)$

Central Difference Formulas

For the frist derivative the formula is:

$$x'(t) = \frac{x(t+h) - x(t-h)}{2h}$$

$$\rightarrow x_i' = \frac{x_{i+1} - x_{i-1}}{2h}$$

For the second derivative the formula is:

$$x''(t) = \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}$$

$$\rightarrow x_i'' = \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}$$

central difference for PDEs are as follows:

$\frac{\partial u}{\partial t} \approx \frac{u_i^{j+1} - u_i^j}{\Delta t}$ and ...

$\frac{\partial^2 u}{\partial t^2} \approx \frac{(u_{i+1}^j - 2u_i^j + u_{i-1}^j) + (u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1})}{2\Delta x^2}$

Matrix Formulas

The formulas also take a matrix form in a linear system $Ax = b$. The pricipal equation of the system is found in the form:

$a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i$

In $Ax = b$ form this looks like:

$$A = \begin{bmatrix} d_1 & c_1 & & \\ a_2 & d_2 & c_2 & \\ & & \ddots & \\ & & a_n & d_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 - a_1\alpha \\ b_2 \\ \vdots \\ b_n - c_n\beta \end{bmatrix}$$

2 Shooting Method Formulas

General System Formulas

Similar to finite methods. The general system is a BVP set up as: $x'' = F(t,x,x')$ with boundary values $x(a) = \alpha$ and $x(b) = \beta$. Here we may also have formulas for hypothesized terminal values at b in the form $x(b) \approx \beta \approx \phi(z)$. z values will be of an IVP form $x'(a) = z_{IVP}$

The error formula for $\phi(z)$ is given by: $\epsilon = |\phi(z) - \beta|$

Iterative z formulas

The general form iterative z formula is as follows:

$$z_{n+1} = z_n + \frac{\beta - \phi(z_n)}{\phi(z_n) - \phi(z_{n-1})}(z_n - z_{n-1})$$

In the simple form the formula is:

$$z_3 = z_2 + \frac{\beta - \phi(z_2)}{\phi(z_n) - \phi(z_1)}(z_2 - z_1)$$

3 Linear Algebra Overview

LU Decomposition Formulas

Here $A = LU$, this would look like the following:

$$A = LU, \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

to make this fit on the cheatsheet I have broken A into several vectors, $A = [a_1|a_2|a_3]$, so these parts of A are:

$$a_1 = \begin{bmatrix} l_{11}u_{11} \\ l_{21}u_{11} \\ l_{31}u_{11} \end{bmatrix}, \quad a_2 = \begin{bmatrix} l_{11}u_{12} \\ l_{21}u_{12} + l_{22}u_{22} \\ l_{31}u_{12} + l_{32}u_{22} \end{bmatrix}$$

$$a_3 = \begin{bmatrix} l_{11}u_{13} \\ l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}$$

Ax = b, LU Decomposition Formulas

Note that before $Ax = b$ and we also have $A = LU$, thus $LUx = b$. With a substitution we get $Ux = y$, so $Ly = b$.

LU Decomposition Inverse Formula

This is similar to normal LU decomposition. The formula takes the form of $AX = I$ where all matrices are n by n, so $X = [x_1|x_2|...|x_n]$ with the standard identity $I = [I_1|I_2|...|I_n]$.

Gaussian Inverse Formula

The gaussian inverse does the following: $[A|I] \rightarrow [I|A^{-1}]$

Cholesky Factorization Formulas

This is similar to LU decomposition, only $U = L^T$. So: $A = LL^T$ which means that:

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix},$$

$$A = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Then solving for the system and getting a cleaner L at the end gives the formula:

$$L = \begin{bmatrix} \sqrt{a_{11}} & 0 & 0 \\ \frac{a_{21}}{l_{11}} & \sqrt{a_{22} - l_{21}^2} & 0 \\ \frac{a_{31}}{l_{11}} & \frac{a_{32} - l_{31}l_{21}}{l_{22}} & \sqrt{A_{33} - l_{31}^2 - l_{32}^2} \end{bmatrix}$$

But the actual Cholesky factorization is just the matrix L.

Gaussian Elimination Formulas

Gassian elimination is an $Ax = b$ solving method. It makes an upper triangual matrix with the form: $[A|b]$

4 Initial Value Problems

RK4

This is the 4th order (RK4) Runge-Kutta method for the Initial Value Problem (IVP):

$x(t+h) = x(t) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

where the following are values of K_n:

$K_1 = hf(t,x)$

$K_2 = hf(t + \frac{1}{2}h, x + \frac{1}{2}K_1)$

$K_3 = hf(t + \frac{1}{2}h, x + \frac{1}{2}K_2)$

$K_4 = hf(t + h, x + K_3)$

first, the K_n values are calculated in succession. They the K_n values are filled into the first formula above.

Modified Euler's Method

the Modified Euler's method is simply:

$K_1 = f(t_n, x_n)$, $K_2 = f(t_n + h, x_n + hK_1)$

$x_{n+1} = x(t_n) + \frac{h/2}{(}K_1 + K_2)$

5 Generally Useful Maths

Trig Properties

$\sin^2 x + \cos^2 x = 1 \quad \sec x = \frac{1}{\cos x}$

$2\sin x = \sin x \cos x$

$\tan x = \frac{\sin x}{\cos x} = \frac{1}{\cot x} \quad \csc x = \frac{1}{\sin x}$

$\frac{d}{dx}\sin x = \cos x \quad \frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx}\tan x = \sec^2 x \quad \frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \quad \frac{d}{dx}\sec x = \sec x \tan x$

Log & Exp Properties

$\log x^n = n\log x \quad \log(\frac{1}{x}) = -\log x$

$\log_a x = \frac{\log_b x}{\log_a x} \quad \frac{d}{dx}e^{ax} = ae^{ax}$

$x^0 = 1 \quad x^n \cdot x^m = x^{n+m} \quad x^{-n} = \frac{1}{x^n}$

$\log_a x^n = n\log_a x \quad \frac{e^{-nx}}{e^x} = e^{-(n+1)x}$

$\log_a(\frac{x}{y}) = \log_a x - \log_a y$

$\log_a(xy) = \log_a x + \log_a y$

$\frac{d}{dx}\ln x = \frac{1}{x}$, $\frac{d}{dx}a^{g(x)} = \ln(a)a^{g(x)}g'(x)$

$\frac{d}{dx}a^{g(x)} = \ln(a)a^{g(x)}g'(x)$, $\frac{d}{dx}b^x = b^x \ln x$

$\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$, $\frac{d}{dx}a^x = a^x \ln a$

$\frac{d}{dx}\log_a(g(x)) = \frac{g'(x)}{ln(a)g(x)}$

Other Derivative Rules

$\frac{d}{dx}f(g(x)) = f'(g(x))g(x)$

$\frac{d}{dx}f(x)/g(x) = \frac{(f'(x)g(x) - g'(x)f(x))}{g(x)^2}$

In Class Terminology

the relative error formula: $\frac{|x-\hat{x}|}{x}$

more generally, with $\hat{x}, \hat{y}$ being rounded terms we get relative error as:

$\frac{(x-y)-(\hat{x}-\hat{y})}{(x-y)} = relative \quad error$

6 Boundary Value Problems

Can be descretized with finite difference methods. The formula goes from t_0, Δt, ... , t_n. The fromulas for the known boundary values are of the form: $x(t_0) = \alpha$ and $x(t_n) = \beta$.

Jacobi Method

Solves $Ax = b$. Solve the system of equations by guessing at x_n valeus. starts with initial guess $x = [0 \ 0 \ 0]^T$, converges on an answer after iteration. Uses the formula:

$x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j \ne i} a_{ij}x_j^{(k)})$, $i = 1, ..., n$.

Gauss-Seidel Method

Just like Jacobi, only it substitutions during iteration. Uses formula: $x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_j^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^n a_{ij}x_j^{(k)})$

7 PDE's

Solved like BVPs.

Classification Formulas

Discriminant gives classification:

$B^2 - 4AC$, with form:

$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial t} + C\frac{\partial^2 u}{\partial t^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial t} + Fu = G$

$B^2 - 4AC = 0$, Parabolic

$B^2 - 4AC > 0$, Hyperbolic

$B^2 - 4AC < 0$, Elliptic

Crank-Nicholson

the general form of the problem uses the equation $s = \frac{\Delta x^2}{\Delta t}$.