UGA CSCI 4150/6150
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1 Finite Difference Methods

General System Formulas

The general system is a BVP set up as: $x'' = F(t, x, x')$ with boundary values $x(a) = \alpha$ and $x(b) = \beta$. Here $t = a + ih$ and $h = \frac{b-a}{n}$ where $(0 \leq i \leq n)$

Central Difference Formulas

For the frist derivative the formula is:

$$x'(t) = \frac{x(t+h) - x(t-h)}{2h}$$

$$\rightarrow x'_i = \frac{x_{i+1} - x_{i-1}}{2h}$$

For the second derivative the formula is:

$$x''(t) = \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}$$

$$\rightarrow x''_i = \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}$$

Matrix Formulas

The formulas also take a matrix form in a linear system $Ax = b$. The pricipal equation of the system is found in the form: $a_i x_{i-1} + d_i x_i + c_i x_{i+1} = b_i$
In $Ax = b$ form this looks like:

$$A = \begin{bmatrix} d_1 & c_1 & & \\ a_2 & d_2 & c_2 & \\ & & \ddots & \\ & & a_n & d_n \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 - a_1\alpha \\ b_2 \\ \vdots \\ b_n - c_n\beta \end{bmatrix}$$

2 Shooting Method Formulas

General System Formulas

Similar to finite methods. The general system is a BVP set up as: $x'' = F(t, x, x')$ with boundary values $x(a) = \alpha$ and $x(b) = \beta$. Here we may also have a first derivative formula for terminal values at b in the form $x(b) = \beta = \phi(z)$. z values will be of an IVP form $x'(a) = z_{IVP}$

Iterative z formulas

The general form iterative z formula is as follows:

$$z_{n+1} = z_n + \frac{\beta - \phi(z_n)}{\phi(z_n) - \phi(z_{n-1})}(z_n - z_{n-1})$$

In the simple form the formula is:

$$z_3 = z_2 + \frac{\beta - \phi(z_2)}{\phi(z_n) - \phi(z_1)}(z_2 - z_1)$$

3 Linear Algebra Overview

LU Decomposition Formulas

Here $A = LU$, this would look like the following:

$$A = LU, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

to make this fit on the cheatsheet I have broken A into several vectors, $A = [a_1|a_2|a_3]$, so these parts of A are:

$$a_1 = \begin{bmatrix} l_{11} \\ l_{21} \\ l_{31} \end{bmatrix}, \quad a_2 = \begin{bmatrix} l_{11}u_{12} \\ l_{21}u_{12} + l_{22}u_{22} \\ l_{31}u_{12} + l_{32}u_{22} \end{bmatrix}$$

$$a_3 = \begin{bmatrix} l_{11}u_{13} \\ l_{21}u_{13} + l_{22}u_{23} \\ l_{31}u_{13} + l_{32}u_{23} + l_{33}u_{33} \end{bmatrix}$$

Ax = b, LU Decomposition Formulas

Note that before $Ax = b$ and we also have $A = LU$, thus $LUx = b$. With a substitution we get $Ux = y$, so $Ly = b$.

LU Decomposition Inverse Formula

Cholesky Factorization Formulas

This is similar to LU decomposition, only $U = L^T$. So: $A = LL^T$ which means that:

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}, L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix},$$

$$A = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Then solving for the system and getting a cleaner L at the end gives the formula:

$$L = \begin{bmatrix} \sqrt{a_{11}} & 0 & 0 \\ \frac{a_{21}}{l_{11}} & \sqrt{a_{22} - l_{21}^2} & 0 \\ \frac{a_{31}}{l_{11}} & \frac{a_{32} - l_{31}l_{21}}{l_{22}} & \sqrt{A_{33} - l_{31}^2 - l_{32}^2} \end{bmatrix}$$

But the actual Cholesky factorization is just the matrix L.