UGA CSCI 4150/6150
Caleb Ashmore Adams

1 Finite Difference Methods

General System Formulas

The general system is a BVP set up as: $x'' = F(t,x,x')$ with boundary values $x(a)=\alpha$ and $x(b)=\beta$. Here $t=a+ih$ and $h=\frac{b-a}{n}$ where $(0\le i\le n)$

Central Difference Formulas

For the frist derivative the formula is:

$$x'(t)=\frac{x(t+h)-x(t-h)}{2h}$$

$$\to x_i' = \frac{x_{i+1}-x_{i-1}}{2h}$$

For the second derivative the formula is:

$$x''(t)=\frac{x(t+h)-2x(t)+x(t-h)}{h^2}$$

$$\to x_i''=\frac{x_{i+1}-2x_i+x_{i-1}}{h^2}$$

central difference for PDEs are as follows:

$\frac{\partial u}{\partial t}\approx \frac{u_i^{j+1}-u_i^j}{\Delta t}$ and ...

$\frac{\partial^2 u}{\partial t^2}\approx \frac{(u_{i+1}^j-2u_i^j+u_{i-1}^j)+(u_{i+1}^{j+1}-2u_i^{j+1}+u_{i-1}^{j+1})}{2\Delta x^2}$

Matrix Formulas

The formulas also take a matrix form in a linear system $Ax=b$. The pricipal equation of the system is found in the form:

$a_ix_{i-1}+d_ix_i+c_ix_{i+1}=b_i$

In $Ax=b$ form this looks like:

$$A=\begin{bmatrix} d_1 & c_1 & & \\ a_2 & d_2 & c_2 & \\ & & \ddots & \\ & & a_n & d_n \end{bmatrix}$$

$$x=\begin{bmatrix} x_1\\ x_2\\ \vdots \\ x_n\end{bmatrix},\quad b=\begin{bmatrix} b_1-a_1\alpha \\ b_2 \\ \vdots \\ b_n-c_n\beta\end{bmatrix}$$

2 Shooting Method Formulas

General System Formulas

Similar to finite methods. The general system is a BVP set up as: $x''=F(t,x,x')$ with boundary values $x(a)=\alpha$ and $x(b)=\beta$. Here we may also have formulas for hypothesized terminal values at b in the form $x(b)\approx\beta\approx\phi(z)$. z values will be of an IVP form $x'(a)=z_{IVP}$

The error formula for $\phi(z)$ is given by: $\epsilon = |\phi(z)-\beta|$

Iterative z formulas

The general form iterative z formula is as follows:

$$z_{n+1}=z_n+\frac{\beta-\phi(z_n)}{\phi(z_n)-\phi(z_{n-1})}(z_n-z_{n-1})$$

In the simple form the formula is:

$$z_3=z_2+\frac{\beta-\phi(z_2)}{\phi(z_n)-\phi(z_1)}(z_2-z_1)$$

3 Linear Algebra Overview

LU Decomposition Formulas

Here $A=LU$, this would look like the following:

$$A=LU,\qquad A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$L=\begin{bmatrix} l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}, U=\begin{bmatrix} u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

to make this fit on the cheatsheet I have broken A into several vectors, $A=[a_1|a_2|a_3]$, so these parts of A are:

$$a_1=\begin{bmatrix} l_{11}u_{11}\\ l_{21}u_{11}\\ l_{31}u_{11}\end{bmatrix},\quad a_2=\begin{bmatrix} l_{11}u_{12}\\ l_{21}u_{12}+l_{22}u_{22}\\ l_{31}u_{12}+l_{32}u_{22}\end{bmatrix}$$

$$a_3=\begin{bmatrix} l_{11}u_{13}\\ l_{21}u_{13}+l_{22}u_{23}\\ l_{31}u_{13}+l_{32}u_{23}+l_{33}u_{33}\end{bmatrix}$$

Ax = b, LU Decomposition Formulas

Note that before $Ax=b$ and we also have $A=LU$, thus $LUx=b$. With a substitution we get $Ux=y$, so $Ly=b$.

LU Decomposition Inverse Formula

This is similar to normal LU decomposition. The formula takes the form of $AX=I$ where all matrices are n by n, so $X=[x_1|x_2|...|x_n]$ with the standard identity $I=[I_1|I_2|...|I_n]$.

Gaussian Inverse Formula

The gaussian inverse does the following: $[A|I]\to[I|A^{-1}]$

Cholesky Factorization Formulas

This is similar to LU decomposition, only $U=L^T$. So: $A=LL^T$ which means that:

$$L=\begin{bmatrix} l_{11} & 0 & 0\\ l_{21} & l_{22} & 0\\ l_{31} & l_{32} & l_{33}\end{bmatrix}, L^T=\begin{bmatrix} l_{11} & l_{21} & l_{31}\\ 0 & l_{22} & l_{32}\\ 0 & 0 & l_{33}\end{bmatrix},$$

$$A=\begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31}\\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{21}l_{31}+l_{22}l_{32}\\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2\end{bmatrix}$$

Then solving for the system and getting a cleaner L at the end gives the formula:

$$L=\begin{bmatrix} \sqrt{a_{11}} & 0 & 0\\ \frac{a_{21}}{l_{11}} & \sqrt{a_{22}-l_{21}^2} & 0\\ \frac{a_{31}}{l_{11}} & \frac{a_{32}-l_{31}l_{21}}{l_{22}} & \sqrt{A_{33}-l_{31}^2-l_{32}^2}\end{bmatrix}$$

But the actual Cholesky factorization is just the matrix L.

Gaussian Elimination Formulas

Gassian elimination is an $Ax=b$ solving method. It makes an upper triangual matrix with the form: $[A|b]$

4 Initial Value Problems

RK4

This is the 4th order (RK4) Runge-Kutta method for the Initial Value Problem (IVP):

$x(t+h)=x(t)+\frac{1}{6}(K_1+2K_2+2K_3+K_4)$

where the following are values of K_n:

$K_1=hf(t,x)$

$K_2=hf(t+\frac{1}{2}h,x+\frac{1}{2}K_1)$

$K_3=hf(t+\frac{1}{2}h,x+\frac{1}{2}K_2)$

$K_4=hf(t+h,x+K_3)$

first, the K_n values are calculated in succession. They the K_n values are filled into the first formula above.

Modified Euler's Method

the Modified Euler's method is simply:

$K_1=f(t_n,x_n)$, $K_2=f(t_n+h,x_n+hK_1)$

$x_{n+1}=x(t_n)+\frac{h/2}{(}K_1+K_2)$

5 Generally Useful Maths

Trig Properties

$\sin^2 x+\cos^2 x=1 \quad \sec x=\frac{1}{\cos x}$

$2\sin x=\sin x\cos x$

$\tan x=\frac{\sin x}{\cos x}=\frac{1}{\cot x} \quad \csc x=\frac{1}{\sin x}$

$\frac{d}{dx}\sin x=\cos x \quad \frac{d}{dx}\cos x=-\sin x$

$\frac{d}{dx}\tan x=\sec^2 x \quad \frac{d}{dx}\cot x=-\csc^2 x$

$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx}\arccos x=\frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}\arctan x=\frac{1}{1+x^2} \quad \frac{d}{dx}\sec x=\sec x\tan x$

Log & Exp Properties

$\log x^n=n\log x \quad \log(\frac{1}{x})=-\log x$

$\log_a x=\frac{\log_b x}{\log_a x} \quad \frac{d}{dx}e^{ax}=ae^{ax}$

$x^0=1 \quad x^n\cdot x^m=x^{n+m} \quad x^{-n}=\frac{1}{x^n}$

$\log_a x^n=n\log_a x \quad \frac{e^{-nx}}{e^x}=e^{-(n+1)x}$

$\log_a(\frac{x}{y})=\log_a x-\log_a y$

$\log_a(xy)=\log_a x+\log_a y$

$\frac{d}{dx}\ln x=\frac{1}{x}$, $\frac{d}{dx}a^{g(x)}=\ln(a)a^{g(x)}g'(x)$

$\frac{d}{dx}a^{g(x)}=\ln(a)a^{g(x)}g'(x)$, $\frac{d}{dx}b^x=b^x\ln x$

$\frac{d}{dx}e^{g(x)}=g'(x)e^{g(x)}$, $\frac{d}{dx}a^x=a^x\ln a$

$\frac{d}{dx}\log_a(g(x))=\frac{g'(x)}{ln(a)g(x)}$

Other Derivative Rules

$\frac{d}{dx}f(g(x))=f'(g(x))g(x)$

$\frac{d}{dx}f(x)/g(x)=\frac{(f'(x)g(x)-g'(x)f(x))}{g(x)^2}$

In Class Terminology

the relative error formula: $\frac{|x-\hat{x}|}{x}$

more generally, with $\hat{x},\hat{y}$ being rounded terms we get relative error as:

$\frac{(x-y)-(\hat{x}-\hat{y})}{(x-y)}=relative\ \ error$

6 Boundary Value Problems

Can be descretized with finite difference methods. The formula goes from t_0, Δt, ... , t_n. The fromulas for the known boundary values are of the form: $x(t_0)=\alpha$ and $x(t_n)=\beta$.

Jacobi Method

Solves $Ax=b$. Solve the system of equations by guessing at x_n valeus. starts with initial guess $x=[0\ 0\ 0]^T$, converges on an answer after iteration. Uses the formula:

$x_i^{(k+1)}=\frac{1}{a_{ii}}(b_i-\sum_{j\neq i}a_{ij}x_j^{(k)})$, $i=1,...,n$.

Gauss-Seidel Method

Just like Jacobi, only it substitutions during iteration. Uses formula: $x_i^{(k+1)}=\frac{1}{a_{ii}}(b_i-\sum_j^{i-1}a_{ij}x_j^{(k+1)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)})$

7 PDE's

Solved like BVPs.

Classification Formulas

Discriminant gives classification:

B^2-4AC, with form:

$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial t}+C\frac{\partial^2 u}{\partial t^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial t}+Fu=G$

$B^2-4AC=0$, Parabolic, $u_t=u_{xx}$

$B^2-4AC>0$, Hyperbolic, $u_{tt}=u_{xx}$

$B^2-4AC<0$, Elliptic, $u_{xx}+u_{yy}=0$

Crank-Nicholson

The PDE will be given in form: $u_t=u_{xx}$, with some range $\alpha_x<x<\beta_x$ and time constraint $t>\gamma_t$. Boundaries will be of the form: $u(x_0,t_0)=\alpha$, $u(x_n,t_n)=\beta$ with some constraint $t>\gamma_t$ and $u(x,t_0)=f(x)$ with some constraint $a_x\le x\le b_x$

remember, equations are of form $u(x,t)$

Note, the forms in the descrete case take $x_0^0,x_1^0,x_2^0,...x_i^j$ where j is time and x is the function.

this gives the equations $x_0^0=u(x_0,t_0)$ and x_0 terms use the equation and form $u(x,t_0)=u(x^0_{x_0+\Delta x})=f(x)$

steps are Δt and Δx, the general form of the problem uses the equation $s=\frac{\Delta x^2}{\Delta t}$.

The matrix form of the problem has the following equation form:

$u_{i+...}^{j+1}+...=u_{i+...}^{j}+D$

such that

$\alpha(u_{i-1}^{j+1})+\beta(u_i^{j+1})+\gamma(u_{i+1}^{j+1})=b_i^j+D$