

Understanding the Kalman Filter

A Simple, Verbose, 1D Guide

Caleb Adams

UGA Small Satellite Research Laboratory

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The use of the Kalman Filter Algorithm is common in robotics as a control method [1]. More importantly, it can be used to control spacecraft - and who doesn't love a good spaceship? Preferably one with an invisibility feild... but we won't be discussing spaceships, or invisibility feilds. For this case we will imagine a humble cart on a track. The cart can control its movment by applying a force. The cart will also have the ability to detect it's distance along the track using a lazer.

1 A Motion Model

It is simple to understand a linear motion model. For every time step, let's call the time step Δt , we move some distance, let's call the distance Δx , across the track. We will imagine that this track is the x axis. The amount of distance (Δx) that the cart moves in a certain amount of time (Δt) is its velocity. We will say the the robots instantaneous change in position, the derivative that results in velocity, is represented as \dot{x} . If we remember the form definition of the derivative [2] we see:

$$\dot{x} = x(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

We can see that any movement that our cart makes along our track is given by some change in time multiplied by the instantaneous change in position over that time, so $\Delta x = \dot{x}\Delta t$. If we recall from the great teacher Newton, we can write an equation of motion [3] as follows:

$$x = x_0 + \dot{x}\Delta t + \frac{1}{2}\ddot{x}(\Delta t)^2$$

Here the x_0 value represents the starting position of the cart along the track. In the simplest cast, the starting position can be 0. We can also see that the term \ddot{x} is the acceleration term a . I will be helpful to again remember $F = ma$. We can rearrange this to see $F = ma \rightarrow F = m\ddot{x} \rightarrow \ddot{x} = \frac{F}{m}$, thus:

$$x = x_0 + \dot{x}\Delta t + \frac{1}{2}\frac{F}{m}(\Delta t)^2$$

We can then view the velocity in the same way. With the term \dot{x}_0 the intial velocity of the system:

$$\dot{x} = \dot{x}_0 + \frac{F}{m}\Delta t$$

Next, we want to view the system in a more compact way. To do this we see that the cart has a state s at any given time t . In otherwords, at any given time the state of the cart can be discribed by its position and velocity. We can represent this state with the existing functons for x and \dot{x} :

$$s = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

And if we continue to expand this equation we get...

$$s = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_0 + \dot{x}\Delta t + \frac{1}{2}\frac{F}{m}(\Delta t)^2 \\ \dot{x}_0 + \frac{F}{m}\Delta t \end{bmatrix}$$

Now We apply some basic linear algebra [4]:

$$\begin{aligned} s &= \begin{bmatrix} x_0 + \dot{x}\Delta t + \frac{1}{2}\frac{F}{m}(\Delta t)^2 \\ \dot{x}_0 + \frac{F}{m}\Delta t \end{bmatrix} \rightarrow s = \begin{bmatrix} x_0 + \dot{x}\Delta t \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\frac{F}{m}(\Delta t)^2 \\ \frac{F}{m}\Delta t \end{bmatrix} \\ \rightarrow s &= \begin{bmatrix} x_0 + \dot{x}\Delta t \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2m}(\Delta t)^2 \\ \frac{1}{m}\Delta t \end{bmatrix} F \rightarrow s = \begin{bmatrix} x_0 + \dot{x}\Delta t \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} F \\ &\rightarrow s = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} F \end{aligned}$$

Thus,

$$s = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} F$$

It is useful to think of our system in this way. It allows us to have a single force F acting upon the cart and allows us to derive its state given an initial x_0 and \dot{x}_0 .

References

- [1] S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics (Intelligent Robotics and Autonomous Agents)*. The MIT Press, 2005.
- [2] M. Spivak, *Calculus*, 4th ed. Publish or Perish, 2008.
- [3] A. N. Whitehead and B. Russell, *Principia Mathematica*. Cambridge University Press, 1925–1927.
- [4] G. Strang, *Introduction to Linear Algebra*, 4th ed. Wellesley, MA: Wellesley-Cambridge Press, 2009.