Understanding the Kalman Filter

A Simple, Verbose, 1D Guide

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The use of the Kalman Filter Algorithm is common in robotics as a control method [1]. More importantly, it can be used to control spacecraft - and who doesn't love a good spaceship? Preferably one with an invisibility feild... but we won't be discussing spaceships, or invisibility feilds. For this case we will imagine a humble cart on a track. The cart can control its movment by applying a force. The cart will also have the ability to detect it's distance along the track using a lazer.

1 A Motion Model

It is simple to understand a linear motion model. For every time step, let's call the time step Δt, we move some distance, let's call the distance Δx, across the track. We will imagine that this track is the x axis. The amount of distance (Δx) that the cart moves in a certain amount of time (Δt) is its velocity. We will say the the robots instantaneous change in position, the derivative that results in velocity, is represented as $\dot{x}$. If we remember the form definition of the derivative [2] we see:

$$\dot{x} = x(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We can see that any movement that our cart makes along our track is given by some change in time multiplied by the instantaneous change in position over that time, so $\Delta x = \dot{x}\Delta t$. If we recall from the great teacher Newton, we can write an equation of motion [3] as follows:

$$x = x_0 + \dot{x}\Delta t + \frac{1}{2}\ddot{x}(\Delta t)^2$$

Here the x_0 value represents the starting position of the cart along the track. In the simplest cast, the starting position can be 0. We can also see that the term $\ddot{x}$ is the acceleration term a. I will be helpful to again remember $F = ma$. We can rearrange this to see $F = ma \to F = m\ddot{x} \to \ddot{x} = \frac{F}{m}$, thus:

$$x = x_0 + \dot{x}\Delta t + \frac{1}{2}\frac{F}{m}(\Delta t)^2$$

We can then view the velocity in the same way. With the term $\dot{x}_0$ the intial velocity of the system:

$$\dot{x} = \dot{x}_0 + \frac{F}{m}\Delta t$$

Next, we want to view the system in a more compact way. To do this we see that the cart has a state s at any given time t. In otherwords, at any given time the state of the cart can be discribed by its position and velocity. We can represent this state with the existing funcitons for x and $\dot{x}$:

$$s = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

And if we continue to expand this equation we get...

$$s = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} x_0 + \dot{x}\Delta t + \frac{1}{2}\frac{F}{m}(\Delta t)^2 \\ \dot{x}_0 + \frac{F}{m}\Delta t \end{bmatrix}$$

Now We apply some basic linear algebra [4]:

$$s = \begin{bmatrix} x_0 + \dot{x}\Delta t + \frac{1}{2}\frac{F}{m}(\Delta t)^2 \\ \dot{x}_0 + \frac{F}{m}\Delta t \end{bmatrix} \rightarrow s = \begin{bmatrix} x_0 + \dot{x}\Delta t \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\frac{F}{m}(\Delta t)^2 \\ \frac{F}{m}\Delta t \end{bmatrix}$$

$$\rightarrow s = \begin{bmatrix} x_0 + \dot{x}\Delta t \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2m}(\Delta t)^2 \\ \frac{1}{m}\Delta t \end{bmatrix} F \rightarrow s = \begin{bmatrix} x_0 + \dot{x}\Delta t \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} F$$

$$\rightarrow s = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} F$$

Thus,

$$\therefore \qquad s = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2m} \\ \frac{\Delta t}{m} \end{bmatrix} F$$

References

[1] S. Thrun, W. Burgard, and D. Fox, *Probabilistic Robotics (Intelligent Robotics and Autonomous Agents)*. The MIT Press, 2005.

[2] M. Spivak, *Calculus*, 4th ed. Publish or Perish, 2008.

[3] A. N. Whitehead and B. Russell, *Principia Mathematica*. Cambridge University Press, 1925–1927.

[4] G. Strang, *Introduction to Linear Algebra*, 4th ed. Wellesley, MA: Wellesley-Cambridge Press, 2009.