SGN – Assignment #1

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1 Periodic orbit

Consider the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$. Note that the CRTBP has an integral of motion, that is, the Jacobi constant

$$J(x, y, z, v_x, x_y, v_z) := 2\Omega(x, y, z) - v^2 = C$$

where $\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$ and $v^2 = v_x^2 + v_y^2 + v_z^2$.

1) Find the coordinates of the five Lagrange points L_i in the rotating, adimensional reference frame with at least 10-digit accuracy and report their Jacobi constant C_i.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.068792441776 \\ y_0 &= 0 \\ z_0 &= 0.071093328515 \\ v_{x0} &= 0 \\ v_{y0} &= 0.319422926485 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit having a Jacobi Constant $C = 3.09$; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM, either approximated through finite differences **or** achieved by integrating the variational equation.
Hint: Consider working on $\varphi(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t)$ and $J(\mathbf{x} + \Delta\mathbf{x})$ and then enforce perpendicular cross of $y = 0$ and Jacobi energy.

The periodic orbits in the CRTBP exist in families. These can be computed by 'continuing' the orbits along one coordinate or one parameter, e.g., the Jacobi energy C. The *numerical continuation* is an iterative process in which the desired variable is *gradually* varied, while the rest of the initial guess is taken from the solution of the previous iteration, thus aiding the convergence process.

3) By gradually decreasing C and using numerical continuation, compute the families of halo orbits until $C = 3.04$.

(8 points)

1.1 Find the coordinates of the five Lagrange points

To set up the 3D Earth-Moon CRTBP, we define a rotating reference frame where the position of the Earth and Moon are fixed, and the object moves under the gravitational influence of these bodies. In such frame the Earth is positioned at $[-\mu, 0, 0]$, the Moon's at $[1-\mu, 0, 0]$ while the centre is the system's barycentre. To find the equilibrium points, we set the partial derivatives of the effective potential Ω with respect to x, y, and z equal to zero, and solve the following system of equations:

$$\Omega = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$$

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$
$$r_2 = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$$

$$\begin{cases} \frac{\partial\Omega}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{\mu(x-1+\mu)}{r_2^3} = 0, \\ \frac{\partial\Omega}{\partial y} = y\left(1 - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0, \\ \frac{\partial\Omega}{\partial z} = z\left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) = 0. \end{cases}$$

We only have five real solutions to this system, which corresponds to the five Lagrangian points: each point can be linked to a certain Jacobi constant J, which acts as a parameter for the point's energy.

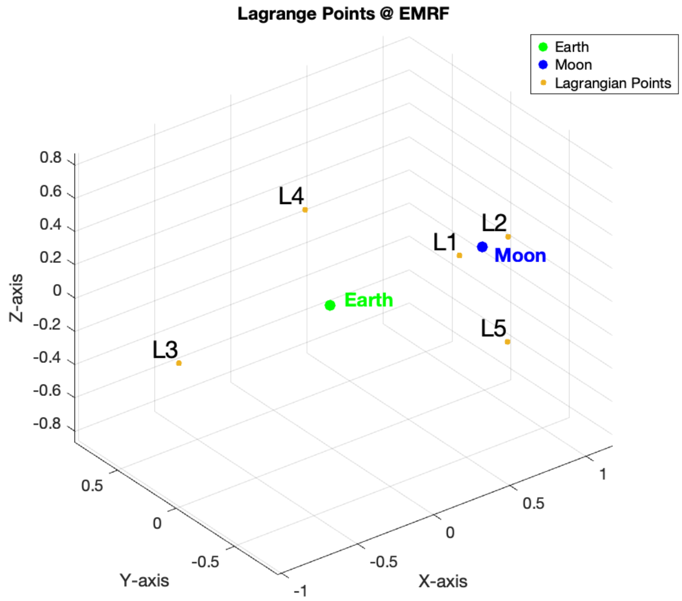

Figure 1: Lagrangian Points

	L_1	L_2	L_3	L_4	L_5
x	0.8369180073	1.1556799131	−1.0050624018	0.4878500000	0.4878500000
y	0.0000000000	0.0000000000	0.0000000000	0.8660254038	−0.8660254038
C	3.2003380950	3.1841582164	3.0241489429	3.0000000000	3.0000000000

Table 1: Lagrangian points coordinates and Jacobi constants

1.2 Find the periodic halo orbit having a Jacobi Constant C = 3.09

The main scheme to compute a periodic halo orbit with a fixed Jacobi constant is based on iteratively adjusting the values of x_0, z_0, and v_{y0}, while constraining v_x and v_z to be zero at the point where the trajectory crosses the symmetry plane $y = 0$. This approach ensures that the orbit satisfies the periodicity condition imposed by the CRTBP symmetry, which requires the trajectory to cross the x-axis perpendicularly. The initial state is thus corrected until the trajectory intersects the $y = 0$ plane with $v_x = v_z = 0$, while simultaneously satisfying the desired Jacobi constant.

This computation is formalized through a differential correction scheme that relies on a first-order expansion of the flow map $\boldsymbol{\varphi}$, which governs the spacecraft's state evolution in time under the CRTBP dynamics. The flow is linearized with respect to both initial condition perturbations $\Delta\mathbf{x}$ and final time variation Δt:

$$\boldsymbol{\varphi}(\mathbf{x}_0 + \Delta\mathbf{x}, t + \Delta t) \approx \boldsymbol{\varphi}(\mathbf{x}_0, t) + \frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{x}}(\mathbf{x}_0, t)\Delta\mathbf{x} + \frac{\partial \boldsymbol{\varphi}}{\partial t}(\mathbf{x}_0, t)\Delta t \tag{1}$$

Here, the partial derivative with respect to the initial state is the *State Transition Matrix* (STM) $\Phi(t_0, t)$, and the time derivative corresponds to the vector field $\mathbf{f}(\mathbf{x}, t)$ defining the CRTBP equations of motion. Rewriting the expansion, we obtain the linear correction relationship:

$$\Phi(t_0, t)\Delta\mathbf{x} + \mathbf{f}(\mathbf{x}_f, t)\Delta t = \mathbf{x}_{f,\text{target}} - \mathbf{x}_f \tag{2}$$

At the heart of the correction loop lies the STM, which captures the sensitivity of the *controlled variables* x_0, z_0, v_{y0}, and T with respect to the deviation from the terminal conditions at the $y = 0$ crossing. These include the requirement that $y_f = 0$, $v_{x,f} = 0$, $v_{z,f} = 0$, and that the Jacobi constant matches the target value.

The resulting update rule at each iteration k is expressed as:

$$\begin{Bmatrix} x_0^{k+1} \\ z_0^{k+1} \\ v_{y0}^{k+1} \\ T^{k+1} \end{Bmatrix} = \begin{Bmatrix} x_0^{k} \\ z_0^{k} \\ v_{y0}^{k} \\ T^{k} \end{Bmatrix} - \begin{bmatrix} \Phi(2,1) & \Phi(2,3) & \Phi(2,5) & 2v_y^k \\ \Phi(4,1) & \Phi(4,3) & \Phi(4,5) & \frac{\partial \Omega}{\partial x} + 2v_y^k \\ \Phi(6,1) & \Phi(6,3) & \Phi(6,5) & \frac{\partial \Omega}{\partial z} \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial z} & \frac{\partial J}{\partial v_y} & 0 \end{bmatrix}^{-1} \begin{Bmatrix} -y_f \\ -v_{x,f} \\ -v_{z,f} \\ J - J_{\text{target}} \end{Bmatrix} \tag{3}$$

The correction loop continues until the terminal errors in v_x, v_z, and the Jacobi constant fall below a predefined tolerance, here set to 10^{-14}.

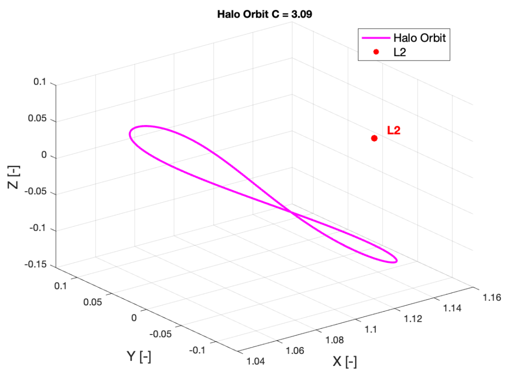

Figure 2: 3D View

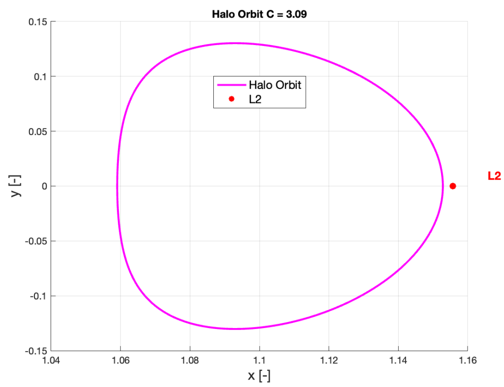

Figure 3: Top View

x	y	z
1.0590402077	0.0000000000	0.0739277378

v_x	v_y	v_z
0.0000000000	0.3469245709	0.0000000000

Table 2: Corrected initial state of the halo orbit with $C = 3.09$

1.3 Compute the families of halo orbits until C = 3.04

The task of finding proper initial condition for the correction loop may be challenging for Jacobi constants different from the given one; trying to find the solution for the periodic Halo orbit with $C = 3.04$, using the given initial conditions, tailored for a $C = 3.09$ may not be feasible. To address this issue, the concept of *numerical continuation* was employed. The core idea behind this method is to gradually vary the Jacobi constant from the initial value to the desired final value by taking intermediate steps: the greater the number of steps, the better the convergence. The solution found for each intermediate value of C is used as the initial guess for the next step, until the desidered family is reached.

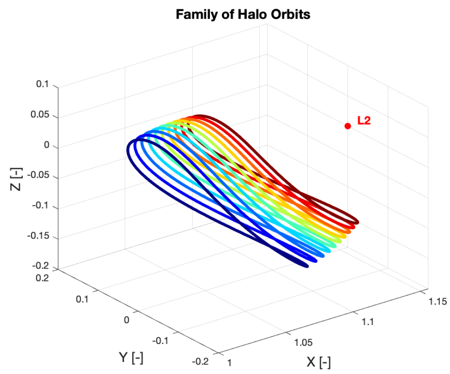

Figure 4: 3D View

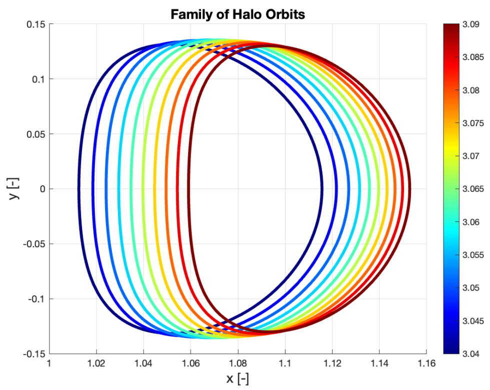

Figure 5: Top VIew

x	y	z
1.0125655235	0.0000000000	0.0672339583

v_x	v_y	v_x
0.0000000000	0.5103251959	0.0000000000

Table 3: Corrected initial state of the halo orbit with $C = 3.04$

2 Impulsive guidance

Consider the two-impulse transfer problem stated in Section 3.1 (Topputo, 2013)*.

1) Using the procedure in Section 3.2, produce a first guess solution using $\alpha = 0.2\pi$, $\beta = 1.41$, $\delta = 4$, and $t_i = 2$. Plot the solution in both the rotating frame and Earth-centered inertial frame (see Appendix 1 in (Topputo, 2013)). Consider the parameters listed in Table 4 and extract the radius and gravitational parameters of the Earth and Moon from the provided kernels and use the latter to compute the parameter μ.

Symbol	**Value**	**Units**	**Meaning**
m_s	3.28900541×10^5	-	Scaled mass of the Sun
ρ	3.88811143×10^2	-	Scaled Sun-(Earth+Moon) distance
ω_s	$-9.25195985 \times 10^{-1}$	-	Scaled angular velocity of the Sun
ω_{em}	$2.66186135 \times 10^{-1}$	s^{-1}	Earth-Moon angular velocity
l_{em}	3.84405×10^8	m	Earth-Moon distance
h_i	167	km	Altitude of departure orbit
h_f	100	km	Altitude of arrival orbit
DU	3.84405000×10^5	km	Distance Unit
TU	4.34256461	days	Time Unit
VU	1.02454018	km/s	Velocity Unit

Table 4: Constants to be considered to solve the PBRFBP. The units of distance, time, and velocity are used to map scaled quantities into physical units.

2) Considering the first guess in 1) and using $\{\mathbf{x}_i, t_i, t_f\}$ as variables, solve the problem in Section 3.1 with simple shooting in the following cases
   a) without providing any derivative to the solver, and
   b) by providing the derivatives and by estimating the state transition matrix with variational equations.
3) Considering the first guess solution in 1) and the procedure in Section 3.3, solve the problem with multiple shooting taking $N = 4$ and using the variational equation to compute the Jacobian of the nonlinear equality constraints.
4) Perform an n-body propagation using the solution $\{\mathbf{x}_i, t_i, t_f\}$ obtained in point 2), transformed in Earth-centered inertial frame and into physical units. To move from 2-D to 3-D, assume that the position and velocity components in inertial frame are $r_z(t_i) = 0$ and $v_z(t_i) = 0$. To perform the propagation it is necessary to identify the epoch t_i. This can be done by mapping the relative position of the Earth, Moon and Sun in the PCRTBP to a similar condition in the real world:
   a) Consider the definition of $\theta(t)$ provided in Section 2.2 to compute the angle $\theta_i = \theta(t_i)$. Note that this angle corresponds to the angle between the rotating frame x-axis, aligned to the position vector from the Earth-Moon System Barycenter (EMB) to the Moon, and the Sun direction.
   b) The angle θ ranges between $[0, 2\pi]$ and it covers this domain in approximately the revolution period of the Moon around the Earth.
   c) Solve a zero-finding problem to determine the epoch at which the angle Moon-EMB-Sun is equal to θ_i, considering as starting epoch `2024 Sep 28 00:00:00.000 TDB`.

*F. Topputo, "On optimal two-impulse Earth–Moon transfers in a four-body model", Celestial Mechanics and Dynamical Astronomy, Vol. 117, pp. 279–313, 2013, DOI: 10.1007/s10569-013-9513-8.

Hints: Exploit the SPK kernels to define the orientation of the rotating frame axes in the inertial frame for an epoch t. Consider only the projection of the EMB-Sun position vector onto the so-defined x-y plane to compute the angle (planar motion).

Plot the propagated orbit and compare it to the previously found solutions. (11 points)

2.1 Produce a first guess solution

The Planar Bi-Circular Restricted Four-Body Model can be used to model an impulsive transfer between the Earth and the Moon. In this specific model, we consider the influence of a third primary, the Sun, which revolves in a circular orbit around the Earth-Moon centre of mass. The Sun's perturbing element can be modelled using three constants: the scaled mass of the Sun m_s, the scaled Sun-(Earth+Moon) distance ρ and the scaled angular velocity of the Sun ω_s.

The 2D equations of motions can be derived as the following, in which Ω_3 is the *effective potential* also found in the C3BP 1.1:

$$\Omega_4(x, y, t) = \Omega_3(x, y) + \frac{m_s}{\mathbf{r}_3(t)} - \frac{m_s}{\rho^2}(x\cos(\omega_s t) + y\sin(\omega_s t))$$

$$\mathbf{r}_3 = \sqrt{(x - \rho\cos(\omega_s t))^2 + (y - \rho\sin(\omega_s t))^2}$$

$$\begin{cases} \ddot{x} - 2\dot{y} = \frac{\partial \Omega_4}{\partial x} \\ \ddot{y} + 2\dot{x} = \frac{\partial \Omega_4}{\partial y} \end{cases}$$

The problem is the following: the spacecraft is stationed in a circular orbit around the Earth, at altitude h_i, and performs a manoeuvre tangential to its trajectory, of magnitude ΔV_i, at time t_i, which injects it on a transfer trajectory. Another tangential impulse of magnitude ΔV_f, performed at time t_f, injects the spacecraft into the final circular orbit around the Moon, denoted by an altitude h_f.

The optimization problem consists of finding the initial state $\{\mathbf{x}_i, t_i, t_f\}$, with $t_f > t_i$, such that the initial condition $\boldsymbol{\psi}_i(\mathbf{x}_i)$ and final condition $\boldsymbol{\psi}_f(\mathbf{x}_f)$ are both satisfied, where $\mathbf{x}_f = \boldsymbol{\phi}(\mathbf{x}_i, t_i; t_f)$, and the total mission cost $\Delta v(\mathbf{x}_i, t_i, t_f)$ is minimized. The objective function and the constraints are provided below:

Objective Function

$$\Delta v(\mathbf{x}_i, t_i, t_f) = \Delta v_i(\mathbf{x}_i) + \Delta v_f(\boldsymbol{\phi}(\mathbf{x}_i, t_i; t_f))$$

$$\Delta v_i = \sqrt{(\dot{x}_i - y_i)^2 + (\dot{y}_i + x_i + \mu)^2} - \sqrt{\frac{1-\mu}{\mathbf{r}_i}}$$

$$\Delta v_f = \sqrt{(\dot{x}_f - y_f)^2 + (\dot{y}_f + x_f + \mu - 1)^2} - \sqrt{\frac{\mu}{\mathbf{r}_f}}$$

Constraints

$$\boldsymbol{\psi}_i(\mathbf{x}_i) : \begin{cases} (x_i + \mu)^2 + y_i^2 - \mathbf{r}_i^2 = 0, \\ (x_i + \mu)(\dot{x}_i - y_i) + y_i(\dot{y}_i + x_i + \mu) = 0 \end{cases}$$

$$\boldsymbol{\psi}_f(\mathbf{x}_f) : \begin{cases} (x_f + \mu - 1)^2 + y_f^2 - \mathbf{r}_f^2 = 0, \\ (x_f + \mu - 1)(\dot{x}_f - y_f) + y_f(\dot{y}_f + x_f + \mu - 1) = 0 \end{cases}$$

Instead of working with the six scalars $\{x_i, t_i, t_f\}$, a transformation has been developed to specify the initial condition as a *four scalars vector*, given by the statement of the problem:

Parameter	Description
α	Angle
β	Initial-to-circular velocity ratio
t_i	Initial time
δ	Transfer duration

Table 5: Initial Guess Parameters

$$v_0 = \beta\sqrt{\frac{1-\mu}{r_i}}$$
$$x_0 = r_i\cos(\alpha) - \mu$$
$$y_0 = r_i\sin(\alpha)$$
$$\dot{x}_0 = -(v_0 - r_i)\sin(\alpha)$$
$$\dot{y}_0 = (v_0 - r_i)\cos(\alpha)$$

$r_{x,0}$ $[DU]$	$r_{y,0}$ $[DU]$	$v_{x,0}$ $[VU]$	$v_{y,0}$ $[VU]$
0.001624	0.010008	-6.302804	8.675065

Table 6: Initial guess in Earth-Moon rotating frame

2.2 Solve the problem with simple shooting

2.2.1 No Gradient

Propagating the first guess solution in the Planar Bi-Circular Restricted Four-Body Model yields the following trajectories, which, obviously, as the constraint are not enforced, do not intersect the final Lunar orbit:

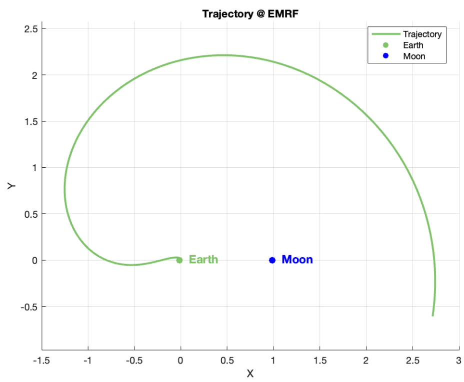

Figure 6: Trajectory @EMRF

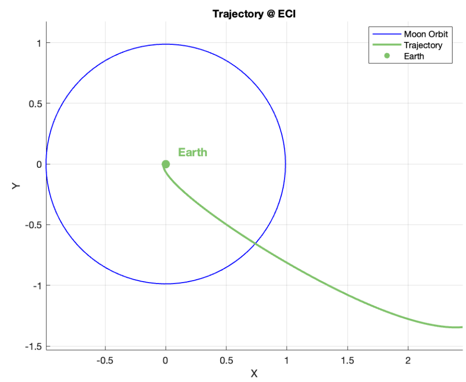

Figure 7: Trajectory @Earth ECI

Instead, the problem was solved using the *MatLab* function *fmincon*, which takes as input the objective function, the initial guess and constraint function. The latter was divided into *equality constraints*, $\psi_f(x_i)$ $\psi_f(x_f)$, and *inequality constraints*, with the only one being that the final time had to be located in the future with respect to the initial time: $t_f - t_i > 0$. To ensure feasible values, bounds are set for the variables: no specific range for position and velocity but initial and final time were searched within: $t_i \in [0, \frac{2\pi}{\omega_s}]$, $t_f \in [t_i, t_i + 23TU]$.
Fmincon primarily outputs the initial state on the parking orbit, as well the time of injection in the transfer orbit and time of insertion in the arrival orbit. From those values, it is possible to retrieve the optimized cost for the mission, which returns $\Delta V_{OPT} = 4.0076 km/s$, and visualize the trajectory in the Earth-Moon rotating frame or, by applying a specific transformation, in the Earth-Centered-Inertial frame: both are provided below.

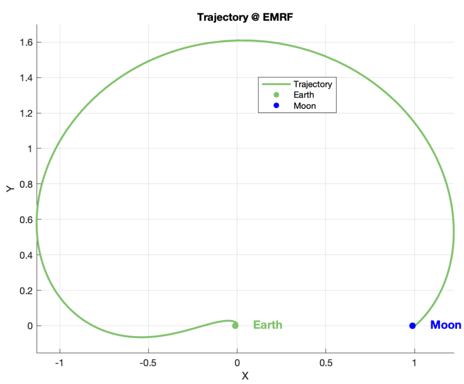

Figure 8: Trajectory @EMRF

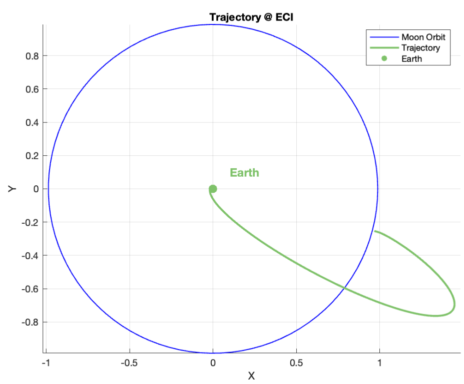

Figure 9: Trajectory @Earth ECI

2.2.2 Gradients

In this case, the optimization process was enhanced by providing both the gradients of the objective function and the constraints to the solver: this proved to drastically achieve convergence in 0.5 s within 13 iterations, and furthermore reduced the optimal mission cost to $\Delta V_{OPT} = 4.0068 km/s$.
The analytical computation of the gradient for ΔV_i and initial constraints, with respect to the initial state, initial time and final time, are detailed inside A.

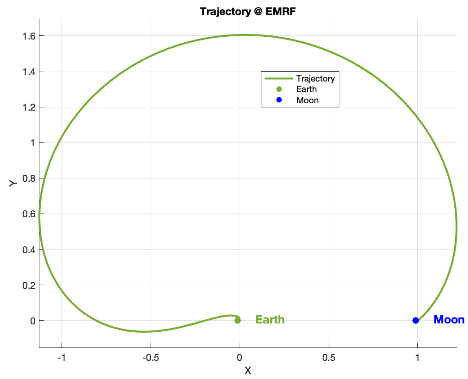

Figure 10: Trajectory @EMRF

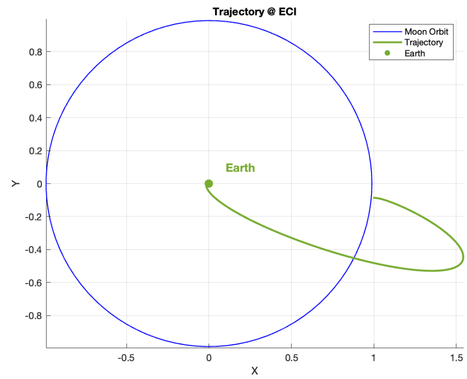

Figure 11: Trajectory @Earth ECI

Gradients	$r_{x,0}$ $[DU]$	$r_{y,0}$ $[DU]$	$v_{x,0}$ $[VU]$	$v_{y,0}$ $[VU]$	t_i $[TU]$	t_f $[TU]$
False	0.001576	0.010074	-6.329271	8.624010	2.022	6.033
True	0.001620	0.010013	-6.290807	8.651506	2.198	6.203

Table 7: Simple shooting solutions in the Earth-Moon rotating frame.

2.3 Solve the problem with multiple shooting

The multiple shooting method divides the entire trajectory into N=4 segments, allowing for a more refined search. At each segment, a boundary condition is applied to ensure that the state at the end of one segment matches the initial state of the next segment, thereby maintaining continuity across the trajectory. To solve the multiple shooting method with $N = 4$, the initial guess used was the one calculated in 2.1, while the initial conditions was the remaining segments were found by dividing the time of flight in three segments and propagating the trajectory until the end of the segment; the final condition of the segment were then used as initial condition for the next one, and so on. This lead to an overall reduction of the total mission cost to $\Delta V_{OPT} = 3.997 km/s$
A special remark must be written on the gradients of the objective function and constraints: both of them were computed analytically using [1], with the former taking dimensions of 16x1 vector while the latter being a 16x14 matrix. The constraints were divided into **equality constraints**, which ensure that the spacecraft satisfies the required initial and final conditions as well as enforcing matching in position and velocity between the segments, and **inequality constraints**, which ensure that the spacecraft remains within safe distances from the Earth and the Moon during the entire trajectory.

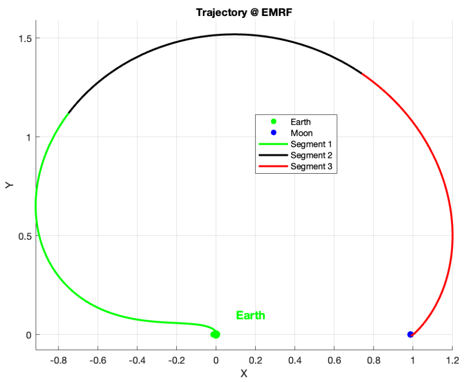

Figure 12: Trajectory @EMRF

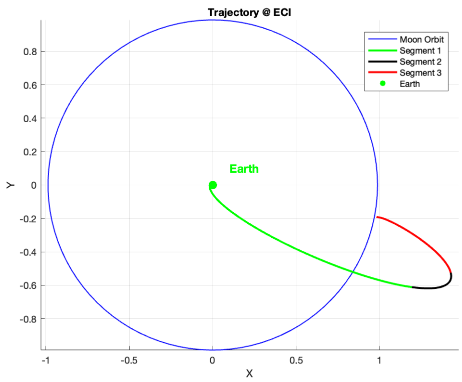

Figure 13: Trajectory @Earth ECI

$r_{x,0}$ $[DU]$	$r_{y,0}$ $[DU]$	$v_{x,0}$ $[VU]$	$v_{y,0}$ $[VU]$	t_i $[TU]$	t_f $[TU]$
0.004103	0.005072	-3.186015	10.208983	2.438	6.097

Table 8: Multiple shooting solution in the Earth-Moon rotating frame.

2.4 Perform an n-body propagation

The last point requested to perform an n-body propagation using the solution found in 2.2.2: this means propagating a trajectory from a 2D PCRTBP solution to a 3D n-body model. This is done by assuming that the z components of position and velocity are both zero. However, we are still missing information on the starting time, that is, the real-world epoch where the actual Sun-EMB-Moon angle matches θ_i: the latter is found solving a zero-finding problem in which the epoch is varied until the angular difference between the analytical angle $\theta(t)$ computed in the PCRTBP model and the one retrieved from SPICE using the Sun and Moon's ephemerides, vanishes. The referred $\theta(t)$ is the angle between:

1. A local rotating frame is built by taking the Moon's position as the x-axis direction, the velocity direction as y-axis, and completing the orthonormal basis via the cross product to find the z-axis.

2. The Sun's position at the same epoch projected onto this locally defined x-y plane,

The angle between the Sun and the x-axis is computed using the `atan2` function. After the correct departure epoch time is found, it used as the starting point for the full 3D trajectory propagation, which is then compared to the idealized CRTBP solution and, ultimately, with all the previously found solutions:

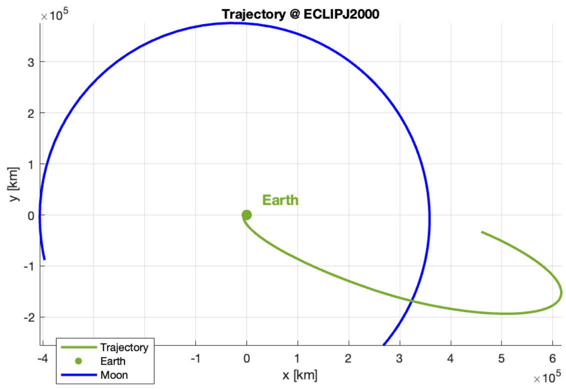

Figure 14: 2D View Trajectory @Earth ECLIPJ2000

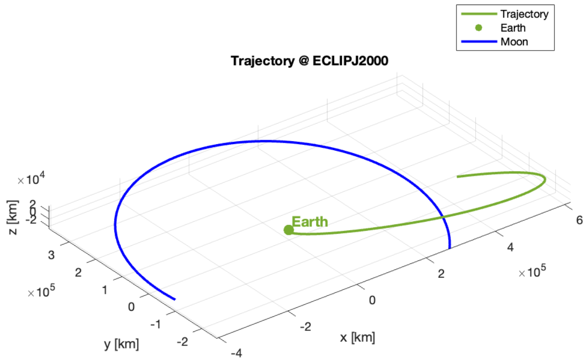

Figure 15: 3D View Trajectory @Earth ECLIPJ2000

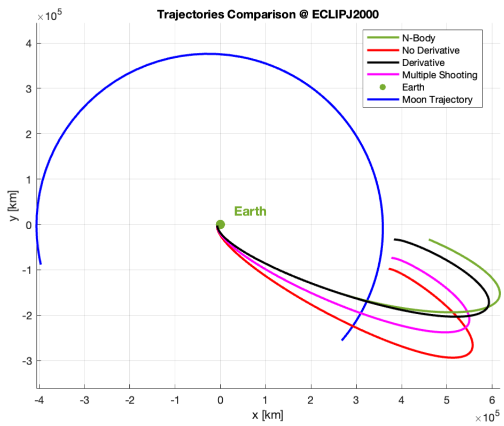

Figure 16: 2D Comparison All Trajectories @Earth ECLIPJ2000

Symbol	**Calendar epoch** (UTC)
t_i	2024-10-12T21:25:28.294
t_f	2024-10-30T06:50:14.761

$r_{x,0}$ $[km]$	$r_{y,0}$ $[km]$	$r_{z,0}$ $[km]$
-6223.46701353	$+2026.19923292$	0.0

$v_{x,0}$ $[km/s]$	$v_{y,0}$ $[km/s]$	$v_{z,0}$ $[km/s]$
-3.39819682	-10.43755495	0.0

Table 9: Initial epoch, final epoch, and initial state in Earth-centered inertial frame.

3 Continuous guidance

A low-thrust option is being considered to perform an orbit raising manoeuvre using a low-thrust propulsion system in Earth orbit. The spacecraft is released on a circular orbit on the equatorial plane at an altitude of 800 km and has to reach an orbit inclined by 0.75 deg on the equatorial plane at 1000 km. This orbital regime is characterized by a large population of resident space objects and debris, whose spatial density q can be expressed as:

$$q(\rho) = \frac{k_1}{k_2 + \left(\dfrac{\rho - \rho_0}{DU}\right)^2}$$

where ρ is the distance from the Earth centre. The objective is to design an optimal orbit raising that minimizes the risk of impact, that is to minimize the following objective function

$$F(t) = \int_{t_0}^{t_f} q(\rho(t))\,\mathrm{d}t.$$

The parameters and reference Distance Unit to be considered are provided in Table 10.

Symbol	Value	Units	Meaning
h_i	800	km	Altitude of departure orbit
h_f	1000	km	Altitude of arrival orbit
Δi	0.75	deg	Inclination change
R_e	6378.1366	km	Earth radius
μ	398600.435	km^3/s^2	Earth gravitational parameter
ρ_0	$750 + R_e$	km	Reference radius for debris flux
k_1	1×10^{-5}	DU^{-1}	Debris spatial density constant 1
k_2	1×10^{-4}	DU^2	Debris spatial density constant 2
m_0	1000	kg	Initial mass
$T_{\max}$	3.000	N	Maximum thrust
$I_{\rm sp}$	3120	s	Specific impulse
DU	7178.1366	km	Distance Unit
MU	m_0	kg	Mass Unit

Table 10: Problem parameters and constants. The units of time TU and velocity VU can be computed imposing that the scaled gravitational parameter $\overline{\mu} = 1$.

1) Plot the debris spatial density $q(\rho) \in [h_i - 100;\, h_f + 100]$ km and compute the initial state and target orbital state, knowing that: i) the initial and final state are located on the x-axis of the equatorial J2000 reference frame; ii) the rotation of the angle Δi is performed around the x-axis of the equatorial J2000 reference frame (RAAN = 0).

2) Adimensionalize the problem using as reference length $DU = \rho_i = h_i + R_e$ and reference mass $MU = m_0$, imposing that $\mu = 1$. Report all the adimensionalized parameters.

3) Using the PMP, write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition. **Hint**: the spacecraft has to reach the target state computed in point 1).

4) Solve the problem considering the data provided in Table 10. To obtain an initial guess for the costate, generate random numbers such that $\lambda_{0,i} \in [-250; +250]$ while $t_f \approx 20\pi$. Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Assess your results exploiting the properties of the Hamiltonian in problems that are not

time-dependent and time-optimal solution. Plot the evolution of the components of the primer vector $\boldsymbol{\alpha}$ in a NTW reference frame[†].

5) Solve the problem for a lower thrust level $T_{\max} = 2.860$ N. Compare the new solution with the one obtained in the previous point. **Hint**: exploit numerical continuation.

(11 points)

[†]The T-axis is aligned with the velocity, the N-axis is normal to the angular momentum, while the W-axis is pointing inwards, i.e., towards the Earth.

3.1 Plot the debris spatial density

The initial point is located on the x-axis of the J2000 equatorial reference frame, on an orbit with altitude of 800 km. Similarly, the final point is located onto the same axis, but on an orbit with altitude of 1000 km and inclined with respect to the equatorial plane of 0.75 deg. As the RAAN of the two points is given to be zero, the can easily compute the final state.

$$\mathbf{r}_i = \begin{bmatrix} r_i \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_i = \sqrt{\frac{\mu}{r_i}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{r}_f = \begin{bmatrix} r_f \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_f = \sqrt{\frac{\mu}{r_f}} \begin{bmatrix} 0 \\ \cos \Delta i \\ \sin \Delta i \end{bmatrix}$$

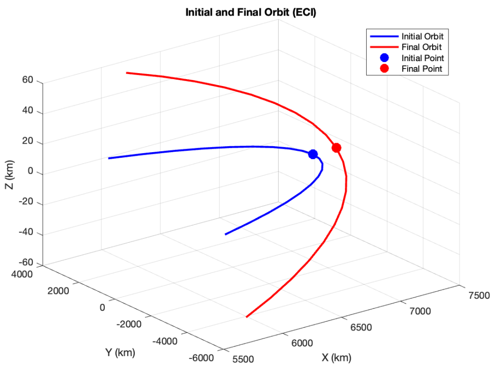

Figure 17: Initial and Final Points @Earth J2000

Furthermore, let q be the spatial density function, which depends on the distance from the centre of the Earth. The debris spatial density, in function of the distance from the centre of the Earth, can be plotted:

$$q(\rho) = \frac{k_1}{k_2 + (\rho - \rho_0)^2}$$

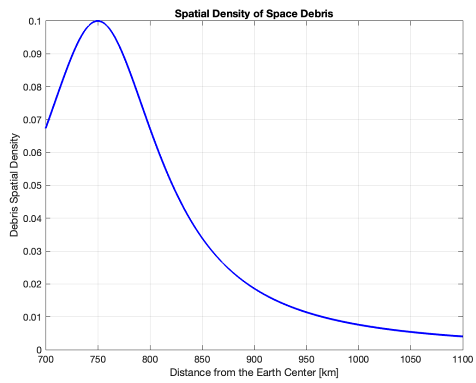

Figure 18: Debris Spatial Density

$r_{x,i}$ $[km]$	$r_{y,i}$ $[km]$	$r_{z,i}$ $[km]$	$v_{x,i}$ $[km/s]$	$v_{y,i}$ $[km/s]$	$v_{z,i}$ $[km/s]$
7178.136600	0000.000000	0000.000000	0.00000000	7.451831	0.00000000

$r_{x,f}$ $[km]$	$r_{y,f}$ $[km]$	$r_{z,f}$ $[km]$	$v_{x,f}$ $[km/s]$	$v_{y,f}$ $[km/s]$	$v_{z,f}$ $[km/s]$
7378.136600	0000.000000	0000.000000	0.00000000	7.349509	0.096210

Table 11: Initial and target state in Earth-centered equatorial J2000 inertial frame.

3.2 Adimensionalize the problem

Starting from the parameters on the left, all the quantities were adimensionalized as the following:

$$\mathrm{TU} = \sqrt{\frac{\mathrm{DU}^3}{\mu}} \quad \mathrm{VU} = \frac{\mathrm{DU}}{\mathrm{TU}} \quad I_{sp}^* = \frac{I_{sp}}{\mathrm{TU}} \qquad T_{max}^* = T_{max}\frac{\mathrm{TU}^2}{\mathrm{MU}\cdot\mathrm{DU}}$$

$$\mathbf{r_i^*} = \frac{\mathbf{r_i}}{DU} \quad \mathbf{v_i^*} = \frac{\mathbf{v_i}}{VU} \quad \mathbf{r_f^*} = \frac{\mathbf{r_f}}{DU} \quad \mathbf{v_f^*} = \frac{\mathbf{v_f}}{VU} \quad g_0^* = g_0\frac{\mathrm{TU}^2}{\mathrm{DU}} \quad \rho_0^* = \frac{\rho_0}{\mathrm{DU}}$$

3.3 Write down the spacecraft equations of motion and costate dynamics using PMP

The problem revolves on finding a trajectory from the initial fixed point to the final fixed one, which minimizes the encounter with debris, only using a set of admissible actions Ω, being $\hat{\alpha}$ the optimal thrust direction and u the throttle factor. We formalize the problem as the following:

$$\min_{(\hat{\alpha},u)\in\Omega} \int_{t_i}^{t_f} q(\rho)\,\mathrm{d}t \quad \text{s.t.} \quad \begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\hat{\alpha},t) \\ \mathbf{x}(t_i) = \mathbf{x}_i \\ \mathbf{v}(t_i) = \mathbf{v}_i \\ \mathbf{r}(t_f) = \mathbf{r}_f \\ \mathbf{v}(t_f) = \mathbf{v}_f \\ \lambda_m(t_f) = 0 \\ H(t_f) = 0 \end{cases}$$

where $\lambda_m(t_f)$ and $H(t_f)$ are the costate of the mass and the Hamiltonian evaluated at the final time. The model used for motion of the spacecraft is the two-body model, in which we can insert the contribution of the thrust and add the dynamic of the mass consumption.

$$f: \begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\dfrac{\mu}{r^3}\mathbf{r} + u\dfrac{T_{\max}}{m}\,\hat{\boldsymbol{\alpha}} \\ \dot{m} = -u\dfrac{T_{\max}}{I_{\mathrm{sp}}g_0} \end{cases}$$

the derivation of the costate's dynamics comes directly from the Hamiltonian:

$$H = q + \boldsymbol{\lambda}^T f$$

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ m \end{bmatrix} \quad \boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda_r} \\ \boldsymbol{\lambda_v} \\ \lambda_m \end{bmatrix}$$

$$-\frac{\partial H}{\partial \mathbf{x}} = \begin{cases} \dot{\boldsymbol{\lambda}}_{\mathbf{r}} = -\dfrac{3\mu}{r^5}(\mathbf{r}\cdot\boldsymbol{\lambda}_{\mathbf{v}})\mathbf{r} + \dfrac{\mu}{r^3}\boldsymbol{\lambda}_{\mathbf{v}} - \dfrac{2k_1\mathbf{r}\cdot(\rho_0 - ||\mathbf{r}||)}{(k_2 + (\rho_0 - ||\mathbf{r}||)^2)^2\,||\mathbf{r}||} \\ \dot{\boldsymbol{\lambda}}_{\mathbf{v}} = -\boldsymbol{\lambda}_{\mathbf{r}} \\ \dot{\lambda}_m = -u\dfrac{T_{\max}}{m^2}\,\boldsymbol{\lambda}_{\mathbf{v}}\cdot\hat{\boldsymbol{\alpha}} \end{cases}$$

the above system of equations can be further simplified by noting that in *time-optimal problems*, the control action u is always 1, while the direction of optimal thrust is directly liked to the primer vector formulation, for which $\hat{\boldsymbol{\alpha}} = -\lambda_v/||\lambda_v||$.

$$
\begin{cases}
\dot{\mathbf{r}} = \mathbf{v} \\
\dot{\mathbf{v}} = -\dfrac{\mu}{r^3}\mathbf{r} + u\dfrac{T_{\max}}{m}\,\hat{\boldsymbol{\alpha}} \\
\dot{m} = -u\dfrac{T_{\max}}{I_{\text{sp}}g_0} \\
\dot{\boldsymbol{\lambda}}_{\mathbf{r}} = -\dfrac{3\mu}{r^5}(\mathbf{r}\cdot\boldsymbol{\lambda}_{\mathbf{v}})\mathbf{r} + \dfrac{\mu}{r^3}\boldsymbol{\lambda}_{\mathbf{v}} - \dfrac{2k_1\mathbf{r}\cdot(\rho_0 - ||\mathbf{r}||)}{(k_2 + (\rho_0 - ||\mathbf{r}||)^2)^2\,||\mathbf{r}||} \\
\dot{\boldsymbol{\lambda}}_{\mathbf{v}} = -\boldsymbol{\lambda}_{\mathbf{r}} \\
\dot{\lambda}_m = -u\dfrac{T_{\max}}{m^2}\,\boldsymbol{\lambda}_{\mathbf{v}}\cdot\hat{\boldsymbol{\alpha}}
\end{cases}
\qquad \text{s.t.} \qquad
\begin{cases}
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \hat{\alpha}, t) \\
\mathbf{x}(t_i) = \mathbf{x}_i \\
\mathbf{v}(t_i) = \mathbf{v}_i \\
\mathbf{r}(t_f) = \mathbf{r}_f \\
\mathbf{v}(t_f) = \mathbf{v}_f \\
\lambda_m(t_f) = 0 \\
H(t_f) = 0
\end{cases}
$$

The condition on the Hamiltonian is necessary in order to fix the final time, and goes by with the nature of having a fixed arrival state, therefore, its value must be always zero. However, as it shown in 20, the real evolution in time of the Hamiltonian will exhibit very small fluctuations around a constant near-zero value.

3.4 Solve the problem

Now that the problem has been identified and the Euler-Lagrange equations have been written for our case, we need to find a correct set of initial conditions $\mathbf{X0} = [\boldsymbol{\lambda_i}, tf]$. As this process is challenging, the costate's guess was generated randomly between [-250; +250] while t_f was stochastically selected, at each run, between $[20\pi; 20.5\pi]$. The problem was solved via the use of *fsolve*, which adjusts the input vector $\mathbf{X0}$ until the Euler-Lagrange equations are solved, i.e., when it finds values for $\mathbf{X0}$ that most closely matches the boundary conditions. As convergence was not achieved with every random guess, the whole process was put inside a while loop, which continues until either a solution is found or the maximum number of iterations is reached.

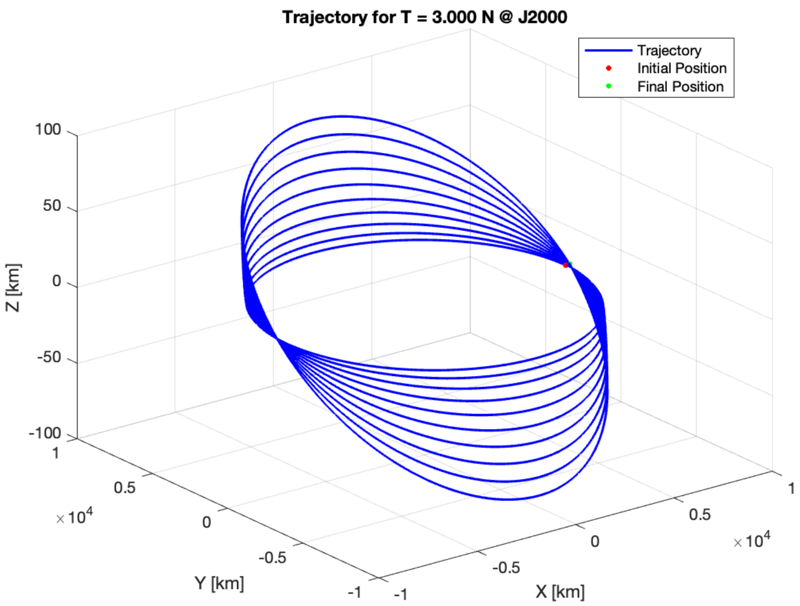

Figure 19: Trajectory for T = 3.000 N @Earth J2000

t_f [mins]	m_f $[kg]$
1035.1974	993.9120

λ_{0,r_x}	λ_{0,r_y}	λ_{0,r_z}	λ_{0,v_x}	λ_{0,v_y}	λ_{0,v_z}	$\lambda_{0,m}$
−214.9811	−10.3658	0.8856	−10.3929	−214.6104	−112.9452	2.5964

Table 12: Optimal orbit raising transfer solution ($T_{\max} = 3.000$ N).

Error	**Value**	**Units**
$\|\|\mathbf{r}(t_f) - \mathbf{r}_f\|\|$	0.0012	km
$\|\|\mathbf{v}(t_f) - \mathbf{v}_f\|\|$	0.0011	m/s

Table 13: Final state error with respect to target position and velocity ($T_{\max} = 3.000$ N).

Furthermore, the results of the Hamiltonian and the evolution of the primer vector in the **NTW** are provided below. Due to the time independence of the Hamiltonian, it remains nearly constant, with a maximum variation from the moving average of $2 \cdot 10^{-10}$.

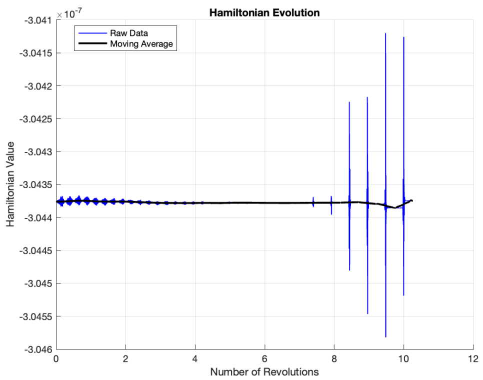

Figure 20: Hamiltonian Behaviour

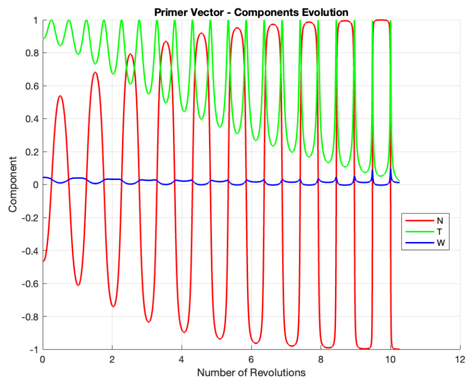

Figure 21: NTW Evolution

3.5 Solve the problem for a lower thrust level T = 2.860 N

In order to find the solution for a lower level of thrust, a numerical continuation approach was employed, gradually decreasing the thrust. To ensure that the solution for each thrust level converges, we start as an initial guess the solution found in the previous point. Then, the solution for each intermediate thrust level is used as guess for the following step, until convergence is achieved for the desidered thrust level. The level of accuracy is dependent on how many intermediate levels we decide to use to go from one thrust level to the other, with smaller steps achieving faster convergence: for this reason, the thrust level was decreased at each step of 0.014 N, corresponding to a 10 segments discretization.

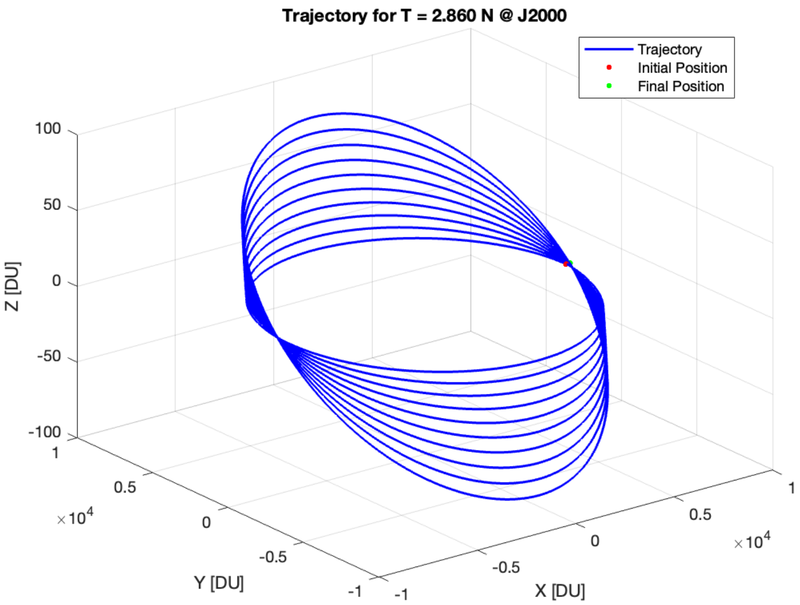

Figure 22: Trajectory for T = 2.860 N @Earth J2000

t_f [mins]	m_f [kg]
1030.9760	994.2198

λ_{0,r_x}	λ_{0,r_y}	λ_{0,r_z}	λ_{0,v_x}	λ_{0,v_y}	λ_{0,v_z}	$\lambda_{0,m}$
−593.1526	−11.5904	2.2973	−11.9293	−592.7887	−920.3691	17.3814

Table 14: Optimal orbit raising transfer solution ($T_{\max} = 2.860$ N).

Error	**Value**	**Units**
$\|\|\mathbf{r}(t_f) - \mathbf{r}_f\|\|$	0.0003	km
$\|\|\mathbf{v}(t_f) - \mathbf{v}_f\|\|$	0.0003	m/s

Table 15: Final state error with respect to target position and velocity ($T_{\max} = 2.860$ N).

References

[1] Francesco Topputo. On optimal two-impulse Earth–Moon transfers in a four-body model. *Celestial Mechanics and Dynamical Astronomy*, 117(3):279–313, November 2013.

Appendices

A Gradients Appendix

$$\frac{\partial \Delta V_i}{\partial \mathbf{x_i}} = \frac{1}{\sqrt{(v_{xi}-y_i)^2+(v_{yi}+x_i+\mu)^2}} \begin{bmatrix} v_{yi}+x_i+\mu \\ y_i - v_{xi} \\ v_{xi}-y_i \\ v_{yi}+x_i+\mu \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{\partial C_1}{\partial \mathbf{x_i}} = \begin{bmatrix} 2\mu+2x_i & 2y_i & 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{\partial C_2}{\partial \mathbf{x_i}} = \begin{bmatrix} v_{xi} & v_{yi} & \mu+x_i & y_i & 0 & 0 \end{bmatrix}$$

Given $\mathbf{p_f} = [x_f, y_f, v_{xf}, v_{yf}]$, the gradient of ΔV_f with respect to $\mathbf{x_f} = [\mathbf{p_f}, t_i, t_f]$ requires the aid of the STM, as well as the state of the system at a specific position and time.

$$\frac{\partial \Delta V_f}{\partial \mathbf{p_f}} = \frac{1}{\sqrt{(v_{xi}-y_i)^2+(v_{yi}+x_i+\mu)^2}} \begin{bmatrix} \mu+v_{yf}+x_f-1 \\ -v_{xf}+y_f \\ v_{xf}-y_f \\ \mu+v_{yf}+x_f-1 \end{bmatrix}$$

$$\begin{aligned} \frac{\partial V_i}{\partial t_i} &= \frac{\partial V_f}{\partial \mathbf{p_f}} \cdot -\Phi(t_i,t_f)\cdot f(t_i,x_i) \\ \frac{\partial V_i}{\partial t_f} &= \frac{\partial V_f}{\partial \mathbf{p_f}} \cdot f(t_f,x_f) \end{aligned}$$

$$\begin{aligned} \frac{\partial C_3}{\partial \mathbf{p_f}} &= \begin{bmatrix} 2\mu+2x_f-2 & 2y_f & 0 & 0 \end{bmatrix} \Phi(t_i,t_f) \\ \frac{\partial C_3}{\partial t_1} &= \begin{bmatrix} 2\mu+2x_f-2 & 2y_f & 0 & 0 \end{bmatrix} - \Phi(t_i,t_f)\cdot f(t_i,x_i) \\ \frac{\partial C_3}{\partial t_f} &= \begin{bmatrix} 2\mu+2x_f-2 & 2y_f & 0 & 0 \end{bmatrix} \cdot f(t_f,x_f) \end{aligned}$$

$$\begin{aligned} \frac{\partial C_4}{\partial \mathbf{p_f}} &= \begin{bmatrix} v_{xf} & v_{yf} & \mu+x_f-1 & y_f \end{bmatrix} \cdot \Phi(t_i,t_f) \\ \frac{\partial C_4}{\partial t_1} &= \begin{bmatrix} v_{xf} & v_{yf} & \mu+x_f-1 & y_f \end{bmatrix} \cdot -\Phi(t_i,t_f)\cdot f(t_i,x_i) \\ \frac{\partial C_4}{\partial t_f} &= \begin{bmatrix} v_{xf} & v_{yf} & \mu+x_f-1 & y_f \end{bmatrix} \cdot f(t_f,x_f) \end{aligned}$$

finally, we obtain the objective function gradient by summing the intermediate gradients, with dimension 6x1 and the constraints gradient, a 18x24 matrix.

$$\nabla \Delta V = \frac{\partial \Delta V_i}{\partial \mathbf{x_i}} + \Phi(t_i,t_f)\cdot \frac{\partial \Delta V_f}{\partial \mathbf{x_f}}$$