Numerical modelling of the atmosphere and oceans
Lecture 4

P.L. Vidale [1]

National Centre for Atmospheric Science and Department of Meteorology
University of Reading, UK
p.l.vidale@reading.ac.uk

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[1] based in part on J. Methven lecture notes from 2013

From Newton to Navier-Stokes

In fluid dynamics we often thing of Newton's 2nd law in this form:

$$\frac{D_a \mathbf{v_a}}{Dt} = \sum \mathbf{F} \tag{1}$$

where the subscript a indicates that we follow the motion as viewed in an inertial system. In fact, Newton's 2nd law and the continuity equation (of mass and entropy, with appropriate *boundary conditions* and *forcing*) are often presented together, as the Navier-Stokes equations:

$$\nabla \cdot \vec{u} = 0$$

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u}\right) = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{F}$$

Figure: The Navier-Stokes equations

which are capable of representing any type of fluid motion, and are appicable to an extremely wide range of scales.
We do not know how to solve these equations, and in fact the Clay Mathematics Institute of Cambridge, Massachusetts (CMI) selected this as one of the seven Millenium Prize Problems, offering \$ 1 million for a solution.

We need to relate Newton's 2nd law, as viewed in an inertial system,

$$\frac{D_a \mathbf{v_a}}{Dt} = \sum \mathbf{F} \tag{2}$$

to a rotating coordinate system point of view. This requires expressing v_a as v by using the relationship for a position vector r.

$$\frac{D_a \mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \Omega \times \mathbf{r} \tag{3}$$

and $\mathbf{v_a} = \mathbf{v} + \Omega \times \mathbf{r}$, so that:

$$\frac{D_a \mathbf{v_a}}{Dt} = \frac{D\mathbf{v_a}}{Dt} + \Omega \times \mathbf{v_a} \tag{4}$$

and finally:

$$\frac{D_a \mathbf{v_a}}{Dt} = \frac{D\mathbf{v}}{Dt} + 2\Omega \times \mathbf{v} - \Omega^2 \mathbf{r} \tag{5}$$

Still Newton's 2nd law, now for fluid motion on Earth

The form of the momentum equations for geophysical fluids is:

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla p}{\rho} + \mathbf{g} + \mathbf{F_r} \tag{6}$$

or, if Earth's rotation is included:

$$\frac{D\mathbf{v}}{Dt} = -\frac{\nabla p}{\rho} + \mathbf{g} + \mathbf{F_r} - 2\Omega \times \mathbf{v} \tag{7}$$

where F_r now represents frictional forces. The system supports a complex variety of phenomena across a huge range of scales.
Flow almost adiabatic with "weak" forcing $\Rightarrow$ stable stratification established.
Earth's rotation has profound influence on motions $\Rightarrow$ layerwise-2D motions.

The system may be simplified by noting the following:

- Earth rotates about its polar axis at rate $\Omega = 7.292 \times 10^{-5}\mathrm{s}^{-1}$,
- Earth is approximately an oblate spheroid,
- Radius to poles $a_{pole} \approx 6357$km, radius to equator $a_{eq} \approx 6378$km. Average, $a \approx 6371$km.
- Atmosphere and oceans are shallow relative to the Earth's radius.

Shallow atmosphere approximation

Also known as the *traditional approximation* since it also applies to the oceans. The deepest motions in the atmosphere have depth scales, $H \approx 20$km. The oceans are even shallower. Therefore $H/a < 1/300$. The shallow limit $H/a \ll 1$ gives:

- radial coordinate $r \to a$ (i.e., constant)
- $\frac{\partial}{\partial r} \to \frac{\partial}{\partial z}$ where z is height relative to Earth's surface
- In order to retain same form for kinetic energy and zonal angular momentum equations, must drop terms containing w in horizontal momentum equations and the horizontal component of Coriolis acceleration.

The resulting *primitive equations* (PEs) are the basis of almost all atmosphere and ocean *general circulation models* (GCMs) except the Met Office Unified Model which does not make a shallow atmosphere approximation.

- Biggest errors are in Tropics associated with neglect of horizontal component of Coriolis acceleration.[2]
- Effects of spherical geopotential approximation are unknown.[3]

[2] White, Hoskins, Roulstone and Staniforth (2005) Consistent approximate models of the global atmosphere: shallow, deep, hydrostatic, quasi-hydrostatic and non-hydrostatic. *Quart. J. Roy. Met. Soc.*, **131**, 2081-2107.

[3] White, Staniforth and Wood (2008) Spheroidal coordinate systems for modelling global atmospheres. *Quart. J. Roy. Met. Soc.*, **134**, 261-270.

Filtering the equations to remove unwanted scales of motion

The problem with our Euler equations is that they support all types of motion, some of which may be unwanted, depending on what scientific questions we are trying to answer.

Example: sound waves.
The governing equations of compressible fluid dynamics contain acoustic waves. In low Mach number flows, typical of atmospheric or oceanic conditions, acoustic waves play no essential role and yet they severely constrain the time step in numerical modeling. Thus, there has long been an interest in approximating the governing equations to eliminate or "filter out" acoustic waves.

Wave type	horizontal dimension	propagation speed	propagation direction	time step required
Sound				
Short Gravity				
Long Gravity				
Rossby				
Kelvin				

Further approximations

Further approximations are usually based on *scaling* where the typical spatial and length scales associated with motions of interest are assumed. For example:

Planar approximation ($L/a \ll 1$) Can use local Cartesian coordinates where $(dx, dy) = (a \cos\phi_0 \, d\lambda, a \, d\phi)$.
Can drop spherical metric terms from PEs.

Anelastic approximation ($\Delta\rho/\rho_0 \ll 1$) Mass conservation equation becomes $\nabla.(\rho_r \mathbf{u}) = 0$ where $\rho_r(z)$ is a reference density. Filters sound waves.

Boussinesq approximation ($\Delta\rho/\rho_0 \ll 1$ and $H \ll H_\rho$) Huge density height scale, H_ρ, implies $w \frac{\partial \rho_r}{\partial z} \ll \rho_r \frac{\partial w}{\partial z}$. In practice, the density variation is only important in the buoyancy term of the (vertical) momentum equation, ρg. The continuity equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{u} = 0 \qquad (8)$$

ends up simplified to its incompressible ($\rho_r \approx$ constant) form: $\nabla \cdot \mathbf{u} = 0$.

Hydrostatic balance ($H/L \ll 1$) Vertical momentum equation reduces to:

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g \qquad (9)$$

Filters vertically-propagating sound waves, but can support Lamb waves, and can distort gravity waves if $L_x \approx L_y$.

PEs on the sphere

In an Eulerian framework, our equation (1) ends up looking like this:

$$\frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla)\, u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \tag{10}$$

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\, v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \tag{11}$$

$$\frac{\partial w}{\partial t} + (\mathbf{v} \cdot \nabla)\, w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \tag{12}$$

We must now consider the fact that we are studying fluids (atmosphere, ocean) on our planet, which is approximately a sphere of radius a, rotating with angular velocity Ω,

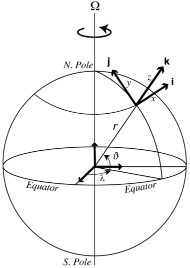


Figure: Schematic of spherical coordinate system

In the spherical coordinate system we use λ (longitude), θ (latitude) and r, to replace x, y, z, where $x = r\cos\theta\lambda$; $y = r\theta$; $z = r - a$, so that:

$$u = \frac{dx}{dt} = r\cos\theta\frac{d\lambda}{dt}; v = \frac{dy}{dt} = r\frac{d\theta}{dt} =; w = \frac{dz}{dt} \tag{13}$$

are how we write our spatial derivatives on the Sphere.

PEs on the Sphere before scale analysis

The momentum equations in spherical coordinates are then given by

$$\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - \left(\frac{u\tan\vartheta}{a}\right)v + \frac{w}{a}u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$
$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + \left(\frac{u\tan\vartheta}{a}\right)u + \frac{w}{a}v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$
$$\frac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w - \frac{u^2+v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z$$

where the new terms (those involving $1/a$) result from consideration of the curvature of Earth.

The full form of the momentum equations in a rotating spherical coordinate system is then

$$\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)v + \frac{w}{a}u + w\cdot 2\Omega\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$
$$\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)u + \frac{w}{a}v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y$$
$$\frac{\partial w}{\partial t} + (\mathbf{v}\cdot\nabla)w - \frac{u^2+v^2}{a} - u\cdot 2\Omega\cos\vartheta = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z$$

where the terms involving Ω are the Coriolis terms. The symbol f is often used to represent the Coriolis term that appears in both horizontal momentum equations ($2\Omega\sin\vartheta$). The equation of state and the continuity, thermodynamic, and constituent equations are not modified by the effects of rotation.

Small quantities, big quantities: Scale Analysis

Cancelling out the smallest terms, we end up with simpler equations and some important approximations, for instance hydrostatic in the vertical momentum equation.

Table 3.1: Characteristic scales for the variables in the horizontal momentum equations for (**A**) a synoptic-scale disturbance in the mid-latitude atmosphere and (**O**) the wind-driven circulation in the mid-latitude ocean.

	U	W	L	H	$^{\delta P_{xy}}/_{\rho}$	$^{\delta P_z}/_{\rho}$	$f0$	T=L/U
A	$10\,\mathrm{m\,s^{-1}}$	$10^{-2}\,\mathrm{m\,s^{-1}}$	$10^{6}\,\mathrm{m}$	$10^{4}\,\mathrm{m}$	$10^{3}\,\mathrm{m^2\,s^{-2}}$	$10^{5}\,\mathrm{m^2\,s^{-2}}$	$10^{-4}\,\mathrm{s^{-1}}$	$10^{5}\,\mathrm{s}$
O	$10^{-1}\,\mathrm{m\,s^{-1}}$	$10^{-5}\,\mathrm{m\,s^{-1}}$	$10^{5}\,\mathrm{m}$	$10\,\mathrm{m}$	$1\,\mathrm{m^2\,s^{-2}}$	$10^{2}\,\mathrm{m^2\,s^{-2}}$	$10^{-4}\,\mathrm{s^{-1}}$	$10^{6}\,\mathrm{s}$

Table 3.2: Scale analysis of the horizontal momentum equations using the characteristic scales listed in Table 3.1.

x-Eq.	$\frac{\partial u}{\partial t}$	$+(\mathbf{v}\cdot\nabla)u$	$-v2\Omega\sin\vartheta$	$-\frac{uv\tan\vartheta}{a}$	$+\frac{uw}{a}$	$+w2\Omega\cos\vartheta$	$=$	$-\frac{1}{\rho}\partial p\partial x$
y-Eq.	$\frac{\partial v}{\partial t}$	$+(\mathbf{v}\cdot\nabla)v$	$+u2\Omega\sin\vartheta$	$+\frac{u^2\tan\vartheta}{a}$	$+\frac{vw}{a}$		$=$	$-\frac{1}{\rho}\partial p\partial y$
Scales	U/T	U^2/L	f_0U	U^2/a	UW$/a$	f_0W		$^{\delta P_{xy}}/_{\rho}$/L)
A $(\mathrm{m\,s^{-2}})$	10^{-4}	10^{-4}	10^{-3}	10^{-5}	10^{-8}	10^{-6}		10^{-3}
O $(\mathrm{m\,s^{-2}})$	10^{-7}	10^{-7}	10^{-5}	10^{-9}	10^{-12}	10^{-8}		10^{-5}

Table 3.3: Scale analysis of the vertical momentum equation using the characteristic scales listed in Table 3.1.

z-Eq.	$\frac{\partial w}{\partial t}$	$+(\mathbf{v}\cdot\nabla)w$	$-\frac{u^2+v^2}{a}$	$-u2\Omega\cos\vartheta$	$=$	$-\frac{1}{\rho}\partial p\partial x$	$-g$
Scales	W/T	UW/L	U^2/a	f_0U		$^{\delta P_z}/_{\rho}$/H	g
A $(\mathrm{m\,s^{-2}})$	10^{-7}	10^{-7}	10^{-5}	10^{-3}		10	10
O $(\mathrm{m\,s^{-2}})$	10^{-9}	10^{-9}	10^{-9}	10^{-5}		10	10

Simplified PEs after scale analysis

$$
\begin{aligned}
\frac{\partial u}{\partial t} + (\mathbf{v}\cdot\nabla)u - \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)v &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x \\
\frac{\partial v}{\partial t} + (\mathbf{v}\cdot\nabla)v + \left(2\Omega\sin\vartheta + \frac{u\tan\vartheta}{a}\right)u &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g
\end{aligned}
$$

The primitive equations are often used in models to simulate large-scale motion in the atmosphere and ocean. In the 1960s, when computers were much less powerful, these equations were the most complex form of the fluid dynamical equations that could be solved numerically by computers in a reasonable amount of time. The formulation of the primitive equations (i.e., the terms that are included) is fundamentally dependent on scale analysis.

Simplified PEs after scale analysis, now back on a PLANE

f-plane: Coriolis term is a constant f_0 over the entire domain
β-plane: Coriolis term changes linearly in the meridional direction

$$\frac{\partial u}{\partial t} + (\mathbf{v} \cdot \nabla)\, u - f_0 v = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + F_x \qquad (14)$$

$$\frac{\partial v}{\partial t} + (\mathbf{v} \cdot \nabla)\, v + f_0 u = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + F_y \qquad (15)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z \qquad (16)$$

and the continuity equation also becomes much simpler, as metric terms drop out. The simpler geometry of the plane may be retained while taking the latitudinal varations of f into account, by defining a β-plane, for which f is calculated by linearising the Coriolis terms around a constant $f_0 = 2\Omega \sin\phi_0$ - the component of the Earth's rotation normal to the Earth's surface at reference latitude ϕ_0.

Shallow water equations (I)

Making the planar, Boussinesq and hydrostatic approximations, and neglecting those unresolved forces, the PEs reduce to:

$$\frac{Du}{Dt} - f_0 v = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} \tag{17}$$

$$\frac{Dv}{Dt} + f_0 u = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} \tag{18}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{19}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20}$$

plus thermodynamic eqn and eqn of state. Remember that $f_0 = 2\Omega \sin\phi_0$ is the Coriolis parameter normal to the Earth's surface at reference latitude ϕ_0.

Pressure and fluid depth are next divided into a reference state and perturbation:

$$p = p_r(z) + p'(x,y,z,t) \quad ; \quad h = H + \eta(x,y,z,t)$$

Shallow water equations (II)

The shallow water equations are obtained by integrating the PEs over a fluid layer of depth h. Integrating hydrostatic balance downwards from the top to any level z:

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial z} = -g \Rightarrow p_{top} - p(x,y,z,t) &= -\rho_o g(h-z) \\ \Rightarrow -\frac{1}{\rho_o}\frac{\partial p}{\partial x} &= -g\frac{\partial h}{\partial x} \end{aligned}$$

assuming that there are no pressure perturbations on the top. The pressure gradient terms become $-g\frac{\partial \eta}{\partial x}$ and $-g\frac{\partial \eta}{\partial y}$, since the reference depth H must be constant. The mass conservation equation integrated vertically is:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} h + [w]_{bot}^{top} &= 0 \\ \left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} h + \frac{Dh}{Dt} &= 0 \\ \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} &= 0 \end{aligned}$$

where (u,v) is now the depth-average velocity and $w_{bot} = 0$ was assumed.

SWEs linearised around a basic state

Shallow-water equations on the plane tangent to the Earth's surface, *linearised* about a resting basic state
$(u = u_r(z) + u'(x, y, z, t); v = v_r(z) + v'(x, y, z, t); h = H + \eta(x, y, z, t))$:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{21}$$

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial \eta}{\partial x}, \tag{22}$$

$$\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial \eta}{\partial y}, \tag{23}$$

where η is the surface elevation, H is fluid depth, (u, v) is the depth-averaged horizontal velocity, f is the Coriolis parameter and g is the gravitational acceleration.

Now discretise the problem using regular grids to describe u, v and η
e.g., η_{ij} = free surface elevation at the ith longitude and jth latitude point.

Horizontal gradients in SWEs: how to discretize in space?

SWEs seem nice and simple, but now we have spatial gradients: $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial \eta}{\partial x}$, $\frac{\partial \eta}{\partial y}$.

How are we going to discretise the problem, using regular grids, to obtain a good simulation of u, v and η? Something like this?

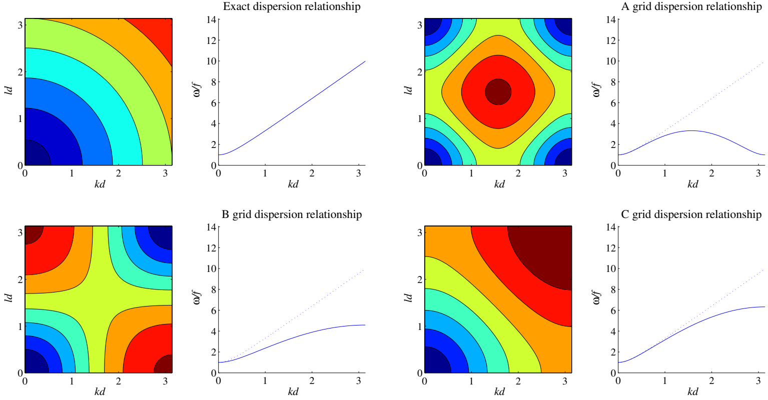
$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{ij} &\approx \left(\frac{u_{i+1,j}+u_{i+1,j-1}}{2}-\frac{u_{i-1,j}+u_{i-1,j-1}}{2}\right)\frac{1}{2\Delta x}\\
\left.\frac{\partial v}{\partial y}\right|_{ij} &\approx \left(\frac{v_{i,j+1}+v_{i-1,j+1}}{2}-\frac{v_{i,j-1}+v_{i-1,j-1}}{2}\right)\frac{1}{2\Delta y}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial \eta}{\partial x}\right|_{ij} &\approx \left(\frac{\eta_{i+1,j}+\eta_{i+1,j+1}}{2}-\frac{\eta_{i-1,j}+\eta_{i-1,j+1}}{2}\right)\frac{1}{2\Delta x}\\
fv\big|_{ij} &\approx f_j\,\frac{v_{i,j}+v_{i,j+1}+v_{i-1,j}+v_{i-1,j+1}}{4}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial \eta}{\partial y}\right|_{ij} &\approx \left(\frac{\eta_{i,j+1}+\eta_{i+1,j+1}}{2}-\frac{\eta_{i,j-1}+\eta_{i+1,j-1}}{2}\right)\frac{1}{2\Delta y}\\
fu\big|_{ij} &\approx f_j\,\frac{u_{i,j}+u_{i+1,j}+u_{i,j-1}+u_{i+1,j-1}}{4}
\end{aligned}
$$

But why? It all seems so arbitrary. There are a few criteria: we want to avoid computational modes and we want waves to propagate/evolve in a realistic way in space and time.

A preview of Lecture 5: graphical representation of inertia-gravity wave dispersion ($R_D/d = 10$), fine grid



Shading shows ω/f_0 ranging from blue (low) to red (high) (contour interval varies between panels). The graphs show solution for $l = 0$. On a fine grid ($d \ll R_D$, high-resolution) the C-grid gives the best results.

Horizontally staggered grids

Need to choose finite difference methods to solve the **shallow-water equations** on the plane tangent to the Earth's surface, *linearised* about a resting basic state:

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{24}$$

$$\frac{\partial u}{\partial t} - f_0 v = -g\frac{\partial \eta}{\partial x}, \tag{25}$$

$$\frac{\partial v}{\partial t} + f_0 u = -g\frac{\partial \eta}{\partial y}, \tag{26}$$

where η is the surface elevation, H is fluid depth, (u, v) is the depth-averaged horizontal velocity, f is the Coriolis parameter and g is the gravitational acceleration.

Now discretise the problem using regular grids to describe u, v and η e.g., η_{ij} = free surface elevation at the ith longitude and jth latitude point.

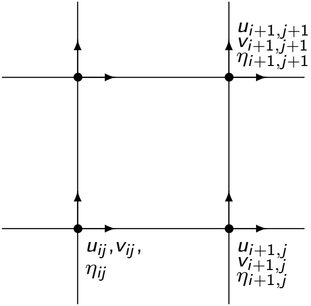
Arakawa[4] proposed a number of staggered finite difference grids that can be used to solve the shallow water equations.

[4] Arakawa A. (1966) Computational design for long-term numerical integration of the equations of fluid motion: Two-dimensional incompressible flow. Part I. *Journal of Computational Physics*, 1, 119-143.

Arakawa's A grid

Simplest and intuitive, but there is the danger of computational modes.

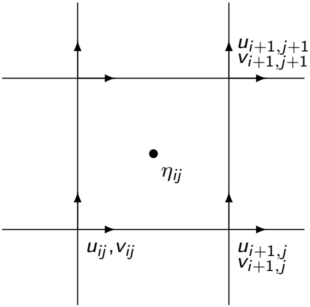
$$
\begin{aligned}
&\eta - grid \qquad & \left.\frac{\partial u}{\partial x}\right|_{ij} &\approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \\
& & \left.\frac{\partial v}{\partial y}\right|_{ij} &\approx \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} \\
&u - grid \qquad & \left.\frac{\partial \eta}{\partial x}\right|_{ij} &\approx \frac{\eta_{i+1,j} - \eta_{i-1,j}}{2\Delta x} \\
& & \left. fv\right|_{ij} &\approx fv_{i,j} \\
&v - grid \qquad & \left.\frac{\partial \eta}{\partial y}\right|_{ij} &\approx \frac{\eta_{i,j+1} - \eta_{i,j-1}}{2\Delta y} \\
& & \left. fu\right|_{ij} &\approx fu_{i,j}
\end{aligned}
$$



Arakawa's B grid

Stagger the variables, eliminate computational mode, but lots of interpolation...

$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{ij} &\approx \left(\frac{u_{i+1,j}+u_{i+1,j+1}}{2}-\frac{u_{i,j}+u_{i,j+1}}{2}\right)\frac{1}{\Delta x}\\
\left.\frac{\partial v}{\partial y}\right|_{ij} &\approx \left(\frac{v_{i,j+1}+v_{i+1,j+1}}{2}-\frac{v_{i,j}+v_{i+1,j}}{2}\right)\frac{1}{\Delta y}\\
\left.\frac{\partial \eta}{\partial x}\right|_{ij} &\approx \left(\frac{\eta_{i,j-1}+\eta_{i,j}}{2}-\frac{\eta_{i-1,j-1}+\eta_{i-1,j}}{2}\right)\frac{1}{\Delta x}\\
\left. fv\right|_{ij} &\approx fv_{i,j}\\
\left.\frac{\partial \eta}{\partial y}\right|_{ij} &\approx \left(\frac{\eta_{i-1,j}+\eta_{i,j}}{2}-\frac{\eta_{i-1,j-1}+\eta_{i,j-1}}{2}\right)\frac{1}{\Delta y}\\
\left. fu\right|_{ij} &\approx fu_{i,j}
\end{aligned}
$$



Arakawa's C grid

Stagger the variables, eliminate computational mode, interpolation only required for Coriolis terms!

$$\eta - grid \qquad \left.\frac{\partial u}{\partial x}\right|_{ij} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x}$$

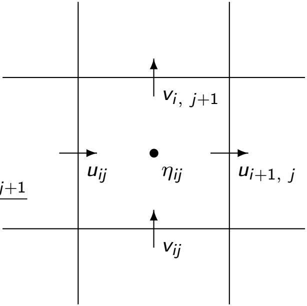
$$\left.\frac{\partial v}{\partial y}\right|_{ij} \approx \frac{v_{i,j+1} - v_{i,j}}{\Delta y}$$

$$u - grid \qquad \left.\frac{\partial \eta}{\partial x}\right|_{ij} \approx \frac{\eta_{i,j} - \eta_{i-1,j}}{\Delta x}$$

$$\left.fv\right|_{ij} \approx f_j \frac{v_{i,j} + v_{i,j+1} + v_{i-1,j} + v_{i-1,j+1}}{4}$$

$$v - grid \qquad \left.\frac{\partial \eta}{\partial y}\right|_{ij} \approx \frac{\eta_{i,j} - \eta_{i,j-1}}{\Delta y}$$

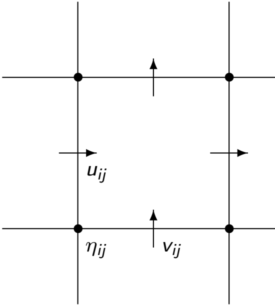
$$\left.fu\right|_{ij} \approx f_j \frac{u_{i,j} + u_{i+1,j} + u_{i,j-1} + u_{i+1,j-1}}{4}$$



Arakawa's D grid

Idea similar to B grid, but we push out u, v instead of η: stagger the variables, eliminate computational mode, but lots of interpolation...

$$
\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{ij} &\approx \left(\frac{u_{i+1,j}+u_{i+1,j-1}}{2} - \frac{u_{i-1,j}+u_{i-1,j-1}}{2}\right)\frac{1}{2\Delta x} \\
\left.\frac{\partial v}{\partial y}\right|_{ij} &\approx \left(\frac{v_{i,j+1}+v_{i-1,j+1}}{2} - \frac{v_{i,j-1}+v_{i-1,j-1}}{2}\right)\frac{1}{2\Delta y} \\
\left.\frac{\partial \eta}{\partial x}\right|_{ij} &\approx \left(\frac{\eta_{i+1,j}+\eta_{i+1,j+1}}{2} - \frac{\eta_{i-1,j}+\eta_{i-1,j+1}}{2}\right)\frac{1}{2\Delta x} \\
\left.fv\right|_{ij} &\approx f_j\frac{v_{i,j}+v_{i,j+1}+v_{i-1,j}+v_{i-1,j+1}}{4} \\
\left.\frac{\partial \eta}{\partial y}\right|_{ij} &\approx \left(\frac{\eta_{i,j+1}+\eta_{i+1,j+1}}{2} - \frac{\eta_{i,j-1}+\eta_{i+1,j-1}}{2}\right)\frac{1}{2\Delta y} \\
\left.fu\right|_{ij} &\approx f_j\frac{u_{i,j}+u_{i+1,j}+u_{i,j-1}+u_{i+1,j-1}}{4}
\end{aligned}
$$



Arakawa's Smörgåsbord

In their discussion of various numerical representations of (28) - (30), AL defined five

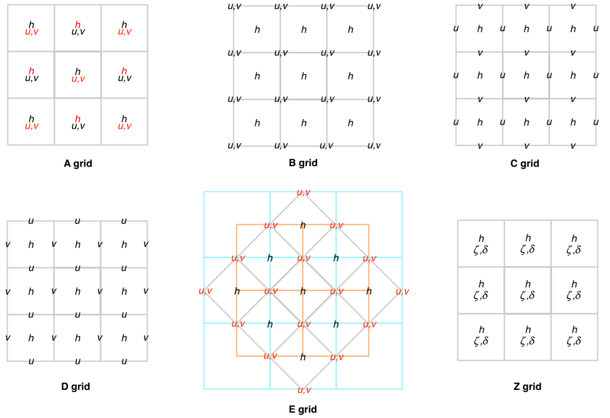


Fig. 8.2: Grids A-E and Z, on a square mesh. The E grid can be obtained by rotating the B grid by 45°. In the sketch of the E grid, the mass variables can be considered to live in the rotated grey boxes, or they can be considered to live in the overlapping blue and orange boxes.

Figure: Many of Arakawa's grid staggers

Excursus: 1D SWEs help to understand grid stagger

Advection

In the good old times when life was easy:

$$\frac{\partial q}{\partial t} = -c\frac{\partial q}{\partial x} \rightarrow q_j^{n+1} = q_j^n - c\frac{\Delta t}{\Delta x}\left(q_{j+1}^n - q_j^n\right)$$

Let us make it a bit more interesting:

$$\frac{\partial q}{\partial t} = -U\frac{\partial q}{\partial x} \rightarrow q_j^{n+1} = q_j^n - U_j^n\frac{\Delta t}{\Delta x}\left(q_{j+1}^n - q_j^n\right)$$

Need something more:

$$\frac{\partial U}{\partial t} = ???$$

Excursus: resolving waves on a grid

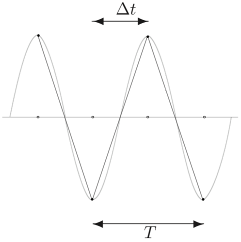


Figure 1-16 Shortest wave (at *cut-off frequency* $\pi/\Delta t$ or period $2\Delta t$) resolved by uniform grid in time.

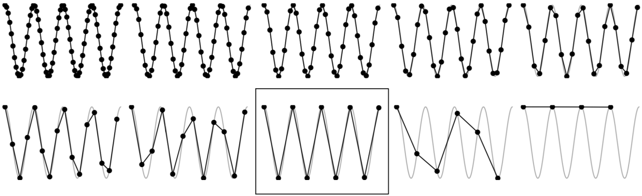


Figure 1-17 Aliasing illustrated by sampling a given signal (gray sinusoidal curve) with an increasing time interval. A high sampling rate (top row of images) resolves the signal properly. The boxed image on the bottom row corresponds to the cut-off frequency, and the sampled signal appears as a seesaw. The last two images correspond to excessively long time intervals that alias the signal, making it appear as if it had a longer period than it actually has.

Excursus: 1D SWEs help to understand grid stagger

Shallow Water Equations

$$\frac{\partial h}{\partial t} + H\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + g\frac{\partial h}{\partial x} = 0$$

Let: $c^2 \equiv gH$ and substitute

$$\rightarrow \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \text{ and } \frac{\partial^2 h}{\partial t^2} = c^2\frac{\partial^2 h}{\partial x^2}$$

both are examples of the wave equation.

Impose the usual solution: $e^{i(kx-\omega t)}$

obtain dispersion equation: $\omega^2 = c^2k^2$

and exact phase speed: $\frac{\omega}{k} = c = \pm\sqrt{gH}$

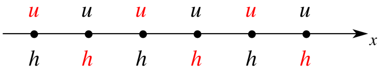
We anticipate two waves: one travelling east and one travelling west.

Excursus: 1D SWEs help to understand grid stagger

Shallow Water Equations: stagger

$$\frac{\partial h_j}{\partial t} + \frac{H}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = 0$$

$$\frac{\partial u_j}{\partial t} + \frac{g}{2\Delta x}\left(h_{j+1}^n - h_{j-1}^n\right) = 0$$



$$h_j^{n+1} = h_j^n - H\frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$$u_j^{n+1} = u_j^n - g\frac{\Delta t}{2\Delta x}\left(h_{j+1}^n - h_{j-1}^n\right)$$

Therefore we anticipate a problematic computational mode

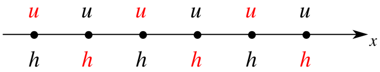
Excursus: 1D SWEs help to understand grid stagger

Shallow Water Equations: computational modes

$$h_j^{n+1} = h_j^n - H\frac{\Delta t}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right)$$

$$u_j^{n+1} = u_j^n - g\frac{\Delta t}{2\Delta x}\left(h_{j+1}^n - h_{j-1}^n\right)$$

Therefore we anticipate a problematic computational mode



If we impose a wave solution:

$$\left(u_j, h_j\right) \sim e^{i\left(kj\Delta x - \omega t\right)}$$

after substitution: $\omega^2 = gH\left(\frac{\sin k\Delta x}{\Delta x}\right)^2$

The dispersion relation is now dependent on wave number!

It **discriminates** ????? waves

Exercise: analytical versus numerical solutions

Start from the two shallow water equations on the previous page. Impose the wave solution proposed on the slide, for u_j and h_j:

$$(u_j, h_j) \sim e^{i(kx_j - \omega t)} = e^{i(kj\Delta x - \omega t)} \tag{27}$$

.

1. Substitute into the centred numerical expressions (in time and space) and show that you end up with an expression that contains a sin function.
2. How does the numerical solution compare to the analytical solution in terms of a) amplitude and b) phase?
3. Plot a wavenumber versus frequency plot for the analytical and numerical solution
4. Expand your thinking to Numerical Weather Prediction models that might be using such schemes: discuss what operational problems we may encounter in the presence of a sin in such numerical predictions.