BDA HM5

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9.1

The shape and rate parameters of a Gamma distribution with mean μ and standard deviation σ are:

$$s = \frac{\mu^2}{\sigma^2} \qquad r = \frac{\mu}{\sigma^2} \tag{1}$$

Given mode ω and standard deviation σ, shape and rate are instead given by:

$$s = 1 + \omega r \qquad r = \frac{\omega + \sqrt{\omega^2 + 4\sigma^2}}{2\sigma^2} \tag{2}$$

For $\mu = 1$ and $\sigma = 10$ we obtain $s = 0.01$ and $r = 0.01$. For $\omega = 1$ and $\sigma = 10$ we obtain $s = 1.105125$ and $r = 0.105125$. As we can see from Fig. 1, when we set the mean to 1 the distribution is highly peaked at small values. Also, comparing the choice $\omega = 1$, values above 75 are favored (i.e., they have higher density). Fig. 2 shows that the choice of the prior on κ also affects the posterior distributions for κ and ω. In particular, setting the mean to 1 leads to a bigger large-values tail in the posterior for κ. In turn, such large κ values affect the estimate of the posterior distribution for individual parameters. Actually, high concentration leads to shrinkage in the θ_s values, see Fig. 3.

9.2

When setting a mode equal to 1 for the prior distribution of κ, the prior distributions for θ_s have rounded shoulders. This is because $\kappa < 1$ values are very unlikely in this case, while they were likely when choosing mean equals to 1. Concentration values $\kappa < 1$ correspond to a u-shaped beta distribution, while values $\kappa > 1$ give a bell shape. If $\kappa < 1$ values have low weight, the bell shape will be dominant and result in the priors of Fig. 4. The choice of mean equals 1 is the least informative one, and could hence be preferable over mode equals 1.

9.3

Fig. 5 shows the results for a Bayesian analysis of the data of Figure 9.12 in the book. Estimates of θ_s are similar to those found by MLE. However, a Bayesian analysis gives us a more complete description of the problem, allowing to obtain full posterior distributions both for hyperparameters κ and ω, informing us about group-level abilities, and for parameters θ_s, describing the abilities of single subjects. Also, We are capable of performing comparisons between subsjects, as well as to average any kind of observable according to the found posterior.

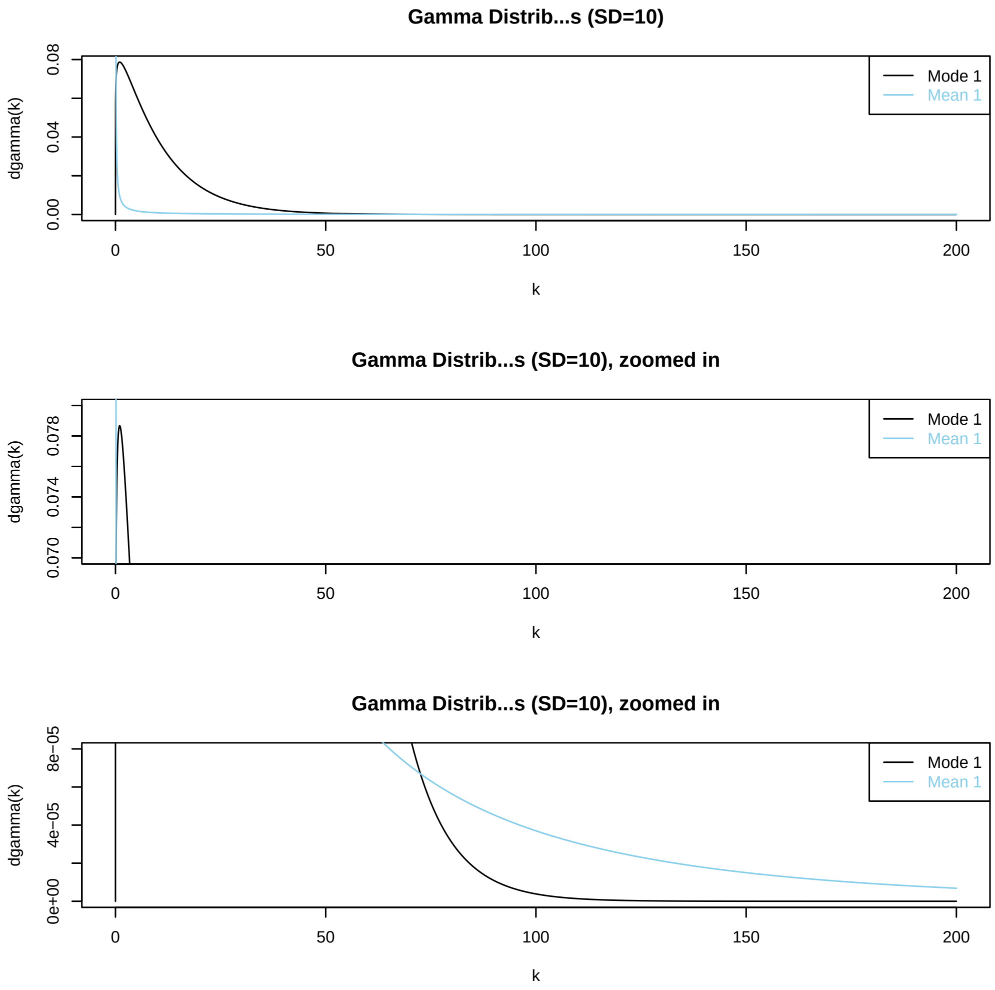
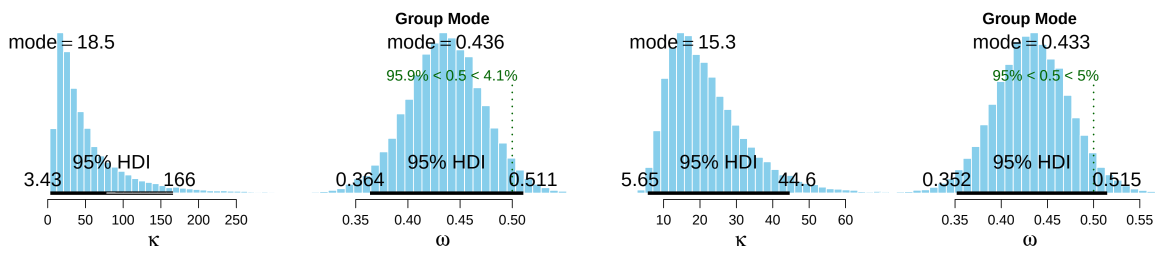
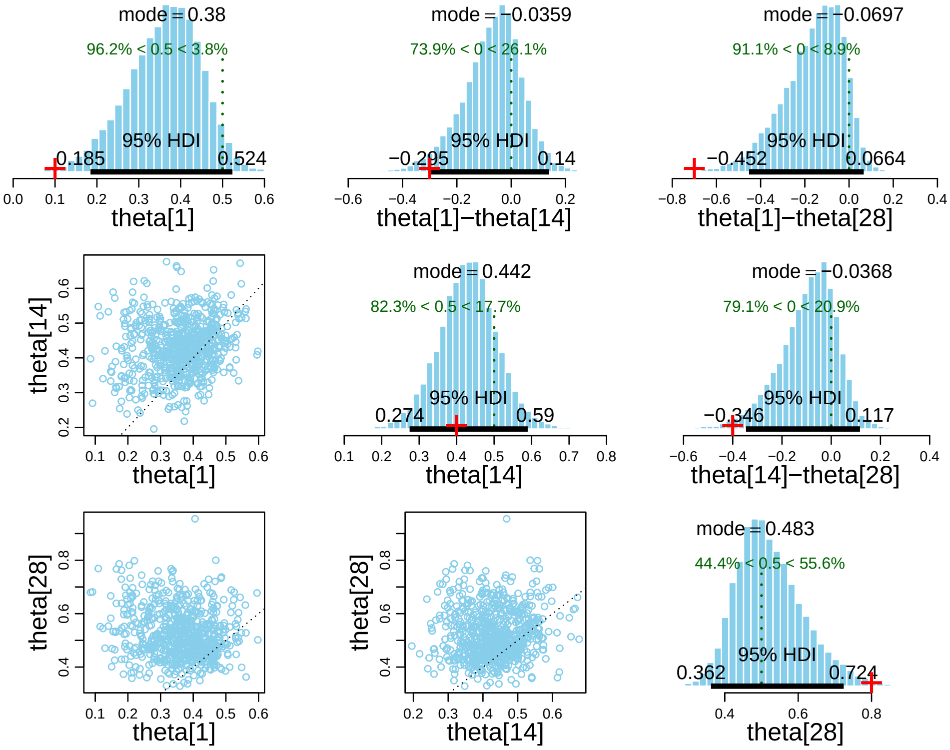


Figure 1: Gamma distribution for the two different parameter choices considered in Exercise 9.

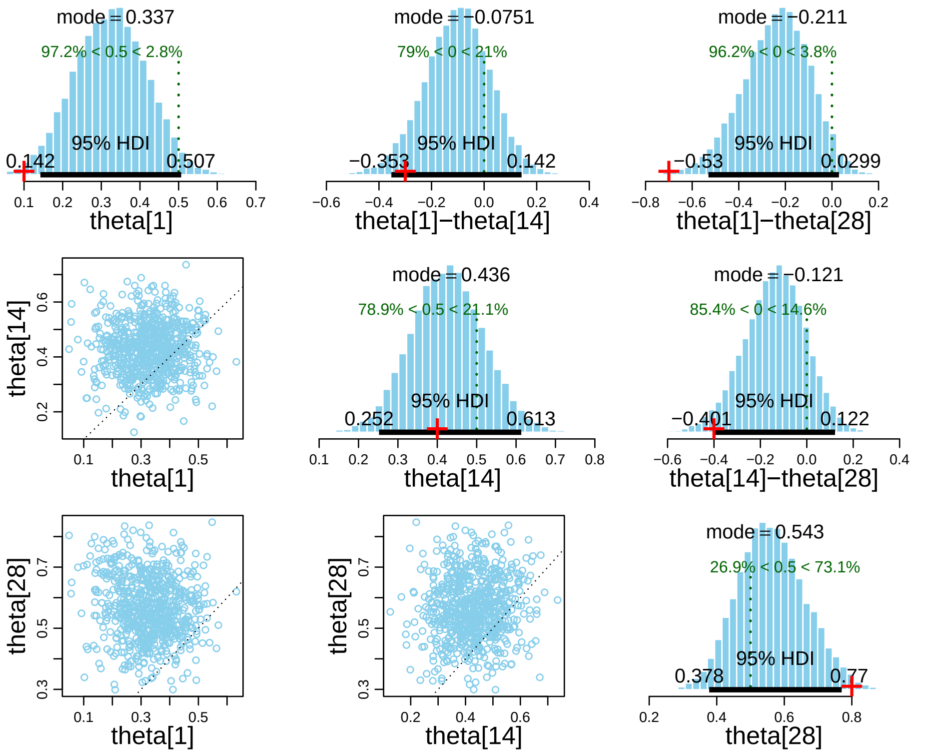


(a) (b)

Figure 2: Posterior distributions for group-level concentration and mode, given a prior on κ with mean 1 (a) and mode 1 (b).

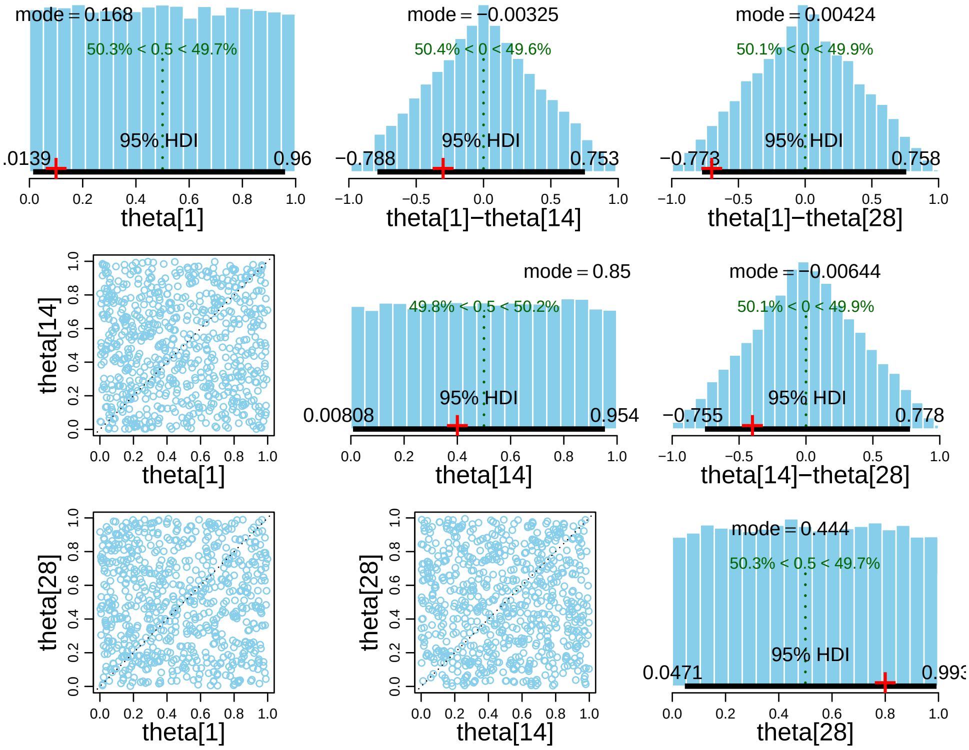


(a)

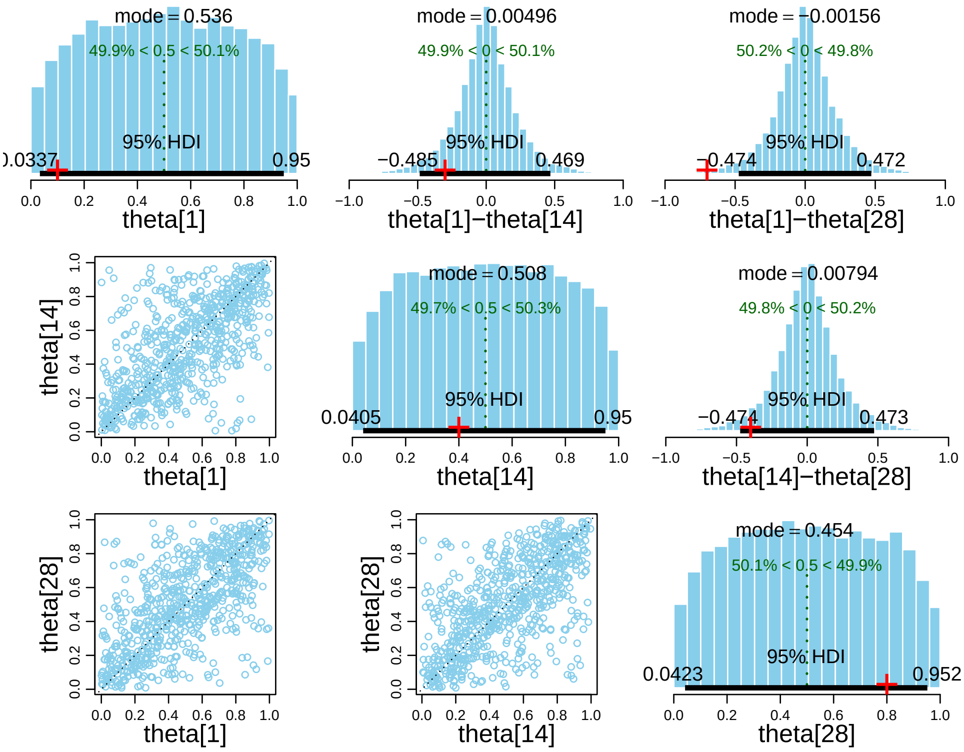


(b)

Figure 3: Posterior distributions for θ_s, given a prior on κ with mean 1 (a) and mode 1 (b).

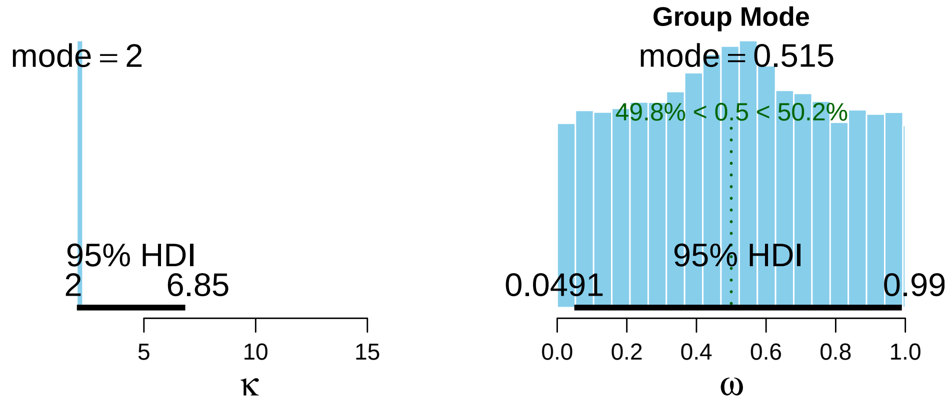


(a)

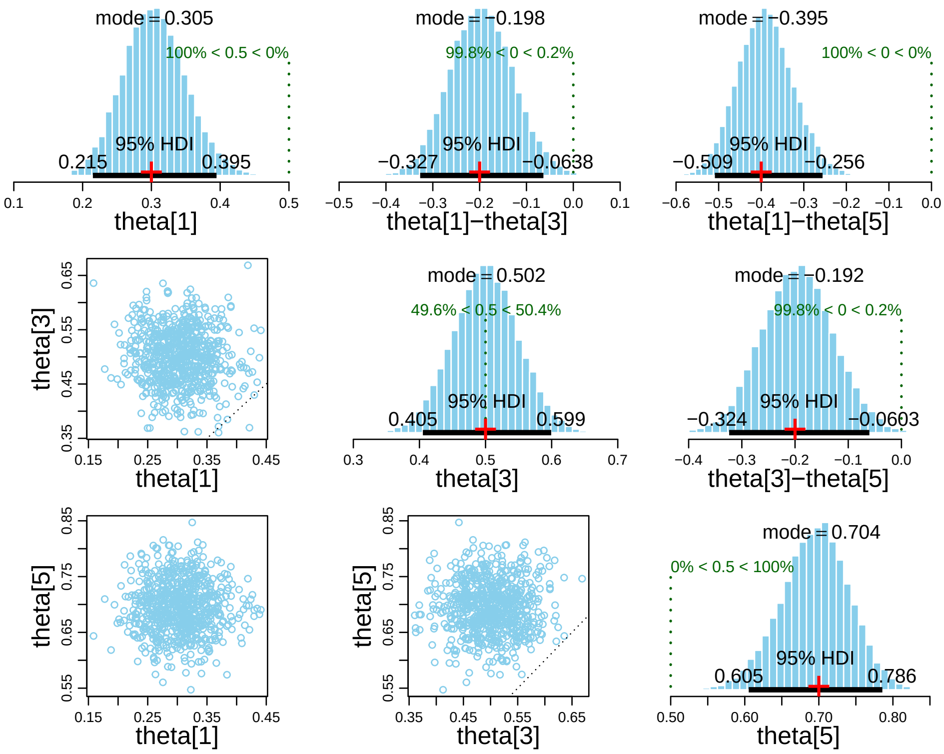


(b)

Figure 4: Prior distributions for θ_s, given a prior on κ with mean 1 (a) and mode 1 (b).



(a)



(b)

Figure 5: Exercise 9.3: Posterior distributions for ω, κ, θ_s, given a prior on κ with mean 1.