

# On few-state (VB-*n*CT) theory

Pierre Beaujean

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## 1 Introduction

In nonlinear optics (NLO), promising molecular architectures often belong to the push-pull family. These structures typically consist of electron donor and acceptor groups connected by a  $\pi$ -conjugated segment that facilitates electronic “communication” between the two moieties.

To elucidate the relationships between NLO properties and structure, simplified models have been developed to capture complex phenomena with minimal parameters. The valence-bond charge-transfer (VB-CT) model, pioneered by Mulliken [1], has been particularly successful in describing the first hyperpolarizabilities of push-pull polyenes. It also applies to other properties, such as UV/VIS absorption spectra and various physico-chemical characteristics [2]. This model represents a system as a superposition of two limiting states, a valence-bond state,  $\phi_{VB}$ , and a charge-transfer state,  $\phi_{CT}$ , connected by electron displacement from the donor to the acceptor (see Fig. 1). Using fundamental parameters, such as the ionization energy of the donor, electron affinity of the acceptor, and the coupling between these states, the VB-CT model provides valuable insight into structure-activity relationships. Indeed, when combined with the sum-over-states (SOS) theory developed by Orr and Ward [3], these models provide estimate of NLO properties and identification of valuable structure-activity relationships.

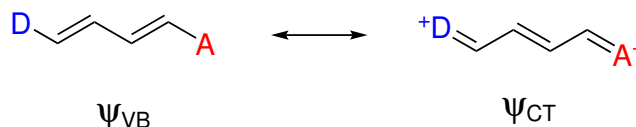


Figure 1: Limiting forms of the VB-CT model.

Despite its simplicity, this model has proven versatile, allowing extensions to systems with multiple donor/acceptor groups through 3-state [4, 5, 6], 4-state [7], and 5-state [8] descriptions. The goal of this document is to quickly review these models few-state models and to introduce a few useful relationships.

## 2 Theory

### 2.1 A model Hamiltonian

Given a set of basis functions  $\{\phi_i\}$ , the energy of a trial function  $\Psi = \sum_i c_i \phi_i$  is always greater than or equal to the true ground state energy,  $\varepsilon_0$ :

$$\varepsilon_0 \leq \varepsilon, \text{ where } \varepsilon = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle},$$

One minimizes the energy of the trial function by setting  $\frac{d\varepsilon}{dc_i} = 0$ . This yields a set of so-called secular equations of the form:

$$\forall k : \sum_i c_i (H_{ki} - \varepsilon_k S_{ki}) = 0, \quad (1)$$

where  $S_{ki} = \langle \phi_k | \phi_i \rangle$  (overlap matrix) and  $H_{ki} = \langle \phi_k | \hat{H} | \phi_i \rangle$  (Hamiltonian matrix). This is a generalized eigenvalue problem. If the basis functions are orthogonal, i.e.,  $S_{ki} = \delta_{k,i}$ , this reduces to a standard eigenvalue problem,  $HC = C\varepsilon$ , where diagonalizing  $H$  gives the energy levels,  $\varepsilon$ .  $C$  is the matrix of coefficients, also referred to as the eigenfunctions of the system.

In the VB- $n$ CT framework, corresponding to a  $(n+1)$ -state model, a set of orthogonal functions is employed consisting of one valence-bond (VB) state,  $\phi_{VB}$ , and  $n$  charge-transfer (CT) states,  $\{\phi_{CT,i} | 0 < i \leq n\}$ . This leads to a model Hamiltonian, given by a  $(n+1) \times (n+1)$  matrix and parameterized as follows:

$$\langle \phi_{VB} | \hat{H} | \phi_{VB} \rangle = E_{VB}, \langle \phi_{VB} | \hat{H} | \phi_{CT,i} \rangle = t, \text{ and } \langle \phi_{CT,i} | \hat{H} | \phi_{CT,j} \rangle = \begin{cases} E_{CT} & \text{if } i = j, \\ T & \text{otherwise,} \end{cases} \quad (2)$$

where  $E_{VB}$  and  $E_{CT}$  denote the energies of the VB and CT states, respectively, while  $t$  and  $T$  are the transfer integrals.

Solving the eigenvalue problem yields a ground state,  $|0\rangle$ , with minimal energy, a set of  $(n-1)$ -fold degenerate excited states,  $\{|e_i\rangle | 0 < i < n\}$  (which does not exist when  $n = 1$ ), and a single non-degenerate excited state,  $|f\rangle$ . Corresponding eigenvalues were provided by Cho *et al.* [8] and are, in all generality, given by:

$$\begin{aligned} E_0 &= \frac{1}{2} \left[ E_{VB} + E_{CT} - (n-1)T - \sqrt{(V - (n-1)T)^2 + 4nt^2} \right], \\ E_{e_i} &= E_{CT} + T, \\ E_f &= \frac{1}{2} \left[ E_{VB} + E_{CT} - (n-1)T + \sqrt{(V - (n-1)T)^2 + 4nt^2} \right], \end{aligned} \quad (3)$$

where  $V = E_{CT} - E_{VB}$ . The associated eigenfunctions are:

$$|0\rangle = \cos \delta |\phi_{VB}\rangle + \frac{\sin \delta}{\sqrt{n}} \sum_{0 < j \leq n} |\phi_{CT,j}\rangle,$$

$$\begin{aligned}
|e_i\rangle &= \frac{1}{\sqrt{i(i+1)}} \left[ i |\phi_{CT,i+1}\rangle - \sum_{0 < j \leq i} |\phi_{CT,j}\rangle \right], \\
|f\rangle &= \sin \delta |\phi_{VB}\rangle - \frac{\cos \delta}{\sqrt{n}} \sum_{0 < j \leq n} |\phi_{CT,j}\rangle,
\end{aligned} \tag{4}$$

where the mixing angle  $\delta \in [0, \pi/2]$  is introduced. For  $0 \leq \delta < \pi/4$ , the ground state is VB-dominated, whereas for  $\pi/4 < \delta \leq \pi/2$ , the CT state dominates. The point  $\delta = \pi/4$  is referred to as the *cyanine limit*.

Minimizing the ground-state energy  $E_g$  with respect to  $\delta$  provides a condition linking the parameters:

$$\frac{\partial E_g(\delta)}{\partial \delta} = 0 \Rightarrow V - (n-1)T = 2t\sqrt{n} \cot(2\delta). \tag{5}$$

Setting the energy origin to:

$$E_{VB} + E_{CT} - (n-1)T = 0, \tag{6}$$

simplifies the eigenvalues to:

$$E_0 = -\sqrt{n} \frac{t}{\sin(2\delta)}, E_{e_i} = nT + t\sqrt{n} \cot(2\delta), \text{ and } E_f = \frac{t}{\sin(2\delta)} \sqrt{n}. \tag{7}$$

To monitor variations in molecular properties, the charge-transfer (CT) character of the ground state,  $\ell_{CT}$ , can be introduced as follows [8, 7, 6]:

$$\ell_{CT} = \frac{1}{n} \sin^2 \delta. \tag{8}$$

With  $\delta \in [0, \pi/2]$ ,  $\ell_{CT}$  varies in the range  $\ell_{CT} \in [0, 1/n]$ . A value of  $\ell_{CT} = 0$  indicates a ground state fully described by the VB form, while  $\ell_{CT} = 1/n$  corresponds to a state entirely characterized by CT. An additional parameter, introduced by Barzoukas and collaborators [9, 10, 11], measures the VB-CT mixing:

$$m_{CT} = -\cos(2\delta) = 2n\ell_{CT} - 1. \tag{9}$$

This parameter,  $m_{CT}$ , ranges from  $-1$  (VB-dominated) to  $+1$  (CT-dominated), effectively capturing the balance between VB and CT character in the ground state.

## 2.2 A set of dipole moments

To utilize the SOS expression, both excitation energies (calculated, in the VB- $n$ CT model, as differences between eigenvalues, given in Eq. (7)) and transition dipoles are required. In general, transition dipoles are computed as follows:

$$\vec{\mu}_{ij} = \langle i | \hat{\mu} | j \rangle = \sum_{\nu\kappa} (C_{i\nu})^\dagger C_{j\kappa} \langle \phi_\nu | \hat{\mu} | \phi_\kappa \rangle, \tag{10}$$

where  $|i\rangle = \sum_{\nu} C_{i\nu} |\phi_{\nu}\rangle$  represents the electronic states of the system,  $C_{i\nu}$  are the coefficients in the state expansion, and  $|\phi_{\nu}\rangle$  are the basis functions.

In the VB- $n$ CT model, the dipole matrix elements  $M_{\mu\nu} = \langle\phi_{\nu}|\hat{\mu}|\phi_{\kappa}\rangle$  are given as follows [12]:

$$\langle\phi_{VB}|\hat{\mu}|\phi_{VB}\rangle = \langle\phi_{VB}|\hat{\mu}|\phi_{CT,i}\rangle = 0, \langle\phi_{CT,i}|\hat{\mu}|\phi_{CT,j}\rangle = \begin{cases} \vec{\mu}_{CT,i} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

This results in a sparse, diagonal dipole matrix parameterized by a select set of CT dipole moments associated to each CT state,  $\{\vec{\mu}_{CT,i} \mid 0 < i \leq n\}$ .

### 2.3 Parameterization of a VB- $n$ CT model

As outlined in Sections 2.1 and 2.2, a VB- $n$ CT model is uniquely defined by a set of  $n$  CT dipole moments,  $\{\vec{\mu}_{CT,i} \mid 0 < i \leq n\}$ , and either of the following parameterizations:

1. Using Eq. (2): specified by  $E_{VB}$ ,  $E_{CT}$  (or equivalently  $V$ ),  $t$ , and  $T$ ;
2. Alternatively, by applying the conditions in Eqs. (5) and (6): specified by  $t$ ,  $T$ , and one of the mixing parameters defined in Section 2.1 (such as  $\delta$ ,  $\ell_{CT}$ , or  $m_{CT}$ ).

This allows a straightforward implementation of this model in Python, which outputs excitation energies and transition dipole moments.

## A Practical expressions for static $\beta$

Let's assume the parameterization from Section 2.3 (with  $m_{CT}$  as the mixing parameter), and a (non-divergent) SOS expression for the static  $\beta$  tensor elements [13]:

$$\beta_{(\zeta\eta\kappa)} = \hbar^{-2} \sum_{\mathcal{P}} \sum_{a'_1, a'_2} \frac{(\zeta\bar{\eta}\kappa)_{a_1 a_2}}{\omega_{a_1} \omega_{a_2}}, \quad (\text{A1})$$

where  $\zeta, \eta, \kappa$  are the Cartesian coordinates  $x, y, z$  (in the molecular frame, the parenthesis indicates that one can freely permute the indices),  $(\zeta\bar{\eta}\kappa)_{a_1 a_2}$  is the notation of Bishop for the product of transition dipole moments,  $\sum_{a'_1, a'_2}$  is a sum over the  $n$  excited states, and  $\sum_{\mathcal{P}}$  is the sum over the different permutations. On that ground, one can derive practical expressions for the components of  $\beta$  in different cases, presented below. For an in-depth derivation, see Chapter 11 of Ref. [14] and the references therein.

**2-state ( $C_{\infty v}$ ) model.** These systems consist of a donor-acceptor (D/A) pair typically located at the ends of a  $\pi$ -conjugated linker, oriented along the  $z$ -axis for convenience. Assuming that the CT dipole moment,  $\mu_{CT}$ , lies entirely along the  $z$ -axis, one obtains the following expression for the only non-null component,  $\beta_{zzz}$  [9, 10, 11]:

$$\beta_{zzz} = -\frac{3}{8} m_{CT} (1 - m_{CT}^2)^2 \frac{\mu_{CT}^3}{t^2}. \quad (\text{A2})$$

**3-state ( $C_{2v}$ ) model.** Assuming that the two CT axes form an angle  $\theta \in [0, \pi/2]$  from the  $z$  axis [6], as represented in Fig. 2, one has:

$$\vec{\mu}_{CT,1} = \mu_{CT} (\sin \theta, 0, \cos \theta) \text{ and } \vec{\mu}_{CT,2} = \mu_{CT} (-\sin \theta, 0, \cos \theta). \quad (\text{A3})$$

There are only two non-null independent components:  $\beta_{zzz}$  and  $\beta_{(zxx)}$ :

$$\begin{aligned} \beta_{zzz} &= -\frac{3}{16} m_{CT} (1 - m_{CT}^2)^2 \frac{\mu_{CT}^3}{t^2} \cos^3 \theta, \\ \beta_{(zxx)} &= \frac{1 - m_{CT}^2}{2} \mu_{CT}^3 \sin^2 \theta \cos \theta \\ &\quad \times \left\{ \frac{1}{\left(2T + t \left[2 \frac{1 - m_{CT}}{1 + m_{CT}}\right]^{1/2}\right)^2} + \frac{2}{2t \left[\frac{2}{1 - m_{CT}^2}\right]^{1/2} \left(2T + t \left[2 \frac{1 - m_{CT}}{1 + m_{CT}}\right]^{1/2}\right)} \right\}. \end{aligned} \quad (\text{A4})$$

Note that when  $\theta = 90^\circ$ , both  $\beta_{zzz}$  and  $\beta_{(zxx)}$  goes to zero, since it corresponds to a situation with a center of inversion ( $D_{\infty h}$ ).

**4-state ( $C_{3v}$ ) system.** Let's, this time, assume that the three dipole moments are at  $120^\circ$  from each other in the  $xy$  plane (and the first one point to the  $y$  direction), but that there is a

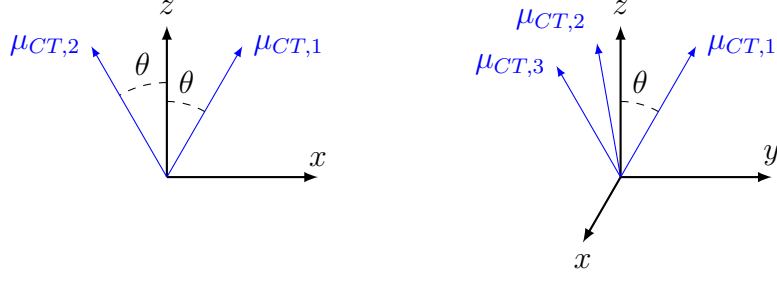


Figure 2: Representation of the charge-transfer dipoles ( $\mu_{CT}$ ) in the 3-state (left) and 4-state (right) models.

$\theta \in [0, \pi/2]$  angle between the dipoles and the  $z$  axis, as in Fig. 2. Thus,

$$\vec{\mu}_{CT,1} = \mu_{CT} (0, \sin \theta, \cos \theta), \vec{\mu}_{CT,2} = \vec{\mu}_{CT,3} = \frac{\mu_{CT}}{2} (\pm \sqrt{3} \sin \theta, -\sin \theta, 2 \cos \theta) \quad (\text{A5})$$

There are three non-null independent components:  $\beta_{zzz}$ ,  $\beta_{(zyy)}$ , and  $\beta_{yyy}$ . Their expressions are:

$$\begin{aligned} \beta_{zzz} &= -\frac{1}{8} m_{CT} (1 - m_{CT}^2)^2 \frac{\mu_{CT}^3}{t^2} \cos^3 \theta, \\ \beta_{(zyy)} &= \frac{1 - m_{CT}^2}{4} \mu_{CT}^3 \sin^2 \theta \cos \theta \\ &\quad \times \left\{ \frac{1}{\left( (3T + t \left[ 3 \frac{1 - m_{CT}}{1 + m_{CT}} \right]^{1/2})^2 \right)} + \frac{2}{2t \left[ \frac{3}{1 - m_{CT}^2} \right]^{1/2} \left( (3T + t \left[ 3 \frac{1 - m_{CT}}{1 + m_{CT}} \right]^{1/2})^2 \right)} \right\}, \\ \beta_{yyy} &= \frac{3}{4} (1 + m_{CT}) \frac{\mu_{CT}^3}{\left( (3T + t \left[ 3 \frac{1 - m_{CT}}{1 + m_{CT}} \right]^{1/2})^2 \right)} \sin^3 \theta. \end{aligned} \quad (\text{A6})$$

The relationship  $\beta_{yyy} = -\beta_{(yxx)}$  holds in this model,<sup>1</sup> as expected from  $C_{3v}$  symmetry. Note that when  $\theta = 90$ , the system correspond to a  $D_{3h}$  symmetry, with  $\beta_{zzz} = \beta_{(zyy)} = \beta_{(zxx)} = 0$ .

**5-states ( $T_d$ ) system.** For the sake of completeness, lets finally consider the  $T_d$  geometry, which was first reviewed by Cho *et al.* [8]. The dipole moments are (arbitrarily) given by

$$\begin{aligned} \vec{\mu}_{CT,1} &= \mu_{CT} \frac{\sqrt{3}}{3} (1, 1, 1), \vec{\mu}_{CT,2} = \mu_{CT} \frac{\sqrt{3}}{3} (1, -1, -1), \\ \vec{\mu}_{CT,3} &= \mu_{CT} \frac{\sqrt{3}}{3} (-1, -1, 1), \text{ and } \vec{\mu}_{CT,4} = \mu_{CT} \frac{\sqrt{3}}{3} (-1, 1, -1). \end{aligned} \quad (\text{A7})$$

<sup>1</sup>Symmetry tables generally reports  $\beta_{yyy} = -\beta_{(yxx)}$ , but it is just a matter of defining the  $\sigma_v$ . Also, this definition is consistent with  $D_{3h}$ .

In this case, there is only one non-null independent component:

$$\beta_{(xyz)} = (1 + m_{CT}) \frac{\sqrt{3}}{3} \frac{\mu_{CT}^3}{\left(4T + 2t \left[\frac{1-m_{CT}}{1+m_{CT}}\right]^{1/2}\right)^2}. \quad (\text{A8})$$

## References

- [1] Robert S. Mulliken. Molecular Compounds and their Spectra. II. *J. Am. Chem. Soc.*, 74(3):811–824, February 1952.
- [2] Christopher J. Bender. Theoretical models of charge-transfer complexes. *Chem. Soc. Rev.*, 15(4):475, 1986.
- [3] B.J. Orr and J.F. Ward. Perturbation theory of the non-linear optical polarization of an isolated system. *Mol. Phys.*, 20:513–526, 1971.
- [4] Sangjoon Hahn, Dongho Kim, and Minhaeng Cho. Nonlinear Optical Properties of the Linear Quadrupolar Molecule: Structure-Function Relationship Based on a Three-State Model. *J. Phys. Chem. B*, 103(39):8221–8229, September 1999.
- [5] M. Barzoukas and M. Blanchard-Desce. Molecular engineering of push–pull dipolar and quadrupolar molecules for two-photon absorption: A multivalence-bond states approach. *J. Chem. Phys.*, 113(10):3951–3959, September 2000.
- [6] Mingli Yang and Benoît Champagne. Large Off-Diagonal Contribution to the Second-Order Optical Nonlinearities of  $\Lambda$ -Shaped Molecules. *J. Phys. Chem. A*, 107(19):3942–3951, May 2003.
- [7] Minhaeng Cho, Hyun-Soo Kim, and Seung-Joon Jeon. An elementary description of non-linear optical properties of octupolar molecules: Four-state model for guanidinium-type molecules. *J. Chem. Phys.*, 108(17):7114–7120, May 1998.
- [8] Minhaeng Cho, Sun-Young An, Hochan Lee, Isabelle Ledoux, and Joseph Zyss. Nonlinear optical properties of tetrahedral donor–acceptor octupolar molecules: Effective five-state model approach. *J. Chem. Phys.*, 116(21):9165–9173, June 2002.
- [9] M. Barzoukas, A. Fort, and M. Blanchard-Desce. A two-form description of push-pull molecules: Correlations between structure, intramolecular charge transfer and (hyper)polarizabilities. *J. Nonlinear Opt. Phys. Mater.*, 05(04):757–765, October 1996.
- [10] M. Barzoukas, C. Runser, A. Fort, and M. Blanchard-Desce. A two-state description of (hyper) polarizabilities of push-pull molecules based on a two-form model. *Chem. Phys. Lett.*, 257(5-6):531–537, August 1996.
- [11] M. Blanchard-Desce and M. Barzoukas. Two-form two-state analysis of polarizabilities of push pull molecules. *J. Opt. Soc. Am. B*, 15(1):302, January 1998.

- [12] Daqi Lu, Guanhua Chen, Joseph W. Perry, and William A. Goddard. Valence-Bond Charge-Transfer Model for Nonlinear Optical Properties of Charge-Transfer Organic Molecules. *J. Am. Chem. Soc.*, 116(23):10679–10685, November 1994.
- [13] David M. Bishop. Explicit nondivergent formulas for atomic and molecular dynamic hyperpolarizabilities. *The Journal of Chemical Physics*, 100(9):6535–6542, May 1994.
- [14] P. Beaujean. *Quantum Chemistry Investigations on Nonlinear Optical Materials: from Reference to Complex Systems*. Presses universitaires de Namur, 2021.