Free Quasi-Symmetric Functions, Product Actions and Quantum Field Theory of Partitions

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Introduction

Dans un papier relativement rcent, Philippe et al. ont soulign l'importance d'indexer les reads afin de rsoudre des problmes de mapping ou de correction, et ont dveloppe un index supportant les requtes suivantes :

- Dans quels reads f apparat ?
- Dans combien de reads f apparat ?
- Quelles sont les occurrences de f ?
- Quel est le nombre d'occurrences de f ?
- Dans quels reads f n'apparat qu'une fois ?
- Dans combien de reads f n'apparat qu'une fois ?
- Quelles sont les occurrences de f dans les reads o f n'apparat qu'une fois ?

Actions of a direct product of permutation groups

Direct product actions

Two pairs (G_1, X_1) and (G_2, X_2), each G_i is a permutation group acting on X_i.

Intransitive action of $G_1 \times G_2$ on $X_1 \sqcup X_2$:

$$(\sigma_1, \sigma_2)x = \begin{cases} \sigma_1 x & \text{if } x \in X_1 \\ \sigma_2 x & \text{if } x \in X_2 \end{cases}.$$

$(G_1, X_1) \nrightarrow (G_2, X_2) := (G_1 \times G_2, X_1 \sqcup X_2)$.

Cartesian action of $G_1 \times G_2$ on $X_1 \times X_2$:

$$(\sigma_1, \sigma_2)(x_1, x_2) = (\sigma_1 x_1, \sigma_2 x_2).$$

$(G_1, X_1) \times\!\!\!\!\nearrow (G_2, X_2) := (G_1 \times G_2, X_1 \times X_2)$.

Explicit realization

Denote

- by $\circ_N$ the natural action of $\mathfrak{S}_n$ on $\{0, \ldots, n-1\}$,
- by $\circ_I$ the intransitive action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \cdots, n+m-1\}$
- by $\circ_C$ the cartesian action of $\mathfrak{S}_n \times \mathfrak{S}_m$ on $\{0, \ldots, nm-1\}$.

More precisely,

$$(\sigma_1, \sigma_2) \circ_I i = \begin{cases} \sigma_1 \circ_N i & \text{if } 0 \le i \le n-1 \\ \sigma_2 \circ_N (i-n) + n & \text{if } n \le i \le n+m-1 \end{cases}.$$

and

$$(\sigma_1, \sigma_2) \circ_C (j + nk) = (\sigma_1 \circ_N j) + n(\sigma_2 \circ_N k)$$

for $0 \le i \le n+m-1$, $0 \le j \le n-1$ and $0 \le k \le m-1$.

Let the map $\nrightarrow$: $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{n+m}$ defined by

$$\sigma_1 \nrightarrow \sigma_2 = \sigma_1 \sigma_2[n]$$

$\sigma_1 = 1320 \in \mathfrak{S}_4$, $\sigma_2 = 534120 \in \mathfrak{S}_6$.

$\sigma_1 \nrightarrow \sigma_2 = 1320978564$, $\sigma_2 \nrightarrow \sigma_1 = 5341207986$

Proposition

$(\sigma_1 \nrightarrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_I i$.

Let the map $\times\!\!\!\!\nearrow$: $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ defined by

$$\sigma_1 \times\!\!\!\!\nearrow \sigma_2 = \prod_{i,j} c_i \times\!\!\!\!\nearrow c'_j$$

where $\sigma_1 = c_1 \cdots c_k$ and $\sigma_2 = c'_1 \cdots c'_{k'}$ are the decompositions of σ_1 and σ_2 in a product of cycles and

$$c \times\!\!\!\!\nearrow c' = \prod_{s=0}^{l\wedge l'-1} (\phi(s,0), \phi(s+1,1) \cdots, \phi(s+l\vee l'-1, l\vee l'-1)),$$

($\wedge :=$ gcd, $\vee :=$ lcm, $c = (i_0, \cdots, i_{l-1})$, $c' = (j_0, \cdots, j_{l'-1})$ are two cycles and $\phi(k, k') = i_{k \bmod l} + n j_{k' \bmod l'}$.)

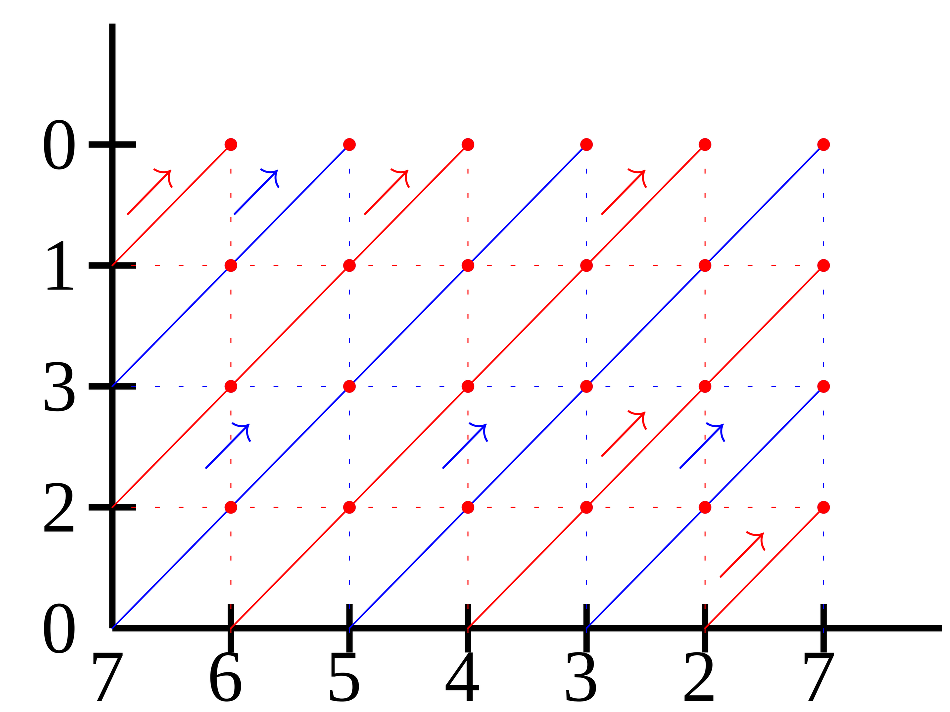
The cartesian action is compatible with the natural action.

Proposition

$(\sigma_1 \times\!\!\!\!\nearrow \sigma_2) \circ_N i = (\sigma_1, \sigma_2) \circ_C i$.

$c_1 = (0, 2, 3, 1)$, $c_2 = (7, 6, 5, 4, 3, 2)$.

$c_1 \times\!\!\!\!\nearrow c_2$:



$(28, 26, 23, 17, 12, 10, 31, 25, 20, 18, 15, 9)$

$(30, 27, 21, 16, 14, 11, 29, 24, 22, 19, 13, 8)$.

Algebraic structure

Proposition *Associativity*

Let $\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$ be 3 permutations

1. $\sigma_1 \nrightarrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \nrightarrow \sigma_2) \nrightarrow \sigma_3$
2. $\sigma_1 \times\!\!\!\!\nearrow (\sigma_2 \times\!\!\!\!\nearrow \sigma_3) = (\sigma_1 \times\!\!\!\!\nearrow \sigma_2) \times\!\!\!\!\nearrow \sigma_3$

Proposition *Semi-distributivity*

$\sigma_1 \in \mathfrak{S}_n$, $\sigma_2 \in \mathfrak{S}_m$ and $\sigma_3 \in \mathfrak{S}_p$

$$\sigma_1 \times\!\!\!\!\nearrow (\sigma_2 \nrightarrow \sigma_3) = (\sigma_1 \times\!\!\!\!\nearrow \sigma_2) \nrightarrow (\sigma_1 \times\!\!\!\!\nearrow \sigma_3)$$