Mathematics. — *A combinatorial problem.* By N. G. DE BRUIJN. (Communicated by Prof. W. VAN DER WOUDE.)

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1. Some years ago Ir. K. POSTHUMUS stated an interesting conjecture concerning certain cycles of digits 0 or 1, which we shall call P_n-cycles [1]. For $n = 1, 2, 3, \ldots$ a P_n-cycle be an ordered cycle of 2^n digits 0 or 1, such that the 2^n possible ordered sets of n consecutive digits of that cycle are all different. As a consequence, any ordered set of n digits 0 or 1 occurs exactly once in that cycle.

For example, a P_3-cycle is $\overrightarrow{00010111}$ [2], respectively showing the triples 000, 001, 010, 101, 011, 111, 110, 100, which are all possible triples indeed.

For $n = 1, 2, 3, 4$, the P_n-cycles can easily be found.

We have only one P_1-cycle, viz. 01, and only one P_2-cycle, viz. 0011. There are two P_3-cycles, viz. 00010111 and 11101000, and sixteen P_4-cycles, eight of which are

0000110100101111	1111001011010000
0000100110101111	1111011001010000
0000101100111101	1111010011000010
0000110101111001	1111001010000110

the remainig eight being obtained by reversing the order of these, respectively.

Ir. POSTHUMUS found the number of P_5-cycles to be 2048, and so he had the following number of P_n-cycles for $n = 1, 2, 3, 4, 5$:

$$\begin{array}{lccccc} & 1 & , 1 & , 2 & , 2^4 & , 2^{11} \;, \\ \text{or} & 2^{2^0-1} & , 2^{2^1-2} & , 2^{2^2-3} & , 2^{2^3-4} & , 2^{2^4-5}. \end{array}$$

Thus he was led to the conjecture, that the number of P_n-cycles be $2^{2^{n-1}-n}$ for general n. In this paper his conjecture is shown to be correct. Its proof is given in section **3**, as a consequence of a theorem concerning a special type of networks, stated and proved in section **2**. In section **4** another application of that theorem is mentioned.

2. We consider a special type of networks, which we shall call T-nets.

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[1]) These arise from a practical problem in telecommunication.

[2]) With this notation, $\overrightarrow{00010111}$, $\overrightarrow{00101110}$, etc., are to be considered as the same cycle. (Properly speaking, the digits must be placed around a circle.) On the other hand we do not identify the cycles $\overrightarrow{00010111}$ and $\overrightarrow{11101000}$, the second of which is obtained by reversing the order of the first one.

Henceforth we simply write 00010111 instead of $\overrightarrow{00010111}$.

A *T-net of order m* will be a network of *m junctions* and *2m one-way roads* (oriented roads), with the property that each junction is the start of two roads and also the finish of two roads. The network need not lie in a plane, or, in other words, viaducts, which are not to be considered as junctions, are allowed. Furthermore we do not exclude roads leading from a junction to that same junction, and we neither exclude pairs of junctions connected by two different roads, either in the same, or in opposite direction. Figs. 1*a* and 1*b* show examples of T-nets, of orders 3 and 6, respectively.

In a T-net we consider closed walks, with the property that any road of the net is used exactly once, in the prescribed direction. Such walks will be called "*complete walks*" of that T-net. Two complete walks are considered to be identical, if, and only if, the sequence of roads [3]) gone through in the first walk is a cyclic permutation of that in the second walk. The nets of figs. 1*a* and 1*b* admit 2 and 8 complete walks, respectively.

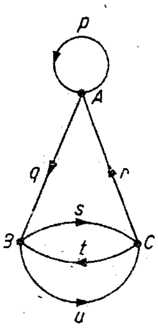

Fig. 1*a*

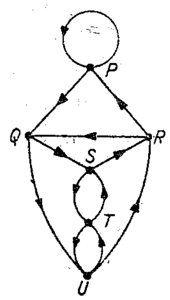

Fig. 1*b*

The number of complete walks of a T-net N be denoted by $|N|$. This number $|N|$ is zero, if N is not connected, that is to say, if N can be divided into two seperate T-nets [4]).

We now describe a process, which we call the "*doubling*" of a T-net, and which is illustrated by the relation between the nets of figs. 1a and 1*b*. Be N a T-net of order m, with junctions $A, B, C, \ldots$, and roads $p, q, r, \ldots$. Then we construct the "doubled" net N^* by taking $2m$ junctions $P, Q, R, \ldots$, corresponding to the roads of N, respectively. We construct a one-way road from a junction P to a junction Q, if the corresponding roads p and q of N have the property, that the finish of p lies in the same junction of N as the start of q. Thus $4m$ roads are obtained in N^*, and it is easy to see that N^* is a T-net; its order is $2m$.

[3]) If we should replace the word "roads" by "junctions" here, this sentence would get another meaning, since two junctions may be connected by two roads in the same direction.

[4]) The converse is also true: for a connected T-net we have $|N| > 0$. However, we do not need this result in the proof of our main theorem.

A remarkably simple relation exists between the numbers of complete walks of N and N^* [5]):

Theorem. If N is a T-net of order m ($m = 1, 2, 3, \ldots$), and N^* is the doubled net, then we have

$$|N^*| = 2^{m-1} \cdot |N|. \qquad (1)$$

Proof. We first consider two cases, in which (1) is easily established.

Case 1. If N is not connected, the same holds for N, and hence $|N| = |N^*| = 0$.

Case 2. We now consider the case, where each junction of N is connected with itself. For any value of m, only one connected net of this type exists, consisting of junctions $A_1, A_2, \ldots, A_m$, connected by roads $A_1A_2, A_2A_3, \ldots, A_{m-1}A_m$, A_mA_1, and $A_1A_1, A_2A_2, \ldots, A_mA_m$ [6]). For this net we have $|N| = 1$, and some quite trivial considerations show that $|N^*| = 2^{m-1}$.

We prove the general case by induction. For $m = 1$ only one T-net is possible, consisting of one junction A and two roads leading from A to A. This net belongs to case 2 mentioned above, and we have $|N| = |N^*| = 1$.

Now suppose (1) to be valid for all T-nets of order $m - 1$ ($m > 1$), and be N a T-net of order m. We may suppose to be able to choose a junction A, not connected with itself, for otherwise N belongs to case 2. Hence we have four different roads p, q, r, s; p and q leading to A, r and s starting from A.

A net N_1 arises from N by omitting A, p, q, r, s, and constructing two new roads, one from the start of p to the finish of r, and one from the start of q to the finish of s.

A second net N_2 arises in a similar way, but now by combining p with s and q with r. This is illustrated by fig. 2; the parts of the nets, which are not drawn, are equal for N, N_1 and N_2.

A complete walk of N corresponds to a complete walk either of N_1, or of N_2, and so we have

$$|N| = |N_1| + |N_2|. \qquad (2)$$

On doubling the nets N_1 and N_2 we obtain nets $N_1{}^*$ and $N_2{}^*$, respectively.

We shall prove

$$|N^*| = 2\,|N^*_1| + 2\,|N^*_2|. \qquad (3)$$

[5]) This relation can also be interpreted without introducing the doubling process. Namely, a complete walk of N^* corresponds to a closed walk through N, with the property that any road of N is used exactly twice in that walk, and such that at any junction each of the four possible combinations of a finish and a start is taken exactly once. We can give an even simpler interpretation in terms of N^*, for a complete walk of N corresponds to a closed walk through N^*, visiting any junction of N^* exactly once. But, since not every T-net can be considered as a N^*, this does not lead to an essential simplification of our theorem.

[6]) AB denotes a one-way road leading from A to B.

N_1^* and N_2^* arise directly from N^* by simple operations. If P, Q, R, S are the junctions of N^* corresponding to the roads p, q, r, s of N, we obtain N_1^* by omitting the roads PR, PS, QR, QS from N^*, and identifying the

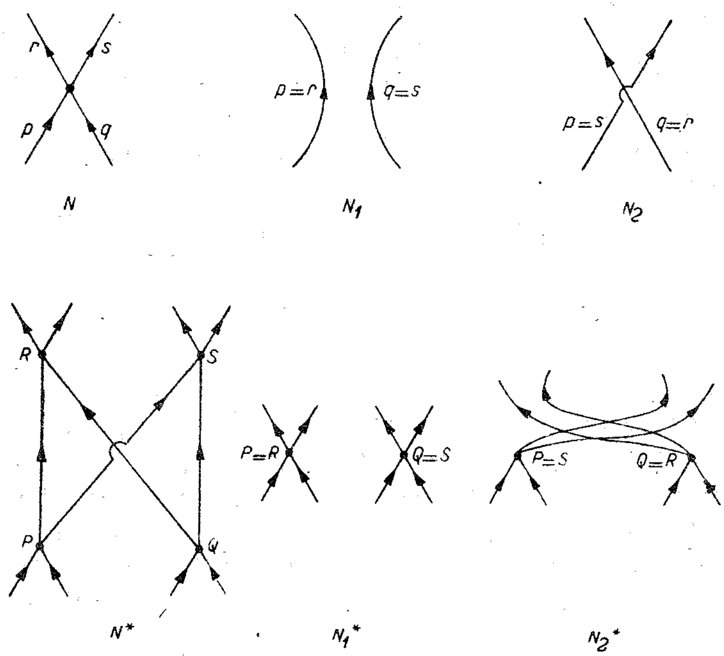

Fig. 2

four junctions two by two: $P = R$ and $Q = S$. N_2^* is obtained analogously ($P = S$ and $Q = R$). Again, fig. 2 shows the corresponding details of N^*, N_1^* and N_2^*.

Henceforth we deal with N^*, N_1^* and N_2^*, and no longer consider N, N_1, N_2.

We now first introduce the term "path". A *path* is an ordered sequence of roads, no two of which are identical, such that the finish of each road is the start of the next one. The last one, however, need not lead to the start of the first one.

A complete walk of N^*, N_1^*, or N^*_2, contains four special paths, each one leading from one of the junctions R, S to one of the junctions P, Q, such that any road of N^*, except PR, PS, QR, QS, belongs to just one of those paths. Choosing a definite set of four paths, according to the conditions just mentioned, we consider all (possibly existing) complete walks of N, N_1 and N_2 containing those paths. The numbers of these complete walks be denoted by n, n_1, n_2, respectively.

The numbers n, n_1, n_2 admit of a simple interpretation. Be N^{**} the net, arising from N^* on replacing each of the four paths by one single road, with the same start and finish as the corresponding path. In the same way nets N_1^{**} and N_2^{**} arise from N_1^* and N_2^*. Evidently $n = |N^{**}|$, $n_1 = |N_1^{**}|$, $n_2 = |N_2^{**}|$. We now show, that

$$n = 2n_1 + 2n_2, \quad \quad (4)$$

for which we have to consider two different cases.

Case A (fig. 3). The paths starting from R lead to different junctions.

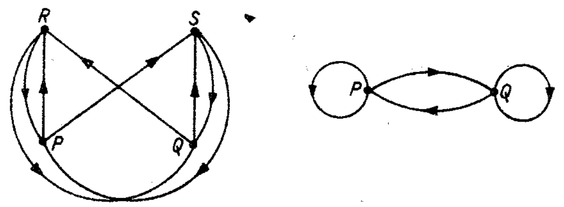

Fig. 3

The four paths thus respectively lead from R to P, from R to Q, from S to P, and from S to Q.

Now N^{**} consists of the junctions P, Q, R, S, with the roads $PR, PS, QR, QS, RP, RQ, SP, SQ$. This net admits four different complete walks. N_1^{**} consists of only two junctions P and Q, with roads PP, PQ, QP, QQ. This net admits only one complete walk. The net N_2^{**} is equivalent to N_1^{**}. Thus we have obtained $n = 4$, $n_1 = 1$, $n_2 = 1$, and (4) holds true.

Case B (fig. 4). The paths starting from R lead to one and the same

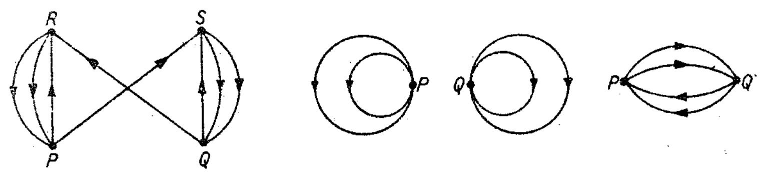

Fig. 4

junction, say P (the same obtains for Q). We now have the four paths RP, RP, SQ, SQ.

Now N^{**} consists of the junctions P, Q, R, S, with roads $PR, PS, QR, QS, RP, RP, SQ, SQ$. This net admits four complete walks. N_1^{**} consists of two junctions P and Q, with roads PP, PP, QQ, QQ, and so it is not connected. N_2^{**} consists of P, Q, with roads PQ, PQ, QP, QP, admitting two complete walks. Now we have $n = 4$, $n_1 = 0$, $n_2 = 2$, and hence (4) holds also true in case B.

Formula (4) being proved for any admissible system of four paths, the truth of (3) is now evident.

Our theorem is an immediate consequence of (3). Namely, N_1 and N_2 being nets of order $m - 1$, our assumption of induction yields

$$|N_1^*| = 2^{m-2}|N_1|, \qquad |N^*_2| = 2^{m-2}|N_2|,$$

and by (3) and (2) we now have

$$|N^*| = 2|N_1^*| + 2|N_2^*| = 2^{m-1}|N_1| + 2^{m-1}|N_2| = 2^{m-1}|N|.$$

3. The theorem of the preceding section provides a proof of POSTHUMUS' conjecture. For $n \geq 2$, N_n be the following network of order 2^n. As junctions we take the ordered n-tuples of digits 0 or 1, and we connect two n-tuples A and B by a one-way road AB, if the last $n-1$ digits of A correspond to the first $n-1$ digits of B. Fig. 5 shows the nets N_2 and N_3.

On "doubling" this net N_n we obtain the net N_{n+1}. Namely, any road AB of N_n (see N_2 in fig. 5) corresponds to an ordered $(n+1)$-tuple,

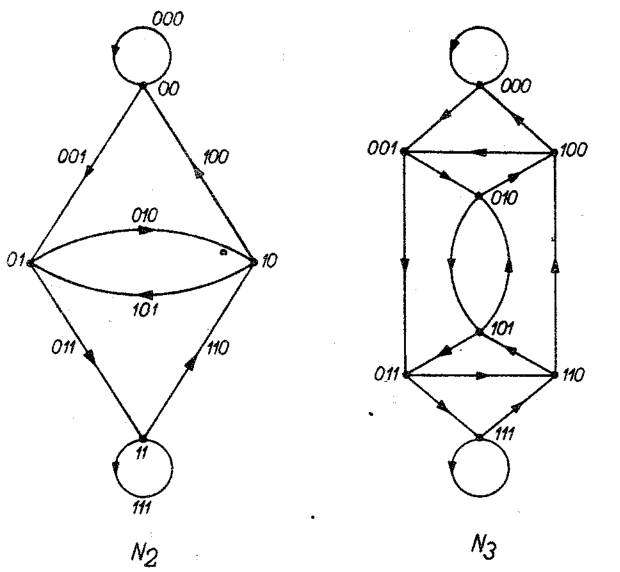

Fig. 5

consisting of the digits of A, followed by the last digit of B (or, what is the same, the first digit of A, followed by the digits of B). Two $(n+1)$-tuples P, Q, turn out to be connected in N_{n+1}, if the last n digits of the first one correspond to the first n digits of the second one, since these n digits characterize the common finish and start of the roads p and q of N_n. Hence $N_n^* = N_{n+1}$.

A complete walk of N_n leads to a P_{n+1}-cycle in the following way. If such a walk consecutively goes through the roads AB, BC, ..., ZA, we write down consecutively, the first digit of A, the first digit of B, ..., the first digit of Z. This sequence, considered as a cycle, is a P_{n+1}-cycle. Namely, on taking the first digits of $n+1$ consecutive junctions $A, B, C, \ldots$ of the walk under consideration, we obtain the $(n+1)$-tuple, belonging to the road AB. The walk in N_n being complete, it is now clear that any $(n+1)$-tuple occurs exactly once in our cycle.

Conversely, any P_{n+1}-cycle arises from a complete walk in N_n by the described process, and different complete walks lead to different P_{n+1}-cycles. Hence the number of P_{n+1}-cycles equals $|N_n|$.

We now prove POSTHUMUS' conjecture by induction. For $n = 1, 2, 3$ its truth is already established in section **1**. Now take $n \geq 3$, and suppose the number of P_n-cycles to be $2^{2^{n-1}-n}$, whence $|N_{n-1}| = 2^{2^{n-1}-n}$. The order of N_{n-1} being 2^{n-1}, the theorem of section **2** yields

$$|N_{n-1}^*| = 2^{2^{n-1}-1} \, . \, |N_{n-1}|,$$

and it follows

$$|N_n| = 2^{2^{n-1}-1} \, . \, 2^{2^{n-1}-n} = 2^{2^n-n-1}.$$

The number of P_{n+1}-cycles equalling $|N_n|$, POSTHUMUS' conjecture turns out to be true.

4. Another application of section **2** is the following one. We call a n-tuple of digits 0, or 2 admissible, if no two consecutive digits are equal; the last digit, however, may be the same as the first one. The number of admissable n-tuples is easily shown to be $3 \, . \, 2^{n-1}$. As a Q_n-cycle we now define an ordered cycle of $3 \, . \, 2^{n-1}$ digits 0,1 or 2, such that any admissable n-tuple is represented once by n consecutive digits of the cycle. For instance twelve Q_3-cycles exist. Two of them are 012010202121 and 012020102121, whereas the other ten are found by applying permutations of the symbols 0,1 and 2.

For general $n > 1$, the number of Q_n-cycles amounts to $3 \, . \, 2^{3.2^{n-2}-n-1}$. A proof can be given completely analogous to that in section **3.**

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N.V. Philips' Gloeilampenfabrieken.