Referee report on the paper

Subquadratic-time algorithm for the diameter and all eccentricities on median graphs

The paper presents an algorithm with complexity $\widetilde{O}(n^{\frac{5}{3}})$ for computing the eccentricities of all vertices of a median graph G with n vertices. This leads to an algorithm with the same complexity for computing diameter and center of median graphs. Median graphs is the most important class of graphs in metric graph theory. This is due to the rich structure and properties of such graphs, their bijections with CAT(0) cube complexes arising in geometric group theory and event structures arising in concurrency.

Recently several papers were devoted to the design of efficient algorithms for distance problems in median graphs. In particular, the paper [13] proposes a linear time algorithm for computation of the Θ-classes, the total distance, and the median function of median graphs and the paper [14] presents a linear time algorithm for the eccentricity problem in median graphs of constant dimension d (the algorithm of [14] is exponential in d).

The paper under revue uses some known properties of median graphs (in particular, the algorithm of [14] of computation of Θ-classes) and extends the approach of [14] to obtain the first subquadratic-time algorithms for the eccentricity problem in all median graphs. This result is interesting and novel. As much I checked the proofs, they are correct. The writing style of the current short version is not optimal since it looks like as a rapid copy-past from the full version. I hope that if accepted, the presentation of the results will improve. Summarizing, I think that the paper meets the standards of STACS and I suggest to be accepted.

General remarks:

(A) As I understand, the first algorithm in the paper works as follows. The median graph G is partitioned into gated/convex subgraphs by considering all Θ-classes which contain at least D edges (for some chosen D). The fact that all Θ-classes of such gated subgraphs have at most D edges imply that the dimension d of each such part is constant. Therefore, the authors can apply to each part the algorithm of [14] for median graphs of constant dimension. Furthermore, the authors improve the complexity of the algorithm of [14] by improving over the exponential constant in d. To do this, they have to solve the eccentricity problem on simplex graphs (sometime also called bouquets of cubes), which arise as stars of vertices. All this is done in Appendices B and C.

The content of Appendices B and C (which is an important constituent of the algorithm) are not treated at all in the main body of the paper. For example the Section 3 is just the formulation of Theorem 3.1 and a link to Appendix B. This also concerns Subsection 4.1 and Appendix C. You speak in Subsections 2.3 and 2.4 about POFs and maximal POFs, but this is not used at all in Section 4 and is used only in the appendices and Section 5.

The subgraphs H occurring in the partition of $G\setminus(E_1\cup\ldots\cup E_q)$ are all convex/gated and for them the conclusions of Lemmas 4.3 and 4.4 are immediate. I don't see any necessity to formulate two such lemmas on 7 lines, while the conclusion is immediate. This also concerns Lemma 4.5: any Θ-class of a d-cube of G contains at least 2^{d-1} edges and this is what you need. I think that you don't need to formulate this lemma (which occupies 5 lines) but simply deduce and use this.

I would use the obtained space to better describe the general algorithm (as I did at the beginning of (A)) and pay more attention what is the value of D in your proof of Lemma 4.6 and what will be the resulting constant dimension of the graphs in the partition.

Of course, I believe that it is better to present in more details the main algorithm of the paper and give details in the main text about the results of Appendices B and C. This can be done by reducing Section 5.

(B) I think that the authors can still improve their use of known properties of median graphs. This will reduce some proofs, as I will show below. For example, one useful property is the downward cube property (see Lemma 3.10 of [13]]): the edges incident to u in any interval $I(u,v)$ define a cube of G. From this property immediately follows that a median graph with n vertices and dimension d contains at most dn edges (a result you mention).

Personally for me, the notion of a POF and the result that pairwise orthogonal Θ-classes define a cube (Theorem A.11) are a kind of folklore. For sure, its proof exists in some paper (for example, Lemma 3.7 of [13] contain this kind of result but not completely the same). I see two proofs of this result. First: the hyperplanes of the CAT(0) cube complex of G are gated (in the l_1-metric), since they pairwise intersect, by the Helly property for gated sets they have a common point. This point is the barycenter of the required cube of G. Second proof: pick a base point v_0 and consider the intersection of all halfspaces defined by the Θ-classes of the POF not containing v_0. Since such halfspaces are gated and pairwise intersect, their intersection is nonempty. Pick in this intersection a vertex closest to v_0. From the choice of v, all neighbors of v in $I(v,v_0)$ belong to the Θ-classes of the POF. By the downward cube property they define a cube.

You can leave the proof of Theorem A.11 but you should mention that the notion of POFs and this kind of result is not completely new. Also you can mention that your notion of orthogonality is the same as crossing, incompatible, or concurrent.

(C) Theorem B.4 is not really a theorem but a lemma and it has a simple proof: Let G be the simplex graph of a graph H. Let C_u and C_v be the cliques of H corresponding to u and v. Then $d(u,v)=|C_u\Delta C_v|$. Consider the clique $C=C_v\setminus C_u$ and suppose C corresponds to a vertex w of G. If $C_u\cap C_v\neq\emptyset$, then $d(u,w)=|C_v\Delta C|=|C_u|+|C|<|C_u\setminus C_v|+|C_v\setminus C_u|=d(u,v)$, contrary to the choice of v as furthest from u vertex.

(D) I don't know why for this paper you have to consider the weighted version of the problem OPP (i.e., WOPP).

(E) Ladder and anti-lader labels: on p.31 you say that "A key characterization on ladder sets states that their Θ-classes are pairwise orthogonal". This trivially holds from the downward cube lemma. Indeed, if $u \in I(u,v)$, then by this lemma each set of neighbors of u in $I(u,v)$ and in $I(u,v_0)$ define cubes.

(F) I think that the notion of star of a vertex used in the proof of Theorem C.10 must be properly defined outside the proof and stated that it induces a gated subgraph.

(G) The simplex graphs have been first defined in [12] and in the paper by Bandelt and Van de Vel. They proved that the initial graph is the crossing graph of its simplex graph. Thus reference to [30] on last paragraph on p.31 must be updated. Also crossing graph (or incompatibility graph) was known much before [29] and [30].

H.-J. Bandelt, M. van de Vel, Embedding topological median algebras in products of dendrons, Proc. London Math. Soc. 58 (1989), 439–453.

Minor remarks:

Abstract

1st paragraph: "longstanding question" is a little bit strong

2nd paragraph: Median graphs constitute... because their structure represent $\to$ because their structure represents

3rd paragraph: this algorithm enumerate $\to$ enumerates

1 Introduction

1.1

1rst paragraph: On sparse graphs, it was shown [39] $\to$ it was shown in [39]

2nd paragraph: I think that you should mention that the median graphs are exactly the 1-skeletons of CAT(0) cube complexes and the domains of event structures and cite [19] and [11]// 4th paragraph: median graphs can be recognised in $O(n^3/2\sqrt(n))$ $\to$ "$O(n^3/2\log(n))$

1.2

4th paragraph: The second and main contribution...which compute $\to$ computes

This framework...its largest Θ-classes $\to$ Θ-class

2 Median graphs

2.1

3rd paragraph: It contains exactly the vertices which metrically between u and v $\to$ which lies

Between Lemmas 3 and 4: $\partial H_i' = N(H_i'')$. Add missing dot

Put differently, set $\partial H_i'$ $\to$ the set

Figure 2 illustrates the notions of $\Theta - class$, halfspace and boundaries $\to$ boundary

2.3

Pairwise orthogonal families: We focus on set $\to$ on the set

Lemma 9: There exists an hypercube $\to$ a hypercube

Lemma 10, 2nd bullet: For any POF X, there is a unique $\rightarrow$ an unique
2nd paragraph after Lemma 11: Each hypercube...and going into v according the v_0-orientation $\rightarrow$ according to
Proof of Theorem 1, 1rst paragraph: "Assume that $Y \subsetneq X_Q$" $\rightarrow \supsetneq$
Proof of Theorem 1, 3rd paragraph: "Then, set $\Cap_{X_Q} \partial H_i''$" $\rightarrow$ Then, the set
Proof of Theorem 1, last line: "In summary, ... with a unique" $\rightarrow$ with an unique
3 Simplex graphs
3.1
Proof of Theorem 2 first line: For each POF X, there is an hypercube $\rightarrow$ a hypercube
3.2.1
2nd paragraph: For each $a \in V(T)$ ... the two edges connectiong it to his children $\rightarrow$ its children
5th paragraph: First, we split $\mathcal{L}$ in two sets: one with POFs containing E_{i_i}, the other with POFs which does not contains $E_{i_i} \rightarrow E_{i_1}$
6th paragraph: some notation $\rightarrow$ some notations
7th paragraph $\rightarrow$ for node a with a universe $\rightarrow$ with an universe
end of 8th paragraph: The leaves of this block both belongs $\rightarrow$ belong
9th paragraph: with the following inputs $\rightarrow$ input
the collection is made up $\rightarrow$ is made up of
2nd paragraph after Corollary 2: Now, we give some notation $\rightarrow$ notations
3.2.2
2nd paragraph after Definition 10: Observe that any pair (a_0, L) is a constraint pair as $R(a) = \emptyset \rightarrow R(a_0)$
3.2.3
Case C^*: We fix $I(a', X') = (a^+, , X) \rightarrow (a^+, X)$
4
4.2
Lemma 20: let $v^* \in \partial H_i''$ be its gate $\rightarrow$ be its gate in H_i''
Proof of Lemma 24 3rd paragraph: the connected components of $G \setminus (E_i \cup E_2 \cup ...E_{i-1}) \rightarrow \cup E_{i-1}$
Proof of Lemma 24 6th paragraph: let us assume H to label $\rightarrow$ H labels
H is a i-node $\rightarrow$ an i-node