Practical Invalid Curve Attack Using Quadratic Twist

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1 Introduction

The so called *Invalid curve attack* is a real threat for cryptographic protocols based on elliptic curves. The attack has first been presented in [2] and the use of twists was described in [4]. OpenPGP.js prior to 4.2.0 was found to be vulnerable[1]. The Node.js secp256k1-node allows bindings to the "Bitcoin curve" `secp256k1` and was found to be vulnerable[2] to small subgroup attacks. Bluetooth was proved to be vulnerable to a "Fixed Coordinate" variant [3]. Edwards model has also been examined in [5]. The SafeCurves website and the associated paper [1] point out as *"An ECC implementor can stop an invalid-curve attack by checking whether the input point Q satisfies the correct curve equation; [...] But this creates a conflict between simplicity and security. An implementation that does not include this check is simpler and more likely to be produced, and will pass typical functionality tests."*

The rest of the paper is organized as follows. Section 2 recalls the basics mathematical concepts used in the sequel, we recall basics facts about discrete logarithm problem (DLP) and twists of elliptic curves. Section 3 presents the general setting of the attack and ways to exploit poor implementation and weak curves. Section 4 is a complete walktrhought an example with a toy curve E. A comprehensive study of morphisms and the quadratic twist E_d allows to solve the challenge DLP not in $E(\mathbb{F}_{q^2})$ and $E_d(\mathbb{F}_{q^2})$ (as usually seen in Ivalid Curve Attack write ups) but instead in $E(\mathbb{F}_q)$ and $E_d(\mathbb{F}_q)$. This paper has an expository role.

2 Background Material

Notations : We will denote by $\mathbb{F}_q$ the finite field with $q = p^n$ elements where $p \geq 5$ and $n \in \mathbb{N} - \{0\}$. We will denote by $E/\mathbb{F}_q$ an elliptic curve defined over $\mathbb{F}_q$. The reader is assumed to be familiar with basic theory of elliptic curves.
Short Weierstrass equations. Since $p \geq 5$, every elliptic curve $E/\mathbb{F}_q$ may be written as

$$E : y^2 = x^3 + ax + b, \;\; a, b \in \mathbb{F}_q.$$

This is a so called *short Weierstrass form* of the curve E *defined over* $\mathbb{F}_q$. The point at infinity of E is denoted by $\mathcal{O}_E$.

Remark 1.

1. The condition $p \geq 5$ is not a restriction in our context since p will usually be a large prime.
2. A short Weierstrass form is not unique. This will be completed in the subsection about twists.

Automorphisms. Let $E_1/\mathbb{F}_q$ and $E_2/\mathbb{F}_q$ be elliptic curves. These curves may be seen over $\overline{\mathbb{F}_q}$ that is, the coefficients of their equation may be seen as lying in $\overline{\mathbb{F}_q}$ instead of in $\mathbb{F}_q$. Every geometric isomorphism of elliptic curve ϕ from $E_1/\overline{\mathbb{F}_q}$ to $E_2/\overline{\mathbb{F}_q}$ has an affine part of the form

$$\phi(x, y) = (u^2x + r, u^3y + su^2x + t). \tag{1}$$

for $u \in \overline{\mathbb{F}_q}^*$, $r, s, t \in \overline{\mathbb{F}_q}$. We will denote geometric isomorphism as $\phi/\overline{\mathbb{F}_q}$. The isomorphism ψ is said to be *defined over* $\mathbb{F}_q$ or *rational* if $u, r, s, t \in \mathbb{F}_q$, we will denote it by $\psi/\mathbb{F}_q$.

Proposition 1. Let $E_i/\mathbb{F}_q$, $i \in \{1; 2\}$ be elliptic curves given by short Weierstrass equations.

$$E_i : y^2 = x^3 + a_ix + b_i, \;\; a_i, b_i \in \mathbb{F}_q.$$

A geometric isomorphism ϕ froom E_1 to E_2 is of the form

$$\phi(x, y) = (u^2x, u^3y).$$

[1] https://www.cve.org/CVERecord?id=CVE-2019-9155
[2] https://nvd.nist.gov/vuln/detail/CVE-2024-48930

Proof. This is included as a first step to fully understand isomorphisms in the quadratic twist case.
Let $(x,y) \in E_1$ and ϕ as given by (1). Applying ϕ to the equation of E_1 and expanding yields

$$\begin{aligned}
&y^2 = x^3 + a_1x + b_1 \\
&\Leftrightarrow (u^3y + su^2x + t)^2 = (u^2x + r)^3 + a_1(u^2x + r) + b_1 \\
&\Leftrightarrow u^6y^2 + s^2u^4x^2 + t^2 + 2u^5sxy + 2u^3ty + 2tsu^2x = u^6x^3 + 3ru^4x^2 + 3r^2u^2x + r^3 + a_1u^2x + a_1r + b_1(*)
\end{aligned}$$

Identifying coefficients of xy and y with those of $y^2 = x^3 + a_2x + b_2$ yields $s = 0, t = 0$ (recall that $u \neq 0$ and $p \neq 2$).

$$(*) \Leftrightarrow u^6y^2 = u^6x^3 + 3ru^4x^2 + 3r^2u^2x + r^3 + a_1u^2x + a_1r + b_1$$

Then, identifying the coefficient of x^2 with the short equation of E_2 yields $r = 0$ (here we use $p \neq 3$). Thus $\phi(x,y) = (u^2x, u^3y)$. We conclude with the following computations that will be used in the sequel.

$$\begin{aligned}
&u^6y^2 = u^6x^3 + a_1u^2x + b_1 \\
&\Leftrightarrow y^2 = x^3 + \frac{a_1}{u^4}x + \frac{b_1}{u^6} \\
&\Leftrightarrow \frac{a_1}{u^4} = a_2, \ \ \frac{b_1}{u^6} = b_2(**)
\end{aligned}$$

□

Proposition 2. Let $E/\mathbb{F}_q$ be an elliptic curves given by short Weierstrass equation $E : y^2 = x^3 + ax + b, \ \ a, b \in \mathbb{F}_q$. The *$j$-invariant of* E is defined to be

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}.$$

Let $E_1/\mathbb{F}_q, E_2/\mathbb{F}_q$ be elliptic curves, there exists an isomorphism $\phi/\overline{\mathbb{F}_q}$ from E_1 to E_2 if and only if $j(E_1) = j(E_2)$.

Remark 2.
1. We insist that the j-invariant classifies **geometric** isomorphism classes of elliptic curves over $\mathbb{F}_q$.
2. Thanks to $p \geq 5$, E has a short equation and $j(E)$ a special form in this case. Thus $j(E) \notin \{0, 1728\} \Leftrightarrow a, b \in \mathbb{F}_q^*$.

2.1 Twists of Elliptic Curves

Twists. Non trivial twists of $E/\mathbb{F}_q$ are elliptic curves $E'/\mathbb{F}_q$ being isomorphic to E when viewed over $\overline{\mathbb{F}_q}$ but not isomorphic to E when viewed over $\mathbb{F}_q$. Let $E/\mathbb{F}_q$ be an elliptic curve in short Weierstrass equation $y^2 = x^3 + ax + b$. Recall that $q = p^n$ and $p \geq 5$, so it is possible to write such an equation for E.

Definition 1. Le $E/\mathbb{F}_q$ be an elliptic curve. A *twist of E* is an elliptic curve $E_t/\mathbb{F}_q$ such that there is a geometric isomorphism $\phi/\overline{\mathbb{F}_q}$ of elliptic curves $\phi : E \simeq E_t$. A twist E_t of E is *trivial* if there exists an isomorphism ψ of elliptic curve defined over $\mathbb{F}_q$.

Definition 2. Let $d \in \mathbb{F}_q^*$. The *twist E_d of E by d* is the elliptic curve given in short Weierstrass equation

$$E_d : y^2 = x^3 + d^2ax + d^3b.$$

Remark 3. We did not specify that E_d is a non trivial. Actually, let δ be a square root of d in $\overline{\mathbb{F}_q}$ i.e. $\delta^2 = d$, then

$$\begin{aligned}
\phi &: E \to E_d \\
(x,y) &\mapsto (\frac{x}{d}, \frac{y}{d\delta})
\end{aligned}$$

is a geometric isomorphism from E to E_d. It matches the relations (**) concluding proof of Proposition 1 with $a_1 = a, b_1 = b, a_2 = ad^2, b_2 = bd^3$ and $d = \frac{1}{u}$.

Proposition 3. Assume that $j(E) \neq 0, 1728$. The twist E_d is trivial if and only if $d \in (\mathbb{F}_q^*)^2$.

Proof. ($\Rightarrow$) Assume that there exists a rational isomorphism ψ from E to E_d. According to Proposition 1

$$\exists u \in \mathbb{F}_q^*, \ \psi(x,y) = (u^2x, u^3y)$$

According to (**), $\frac{a}{u^4} = ad^2$ and $\frac{b}{u^6} = bd^3$. Recall that since $p \geq 5$, the assumption about $j(E)$ is equivalent to $a, b \neq 0$. Thus $\frac{1}{u^4} = d^2$, $\frac{1}{u^6} = d^3$ and $d = \frac{d^3}{d^2} = \frac{1}{u^2} \in (\mathbb{F}_q^*)^2$.

($\Leftarrow$) Conversely, let $\delta \in \mathbb{F}_q^*$ such that $\delta^2 = d$. Then a rational isomorphism from E to E_d is

$$\psi(x,y) = (\frac{x}{d}, \frac{y}{d\delta})$$

□

Proposition 4. Assume that $E/\mathbb{F}_q$ has $j(E) \neq 0, 1728$. Then a twist $E_t/\mathbb{F}_q$ of $E/\mathbb{F}_q$ is either trivial or E_d for some $d \in (\mathbb{F}_q^*)\backslash(\mathbb{F}_q^*)^2$.

Proof. Assume that $E_t/\mathbb{F}_q$ is a non trivial twist of $E/\mathbb{F}_q$ with isomorphism $\phi : E \to E_t$ given by $\phi(x, y) = (u^2 x, u^3 y)$, $u \in \overline{\mathbb{F}_q}$, $u \notin \mathbb{F}_q$. Let $E_t : y^2 = x^3 + a_t x + b_t$, $a_t, b_t \in \mathbb{F}_q$, thus (**) yields

$$a_t = \frac{a}{u^4}, b_t = \frac{b}{u^6}$$

Then $u^2 = \frac{b a_t}{a b_t} \in \mathbb{F}_q$, i.e. $u \notin \mathbb{F}_q$ but $u^2 \in \mathbb{F}_q$. This means that $u \in \mathbb{F}_{q^2}\backslash\mathbb{F}_q$. Let $d := \frac{1}{u^2}$, then $a_t = d^2 a, b_t = d^3 b$ and $\phi(x, y) = \left(\frac{x}{d}, \frac{uy}{d}\right)$. □

Remark 4. Proposition 3 is wrong if $p < 5$. For example, let $p = q = 3$, $E : y^2 = x^3 + x + 1$ and $d = 2$. In this case, E is isomorphic to $E_d : y^2 = x^3 + x + 2$ with ψ given by constants $(u, r, s, t) = (2, 2, 0, 0)$. So E_d is a trivial twist of E but d is non square modulo 3. Note that the definition for the j-invariant when $p = 3$ gives here $j(E) = 0$.

Order of group of rational points. The group of rational points $E(\mathbb{F}_q)$ has order $\sharp E(\mathbb{F}_q) = q + 1 - t$ where t is the *Trace of Frobenius*. An extensive description of the Frobenius endomorphism is out scope for this paper, we only need some basics facts we recall below.

Proposition 5. 1. **(Hasse Bound)** One has $|t| \leq 2\sqrt{q}$.

2. One has $\sharp E_d(\mathbb{F}_q) = q + 1 + t$, thus $|\sharp E_d(\mathbb{F}_q) - \sharp E(\mathbb{F}_q)| = |2t| \leq 4\sqrt{q}$.

Proof. 1. The proof of Hasse Bound is technical, the interessted reader may refer to V Theorem 1.1 from [6]

2. We include this proof to give some insight into how points distribute over $E(\mathbb{F}_q)$, $E_d(\mathbb{F}_q)$. We will prove that

$$\sharp E(\mathbb{F}_q) + \sharp E_d(\mathbb{F}_q) = 2q + 2 \quad (\dagger)$$

substituting $\sharp E(\mathbb{F}_q) = q + 1 - t$ in (†) gives the result.

Let $f(x) = x^3 + ax + b$, $d \in \mathbb{F}_q^*$ be non square in $\mathbb{F}_q$ and $E_d : Y^2 = X^3 + d^2 aX + d^3 b$. The following change of variables on E_d (take care, this is not an automorphism of E_d but a tool to count points)

$$X = dx, Y = dy$$

yields $Y^2 = X^3 + d^2 aX + d^3 b \Leftrightarrow d^2 y^2 = d^3 x^3 + d^3 ax + d^3 b \Leftrightarrow y^2 = df(x)$.

- Let $x \in \mathbb{F}_q$ such that $f(x) = 0$, then $(x, 0) \in E(\mathbb{F}_q)$ and $(dx, 0) \in E_d(\mathbb{F}_q)$. Each curve gets one point.
- Let $x \in \mathbb{F}_q$ such that $f(x) \in (\mathbb{F}_q^*)^2$. Then $y^2 = f(x)$ has two solutions. Since d is non square, $df(x)$ is non square and $y^2 = df(x)$ has no solution in $\mathbb{F}_q$. So E gets two points and E_d zero.
- Let $x \in \mathbb{F}_q$ such that $f(x) \notin (\mathbb{F}_q^*)^2$. Then $df(x) \in (\mathbb{F}_q^*)^2$ and $y^2 = df(x)$ has two solution giving rise to two points on E_d. Here, E gets zero point.

Let $E_a(\mathbb{F}_q) = E(\mathbb{F}_q)\backslash\{\mathcal{O}_E\}$ be the affine part of E and $E_{d,a}(\mathbb{F}_q) = E_d(\mathbb{F}_q)\backslash\{\mathcal{O}_{E_d}\}$ be the affine part of E_d. Each $x \in \mathbb{F}_q$ contributes for 2 points in the affine part of $E_a(\mathbb{F}_q) \cup E_{d,a}(\mathbb{F}_q)$, note that $E_a(\mathbb{F}_q) \cap E_{a,d}(\mathbb{F}_q) = \emptyset$. Counting points at infinity once for each curve yields (†).

□

2.2 Discrete Logarithm Problem

Definition 3. Let G be a group in multiplicative notation. The **Discrete Logarithm Problem** (DLP) is : given $b, h \in G$ find $a \in \mathbb{Z}$ such that $h = b^a$.

Remark 5. 1. The group law on an elliptic curve being usually written in additive notation, DLP for elliptic curves is rephrased as : given $P, B \in E(\mathbb{F}_q)$ find $a \in \mathbb{Z}$ such that $P = aB$.

2. The DLP has a solution if and only if P is in the subgroup $<B>$ generated by B.

The following discussion gives the necessaries notions when dealing with generic methods to solve the DLP that will be used in the last section. The generic `discrete_log` method from Sage uses a combination of Pohlig-Hellman, Baby Step Giant Step (BSGS) , Pollard's kangaroo (i.e. Pollard's Lambda), and Pollard's Rho.

- The Pohlig-Helmann method. Let n be the order of a point $B \in E(\mathbb{F}_q)$ and $n = \prod_{i=1}^{m} p_i^{n_i}$ be its prime factorization. The subgroup H generated by B is cyclic of order n, thus has a unique cyclic subgroup H_i of order $p_i^{n_i}$ for each $i \in [|1; m|]$. By means of the Chinese Remainder Theorem (CRT), solving the DLP in H boils down to solve it in each H_i. The subgroups H_i have order a prime power, which may still be quite large. One can reduce the DLP from H_i to subgroups $\tilde{H}_i$ of order **exactly** p_i. We restrict ourselves to solving the DLP in the $\tilde{H}_i$'s.

- BSGS method is a collision finding algorithm to solve DLP that requires $\mathcal{O}(\sqrt{p_i})$ running time and $\mathcal{O}(\sqrt{p_i})$ storage. This gives an bound on the size of the p_i's for which we can hope to solve the DLP with this method.

- Pollard's Rho (resp. Lambda) algorithm is solving the DLP and has $\mathcal{O}(\sqrt{p_i})$ (resp. $\mathcal{O}(\sqrt{p_i})$) time complexity but $\mathcal{O}(1)$) (resp. $\mathcal{O}(\ln p_i)$) space complexity.

3 Invalid Curve Attack

3.1 General Setting

Invalid Cuve Attack is presented in [4]. Let $E/\mathbb{F}_q : y^2 = x^3 + ax + b$ be the secure public curve of the cryptosystem on which DLP is (assumed) hard.

Main idea : The vulnerabilty lies in the fact that a malicious user may send a point that is **not** on $E/\mathbb{F}_q$ but on a weaker curve, say a quadratic twist, and then exploit it to solve an easier DLP. This is mainly due to b not beeing used for scalar multiplication on E in short Weierstrass form. Let's break it down.

1. Mallory may send an honest $P = (x, y)$ on E and gets back $B := \mathtt{k}.P$ from Bob. Mallory wants to recover the secret key `k`. But Mallory may also send a malicious point $Q = (\tilde{x}, \tilde{y})$ on another curve $\tilde{E} : y^2 = x^3 + ax + \tilde{b}$ (note that the a coefficient remains unchanged). Since formulae for computing $\mathtt{k}.Q$ do not involve $b, \tilde{b}$, Bob will correctly compute $\mathtt{k}.Q \in \tilde{E}$, believing he just made a computation on E.

2. Varying the curve $\tilde{E}$, i.e. varying $\tilde{b} \in \mathbb{F}_q$, Mallory collects various $\mathtt{k}.Q_i$ for $Q_i \in \tilde{E}_i$. Chosing wisely the $\tilde{E}_i$'s and Q_i's may produce several easy DLP.

3. Say that Mallory chooses $\tilde{E}_i$ and $Q_i \in \tilde{E}_i$ of order n_i, let p_i be a prime divisor of n_i. Sending Q_i to Bob, Mallory receives $B_i := \mathtt{k}.Q_i$. Note that $\hat{Q}_i = \frac{n_i}{p_i} Q_i$ has order p_i. Thus computing $\hat{B}_i = \frac{n_i}{p_i} B_i$ yields

$$\hat{B}_i = \mathtt{k}.\hat{Q}_i$$

   Mallory solves this DLP in group of order p_i. At this point Mallory knows `k` modulo p_i.

4. Iterating this process over various prime factors p_i of n_i, for various n_i on selected $\tilde{E}_i$ gives `k` modulo many primes. Then by means of the CRT, Mallory is able to recover `k` modulo **the product** of the p_i's. If `k` is known to be a, say, 2^{256} bits key and the product of the p_i is greater than 2^{256} then Mallory actually recovered `k`.

Remark 6. 1. Checking that the received point Q acutally lies on E stops the attack.

2. According to [1], a curve supporting *"simple, fast, constant-time single-coordinate single-scalar multiplication [...]drastically limits the power of invalid-curve attacks"* due to the fact that for a given $x \in \mathbb{F}_q$, formulae for single-coordinate ladders work for the original curve and the quadratic twist. So if the implementation is based on such ladders (e.g. Montgomery ladders), Mallory has only access to points on E and E_d but on no other curve $\tilde{E}$ as above.

3. Even if $\sharp E(\mathbb{F}_q) = hp$ with p prime and h a small cofactor (a classical setting in cryptographic applications) then $\sharp E_d(\mathbb{F}_q)$ might have a prime factorization with many small primes (such numbers are called *smooth*). For example **brainpoolP256t1** curve has prime order but its quadratic twist E_t has smooth order since it factors in a product of 7 primes of which 6 have bit length less than 42 (the last prime factor has bit length 89).

4 Walkthrough Example

In this section we give full details to implement the attack on a toy curve in a fake game between Bob and Mallory. The full code is available in `FastInvalidAttack.py`, containing some more explanations and comments.

Setting : First of all, check that the challenge has certain properties.

- You are given some ECDH challenge, assume moreover that the curve E is in short Weierstrass form.

- The implementation on Bob's side does not checl whether the user passes a point on E.
- If the challenge uses two coordinates to compute on E, this is probably a Small Subgroup Attack. So one may choose x such that it lifts as $Q_i = (x, y)$ of smooth order on a weak curve E_i. This is not the situation we are considering.
- If the challenge uses single coordinate to compute on E (with some sort of Montgomery scalar multiplication) this may be an Invalid Curve Attack restricting the choice of the weak E_i to the quadratic twist E_d.

Assume we are given a challenge where one plays the role of a malicious user Mallory interacting with Bob. Mallory sends a point P to Bob who computes $\mathtt{k}.P$ for some secret key `k`. Mallory goal is to recover `k`. Assume moreover that Bob uses some sort of single-coordinate scalar multiplication. Let's break down the attack in comprehensive blocks.

1. Check the prime factorizations of $\sharp E(\mathbb{F}_q), \sharp E_d(\mathbb{F}_q)$. Check that they are smooth enough, i.e. have many prime factors with bit length less than, say, 40. Moreover check that these orders have **different** small prime factors. This implies that sending to Bob a point on E_d yields a DLP on E_d and thus Mallory gets to know the secret key `k` modulo some new prime factors compared to only sending points that lie on E.
2. One chooses a point $G_0 \in \mathrm{E}_d(\mathbb{F}_q)$ by lifting some $x \in \mathbb{F}_q$, this is done by trial and error until one finds x such that $f(x) \notin (\mathbb{F}_q)^2$.
3. Since $\sharp E_d(\mathbb{F}_q)$ has only one unhandleable big prime factor p_b for DLP, we restrict to the subgroup of order $n_0 = \frac{\sharp E_d(\mathbb{F}_q)}{p_b}$ by setting $\hat{G}_0 := p_b G_0$.
4. One chooses a point $G_1 \in \mathrm{E}(\mathbb{F}_q)$ by lifting some $x \in \mathbb{F}_q$, this is done by trial and error until one finds x such that $f(x) \in (\mathbb{F}_q)^2$.
5. Since $\sharp E(\mathbb{F}_q)$ has only one unhandleable big prime factor p'_b for DLP, we restrict to the subgroup of order $n_1 = \frac{\sharp E(\mathbb{F}_q)}{p'_b}$ by setting $\hat{G}_1 := p'_b G_1$.
6. Mallory sends the x-coordinates of $\hat{G}_0$ and $\hat{G}_1$ to Bob and gets back `pubkey0` and `pubkey1`. Those are x-coordinates of points $P_0 \in E_d(\mathbb{F}_q)$ and $P_1 \in E(\mathbb{F}_q)$ such that

   $$P_i = \mathtt{k}\hat{G}_i$$

   This is where using morphisms between E and E_d allows Mallory to dodge working in $\mathbb{F}_{q^2}$. Note that Bob uses a single-coordinate scalar multiplication (and no extra bit to identify y), Mallory knows the discrete logs only up to sign. Indeed, P_i and $-P_i$ have the same x-coordinate due to the short Weierstrass form, Mallory may have recovered `k` or `-k` (modulo n_i).
7. Mallory solves these two DLP to recover `k` modulo n_0 and n_1. By means of the CRT, Mallory recovers `k` modulo $n_0 n_1$ (note that n_0, n_1 are coprime). Since $n_0 \times n_1 \geq \sharp E(\mathbb{F}_q)$, Mallory has actually recovered `k`.

Remark 7. Note that points 2. and 3. above are usually done in Invalid Curve Attack write ups in $E_d(\mathbb{F}_{q^2})$, thus the DLP involving G_0 is solved over $\mathbb{F}_{q^2}$ implying a significant slow down of the attack. If the challenge has to be solved under some time constraints, this could be an hindering.

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