

# Practical Invalid Curve Attack Using Quadratic Twist

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## 1 Introduction

The so called *Invalid curve attack* is a real threat for cryptographic protocols based on elliptic curves. The attack has first been presented in [2] and the use of twists was described in [4]. OpenPGP.js prior to 4.2.0 was found to be vulnerable<sup>1</sup>. The Node.js secp256k1-node allows bindings to the "Bitcoin curve" `secp256k1` and was found to be vulnerable<sup>2</sup> to small subgroup attacks. Bluetooth was proved to be vulnerable to a "Fixed Coordinate" variant [3]. Edwards model has also been examined in [5]. The SafeCurves website and the associated paper [1] point out as "An ECC implementor can stop an invalid-curve attack by checking whether the input point  $Q$  satisfies the correct curve equation; [...] But this creates a conflict between simplicity and security. An implementation that does not include this check is simpler and more likely to be produced, and will pass typical functionality tests."

The rest of the paper is organized as follows. Section 2 recalls the basics mathematical concepts used in the sequel, we recall basics facts about discrete logarithm problem (DLP) and twists of elliptic curves. Section 3 presents the general setting of the attack and ways to exploit poor implementation and weak curves. Section 4 is a complete walktrhought an example. This paper has an expository role.

## 2 Background Material

**Notations :** We will denote by  $\mathbb{F}_q$  the finite field with  $q = p^n$  elements where  $p \geq 5$  and  $n \in \mathbb{N} - \{0\}$ . We will denote by  $E/\mathbb{F}_q$  an elliptic curve defined over  $\mathbb{F}_q$ . The reader is assumed to be familiar with basic theory of elliptic curves.

**Short Weierstrass equations.** Since  $p \geq 5$ , every elliptic curve  $E/\mathbb{F}_q$  may be written as

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{F}_q.$$

This is a so called *short Weierstrass form* of the curve  $E$  defined over  $\mathbb{F}_q$ .

**Remark 1.** 1. The condition  $p \geq 5$  is not a restriction in our context since  $p$  will usually be a large prime.

2. A short Weierstrass form is not unique. This will be completed in the subsection about twists.

**Automorphisms.** Let  $E_1/\mathbb{F}_q$  and  $E_2/\mathbb{F}_q$  be elliptic curves. These curves may be seen over  $\overline{\mathbb{F}_q}$  that is, the coefficients of their equation may be seen as lying in  $\overline{\mathbb{F}_q}$  instead of in  $\mathbb{F}_q$ . Every geometric isomorphism of elliptic curve  $\phi$  from  $E_1/\overline{\mathbb{F}_q}$  to  $E_2/\overline{\mathbb{F}_q}$  has an affine part of the form

$$\phi(x, y) = (u^2x + r, u^3y + su^2x + t). \quad (1)$$

for  $u \in \overline{\mathbb{F}_q}^*$ ,  $r, s, t \in \overline{\mathbb{F}_q}$ . We will denote geometric isomorphism as  $\phi/\overline{\mathbb{F}_q}$ . The isomorphism  $\psi$  is said to be *defined over  $\mathbb{F}_q$*  or *rational* if  $u, r, s, t \in \mathbb{F}_q$ , we will denote it by  $\psi/\mathbb{F}_q$ .

**Proposition 1.** Let  $E_i/\mathbb{F}_q$ ,  $i \in \{1; 2\}$  be elliptic curves given by short Weierstrass equations.

$$E_i : y^2 = x^3 + a_i x + b_i, \quad a_i, b_i \in \mathbb{F}_q.$$

A geometric isomorphism  $\phi$  from  $E_1$  to  $E_2$  is of the form

$$\phi(x, y) = (u^2x, u^3y).$$

*Proof.* This is included as a first step to fully understand isomorphisms in the quadratic twist case.

Let  $(x, y) \in E_1$  and  $\phi$  as given by (1). Applying  $\phi$  to the equation of  $E_1$  and expanding yields

$$\begin{aligned} y^2 &= x^3 + a_1x + b_1 \\ \Leftrightarrow (u^3y + su^2x + t)^2 &= (u^2x + r)^3 + a_1(u^2x + r) + b_1 \\ \Leftrightarrow u^6y^2 + s^2u^4x^2 + t^2 + 2u^5sxy + 2u^3ty + 2tsu^2x &= u^6x^3 + 3ru^4x^2 + 3r^2u^2x + r^3 + a_1u^2x + a_1r + b_1(*) \end{aligned}$$

<sup>1</sup><https://www.cve.org/CVERecord?id=CVE-2019-9155>

<sup>2</sup><https://nvd.nist.gov/vuln/detail/CVE-2024-48930>

Identifying coefficients of  $xy$  and  $y$  with those of  $y^2 = x^3 + a_2x + b_2$  yields  $s = 0, t = 0$  (recall that  $u \neq 0$  and  $p \neq 2$ ).

$$(*) \Leftrightarrow u^6 y^2 = u^6 x^3 + 3ru^4 x^2 + 3r^2 u^2 x + r^3 + a_1 u^2 x + a_1 r + b_1$$

Then, identifying the coefficient of  $x^2$  with the short equation of  $E_2$  yields  $r = 0$  (here we use  $p \neq 3$ ). Thus  $\phi(x, y) = (u^2 x, u^3 y)$ . We conclude with the following computations that will be used in the sequel.

$$\begin{aligned} u^6 y^2 &= u^6 x^3 + a_1 u^2 x + b_1 \\ \Leftrightarrow y^2 &= x^3 + \frac{a_1}{u^4} x + \frac{b_1}{u^6} \\ \Leftrightarrow \frac{a_1}{u^4} &= a_2, \quad \frac{b_1}{u^6} = b_2 (**) \end{aligned}$$

□

**Proposition 2.** Let  $E/\mathbb{F}_q$  be an elliptic curves given by short Weierstrass equation  $E : y^2 = x^3 + ax + b$ ,  $a, b \in \mathbb{F}_q$ . The  $j$ -invariant of  $E$  is defined to be

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}.$$

Let  $E_1/\mathbb{F}_q, E_2/\mathbb{F}_q$  be elliptic curves, there exists an isomorphism  $\phi/\overline{\mathbb{F}_q}$  from  $E_1$  to  $E_2$  if and only if  $j(E_1) = j(E_2)$ .

**Remark 2.** 1. We insist that the  $j$ -invariant classifies **geometric** isomorphism classes of elliptic curves over  $\mathbb{F}_q$ .  
2. Thanks to  $p \geq 5$ ,  $E$  has a short equation and  $j(E)$  a special form in this case. Thus  $j(E) \notin \{0, 1728\} \Leftrightarrow a, b \in \mathbb{F}_q^*$ .

## 2.1 Twists of Elliptic Curves

**Twists.** Non trivial twists of  $E/\mathbb{F}_q$  are elliptic curves  $E'/\mathbb{F}_q$  being isomorphic to  $E$  when viewed over  $\overline{\mathbb{F}_q}$  but not isomorphic to  $E$  when viewed over  $\mathbb{F}_q$ . Let  $E/\mathbb{F}_q$  be an elliptic curve in short Weierstrass equation  $y^2 = x^3 + ax + b$ . Recall that  $q = p^n$  and  $p \geq 5$ , so it is possible to write such an equation for  $E$ .

**Definition 1.** Let  $E/\mathbb{F}_q$  be an elliptic curve. A *twist* of  $E$  is an elliptic curve  $E_t/\mathbb{F}_q$  such that there is a geometric isomorphism  $\phi/\overline{\mathbb{F}_q}$  of elliptic curves  $\phi : E \simeq E_t$ . A twist  $E_t$  of  $E$  is *trivial* if there exists an isomorphism  $\psi$  of elliptic curve defined over  $\mathbb{F}_q$ .

**Definition 2.** Let  $d \in \mathbb{F}_q^*$ . The *twist*  $E_d$  of  $E$  by  $d$  is the elliptic curve given in short Weierstrass equation

$$E_d : y^2 = x^3 + d^2 ax + d^3 b.$$

**Remark 3.** We did not specify that  $E_d$  is a non trivial. Actually, let  $\delta$  be a square root of  $d$  in  $\overline{\mathbb{F}_q}$  i.e.  $\delta^2 = d$ , then

$$\begin{aligned} \phi : E &\rightarrow E_d \\ (x, y) &\mapsto \left(\frac{x}{d}, \frac{y}{d\delta}\right) \end{aligned}$$

is a geometric isomorphism from  $E$  to  $E_d$ . It matches the relations (\*\*) concluding proof of Proposition 1 with  $a_1 = a, b_1 = b, a_2 = ad^2, b_2 = bd^3$  and  $d = \frac{1}{u}$ .

**Proposition 3.** Assume that  $j(E) \neq 0, 1728$ . The twist  $E_d$  is trivial if and only if  $d \in (\mathbb{F}_q^*)^2$ .

*Proof.* ( $\Rightarrow$ ) Assume that there exists a rational isomorphism  $\psi$  from  $E$  to  $E_d$ . According to Proposition 1

$$\exists u \in \mathbb{F}_q^*, \psi(x, y) = (u^2 x, u^3 y)$$

According to (\*\*),  $\frac{a}{u^4} = ad^2$  and  $\frac{b}{u^6} = bd^3$ . Recall that since  $p \geq 5$ , the assumption about  $j(E)$  is equivalent to  $a, b \neq 0$ . Thus  $\frac{1}{u^4} = d^2, \frac{1}{u^6} = d^3$  and  $d = \frac{d^3}{d^2} = \frac{1}{u^2} \in (\mathbb{F}_q^*)^2$ .

( $\Leftarrow$ ) Conversely, let  $\delta \in \mathbb{F}_q^*$  such that  $\delta^2 = d$ . Then a rational isomorphism from  $E$  to  $E_d$  is

$$\psi(x, y) = \left(\frac{x}{d}, \frac{y}{d\delta}\right)$$

□

**Proposition 4.** Assume that  $E/\mathbb{F}_q$  has  $j(E) \neq 0, 1728$ . Then a twist  $E_t/\mathbb{F}_q$  of  $E/\mathbb{F}_q$  is either trivial or  $E_d$  for some  $d \in (\mathbb{F}_q^*) \setminus (\mathbb{F}_q^*)^2$ .

*Proof.* Assume that  $E_t/\mathbb{F}_q$  is a non trivial twist of  $E/\mathbb{F}_q$  with isomorphism  $\phi : E \rightarrow E_t$  given by  $\phi(x, y) = (u^2x, u^3y)$ ,  $u \in \overline{\mathbb{F}_q}$ ,  $u \notin \mathbb{F}_q$ . Let  $E_t : y^2 = x^3 + a_tx + b_t$ ,  $a_t, b_t \in \mathbb{F}_q$ , thus (\*\*) yields

$$a_t = \frac{a}{u^4}, b_t = \frac{b}{u^6}$$

Then  $u^2 = \frac{ba_t}{ab_t} \in \mathbb{F}_q$ , i.e.  $u \notin \mathbb{F}_q$  but  $u^2 \in \mathbb{F}_q$ . This means that  $u \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . Let  $d := \frac{1}{u^2}$ , then  $a_t = d^2a, b_t = d^3b$  and  $\phi(x, y) = (\frac{x}{d}, \frac{uy}{d})$ .  $\square$

**Remark 4.** Proposition 3 is wrong if  $p < 5$ . For example, let  $p = q = 3$ ,  $E : y^2 = x^3 + x + 1$  and  $d = 2$ . In this case,  $E$  is isomorphic to  $E_d : y^2 = x^3 + x + 2$  with  $\psi$  given by constants  $(u, r, s, t) = (2, 2, 0, 0)$ . So  $E_d$  is a trivial twist of  $E$  but  $d$  is non square modulo 3. Note that the definition for the  $j$ -invariant when  $p = 3$  gives here  $j(E) = 0$ .

**Order of group of rational points.** The group of rational points  $E(\mathbb{F}_q)$  has order  $\#E(\mathbb{F}_q) = q + 1 - t$  where  $t$  is the *Trace of Frobenius*. An extensive description of the Frobenius endomorphism is out scope for this paper, we only need some basics facts we recall below.

**Proposition 5.** 1. (**Hasse Bound**) One has  $|t| \leq 2\sqrt{q}$ .

2. One has  $\#E_d(\mathbb{F}_q) = q + 1 + t$ , thus  $|\#E_d(\mathbb{F}_q) - \#E(\mathbb{F}_q)| = |2t| \leq 4\sqrt{q}$ .

*Proof.* 1. The proof of Hasse Bound is technical, the interested reader may refer to V Theorem 1.1 from [6]

2. We include this proof to give some insight into how points distribute over  $E(\mathbb{F}_q)$ ,  $E_d(\mathbb{F}_q)$ . We will prove that

$$\#E(\mathbb{F}_q) + \#E_d(\mathbb{F}_q) = 2q + 2 \quad (\dagger)$$

substituting  $\#E(\mathbb{F}_q) = q + 1 - t$  gives the result.

Let  $f(x) = x^3 + ax + b$ ,  $d \in \mathbb{F}_q^*$  be non square in  $\mathbb{F}_q$  and  $E_d : Y^2 = X^3 + d^2aX + d^3b$ . The following change of variables on  $E_d$  (take care, this is not an isomorphism of  $E_d$ )

$$X = dx, Y = dy$$

yields  $Y^2 = X^3 + d^2aX + d^3b \Leftrightarrow d^2y^2 = d^3x^3 + d^3ax + d^3b \Leftrightarrow y^2 = df(x)$ .

- Let  $x \in \mathbb{F}_q$  such that  $f(x) = 0$ , then  $(x, 0) \in E(\mathbb{F}_q)$  and  $(dx, 0) \in E_d(\mathbb{F}_q)$ . Each curve gets one point.
- Let  $x \in \mathbb{F}_q$  such that  $f(x) \in (\mathbb{F}_q^*)^2$ . Then  $y^2 = f(x)$  has two solutions. Since  $d$  is non square,  $df(x)$  is non square and  $y^2 = df(x)$  has no solution in  $\mathbb{F}_q$ . So  $E$  gets two points and  $E_d$  zero.
- Let  $x \in \mathbb{F}_q$  such that  $f(x) \notin (\mathbb{F}_q^*)^2$ . Then  $df(x) \in (\mathbb{F}_q^*)^2$  and  $y^2 = df(x)$  has two solution giving rise to two points on  $E_d$ . Here,  $E$  gets zero point.
- Count the point at infinity once for each curve.

Thus each  $x \in \mathbb{F}_q$  contributes for 2 points in  $E(\mathbb{F}_q) \cup E_d(\mathbb{F}_q)$  plus the points at infinity, yielding  $(\dagger)$ .  $\square$

## 2.2 Discrete Logarithm Problem

**Definition 3.** Let  $G$  be a group in multiplicative notation. The **Discrete Logarithm Problem** (DLP) is : given  $b, h \in G$  find  $a \in \mathbb{Z}$  such that  $h = b^a$ .

**Remark 5.** 1. The group law on an elliptic curve being usually written in additive notation, DLP for elliptic curves is rephrased as : given  $P, B \in E(\mathbb{F}_q)$  find  $a \in \mathbb{Z}$  such that  $P = aB$ .

2. The DLP has a solution if and only if  $P$  is in the subgroup  $\langle B \rangle$  generated by  $B$ .

The following discussion gives the necessities notions when dealing with generic methods to solve the DLP that will be used in the last section. The generic `discrete_log` method from Sage uses a combination of Pohlig-Hellman, Baby Step Giant Step (BSGS), Pollard's kangaroo (i.e. Pollard's Lambda), and Pollard's Rho.

- The Pohlig-Hellman method. Let  $n$  be the order of a point  $B \in E(\mathbb{F}_q)$  and  $n = \prod_{i=1}^m p_i^{n_i}$  be its prime factorization. The subgroup  $H$  generated by  $B$  is cyclic of order  $n$ , thus has a unique cyclic subgroup  $H_i$  of order  $p_i^{n_i}$  for each  $i \in [1; m]$ . By means of the Chinese Remainder Theorem (CRT), solving the DLP in  $H$  boils down to solve it in each  $H_i$ . The subgroups  $H_i$  have order a prime power, which may still be quite large. One can reduce the DLP from  $H_i$  to subgroups  $\tilde{H}_i$  of order **exactly**  $p_i$ . We restrict ourselves to solving the DLP in the  $\tilde{H}_i$ 's.
- BSGS method is a collision finding algorithm to solve DLP that requires  $\mathcal{O}(\sqrt{p_i})$  running time and  $\mathcal{O}(\sqrt{p_i})$  storage. This gives an bound on the size of the  $p_i$ 's for which we can hope to solve the DLP with this method.
- Pollard's Rho (resp. Lambda) algorithm is solving the DLP and has  $\mathcal{O}(\sqrt{p_i})$  (resp.  $\mathcal{O}(\sqrt{p_i})$ ) time complexity but  $\mathcal{O}(1)$  (resp.  $\mathcal{O}(\ln p_i)$ ) space complexity.

### 3 Invalid Curve Attack

#### 3.1 General Setting

Invalid Curve Attack is presented in [4]. Let  $E/\mathbb{F}_q : y^2 = x^3 + ax + b$  be the secure public curve of the cryptosystem on which DLP is (assumed) hard.

**Main idea :** The vulnerability lies in the fact that a malicious user may send a point that is **not** on  $E/\mathbb{F}_q$  but on a weaker curve, say a quadratic twist, and then exploit it to solve an easier DLP. This is mainly due to  $b$  not being used for scalar multiplication on  $E$  in short Weierstrass form. Let's break it down.

1. Mallory may send an honest  $P = (x, y)$  on  $E$  and gets back  $B := k.P$  from Bob. Mallory wants to recover the secret key  $k$ . But Mallory may also send a malicious point  $Q = (\tilde{x}, \tilde{y})$  on another curve  $\tilde{E} : y^2 = x^3 + ax + \tilde{b}$  (note that the  $a$  coefficient remains unchanged). Since formulae for computing  $k.Q$  do not involve  $b, \tilde{b}$ , Bob will correctly compute  $k.Q \in \tilde{E}$ , believing he just made a computation on  $E$ .
2. Varying the curve  $\tilde{E}$ , i.e. varying  $\tilde{b} \in \mathbb{F}_q$ , Mallory collects various  $k.Q_i$  for  $Q_i \in \tilde{E}_i$ . Choosing wisely the  $\tilde{E}_i$ 's and  $Q_i$ 's may produce several easy DLP.
3. Say that Mallory chooses  $\tilde{E}_i$  and  $Q_i \in \tilde{E}_i$  of order  $n_i$ , let  $p_i$  be a prime divisor of  $n_i$ . Sending  $Q_i$  to Bob, Mallory receives  $B_i := k.Q_i$ . Note that  $\hat{Q}_i = \frac{n_i}{p_i} Q_i$  has order  $p_i$ . Thus computing  $\hat{B}_i = \frac{n_i}{p_i} B_i$  yields

$$\hat{B}_i = k.\hat{Q}_i$$

Mallory solves this DLP in group of order  $p_i$ . At this point Mallory knows  $k$  modulo  $p_i$ .

4. Iterating this process over various prime factors  $p_i$  of  $n_i$ , for various  $n_i$  on selected  $\tilde{E}_i$  gives  $k$  modulo many primes. Then by means of the CRT, Mallory is able to recover  $k$  modulo **the product** of the  $p_i$ 's. If  $k$  is known to be a, say,  $2^{256}$  bits key and the product of the  $p_i$  is greater than  $2^{256}$  then Mallory actually recovered  $k$ .

**Remark 6.** 1. Checking that the received point  $Q$  actually lies on  $E$  stops the attack.

2. According to [1], a curve supporting "simple, fast, constant-time single-coordinate single-scalar multiplication [...] drastically limits the power of invalid-curve attacks" due to the fact that for a given  $x \in \mathbb{F}_q$ , formulae for single-coordinate ladders work for the original curve and the quadratic twist. So if the implementation is based on such ladders (e.g. Montgomery ladders), Mallory has only access to points on  $E$  and  $E_d$  but on no other curve  $\tilde{E}$  as above.
3. Even if  $\#E(\mathbb{F}_q) = hp$  with  $p$  prime and  $h$  a small cofactor (a classical setting in cryptographic applications) then  $\#E_d(\mathbb{F}_q)$  might have a prime factorization with many small primes (such numbers are called *smooth*). For example **brainpoolP256t1** curve has prime order but its quadratic twist  $E_t$  has smooth order since it factors in a product of 7 primes of which 6 have bit length less than 42 (the last prime factor has bit length 89).

### 4 Walkthrough Example

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