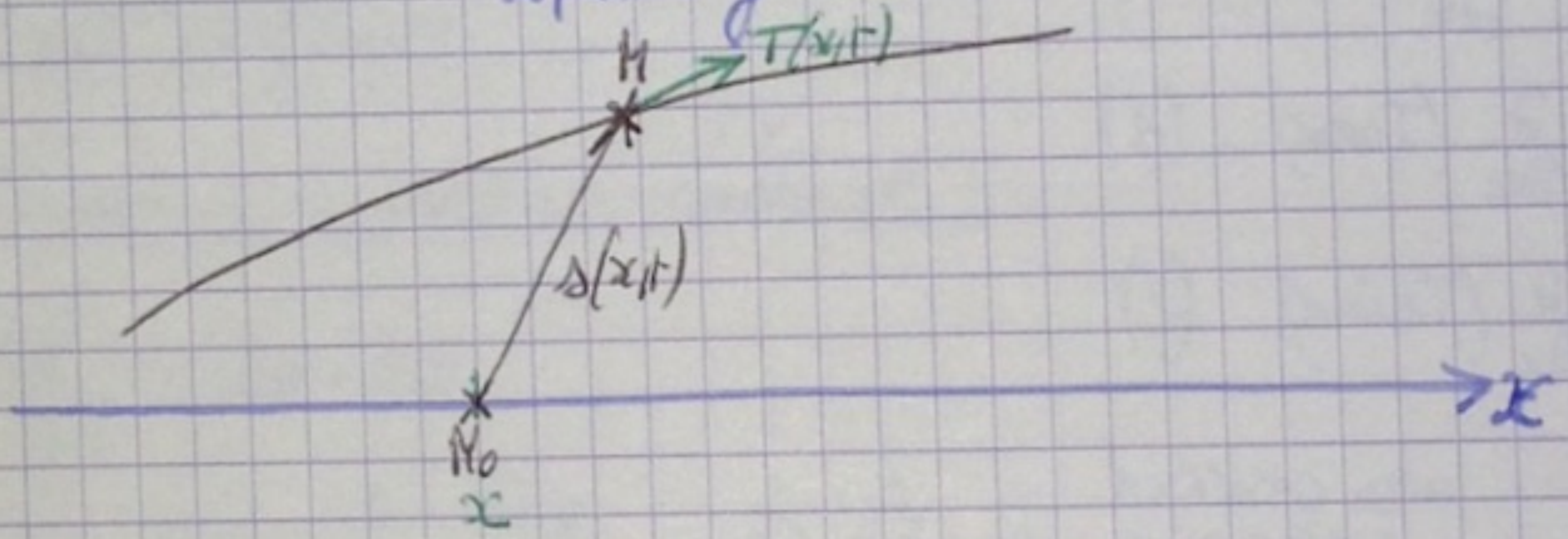
Equation Corde vibrante

1- Modélisation :

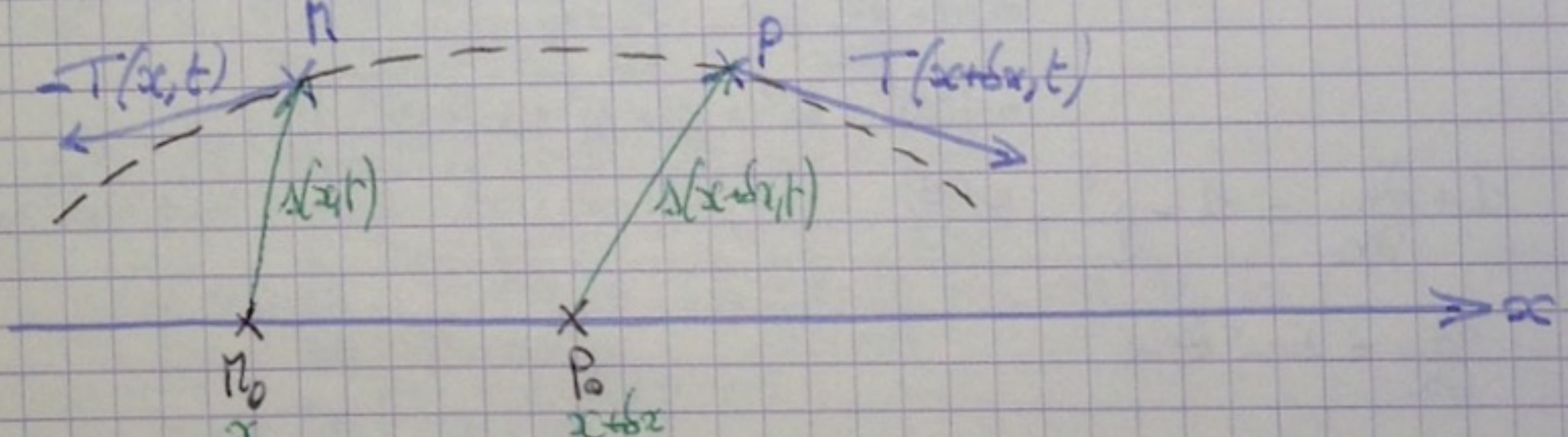
* Corde prétendue suffisamment pour négliger le poids (corde droite pas ◡)
* Corde infiniment souple (sans raideur) : elle se courbe sans aucun effort
  ⇒ elle ne peut transmettre d'effort transverse par rapport à sa direction
  ⇒ la force interne entre éléments adjacents (tension) a pour direction celle de la tangente locale à la corde.

tension $\vec{T}$ au point M : force qu'exerce la partie droite de la corde sur la partie gauche



T_0 : tension de la corde au repos.

2- Equation générale de la corde.



$\overrightarrow{\delta F}$: force sur le petit élément MP de corde

$$\overrightarrow{\delta F} = \vec{T}(x+\delta x,t) - \vec{T}(x,t)$$

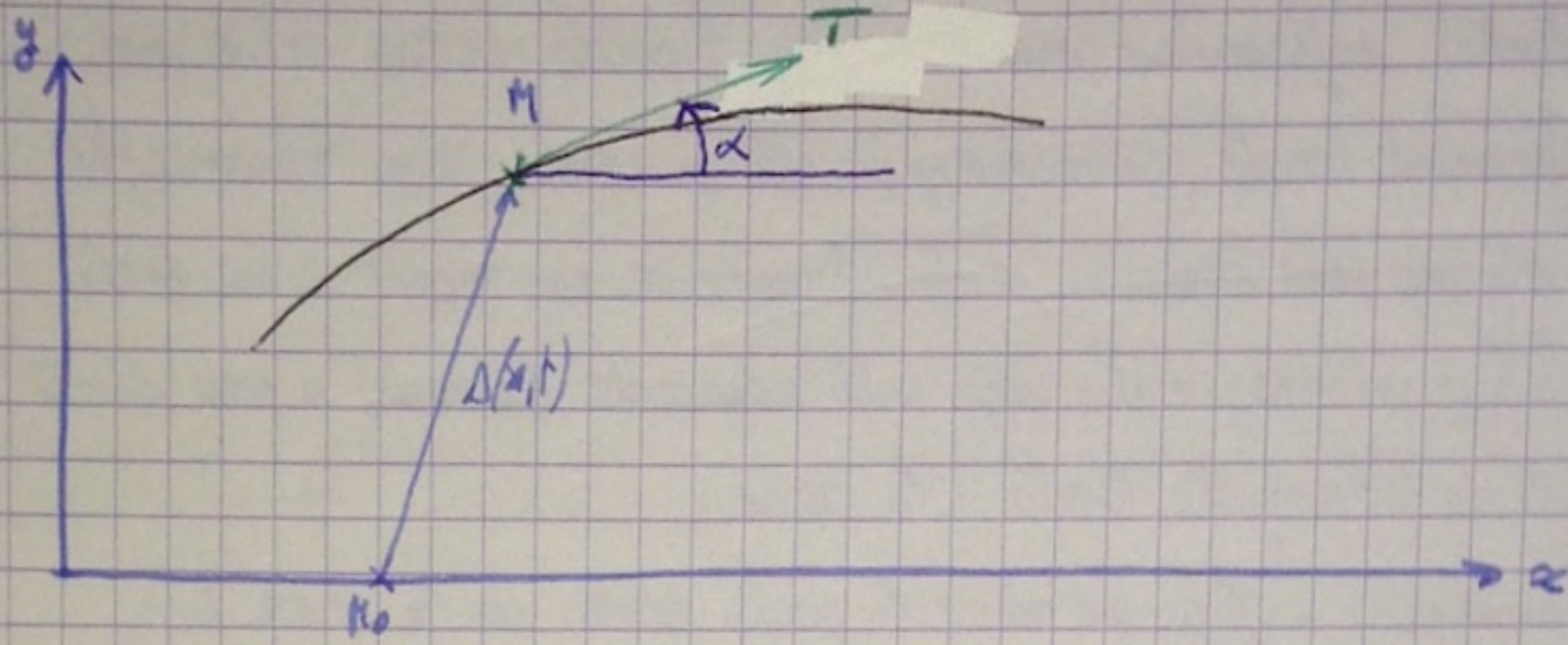
$$\mu\,\delta x\,\frac{\partial^2}{\partial t^2}\vec{s}\left(x+\frac{\delta x}{2},t\right) = \vec{T}(x+\delta x,t) - \vec{T}(x,t)$$

$$= \frac{\partial \vec{T}}{\partial x}\,\delta x$$

$\delta x \to 0$ $$\mu\,\frac{\partial^2}{\partial t^2}\vec{s}(x,t) = \frac{\partial \vec{T}}{\partial x}$$

Mouvement plan : $\vec{s}(x,t) = u(x,t)\vec{e_x} + y(x,t)\vec{e_y}$

3.- Approximation des petites amplitudes



$$\mu \frac{\partial^2 \vec{s}}{\partial t^2} = \frac{\partial \vec{T}}{\partial x} \qquad \vec{T} = T\cos\alpha\,\vec{u_x} + T\sin\alpha\,\vec{u_y}$$

$$\mu \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(T\cos\alpha) \quad (1)$$

$$\mu \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}(T\sin\alpha) \quad (2)$$

H_1. $|\alpha| \ll 1 \implies \cos\alpha \approx 1$
$\implies \sin\alpha \approx \alpha$

Evaluons $\sin\alpha$: $\vec{OM} = \vec{OM_0} + \vec{M_0M}$
α $\qquad = x\vec{u_x} + \vec{s}(x,t)$

tangente : $\frac{\partial \vec{OM}}{\partial x} = \vec{u_x} + \frac{\partial \vec{s}}{\partial x}$ $\qquad \left[\vec{MM'} = \vec{OM}(x+dx) - \vec{OM}(x) = \frac{\partial \vec{OM}}{\partial x}dx \right.$

$$= \left(1 + \frac{\partial u}{\partial x}\right)\vec{e_x} + \frac{\partial y}{\partial x}\vec{e_y}$$

$$= \left[\cos\alpha\ \vec{e_x} + \sin\alpha\,\vec{e_y}\right] \times \left\| \frac{\partial \vec{OM}}{\partial x} \right\|$$

$\hookrightarrow = 1$ au 1er ordre en α

$$1 + \frac{\partial u}{\partial x} = 1 - \frac{\alpha^2}{2} + o(\alpha^3)$$

$$\frac{\partial y}{\partial x} = \sin\alpha \approx \alpha$$

H_2 : Loi de Hooke $\quad \frac{T(x,t) - T_0}{S} = E\,\frac{\delta l - \delta x}{\delta x}$

$$\delta l = \|MM'\| = \delta x \sqrt{\left(1 + \frac{\partial u}{\partial x}\right)^2 + \frac{\partial y}{\partial x}^2} = \delta x \text{ au premier ordre en } \alpha$$

Donc au premier ordre en α, $T(x,t) = T_0$

(1) : $\mu \frac{\partial^2 u}{\partial t^2} = 0$

$u = At + B \Rightarrow u = 0$

(2) $\mu \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2}$

$\frac{\mu_0}{T_0} \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0$

$c^2 = \frac{T_0}{\mu_0} \Rightarrow c = \sqrt{\frac{T_0}{\mu}}$ ← raideur, tension ; ← inertie

Aspect énergétique.

Théorème de la puissance cinétique.

$\frac{\partial E_c}{\partial t} = \mathcal{P}_{ext} + \mathcal{P}_{int}$

$E_c = \frac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 \delta x \Rightarrow \frac{\partial E_c}{\partial t} = \mu \frac{\partial y}{\partial t} \frac{\partial^2 y}{\partial t^2} \delta x$

$\mathcal{P}_{ext} = \vec{T}(x+dx,t)\vec{v}(x+dx,t) - \vec{T}(x,t)\vec{v}(x,t)$

$\mathcal{P}_{ext} = T_0 \left[\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}(x+dx,t) - \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}(x,t) \right]$

$\begin{cases} \vec{T} = T_0(\cos\alpha\,\vec{e_x} + \sin\alpha\,\vec{e_y}) \\ \quad = T_0(\cos\alpha\,\vec{e_x} + \frac{\partial y}{\partial x}\vec{e_y}) \\ \vec{v} = \frac{\partial y}{\partial t}\vec{u_y} \end{cases}$

$\mathcal{P}_{ext} = T_0 \frac{\partial}{\partial x}\left[\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right]\delta x$

$\mu \frac{\partial y}{\partial t}\frac{\partial^2 y}{\partial t^2}\delta x = T_0 \frac{\partial}{\partial x}\left[\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right]\delta x + \mathcal{P}_{int}$

$\mathcal{P}_{int} = \cancel{\mu \frac{\partial y}{\partial t} \times \frac{T_0}{\mu}\frac{\partial^2 y}{\partial t^2}\delta x - T_0 \frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t}\delta x} - T_0 \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x \partial t}\delta x$

$= \frac{\partial}{\partial t}\left[-\frac{T_0}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]$

$$\boxed{E_p = \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2}$$