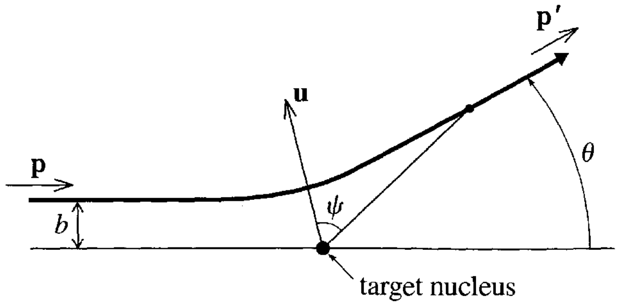

Figure 14.11 The Rutherford scattering of an alpha particle off a fixed atomic nucleus. The orbit is a hyperbola, which is symmetric about the line labelled by the fixed unit vector **u**. The position of the particle can be labelled by its angle ψ measured from **u**. As the particle moves away ($t \to \infty$), $\psi \to \psi_o$, and as $t \to -\infty$, $\psi \to -\psi_o$. Therefore the scattering angle is $\theta = \pi - 2\psi_o$.

the direction of **u**, and it is convenient to label the alpha's position by the polar angle ψ, measured from **u** (see Figure 14.11). Let us denote by ψ_o the limit of ψ as the scattered alpha moves far away, so that the total angle subtended by the alpha's orbit is $2\psi_o$ and the scattering angle is

$$\theta = \pi - 2\psi_o. \tag{14.27}$$

Our job now is to relate the scattering angle θ to the impact parameter b. We can do this by evaluating in two ways the change in the momentum of the projectile,

$$\Delta\mathbf{p} = \mathbf{p}' - \mathbf{p}, \tag{14.28}$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the momentum long before and long after the encounter. First, by conservation of energy, $\mathbf{p}$ and $\mathbf{p}'$ have equal magnitudes, so that the triangle shown in Figure 14.12 is isosceles, and

$$|\Delta\mathbf{p}| = 2p\sin(\theta/2). \tag{14.29}$$

On the other hand, from Newton's second law, $\Delta\mathbf{p} = \int \mathbf{F}dt$. Comparing Figures 14.12 and 14.11, you can see that $\Delta\mathbf{p}$ is in the same direction as the unit vector **u**. Thus

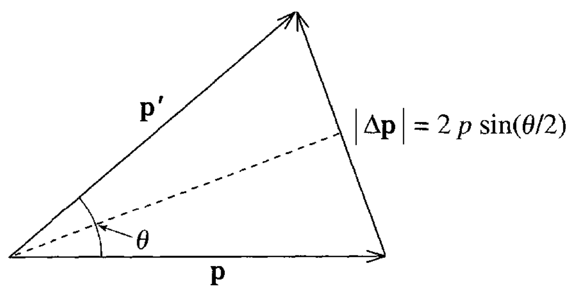

Figure 14.12 The change in momentum of the projectile is $\Delta\mathbf{p} = \mathbf{p}' - \mathbf{p}$. Since, $|\mathbf{p}| = |\mathbf{p}'|$, it is easily seen that $|\Delta\mathbf{p}| = 2p\sin(\theta/2)$.

the magnitude of $\Delta\mathbf{p}$ is given by the same integral, with $\mathbf{F}$ replaced by its component F_u in the direction of $\mathbf{u}$,

$$|\Delta\mathbf{p}| = \int_{-\infty}^{\infty} F_u \, dt.$$

From Figure 14.11 you can see that $F_u = (\gamma/r^2)\cos\psi$. Using the now-familiar trick, we can write $dt = d\psi/\dot{\psi}$, where, since $mr^2\dot{\psi} = \ell = bp$ (see Figure 14.11 again), we can replace $\dot{\psi}$ by bp/mr^2. Putting all of this together, we find

$$|\Delta\mathbf{p}| = \int_{-\psi_o}^{\psi_o} \frac{\gamma\cos\psi}{r^2}\frac{d\psi}{bp/mr^2} = \frac{\gamma m}{bp} 2\sin\psi_o = \frac{2\gamma m}{bp}\cos(\theta/2). \tag{14.30}$$

[To understand the limits in the integral, recall that as $t \to \pm\infty$, so $\psi \to \pm\psi_o$. In the last step I used (14.27) to replace ψ_o by $(\pi - \theta)/2$ and hence $\sin\psi_o$ by $\cos(\theta/2)$.] Equating the two expressions (14.29) and (14.30) for $|\Delta\mathbf{p}|$, we can solve for b to give

$$b = \frac{\gamma m}{p^2}\frac{\cos(\theta/2)}{\sin(\theta/2)} = \frac{\gamma}{mv^2}\cot(\theta/2) \tag{14.31}$$

where in the last equality I replaced p by mv, and v is the projectile's incident speed.

Having found the impact parameter b as a function of the scattering angle θ, we can now use the result (14.23) to give the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta}\cdot b \cdot \left|\frac{db}{d\theta}\right| = \frac{1}{2\sin(\theta/2)\cos(\theta/2)}\cdot\frac{\gamma}{mv^2}\cot(\theta/2)\cdot\frac{\gamma}{mv^2}\frac{1}{2\sin^2(\theta/2)}$$

or, replacing γ by kqQ,

$$\frac{d\sigma}{d\Omega} = \left(\frac{kqQ}{4E\sin^2(\theta/2)}\right)^2 \tag{14.32}$$

where E is the energy of the incident projectiles, $E = \frac{1}{2}mv^2$. This is the celebrated **Rutherford scattering formula**. It gives the differential cross section for scattering of a charge q, with energy E, off a fixed target of charge Q. While it is still today, nearly a century after its derivation by Rutherford, a much-used result, its great historical importance is that it was used to prove the existence of the atomic nucleus, as we now discuss briefly.[9]

[9] Since the atom is a microscopic system, for which quantum, not classical, mechanics should be used, you may be surprised that the classical Rutherford formula worked so well for Rutherford and his assistants. It is one of the most amazing accidents in the history of physics that the quantum formula for scattering of two charged particles agrees exactly with Rutherford's classical formula. (This is certainly not true for other force laws.)

The Experiment of Geiger and Marsden

The best known and most important Rutherford-scattering experiment was performed by Rutherford's assistants Hans Geiger (inventor of the Geiger counter, 1882–1945) and Ernest Marsden (1889–1970) and published in 1913. Their goal was to test Rutherford's "planetary" model of the atom, according to which most of the atomic mass is concentrated in a tiny, positively charged nucleus.[10] As we have seen, this model leads to the cross section (14.32) for scattering of alpha particles, with several very specific predictions: The scattering probablility should be inversely proportional to $\sin^4 \theta/2$, inversely proportional to the energy squared E^2, and proportional to the nuclear charge squared (Q^2). Geiger and Marsden were able to verify all of these predictions with amazing precision, and hence to contribute to the rapid acceptance of Rutherford's nuclear atom. They used alpha particles coming from radon gas ("radium emanation" as it was called then), with energy around 6.5 MeV. (1 MeV = 10^6 electron volts, and 1 eV = 1.6×10^{-19} joules.) They directed a narrow "pencil" of these at a thin metal foil and counted the scattered particles using a small zinc sulphide screen. Any alpha particle striking this screen caused a tiny flash of light or "scintillation," which could be observed through a microscope. In this way, it was possible to count up to about 90 alpha particles per minute (a job needing great patience and concentration!). To observe the angular dependence of the scattering, they could swing the screen and microcope around to angles in the range $5° \leq \theta \leq 150°$. To test the dependence on incident energy, they passed the incident particles through thin sheets of mica, to slow them down and hence vary their energy. And to test the dependence on nuclear charge, they used various different target foils (gold, platinum, tin, silver, copper, and aluminum).

EXAMPLE 14.6 Angular Dependence

To isolate its angular dependence write the Rutherford cross section (14.32) as

$$\frac{d\sigma}{d\Omega}(\theta) = \frac{\sigma_0(E)}{\sin^4 \theta/2} \tag{14.33}$$

and find $\sigma_0(E)$ for scattering of 6.5 MeV alphas off gold. Find the differential cross section at 150° and 5° (Geiger and Marsden's largest and smallest angles). Find the number of alphas they would have had to count in a minute assuming the following values: The number of incident alphas in one minute, $N_{\text{inc}} = 6 \times 10^8$; the thickness of the gold foil, $t = 1\,\mu$m; area of zinc sulphide screen = 1 mm^2; and distance of screen from target = 1 cm. Make a useable plot of the differential cross section as a function of scattering angle θ.

[10] Initially, the sign of the nuclear charge (positive or negative) was not clear, but it was soon found to be positive, with an equal negative charge carried by the orbiting electrons.

The charge of the alpha particle is $q = 2e$ and that of the gold nucleus is $Q = 79e$, so

$$\sigma_o(E) = \left(\frac{2 \times 79 \times ke^2}{4E}\right)^2.$$

This is easily evaluated in SI units, though a slicker way is to use the useful combination $ke^2 = 1.44$ MeV·fm (where fm stands for femtometer or 10^{-15} m). Either way, we find that

$$\sigma_o = 76.6 \times 10^{-30}\ \mathrm{m}^2/\mathrm{sr} = 0.766\ \mathrm{barns/sr}.$$

Substituting into (14.33) we get

$$\frac{d\sigma}{d\Omega}(150°) = 0.88\ \mathrm{barns/sr} \quad \text{and} \quad \frac{d\sigma}{d\Omega}(5°) = 2.1 \times 10^5\ \mathrm{barns/sr}. \tag{14.34}$$

The huge difference between these — more than 5 orders of magnitude — presents considerable practical difficulties, as we shall see. Before we can substitute into (14.17) to give the actual numbers counted we need to calculate n_{tar} and $d\Omega$. As usual, we can find n_{tar} in terms of the density of gold (specific gravity 19.3) and its atomic mass (197):

$$n_{\text{tar}} = \frac{\varrho t}{m} = \frac{(19.3 \times 10^3\ \mathrm{kg/m^3}) \times (10^{-6}\ \mathrm{m})}{197 \times 1.66 \times 10^{-27}\ \mathrm{kg}} = 5.90 \times 10^{22}\ \mathrm{m}^{-2}.$$

Geiger and Marsden's screen had area $A = 1\ \mathrm{mm}^2$ and was at a distance $r = 10$ mm from the target. Therefore, it subtended a solid angle

$$d\Omega = \frac{A}{r^2} = 0.01\ \mathrm{sr}.$$

Putting all of this together, we find for the number of alphas hitting their screen at 150° in a minute

$$\begin{aligned} N_{\text{sc}}(\text{at } 150°) &= N_{\text{inc}}\, n_{\text{tar}} \frac{d\sigma}{d\Omega}(150°)\, d\Omega \\ &= (6 \times 10^8) \times (5.90 \times 10^{22}\ \mathrm{m}^{-2}) \times (0.88 \times 10^{-28}\ \mathrm{m^2/sr}) \times (0.01\ \mathrm{sr}) \\ &= 31, \end{aligned}$$

a number that they could count easily and accurately. On the other hand, the same calculation gives

$$N_{\text{sc}}(\text{at } 5°) = 7.5 \times 10^6,$$

a number that they could not possibly count or even estimate. Obviously measuring the cross section at small angles required them to use a much, much weaker source than at large angles.

Because of the huge variation of the cross section as the scattering angle varies, a straightforward linear plot of $d\sigma/d\Omega$ is not especially useful. If we choose a scale to show the small angles, the cross section for large angles will