Runge-Kutta Differential Equation Solver for Orbital Problems*

Pierre Hurlin†
Maastricht University

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Abstract

The goal of this study is to implement a third-order Runge-Kutta differential equation solver and to apply it to the classical mechanical problem of a satellite orbiting a large central body. Solving this problem by numerical analysis has the advantage to eliminate the tediousness of the analytical solution. However, it raises the question of the accuracy of the numerical solution. We provide various assessments of this accuracy by comparing the numerical solution to the analytical solution deduced from the Kepler's equation. We illustrate the influence of the parameter (standard gravitational parameter, etc.) and the choice of initial conditions on the simulated orbit. Finally, we check the solver by computing the solution for an elliptical orbit over a full period for different step-sizes, and we show that the error has the expected order.

*This study has been realized as a project for the first year of the Bachelor Data Science and Artificial Intelligence, DKE, Maastricht University . The corresponding code has been realized both with Matlab and Java. All the figures reported in this document have been generated with the Matlab code. The schemes have been drawn with Word.

†Maastricht University, Department of Data Science and Knowledge Engineering (DKE). Email: p.hurlin@student.maastrichtuniversity.nl

Contents

1 Introduction

The goal of this study is to implement a third-order Runge-Kutta differential equation solver and to apply it to the classical mechanical two-body or one-body problem, i.e., the problem which describes the orbit of a body subject to a central force. We will consider the case of a satellite orbiting a large central body, e.g., that of Titan around Saturn. By considering the vectorial differential equation deduced from the Newton's law of universal gravitation, we compute an approximated orbit for the satellite. We test our solver by considering an initial position for the satellite located at its periapsis (perikrone in the specific case Saturn), i.e., the nearest point reached by Titan with respect to Saturn.

The advantage of the two-body or one-body problem is that it allows to derive an analytical solution. Indeed, if we neglect all perturbations, it is possible to show that the true orbit is elliptical with a constant period. This theoretical solution corresponds to the solution of the Kepler's equation. In astronomy, the Kepler equation is a formula linking, in an orbit, the eccentricity and the eccentric anomaly to the mean anomaly[1]. The importance of this equation is that it makes it possible to pass from the dynamic parameters of the movement of a satellite (the mean anomaly) to the geometric parameters (the eccentric anomaly).

Solving this mecanical problem by numerical analysis has the advantage to eliminate the tediousness of the analytical solution. However, we have to check the accuracy of the approximate solution provided by the third-order Runge-Kutta solver. For that, we systematically compare the numerical and analytical solutions. We represent the 2-dimensions orbits for various values of the parameters and choices of the initial values. We discuss the influence of the eccentricity and standard gravitational parameters. The eccentricity parameter controls for the flatness of the elliptical orbit. The standard gravitational parameter is defined as the product of the gravitational constant by the mass of the central body, i.e., Saturn in our case. Finally, we check the solver by computing the solution for an elliptical orbit over a full period for different step-sizes and we show that the error has the expected order. Our study show that the third-order Runge-Kutta solver provides approximate, yet still very accurate results for the satellite's orbit.

The rest of the paper is structured as follows. Section 2 presents the method, including the framework and notations, the Kepler's equation, the Newton's law of universal gravitation, and the third-order Runge Kutta method used to solve the ordinary differential equations. Section 3 presents the experiments. The results are presented and discussed in Sections 4 and 5. The section 6 concludes.

[1] Source: Wikipedia, article "Kepler's equation".

2 Methods

The orbit of a body in space is governed by the laws of planetary motion as described by Kepler and the laws of gravitational motion as described by Newton. Here, we will consider the problem of a satellite orbiting a large central body, e.g., that of Titan around Saturn.

2.1 Framework and notations

Kepler's three laws of planetary motion can be stated as follows[2]: (1) All planets move about the Sun in elliptical orbits, having the Sun as one of the foci, (2) A radius vector joining any planet to the Sun sweeps out equal areas in equal lengths of time, (3) The squares of the periods of revolution of the planets are directly proportional to the cubes of their mean distances from the Sun.

Extending the first Kepler's law of planetary motion to any satellite, we assume that that a large central body, say Saturn, is located at the focus of the elliptical orbit of its satellite, say Titan, as represented on Figure 1. The longest axis of the ellipse is called the major axis, while the shortest axis is called the minor axis. Half of the major axis is termed a semi-major axis. We denote by a the semi-major axis, i.e., the distance $a = OA$, and b the semi-minor axis, i.e., $b = OB$. The apsides refer to the farthest (A') and nearest (A) points reached by Titan with respect to Saturn. The points A' and A are respectively called periapsis (perikrone or perisaturnium in the specific case of Saturn) and apoapsis (apokrone or aposaturnium for Saturn)[3].

The points S and S' are each called a focus, and together called foci. By definition, the sum of the distances to the foci from any point on the ellipse is always a constant, with $SP + SP' = 2a$. The focus S is located at a distance c from the intersection (point O) of the minor and major axes, with $c = \sqrt{a^2 - b^2}$. Thus, the Cartesian equation of an ellipse centered at the origin O, is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

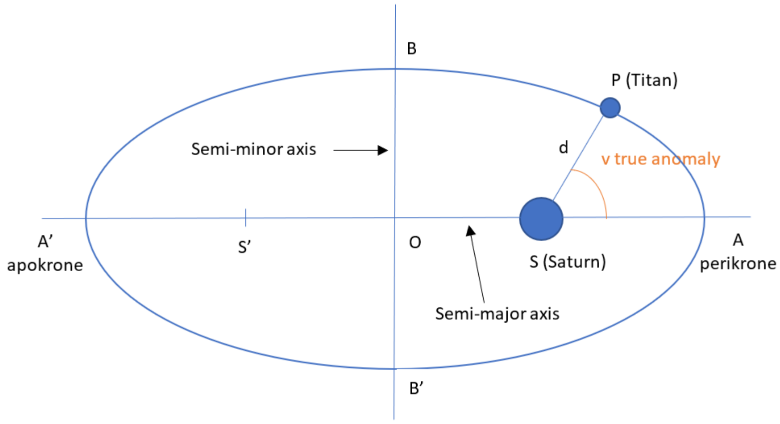
In the sequel, we will we consider the focus S, namely Saturn, as the cardinal point of reference. Then, the Cartesian equation of the ellipse becomes:

$$\frac{(x-c)^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{2}$$

[2] Source: Britannica Encyclopedia, article "Kepler's laws of planetary motion".

[3] Periapsis and apoapsis also denote the distances SA and SA', i.e., the smallest and largest distances between the satellite, namely Titan, and its host body, namely Saturn. Source: Wikipedia, article "Apsis".

Figure 1: Elliptical orbit of Titan



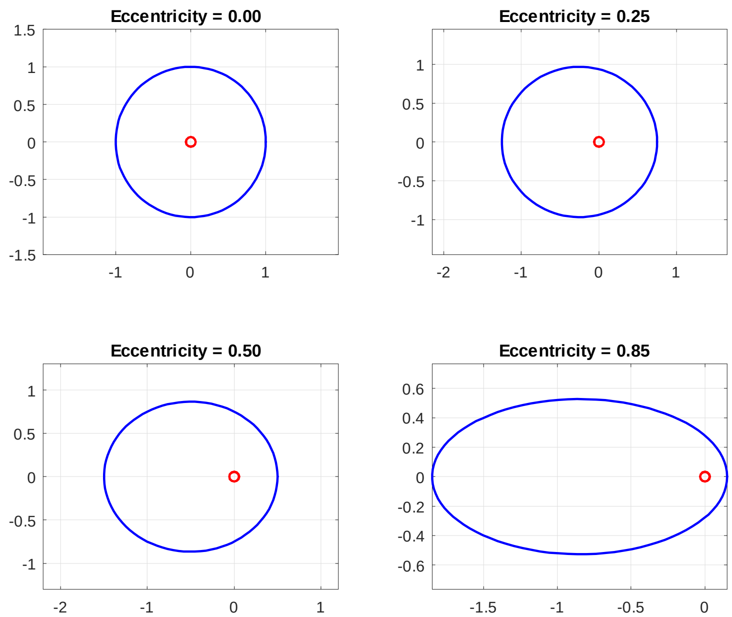
The flatness of the ellipse is measured by its eccentricity, denoted e. The eccentricity can be defined as the ratio of the distance c between the center of the ellipse and each focus to the length of the semi-major axis a, i.e., $e = c/a$, with $0 \leq e < 1$. For a perfect circle, a and b are the same such that the eccentricity is zero. As the eccentricity tends toward 1, the ellipse gets a more elongated shape. Figure 2 display four ellipses with a semi-major axis a equal to 1 and an eccentricity parameter respectively equal to 0, 0.25, 0.5 and 0.85. In each case, the focus S is represented by a red circle. Notice that if the eccentricity is equal to one, the orbit is no longer an ellipse but a parabola, this is why we exclude this case here. In reality, the Titan's orbit has an eccentricity of 0.0288[4], so its orbit is very close to a perfect circle.

2.2 The Kepler's equation

In astronomy, the Kepler equation is a formula linking, in an orbit, the eccentricity e and the eccentric anomaly u to the mean anomaly M_a[5]. Before to study the kinematics of Titan, we first introduce the concepts of mean anomaly and eccentric anomaly. As represented on Figure 3, we consider an auxiliary circle of radius a and center O. The eccentric anomaly u is one of the angles of a right triangle with one vertex at the center of the ellipse O, its adjacent side lying on the major axis, having hypotenuse a (equal to the semi-major axis of the ellipse), and opposite side (perpendicular to the major axis and touching

[4] Source: Britannica Encyclopedia, article "Titan, astronomy".
[5] This presentation of the Keppler equation is derived from [4].

Figure 2: Ellipses for various eccentricity parameters



the point P' on the auxiliary circle of radius a) that passes through the point P[6]. Thus, the eccentric anomaly corresponds to the angle $u = \widehat{P'OA}$. The true anomaly, denoted v, is the angle that the vector SP makes with the direction of the perikrone SA. So, we have $v = \widehat{PSA}$. Notice that the Cartesian equations of the ellipse (equations 1 and 2) can be rewritten as a function of the eccentric anomaly as:

$$(x, y) = (a\cos(u)\,, b\sin(u)), \qquad 0 \le u < 2\pi \tag{3}$$

if the origin corresponds to the point O, or as:

$$(x, y) = (a\cos(u) - c, b\sin(u)), \qquad 0 \le u < 2\pi \tag{4}$$

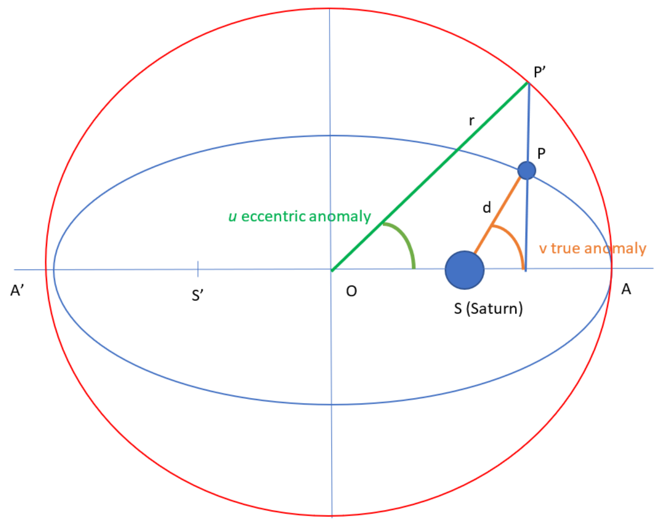
if the origin corresponds to the focus S, i.e., the position of Saturn. Notice that in our code, we use the equation 4 to display the true elliptical orbit.

The true anomaly and the eccentric anomaly are related as follows:

$$\tan\left(\frac{v}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right) \tag{5}$$

[6]Source: Wikipedia, article "eccentric anomaly".

Figure 3: Eccentric and true anomalies



Finally, if the focus S is the origin, the link between the cartesian coordinates (x, y) and the eccentric anomaly is given by:

$$x = a\left(\cos\left(u\right) - e\right) \qquad y = a\sin\left(u\right)\sqrt{1-e^2} \tag{6}$$

The Keppler equation relates various geometric properties of the orbit of a body subject to a central force. This equation is given by:

$$u - e\sin\left(u\right) = \frac{2\pi}{T}\left(t - t_0\right) = M_a \tag{7}$$

where T is the period, i.e., the duration that it takes for the satellite to complete a full orbit, t is time and t_0 is being the time of the transition to perikrone. The quantity $2\pi/T\left(t-t_0\right)$ is called mean anomaly and is denoted by M_a[7]. The period T is equal to:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{8}$$

Thus, given the Kepler equation (7) and the link between the true and the eccentric anomalies (equation 5), or the Cartesian coordinates (equation 6), it is possible to get the position (x, y) of the satellite Titan a each time t on the elliptical orbit. The Kepler equation can be solved by a fixed-point method.

[7]The mean anomaly is often denoted by M, but here we denote it as M_a to make the distinction with the mass of Titan also denoted M in the statement of the resit.

By considering an initial condition $u_0 = M_a/(1-e)$, we compute the recursive series $u_{n+1} = e\sin(u_n) + M_a$ until $|u_{n+1} - u_n| < \varepsilon$, with $\varepsilon = 0.0001$.

Finally, we could express the velocity of Titan as follows. When the distance between Saturn and Titan is equal to $d = a\,(1 - e\cos(u)) = \sqrt{x^2+y^2}$, we have:

$$v^2 = \mu\left(\frac{2}{d} - \frac{1}{a}\right) \tag{9}$$

with $v = \sqrt{v_x^2 + v_y^2}$ and where v_x and v_y respectively denote the horizontal (in the major-axis direction) and vertical (in the minor-axis direction) velocity. Given this formula, we can compute for instance the velocity at the perikrone. When Titan reaches the perikrone (point A), $d = a\,(1-e)$, its velocity is equal to[8]:

$$v^2_{\text{perikrone}} = \mu\left(\frac{2}{a\,(1-e)} - \frac{1}{a}\right) = \frac{\mu}{a}\left(\frac{1+e}{1-e}\right) \tag{10}$$

$$v^2_{x,\text{perikrone}} = 0, \qquad v^2_{y,\text{perikrone}} = \frac{\mu}{a}\left(\frac{1+e}{1-e}\right) \tag{11}$$

This relationship will be used in our code to fix the initial conditions of the ODE which represents the Newton's law of universal gravitation.

2.3 Newton's law of universal gravitation

Newton was able to solidify the relationship between falling objects near to the Earth and the motion of planetary orbits by using the inverse-square relationship that Kepler had suggested in his third law. Newton's law of universal gravitation is usually stated as that every particle attracts every other particle in the universe with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their center[9]. The equation for universal gravitation thus takes the form:

$$F = G\frac{M\ m}{r^2}, \tag{12}$$

where F is the gravitational force acting between two objects, M and m are the masses of central body and its orbiting satellite, r is the distance between the centers of their masses, and G is the gravitational constant. In the sequel, we

[8] Similarly when Titan reaches the apokrone (point A'), $d = a(1+e)$, and its velocity satisfies:

$$v^2_{\text{apokrone}} = \frac{\mu}{a}\left(\frac{1-e}{1+e}\right)$$

$$v^2_{x,\text{apokrone}} = 0, \qquad v^2_{y,\text{apokrone}} = -\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)$$

[9] The sources of this section are three articles issued from Wikipedia: Newton's law of universal gravitation, Gravitational constant, and Kepler orbit.

will assume that compared to the central body, the mass of the orbiting satellite is insignificant, i.e., $M >> m$ and we will neglect the latter. This assumption is appropriate in our case as the mass of Titan is equal to 1.345×10^{23} kg, whereas the mass of Saturn is equal to 5.6834×10^{26} kg. Denote by μ the product of the gravitational constant by the mass of Saturn, such that $\mu = GM$.

In a vectorial form, the Newton's law of universal gravitation becomes:

$$\mathbf{F} = \frac{\mathsf{d}^2 \mathbf{x}}{\mathsf{d}t^2} = -G \frac{M}{|\mathbf{x}|^3} \mathbf{x} = -\frac{\mu}{|\mathbf{x}|^3} \mathbf{x}, \tag{13}$$

where quantities in bold represent vector, $\mathbf{F}$ denotes the gravitational force, $\mathbf{x}$ is the position vector of the satellite, and $|\mathbf{x}| = d$ is the distance between Saturn and its satellite. The acceleration of the satellite is parallel and directly proportional to the net force acting on the satellite, is in the direction of the net force, and is inversely proportional to the mass of the central body.

This formula translates into two formulas, one for each component:

$$\frac{\mathsf{d}^2 x}{\mathsf{d}t^2} = -\frac{\mu}{d^3} x, \qquad \frac{\mathsf{d}^2 y}{\mathsf{d}t^2} = -\frac{\mu}{d^3} y, \tag{14}$$

with $d = \sqrt{x^2 + y^2}$ if the focus S is the origin of the Cartesian plane. As the acceleration is the derivative of the velocity with respect to time, we have a system of four equations:

$$v_x = \frac{\mathsf{d}x}{\mathsf{d}t}, \qquad v_y = \frac{\mathsf{d}y}{\mathsf{d}t}, \tag{15}$$

$$\frac{\mathsf{d}^2 x}{\mathsf{d}t^2} = \frac{\mathsf{d}v_x}{\mathsf{d}t} = -\frac{\mu}{(x^2 + y^2)^{3/2}} x, \qquad \frac{\mathsf{d}^2 y}{\mathsf{d}t^2} = \frac{\mathsf{d}v_y}{\mathsf{d}t} = -\frac{\mu}{(x^2 + y^2)^{3/2}} y. \tag{16}$$

Let us define a vector $\mathbf{z} = (x : y : v_x : v_y)$ associated to the localisation and the velocity of Titan, we get an ordinary differential equation (ODE) such that:

$$\dot{\mathbf{z}} = \frac{\mathsf{d}\mathbf{z}}{\mathsf{d}t} = \mathbf{f}(t, \mathbf{z}), \quad \mathbf{z}(t_0) = \mathbf{z}_0 \tag{17}$$

with $\mathbf{f}(t, \mathbf{z}) = (f_1(t, \mathbf{z}) : f_2(t, \mathbf{z}) : f_3(t, \mathbf{z}) : f_4(t, \mathbf{z}))$ and

$$\dot{x} = f_1(t, \mathbf{z}) = v_x, \tag{18}$$

$$\dot{y} = f_2(t, \mathbf{z}) = v_y, \tag{19}$$

$$\dot{v}_x = f_3(t, \mathbf{z}) = -\frac{\mu}{(x^2 + y^2)^{3/2}} x, \tag{20}$$

$$\dot{v}_y = f_4(t, \mathbf{z}) = -\frac{\mu}{(x^2 + y^2)^{3/2}} y. \tag{21}$$

with $\mathbf{z}_0 = (x_0, y_0, v_{x,0}, v_{y,0})$ a vector of initial conditions, and where quantities in bold represent a vector. Several methods can be used to solve this ODE and thus get the trajectory (position and velocity) of Titan as function of time (see [3] for a complete overview). Here, we will focus on the third-order Rung-Kutta equation solver.

2.4 Runge-Kutta differential equation solver

Runge–Kutta methods are numerical methods used in temporal discretization to approximate the solution of an ODE. The Runge–Kutta family includes many iterative methods (Euler Methods, mid-point method, second-order Runge-Kutta method, third-order Runge-Kutta method, fourth-order Runge-Kutta method or RK4, etc.). Here, we will focus on third-order Runge-Kutta methods. The presentation of the Runge-Kutta methods is reproduced from [3].

In our context, the objective is to find a vectorial function $\mathbf{z} : [t_0, t_f] \to \mathbb{R}^4$ satisfying :

$$\dot{\mathbf{z}} = \frac{\mathsf{d}\mathbf{z}}{\mathsf{d}t} = \mathbf{f}\left(t, \mathbf{z}\right), \quad \mathbf{z}\left(t_0\right) = \mathbf{z}_0, \tag{22}$$

where t_0 is the initial time, $\mathbf{z}_0$ is the initial state, and t_f is the final time. A numerical solution approximates $\mathbf{z}$ at points $t_0 < t_1 < \ldots < t_n = t_f$ as

$$\mathbf{z}\left(t_i\right) = \mathbf{w}_i, \quad \forall i = 0, 1, \ldots, n \tag{23}$$

with a constant step size h such that $t_i = t_0 + ih$ and $h = \left(t_f - t_0\right)/n$.

A general third-order method is defined on the following iterative algorithm, for each step $i \geq 1$:

$$\mathbf{k}_{i,1} = h\mathbf{f}\left(t_i, w_i\right), \tag{24}$$

$$\mathbf{k}_{i,2} = h\mathbf{f}\left(t_i + c_2 h, \mathbf{w}_i + a_{21}\mathbf{k}_{i,1}\right), \tag{25}$$

$$\mathbf{k}_{i,3} = h\mathbf{f}\left(t_i + c_3 h, \mathbf{w}_i + a_{31}\mathbf{k}_{i,1} + a_{32}\mathbf{k}_{i,2}\right), \tag{26}$$

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \left(b_1\mathbf{k}_{i,1} + b_2\mathbf{k}_{i,2} + b_3\mathbf{k}_{i,3}\right). \tag{27}$$

where the parameters c_2 and c_3 are chosen freely and where the other parameters satisfy the following constraints :

$$b_3 = (3c_2 - 2)/(6c_3(c_2 - c_3)), \tag{28}$$

$$b_2 = (3c_3 - 2)/\left(6c_2(c_3 - c_2)\right), \tag{29}$$

$$b_1 = 1 - b_2 - b_3, \tag{30}$$

$$a_{32} = 1/\left(6c_2 b_3\right), \ a_{31} = c_3 - a_{32}, \ a_{21} = c_2. \tag{31}$$

In our code, it is possible to consider any third-order Runge-Kutta method by choosing the parameters c_2 and c_3. By default, we consider here the Heun's third order method with $c_1 = 1/3$ and $c_2 = 2/3$. Given, the previous constraints, we have $b_1 = 1/4$, $b_2 = 0$, $b_3 = 3/4$, $a_{21} = 1/3$, $a_{31} = 0$, and $a_{32} = 2/3$. Thus, the system associated to the Heun's method becomes:

$$\mathbf{k}_{i,1} = h\mathbf{f}\left(t_i, \mathbf{w}_i\right), \tag{32}$$

$$\mathbf{k}_{i,2} = h\mathbf{f}\left(t_i + \frac{1}{3}h, \mathbf{w}_i + \frac{1}{3}\mathbf{k}_{i,1}\right), \tag{33}$$

$$\mathbf{k}_{i,3} = h\mathbf{f}\left(t_i + \frac{2}{3}h, \mathbf{w}_i + \frac{2}{3}\mathbf{k}_{i,2}\right), \tag{34}$$

$$\mathbf{w}_{i+1} = \mathbf{w}_i + \frac{1}{4}\left(\mathbf{k}_{i,1} + 3\mathbf{k}_{i,3}\right). \tag{35}$$

Notice that with the Heun's third order method, the local error is $O(h^4)$, whereas the global error is $O(h^3)$.

3 Experiments

Following the statement of the resit exam, the goal of this study is threefold:

1. To test a Runge–Kutta third order solver on the orbit problem in 2-dimensions with initial position $(x_0, 0)$ and velocity $(0, v_0)$.
2. To try different parameters and initial conditions, including $x_0 = v_0 = 1$ and $\mu = 1, 2$.
3. To check the Heun's third order solver by computing the solution for an elliptical orbit over a full period for different step-sizes h, and to show that the error has the expected order.

3.1 Tests of the Runge–Kutta third order solver

First, we consider three experiments (respectively denoted A, B, and C) in order to test our Runge–Kutta third order solver. The corresponding parameters are fixed to the values reported in Table 1. In the three experiments, the standard gravitational parameter $\mu = GM$, i.e., the product of the gravitational constant G and the mass M of Saturn, is normalised to 1. In fact, in case of Saturn, the standard gravitational parameter μ is equal[10] to $3.7931187 \times 10^{16} m^3 s^{-2}$. The normalization $\mu = 1$ avoids to consider large values in the numerical simulation. The semi-major axis of the elliptical orbit is also normalised to 1. In reality, the semi-major axis of Titan[11] is equal to $1,221,870$ km. Finally, we consider 3 values for the eccentricity parameter, namely $e = 0$ (experiment A), $e = 0.5$

[10] Source: Wikipedia, article "standard gravitational parameter".
[11] Source: Wikipedia, article "Titan (moon)".

(experiment B), and $e = 0.8$ (experiment C). Remind that when $e = 0$, the orbit is a circle of radius a. In reality, the orbit of Titan is close to a perfect circle as its eccentricity is 0.0288. For all the experiments the step size h is fixed to 0.1, the initial time t_0 is 0, and the final time t_f is fixed to the period T. The period (cf. equation 8) is equal to $2\pi = 6.2832$ in all cases, as it only depends on parameters μ and a, which are common to all experiments. Finally, we report the semi-minor axis b and the distance between O and the focus S, namely c, deduced from other parameters.

Table 1: Parameters of experiments A, B, and C

	Experiment A	Experiment B	Experiment C
μ	1	1	1
a	1	1	1
e	0	0.5	0.8
h	0.1	0.1	0.1
b	1	0.8660	0.6000
c	0	0.5000	0.8000
T	2π	2π	2π
t_0	0	0	0
t_f	2π	2π	2π

The vector of initial conditions $\mathbf{z}_0 = (x_0, y_0, v_{x,0}, v_{y,0})$ is fixed accordingly the statement of the exam resit. We consider a Cartesian plan where the focus S, namely Saturn, is the origin (cf. Figure 4). At the initial state, Titan is at the perikrone and its coordinates are thus equal to:

$$x_0 = q = a - c = a\,(1 - e)\,, \qquad y_0 = 0 \tag{36}$$

When Titan is at the perikrone, its horizontal velocity (in the major axis direction) is null, i.e., $v^2_{x,\text{perikrone}} = 0$. Thus, its vertical velocity (in the minor axis direction) is equal to (cf. equation 10):

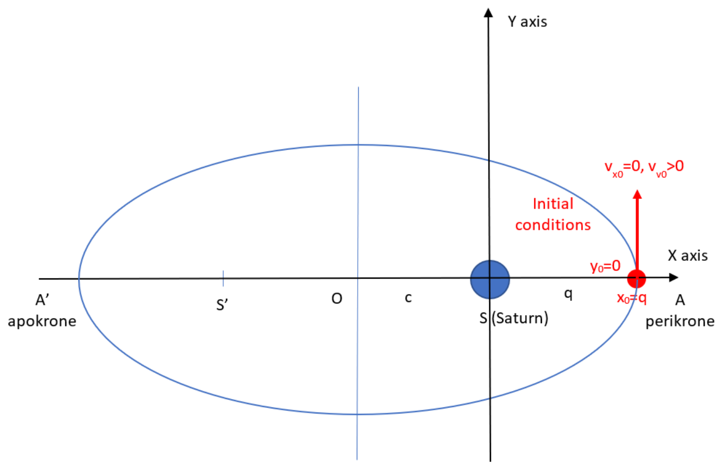
$$v^2_{y,\text{perikrone}} = \frac{\mu}{a}\left(\frac{1+e}{1-e}\right) \tag{37}$$

Thus, the vector of initial conditions is equal to:

$$\mathbf{z}_0 = \begin{pmatrix} a\,(1-e) & 0 & 0 & \frac{\mu}{a}\left(\frac{1+e}{1-e}\right) \end{pmatrix} \tag{38}$$

This initial condition is a necessary condition to get an orbit that converges.

Figure 4: Initial conditions



3.2 Sensivity analysis to the parameter values and initial conditions

The second set of experiments consists in modifying the value of the parameter μ and the initial conditions $\mathbf{z}_0$. First, we repeat experiments A to C, by changing the value of the standard gravitational parameter $\mu = GM$ from 1 to 2 for experiments A to B, and to 0.1 or 0.01 for experiment C. As the gravitational constant G is universal, it comes down to consider a planet with a larger/lower mass than Saturn. Whatever, we artificially increase the gravity force for experiments A and B, and decrease it for C. These experiments will be referred as A', B', C', and C" respectively.

Finally, we repeat the experiment B (eccentricity $e = 0.5$ and $\mu = 1$), but we change the initial condition. In experiment B, the vector of initial conditions was given by the relationship:

$$\mathbf{z}_0 = \left(\begin{array}{cccc} a(1-e) & 0 & 0 & \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} \end{array} \right) = \left(\begin{array}{cccc} 0.5 & 0 & 0 & \sqrt{3} \end{array} \right) \tag{39}$$

We now consider a vector of initial condition given by[12]:

$$\mathbf{z}_0 = \left(\begin{array}{cccc} 0.5 & 0 & 0 & 1 \end{array} \right) \tag{40}$$

The only change in experiment B" compared to experiment B, is that the initial velocity in the y-dimension (minor axis), $v_{y,0} = 1$, is smaller than the velocity

[12] Notice that an initial condition $x_0 = a = 1$ (as required in the statement) in the Cartesian plan with the origin at O is equivalent to $x_0 = a(1-e) = 0.5$ when the origin is located at the focus S. Both values mean that Titan is initially located at its perikrone, as soon as $y_0 = 0$.

required to stay on the elliptical orbit (cf. Figure 4). This experiment will be referred as experiment B".

3.3 Error analysis

Finally, we assess the order of the global and local errors produced by our Runge–Kutta third order solver. The local (truncation) error is the error that causes during a single iteration, assuming perfect knowledge of the true solution at the previous iteration. The global (truncation) error is the accumulation of the local truncation error over all of the iterations, assuming perfect knowledge of the true solution at the initial time step. Remind that for a third-order method, the local error is $O(h^4)$, whereas the global error is $O(h^3)$. Here, we limit our analysis to the global errors.

To assess the global errors, we consider the experiment B ($e = 0.5$, $\mu = 1$) and three step-sizes, namely $h = 0.1$, $h = 0.05$, and $h = 0.01$. At each period $t = t_0, \ldots, T$, the global errors are obtained by comparing the simulated value $\mathbf{z}(t)$ to the theoretical value $\mathbf{z}_t$. The latter is deduced by solving the Kepler's equation (equation 7), and by converting the eccentric anomaly u in cartesian coordinates (x, y) according to equation (6). We consider two assesment criteria: the mean absolute error and the mean relative error. By comparing the errors for different values of h, we will check if the global errors of the Heun's third order method are $O(h^3)$, as previously mentioned.

4 Results

In the experiment A, i.e., when the eccentricity parameter is null, we know that the exact solution for the Titan's orbit is a circle of radius equal to $a = 1$ with a center at point O, as represented on Figure 5. Notice that when $e = 0$, the foci S and S', and the point O are merged. In this case, the vector of initial conditions for the third-order Runge-Kutta solver is equal to:

$$\mathbf{z}_0 = \begin{pmatrix} a(1-e) & 0 & 0 & \frac{\mu}{a}\left(\frac{1+e}{1-e}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \tag{41}$$

We display in Table 2, the ten first values of the vector $\mathbf{z}(t_i)$ obtained with the Heun's third order method with a step-size h equal to 0.1. Due to lack of space, we do not report here the vectors $\mathbf{k}_{i,1}$, $\mathbf{k}_{i,2}$, and $\mathbf{k}_{i,3}$, but only the result for $\mathbf{w}_i = \mathbf{z}(t_i)$. The first line ($i = 0,\ t = 0$) corresponds to the initial condition. Then, for $t > 0$ we observe that the coordinate x decreases, while y increases with time. The velocity in the x-axis becomes negative and v_y decreases, while staying positive. These variations thus produce a circular trajectory.

Table 2: Results of the Heun's third order method (Experiment A)

i	t	x	y	v_x	v_y
0	0	1	0	0	1
1	0.1	0.99501	0.09983	−0.09983	0.99500
2	0.2	0.98007	0.19867	−0.19867	0.98006
3	0.3	0.95535	0.29552	−0.29551	0.95533
4	0.4	0.92108	0.38942	−0.38941	0.92105
5	0.5	0.87760	0.47943	−0.47941	0.87757
6	0.6	0.82536	0.56464	−0.56462	0.82532
7	0.7	0.76487	0.64422	−0.64419	0.76483
8	0.8	0.69675	0.71736	−0.71733	0.69669
9	0.9	0.62165	0.78333	−0.78329	0.62160

Figure 5 displays the theoretical orbit associated to experiment A as well as the simulated trajectory obtained with our Heun's third order solver. We can verify that the simulated trajectory is close to the theoretical orbit.

We get similar results in experiment B when the eccentricity parameter e is fixed to 0.5 (cf. Figure 6). Then, the theoretical orbit is an ellipse with a focus $S(0,0)$ which is different from the center of the auxiliary circle of radius a, i.e., the point $O(-0.5,0)$. We do not report here the initial conditions and the first simulated values $\mathbf{z}(t_i)$ obtained with the Heun's third order solver (these values are reported in our code), but we can observe that these simulated values are also close to the theoretical orbit.

The results are different when eccentricity parameter increases. In the experiment C (cf. Figure 7) with $e = 0.8$, the simulated trajectory does no longer to the theoretical elliptical orbit. Titan tends to crash on Saturn. This result is due to the fact that the errors (cf. next section) made by the solver for the first steps, places Titan on a trajectory where the velocity is not sufficient to offset the gravitational force of Saturn.

Figure 5: Experiment A: eccentricity $e = 0$

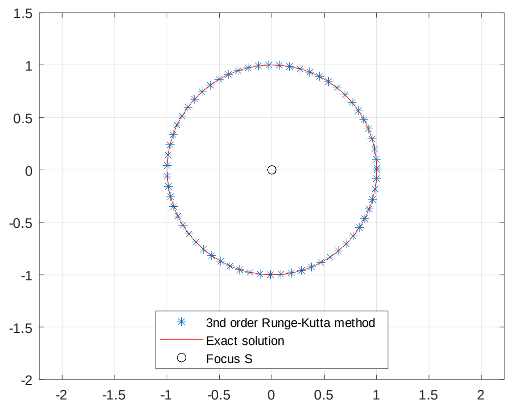


Figure 6: Experiment B: eccentricity $e = 0.5$

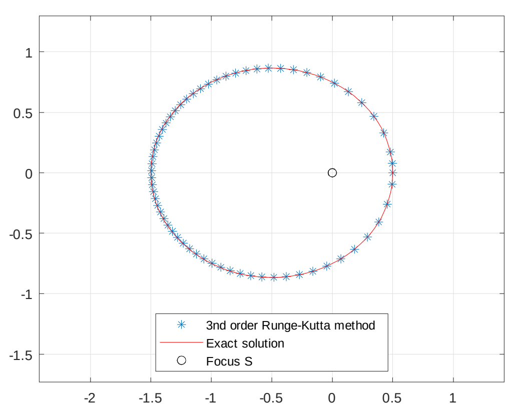
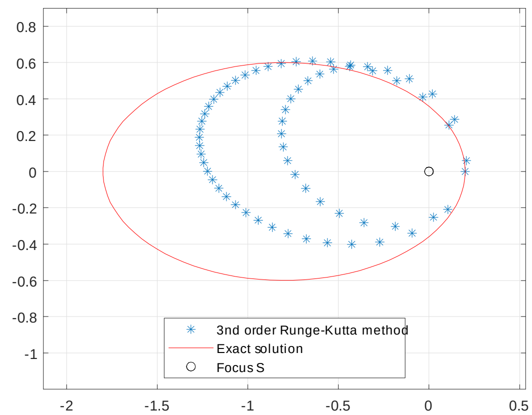
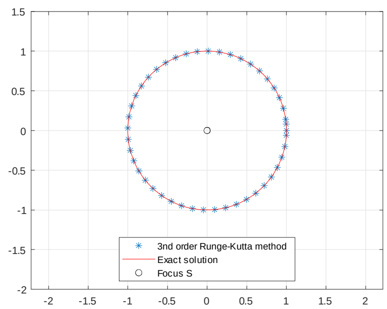


Figure 7: Experiment C: eccentricity $e = 0.8$

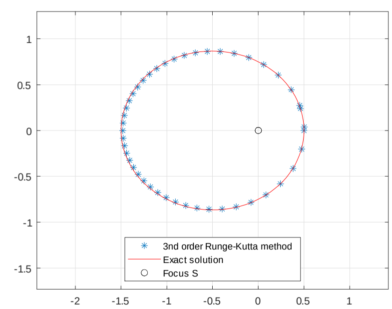


We now assess the influence of the standard gravitational parameter $\mu = GM$. The results are reported on the figures corresponding to experiments A', B', C', and C". When the eccentricity parameter is relatively low (experiments A' and B'), increasing the gravitational force μ from 1 to 2 seems to not change the simulated trajectory, which roughly correspond to the theoretical one. But, if we compare experiments B ($e = 0.5$, $\mu = 1$, figure 6) and B' ($e = 0.5$, $\mu = 1$), we observe that in the later, the simulated trajectory is closer to the focus, especially when Titan reaches the apokrone. As the gravitational force is higher in B' than in B, the net effect of the two forces, i.e., velocity and gravity, is more in favour of the later. The errors (not reported) are higher in experiment B' than in experiment B. The influence of the standard gravitational parameter is much clear when we compare the experiments C ($e = 0.8$, $\mu = 1$, figure 7), C' ($e = 0.8$, $\mu = 0.1$), and C" ($e = 0.8$, $\mu = 0.01$). Decreasing the standard gravitational parameter μ allows reducing the impact of the errors made by the solver for the first steps. When, $\mu = 1$ Titan tends to crash on Saturn as the errors place Titan on a trajectory where the velocity is not sufficient to offset the gravitational force of Saturn. When $\mu = 0.1$ or $\mu = 0.01$, the impact of the first errors decreases and the simulated trajectory tends to the theoretical elliptical orbit.

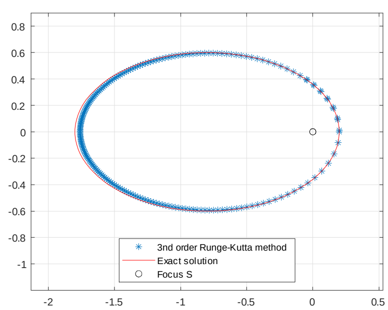
Experiment A': $e = 0, \mu = 2$



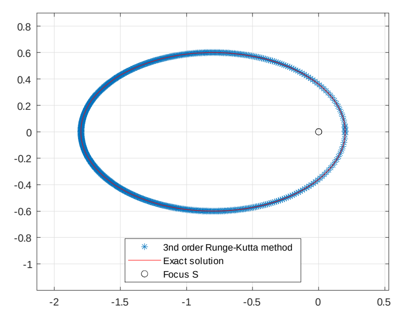
Experiment B': $e = 0.5$, $\mu = 2$



Experiment C': $e = 0.8$, $\mu = 0.01$



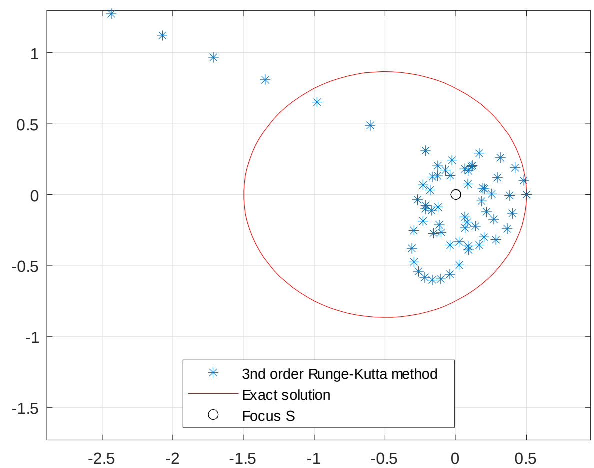
Experiment C": $e = 0.8$, $\mu = 0.01$



The experiment B" ($e = 0.5$, $\mu = 1$, $v_{y,0} = 1$) is designed to assess the influence of the initial conditions . Compared to the initial conditions used in experiment B, Titan is still initially located at its perikrone ($x_0 = 0.5$, $y_0 = 0$), but its initial velocity in the y-dimension (minor axis), i.e., $v_{y,0} = 1$, is larger than the velocity required to stay on the elliptical orbit, i.e., $v_{y,\text{perikrone}} = \sqrt{3}$. We can verify on Figure 8 that the simulated trajectory obtained in experiment B" does not converge. This illustrates the importance of the vector of initial conditions when one solves the ODE associated to the orbit of a satellite.

Finally, as required in the statement, we check our solver by computing the solution for an elliptical orbit over a full period for different step-sizes h, and we show that the error has the expected order. First, Table 3 reports the true and simulated values of the coordinates of Titan for experiment B, with $e = 0.5$ and $\mu = 1$, for $t_0, t_1, \ldots, t_9$. The true position (x_t^*, y_t^*) of Titan is obtained by solving the Kepler's equation (equation 7) and then, by converting the eccentric anomaly u in cartesian coordinates (x, y) according to equation (6). The simulated values $(x(t), y(t))$ are obtained with the Heun's solver for a step-size $h = 0.1$. The two last colums report the relative errors, in percentage,

Figure 8: Experiment B": influence of the initial velocity $v_{y,0}$



respectively defined as:

$$\varepsilon_{x,t} = \frac{|x\,(t) - x_t^*|}{|x_t^*|} \times 100 \qquad \varepsilon_{y,t} = \frac{|y\,(t) - y_t^*|}{|y_t^*|} \times 100$$

Table 3: Error analysis for the Heun's third order method (Experiment B, $h = 0.1$)

i	t	$x(t)$	$y(t)$	x_t^*	y_t^*	$\varepsilon_{x,t}$ (%)	$\varepsilon_{y,t}$ (%)
0	0	0.5	0	0.5	0	0	–
1	0.1	0.48039	0.17094	0.48031	0.17101	0.017349	0.039689
2	0.2	0.42503	0.32932	0.42483	0.32943	0.048450	0.031661
3	0.3	0.34242	0.46697	0.34203	0.46715	0.112600	0.040163
4	0.4	0.24202	0.58086	0.24150	0.58106	0.212170	0.034521
5	0.5	0.13169	0.67157	0.13105	0.67181	0.486750	0.036886
6	0.6	0.01707	0.74134	0.01636	0.74164	4.285500	0.039245
7	0.7	−0.09808	0.79292	−0.09883	0.79329	0.761160	0.046448
8	0.8	−0.21140	0.82891	−0.21218	0.82938	0.369000	0.057089
9	0.9	−0.32139	0.85162	−0.32218	0.85222	0.246730	0.071213

We observe that the relative errors are always inferior to 5%. The maximum relative error is reached when Titan is located at the vertex of the semi-minor axis (point B).

To check if the global error has the expected order, we repeat the experiment B ($e = 0.5$, $\mu = 1$) with three step-sizes, namely $h = 0.1$, $h = 0.05$, and $h = 0.01$. Then, we compare the errors using two criteria, i.e., the mean absolute error and the mean relative error defined as:

$$MAE_x = \frac{1}{n_h} \sum_{i=1}^{n_h} \left| x(t_0 + ih) - x^*_{t_0+ih} \right| \tag{42}$$

$$MAE_y = \frac{1}{n_h} \sum_{i=1}^{n_h} \left| y(t_0 + ih) - y^*_{t_0+ih} \right| \tag{43}$$

$$MRE_x = \frac{1}{n_h} \sum_{i=1}^{n_h} \frac{\left| x(t_0 + ih) - x^*_{t_0+ih} \right|}{\left| x^*_{t_0+ih} \right|} \times 100 \tag{44}$$

$$MRE_y = \frac{1}{n_h} \sum_{i=1}^{n_h} \frac{\left| y(t_0 + ih) - y^*_{t_0+ih} \right|}{\left| y^*_{t_0+ih} \right|} \times 100 \tag{45}$$

where $n_h = floor((T - t_0)/h + 1)$ denotes the number of steps used in the solver to complete a full period. The results are reported in Table 4.

Table 4: Error analysis for the Heun's third-order method (Experiment B, $h = 0.1$)

Step size	$h = 0.1$	$h = 0.05$	$h = 0.01$
n_h	63	126	629
MAE_x	0.010740	0.001302	0.000017
MAE_y	0.007909	0.009650	0.000018
MRE_x (%)	4.1667	0.5563	0.0141
MRE_y (%)	5.6562	0.7165	0.0183

We observe that halving the step size from $h = 0.1$ to $h = 0.05$, divides the mean absolute error (or the mean relative error) by a factor $2^3 = 8$. This result was expected as the Hein's solver is a third order method. This confirms that the global error is $O(h^3)$, as it was expected for all the third-order Runge Kutta methods.

5 Discussion

All these experiments show that the third-order Runge-Kutta differential equation solver provides an accurate approximate solution to the classical mechanical problem of a satellite orbiting a large central body. To improve the accuracy of the simulated orbit, a solution consists in reducing the step-size. However, reducing the step-size h implies to increases the number of steps n_h for the iterative algorithm required to complete a full period T. Table 4 shows that when

$h = 0.1$, only 63 iterations are required to cover a period T. But, for $h = 0.01$, the number of iterations increases to 629, involving more computing time. This illustrates the trade-off between accuracy and computing time.

A solution to overcome this trade-off consists in using a solver with adaptative time step h_i. Rather than considering the same step-size for all the evaluations, it consists in reducing the step-size when the velocity increases, and on the contrary to increases the step-size when the velocity is low. This approach allows to decrease the computing time as it increases the number of iterations only when it is necessary, i.e., when the satellite moves quickly.

6 Conclusion

The approach developed in this document can be generalized to a n-body problem to develop a simulator for the movement of n planets and satellites. This is what has been done by [5] in her master thesis. The author proposed an extensible simulator based on a fourth-order Runge Kutta solver. This simulator allows to organize the celestial bodies to be studied into groups, to calculate their positions, and to graphically visualize their movement using the computed positions.

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