Credit Supply Risk and Precautionary Savings *

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Abstract

Changes in credit supply induce large and frequent variations in households' access to unsecured debt. They generate a novel *financial* precautionary motive, which compounds the microeconomic motive associated with idiosyncratic income risk, as borrowers deleverage and accumulate safe assets to hedge against them. Using a structural model, I estimate that this motive is an important driver of household balance sheets over the business cycle. It also helps explains the historically low level of interest rates in the last decade despite consumption growth, solving a "post-Great Recession risk-free rate puzzle".

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Keywords: Credit supply; precautionary savings; household debt; interest rate; structural estimation.

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1 Introduction

The aggregate supply of credit to households is a central indicator in macro-finance. Its changes are large, distinct from individual credit risk factors, and forecast numerous real and financial variables (Schularick and Taylor (2012)). Not only did credit supply massively contract after the Great Recession (Mian, Rao and Sufi (2013)). Its changes are also frequent in long time series (Mian, Sufi and Verner (2017)), including the recent Covid-19 recession (Cherry, Jiang, Matvos, Piskorski and Seru (forthcoming)), and they continuously affect households' ability to borrow over the business cycle (Ludvigson (1999), Agarwal, Chomsisengphet, Mahoney and Stroebel (2018), Diamond and Landvoigt (forthcoming)).

Yet, while existing work has focused on how the level of borrowing constraints affects household debt (Favilukis, Ludvigson and Nieuwerburgh (2017), Guerrieri and Lorenzoni (2017), Justiniano, Primiceri and Tambalotti (2019)), little is known about the effect of their *volatility* on households' entire balance sheets, including consumption and savings. Do such changes represent an important source of risk for households, and how does it compare with canonical idiosyncratic and aggregate income risk?

This paper uses a general equilibrium model with heterogeneous households and incomplete markets, and a novel decomposition of precautionary motives, to estimate the precautionary savings associated with credit supply risk. When borrowing constraints shrink in response to common shocks affecting lenders, households' ability to insure and smooth consumption deteriorates. As a result, they hedge against changes in credit supply itself by deleveraging and accumulating savings, creating downward pressure on interest rates, which feed back into their balance sheets by making savings less attractive.

Identifying the effect of aggregate risk on precautionary savings is a challenge in business cycle models with heterogeneous households and incomplete markets. In workhorse models with aggregate income risk (Krusell and Smith (1998)), households' savings depend on the volatility of idiosyncratic income as in Aiyagari (1994), but not on the volatility of aggregate income because certainty equivalence applies to shocks with low variance and uniform effects across households. In models where aggregate shocks increase idiosyncratic risk and have large distributional effects (Constantinides and Duffie (1996)), it is hard to disentangle their effect from idiosyncratic shocks alone.

The main contribution of this paper is a decomposition of the precautionary motives arising from different idiosyncratic and aggregate risks which solves this problem. I highlight three main results in a business cycle model with heterogeneous households and incomplete markets, extended with aggregate shocks to productivity and borrowing constraints. First, changes in households' borrowing constraints are a large source of risk. The resulting precautionary savings depress the interest rate by one fourth as much as idiosyncratic income risk in equilibrium, and much more than aggregate income risk whose effect is close to zero. Second, they generate a low debt and low rate environment which helps explain post-credit crisis periods. Third, they contribute to half of the volatility of consumption and the interest rate over the business cycle. Thus, persistently tight borrowing constraints can explain why rates remained low despite consumption recovering after the Great Recession.

The economy consists of infinitely-lived households, who differ in idiosyncratic productivity, and unsecured debt and risk-free bond holdings summarized by their net asset position. Every period, they consume, save in risk-free assets or borrow subject to a credit limit, and elastically supply labor to competitive firms. A government issues risk-free assets and raises progressive taxes to finance existing debt and transfers to households. The real risk-free rate clears the market for safe assets, and the wage clears the labor market. The model features four forms of insurance, which affect the demand for risk-free assets and the interest rate. On the financial side, households can use unsecured debt and risk-free assets to smooth shocks. On the real side, they can adjust their labor supply and receive government transfers when their income falls. I calibrate the model to match the level and cross-sectional moments of unsecured household credit in the U.S. to focus on how households use debt to smooth consumption as in Guerrieri and Lorenzoni (2017). The innovation is that borrowing constraints and total factor productivity (TFP) are subject to continuous, mean-reverting aggregate shocks which affect households' borrowing capacities and wages.

My results rely on a novel nonlinear solution method for this class of models. Unlike in models with fixed borrowing constraints or unexpected shocks, households know the stochastic process for aggregate shocks as in Krusell and Smith (1998). I solve for first- and second-order approximations of the dynamics of the economy around its stationary steady state, summarized by the set of equilibrium conditions. For each shock, second-order terms measure, first, the departure from certainty equivalence due to the precautionary motive, and second, the nonlinearity of impulse responses.

Because of the precautionary motive due to credit supply risk, household debt and the interest rate are lower than when borrowing constraints are fixed or when their changes are fully unexpected. In the stochastic steady state associated with a long-run simulation of the economy, they generate a low debt and low rate environment similar to the post-Great Recession period. Furthermore, this response is state-contingent as it depends on the volatility of credit supply shocks. When their volatility is calibrated to match the risk-free rate in historical data from 1973 to 2005, debt to GDP and the risk-free rate are 3% lower than in an economy with fixed borrowing constraints. When it is calibrated based on the recent period after 2005, they are 25% lower.

I then decompose the sources of these variations. Relative to an economy with fixed credit limits, the lower rate results from the cross-sectional distribution of net asset positions contracting around its mode, an equilibrium response to higher savings from households close to their credit limits. Precautionary savings consist of three components: a microeconomic motive due to idiosyncratic earnings risk, a macroeconomic motive due to TFP shocks, and a novel *financial* motive due to credit supply shocks. First, absent aggregate shocks, idiosyncratic income risk increases the demand for risk-free bonds, and lowers the rate of return that households require to hold them. This is the standard precautionary motive of economies with incomplete markets and heterogeneous investors. It arises because of the "prudence" property of the utility function, and because idiosyncratic income shocks induce some households to become borrowing-constrained, thus unable to smooth consumption. It accounts for the largest contribution to the low risk-free rate level, compared to an economy without shocks (2.40% annually vs. 8.33%), but it fails to explain its very low level in the last decade. Second, TFP risk affects labor earnings and employment, but has close to zero effect on the risk-free rate, household debt, and consumption. This is consistent with with low welfare costs of business cycles (e.g., Lucas (1985)). Third, credit supply shocks generate a sizable precautionary motive, which represents one third of the micro motive, and further lowers the risk-free rate closer to its post-Great Recession average (1.8%).

To quantify the contributions of credit supply shocks to financial and macroeconomic volatility, I perform a variance decomposition analysis. I estimate that credit shocks contribute to about half of the volatility of macro-finance aggregates over time. In particular, they explain 60% of the variation in the risk-free rate over the business cycle. TFP shocks account for a smaller fraction of this volatility. These two shocks are critical to match jointly household balance sheet, interest rate, and macroeconomic moments. First, the level of real risk-free rates in the data (Treasury rates net of inflation expectations) is lower than implied by the Euler equation of unconstrained households and realistic values for their discount rate, risk aversion, and income risk. Second, real rates and household debt are persistent and procyclical. Third, the volatility of risk-free rates relative to output is low, except in the post-Great Recession period when they fell persistently. Fourth, in the same period, rates have remained low despite aggregate consumption recovering.

The model explains the short-run dynamics of bond prices and macroeconomic moments in response to TFP and credit supply shocks. Using nonlinear impulse functions, I show that credit shocks decrease household debt and increase net bond holdings through two channels. A first-order effect: in level, they induce credit-constrained households to deleverage, and those close to the constraint to increase precautionary savings. Then, a novel second-order effect: their volatility makes constraints more likely to bind going forward, further increasing precautionary savings.

This results in turn in lower output, and generates procyclical household debt and interest rate – a puzzle for models without credit supply shocks. With elastic labor supply, low risk-free rates create an intertemporal substitution effect that increases households' consumption of leisure. In particular, it decreases the labor supply of unconstrained households, which are more productive on average. In addition, a wealth effect induces those households to further decrease their labor supply, as they consume more goods.[1] Unlike in a perfect foresight model with fixed credit limits and unexpected shocks, credit contractions lead to persistently lower employment, and generate the positive correlation between risk-free rates, household debt, and employment. Aggregate employment reflects the increase in hours worked by constrained households to pay off their debt, and the decrease in hours worked by unconstrained households. With perfect foresight, the increase dominates because households close to the constraint perfectly anticipate that they will be constrained in the future, and choose to strongly increase their hours. When households have (rational but imperfect) expectations about future credit shocks, the decrease in hours dominates because households expect credit shocks to mean-revert.

Lastly, I show that both TFP and credit supply shocks are key to explain why real risk-free rates have remained persistently low after the Great Recession, despite consumption recovering. This feature is a puzzle for models with only aggregate income risk, where agents should borrow against future income, leading instead to higher rates. I apply a particle filter to the model, to estimate the sequence of structural credit and TFP shocks that generated the observed paths for the risk-free rate and aggregate consumption from 2006 to 2017. This is, to my knowledge, the only paper to do so in a general equilibrium model with heterogeneous agents, incomplete markets, and aggregate risk.

A persistent tightening of borrowing constraints, equivalent to a 15% decrease in the common component of households' credit limits, explains the drop in interest rate from 2.5% to -1.5% (annual 5-Year Treasury Inflation-Indexed Security with constant maturity yield). Its effect is exacerbated by a simultaneous short-lived drop in TFP, which households would like to, but cannot, smooth by issuing more debt. Combined, both shocks generate a temporary decline in consumption during the years of the Great Recession, which vanishes when TFP reverts to its pre-crisis level. In contrast, the implied measure

[1] This effect is absent with preferences of the Greenwood, Hercowitz and Huffman (1988) type.

of borrowing constraint tightness stays persistently low throughout the sample, limiting household indebtedness. It explains the large drop in the real risk-free rate, and then its low volatility around a low level. These results are consistent with empirical estimates for TFP and credit standards. While the latter can be hard to interpret, my approach provides a structural estimate of the common component of credit standards that controls households' access to credit.

This paper relies on a large literature which focuses on idiosyncratic income risk with fixed credit limits but abstracts from their variation over time (Deaton (1991), Gross and Souleles (2002), Parker and Preston (2005), Baker (2018)). Borrowing constraints vary significantly over the business cycle (Ludvigson (1999), Fulford (2015), Becard and Gauthier (forthcoming), Dempsey and Ionescu (2020)) and in response to policy (Agarwal et al. (2018)). Anticipation of future borrowing constraints lower households' demand for risky assets (Guiso, Jappelli and Terlizzese (1996)), and changes in private and public insurance increase precautionary savings (Hubbard, Skinner and Zeldes (1995)). However, the effect of time-varying borrowing constraints on households' entire balance sheets, including precautionary savings, is unclear.

My results extends theoretical evidence focusing on idiosyncratic income risk (Aiyagari (1994), Constantinides and Duffie (1996), Heaton and Lucas (1996)). My findings are consistent with evidence that aggregate risk increases the value of safe assets (Pflueger, Siriwardane and Sunderam (2020), Hartzmark (2016)). While existing models have focused on income risk (Guvenen (2009), Favilukis (2013)), my results highlight the role of credit supply. As in Jermann and Quadrini (2012), the model has heterogeneous agents, incomplete markets, and aggregate financial shocks. They show that financial shocks are key to explain the countercyclicality of firm debt, but abstract from household balance sheet and interest rate moments. That changes in households' ability to insure should affect asset prices, including interest rates, is an idea that underpins most of recent macrofinance (e.g., Cochrane (2017)). While most of it focuses on the covariance of stock returns with an investor's portfolio, the novelty of my approach is to highlight that a lesser ability to *borrow* generates a higher demand for risk-free assets which lowers their equilibrium return and feeds back into their balance sheets.

Finally, this paper proposes the first solution for models with heterogeneous agents, incomplete markets, and aggregate risk (e.g. Favilukis (2013), Favilukis et al. (2017)), which allows to structurally estimate the paths of shocks driving equilibrium prices and quantities using particle filtering. By allowing for multiple continuous aggregate shocks, it improves approaches building on Krusell and Smith (1998) which are limited to studying discrete Markov chains with few states because of the curse of dimensionality.[2]

Outline The rest of the paper is organized as follows. Section 2 presents the model, and Section 3 how its solution generates a decomposition of precautionary motives and of the nonlinear dynamics of the economy. Section 4 describes the calibration. Section 5 analyzes the long-run and short-run comovements of bond pricing, household balance sheet, and macroeconomic moments. Section 6 structurally estimates the credit supply and TFP shocks, which explain the puzzling behavior of interest rates and consumption in the post-Great Recession period. Section 7 concludes.

2 Model

This section describes a discrete time, equilibrium model of unsecured credit with heterogeneous households, incomplete markets, and two sources of aggregate risk: shocks to total factor productivity and to households' borrowing constraints.

2.1 Households

The economy is populated by a continuum of measure 1 of heterogeneous, risk-averse households. Households face idiosyncratic labor income risk. They consume a single final good produced by competitive firms, and elastically supply efficiency units of labor to those firms. Firm's profit are redistributed equally. To smooth consumption over time and share risk, households borrow and save using risk-free assets. When borrowing, they face stochastic borrowing constraints which consist of a aggregate component depending on credit supply and of an idiosyncratic component depending on individual creditworthiness, itself a function of households' income. Households face distortionary, progressive taxes on labor income, and receive lump-sum transfers from the government conditional on their resources.

Households choose consumption, labor supply, and net asset holdings, to maximize the expected discounted value of current and future utility flows from consumption net

[2] A technical contribution is to make second-order perturbations computationally tractable, by combining the projection and perturbation approach of Reiter (2009) with the `gensys2` algorithm of Kim, Kim, Schaumburg and Sims (2008), to reduce the dimensionality of the system of equations for second-order coefficients – and using automatic differentiation to generate that system.

of the disutility of working:

$$\max_{\{c_{it},n_{it},b_{it+1}\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_{it}^{1-\gamma}}{1-\gamma} - \psi \frac{n_{it}^{1+\eta}}{1+\eta}\right], \tag{1}$$

subject to budget and borrowing constraints

$$\begin{aligned} &c_{it} + \frac{b_{it+1}}{1+r_t} + \tau_t\left(\theta_{it}, n_{it}\right) \leq w_t \theta_{it} n_{it} + b_{it} + T\left(\theta_{it}\right) + \pi_t \\ &b_{it+1} \geq -\overline{\phi}_t \phi\left(\theta_{it}\right). \end{aligned} \tag{2}$$

Their (logarithm) idiosyncratic productivity θ follows an AR(1) process, chosen to replicate the evolution of the wages of employed individuals in the U.S. Its volatility is a decreasing function of TFP, z_t, chosen to match estimates for the countercyclicality of individual income (Storesletten, Telmer and Yaron (2004)). In the numerical solution, income is discretized using the Rouwenhorst method as a finite Markov chain $\Theta\left(z_t\right) = \left\{\underline{\theta}\left(z_t\right), ..., \overline{\theta}\left(z_t\right)\right\}$ with transition matrix $\Pi_\theta\left(z_t\right)$.[3]

$$\begin{aligned} &\log \theta_{it} = \rho_\theta \log \theta_{it-1} + \sigma_\theta\left(z_t\right) \epsilon_{it}^{\theta}, \;\; \epsilon^\theta \sim \mathcal{N}(0,1) \\ &\log z_t = \rho_z \log z_{t-1} + \epsilon_t^z \end{aligned} \tag{3}$$

Their borrowing capacities depend on economy-wide variables (e.g., nationwide lending standards) and individual variables (e.g., income, debt and assets, credit score). The aggregate component of borrowing constraints is a mean-reverting AR(1) process with mean $\overline{\phi}$. A shock $\epsilon^\phi < 0$ induces a tightening of all borrowing constraints, $\epsilon^\phi \geq 0$ a relaxation. The $\phi\left(.\right)$ function, which multiplies $\overline{\phi}$, maps individual productivity to credit limits. It is calibrated to match the cross-sectional distribution of unsecured household debt.

The innovations to TFP and credit shocks follow a bivariate normal distribution with mean zero,[4]

$$\begin{aligned} &\log \overline{\phi}_t - \log \overline{\phi} = \rho_\phi \left(\log \overline{\phi}_{t-1} - \log \overline{\phi}\right) + \epsilon_t^\phi \\ &\begin{pmatrix} \epsilon^\phi \\ \epsilon^z \end{pmatrix} \overset{iid}{\sim} \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \sigma_\phi^2 & \sigma_\phi \sigma_z \rho_{\phi z} \\ \sigma_\phi \sigma_z \rho_{\phi z} & \sigma_z^2 \end{pmatrix}\right) \end{aligned} \tag{4}$$

The solution of the model accommodates alternative specifications for stochastic bor-

[3]The Rouwenhorst delivers a more accurate income dynamics than the Tauchen method when the process is persistent.

[4]For the problem to be well-defined, the support of ϵ must be bounded. If not, the household's choice becomes empty for large values of ϵ, which can be interpreted as the household defaulting on his debt. Even if this event is unlikely, it would be inconsistent with one-period bonds being risk-free. Therefore during the simulation I check that households' choice sets are nonempty.

rowing constraints, for instance with additive shocks, or individual sensitivities to shocks. My specification nests the model of Ludvigson (1999), who studies borrowing constraints that vary stochastically with individual income, and of Fulford (2015), which studies access to credit at the extensive margin, both with a fixed interest rate.[5]

Finally, progressive labor income taxes consist of a time-varying intercept and a productivity-dependent slope.

$$\tau_t\left(\theta_{it}, n_{it}\right) = \tau_{0t} + \tau_1\left(\theta_{it}\right) w_t \theta_{it} n_{it} \tag{5}$$

Transfers $T\left(\theta_{it}\right)$ are progressive too.

2.2 Firms

A continuum of competitive firms hires efficient units of labor from households every period, and combines them using a decreasing returns to scale production technology subject to a TFP shock. Firms are owned by households equally and their shares are non-tradable.[6] They solve a static profit maximization problem:

$$\max_{N_t} \pi_t = z_t N_t^{\alpha} - w_t N_t \tag{6}$$

For simplicity, there is no capital. In equilibrium, firms' profits and the wage bill are constant fractions of output, thus the firm sector transmits TFP shocks to households' wages and profit shares with an elasticity of one:

$$\begin{aligned} \pi_t^* &= (1-\alpha) Y_t^* = (1-\alpha) z_t \; {N_t^*}^{\alpha} \\ w_t N_t^* &= \alpha Y_t^* = \alpha z_t \; {N_t^*}^{\alpha} \end{aligned} \tag{7}$$

2.3 Government

The government issues one-period, risk-free bonds in exogenous, positive supply. To finance transfers to households and outstanding debt, it raises a progressive tax on households' labor income and issues new bonds. The progressive income tax is modeled as an affine function of labor income, whose slope depends on households' productivity. The

[5]A limitation of this setting is the lack of a financial accelerator mechanism due to the absence of a tradable durable good used as collateral in borrowing constraints.

[6]This implies that my estimates of precautionary motives can be interpreted as a lower bound. Relative to the data, this assumption makes credit constraints less stringent for poor households, who have larger precautionary motives. It allows to clarify the exposition of those motives, and can be relaxed at the expense of increasing the state space. A tractable alternative would be to assume an exogenous distribution of firm shares depending on income and savings.

intercept of the tax function adjusts such that the government's budget constraint holds every period.[7]

$$\int \tau_t\left(\theta, n\left(\theta, b\right)\right) d\lambda_t\left(\theta, b\right) + \frac{B_{t+1}}{1+r_t} = \int T\left(\theta\right) d\lambda_t\left(\theta, b\right) + B_t \tag{8}$$

Government taxes are an equilibrium object that must be solved for as a fixed point consistent with the household problem. Given prices (risk free rate, wage), they depend on households' policy functions (labor supply) and the cross-sectional distribution. Policy functions and the distribution, in turn, depend on taxes. The quantity of government bonds is assumed to be fixed, and is chosen to match unsecured debt and liquid assets to GDP ratios in the deterministic steady state of the model.

2.4 Equilibrium

With aggregate credit and TFP shocks, the household's problem is similar to Krusell and Smith (1998). The state space includes the cross-sectional distribution of households across productivity and bonds, $\{\lambda_t(\theta, b)\}$. Households know the current price of bonds $q_t = 1/(1+r_t)$, but must forecast the price of bonds next period to allocate their consumption between t and $t+1$. Since government bonds are in fixed supply, it amounts to forecasting next period demand for bonds, which depends on the distribution of net bond positions, which is time-varying because of aggregate shocks affecting wages and credit limits. They must also forecast the current wage, determined by the intersection of firms' labor demand and households' labor supply, itself a function of the wage. Finally, they forecast taxes, which depend on the wage and labor supply. The rational expectation equilibrium of the model is a fixed point from households' expectations to realized values. Given their anticipation of next period distribution and the current wage, households make savings and labor choices in the current period that generate that distribution next period and that wage in the current period.

Definition 1 (competitive equilibrium)**.** A competitive equilibrium is a sequence of time-varying households' policy functions

$$\left\{c_t\left(\theta, b\right), b'_t\left(\theta, b\right), n_t\left(\theta, b\right)\right\}$$

[7] Heathcote, Storesletten and Violante (2017) model a more realistic progressive tax schedule, but use variations in transfers to satisfy the government budget constraint. Since the distribution of transfers $T\left(\theta_{it}\right)$ across households is key to understand how much insurance households get from credit, I choose to take that distribution in U.S. data as given, and assume that the slope of income taxes does not depend on endogenous labor supply.

firms' policy functions N_t, prices r_t, w_t, government taxes τ_t and aggregate shocks $\overline{\phi}_t, z_t$, such that:

(i) Given prices, taxes and aggregate shocks, households' optimality conditions hold:

$$\begin{aligned} c_t(\theta,b)^{-\gamma} &= \beta(1+r_t)\mathbb{E}_t\left[c_{t+1}(\theta,b)^{-\gamma}\right] + \mu_t(\theta,b) \\ (1-\tau_1(\theta))\, w_t\theta c_t(\theta,b)^{-\gamma} &= \psi n_t(\theta,b)^{\eta} \end{aligned} \tag{9}$$

where $\mu_t(\theta,b)$ denotes the multiplier on agent (θ,b)'s borrowing constraint.

(ii) Given prices and aggregate shocks, firms' optimality conditions hold:

$$\alpha z_t N_t^{\alpha-1} = w_t \tag{10}$$

(iii) Given prices and aggregate shocks, the government's budget constraint holds:

$$\tau_{0t} = \int T(\theta)\, d\lambda_t(\theta,b) + B\frac{r_t}{1+r_t} - w_t \int \tau_1(\theta)\, \theta n_t(\theta,b)\, d\lambda_t(\theta,b) \tag{11}$$

(iv) The goods, bonds and labor markets clear:

$$\begin{aligned} &\textstyle\int c_t(\theta,b)\, d\lambda_t(\theta,b) = z_t N_t^{\alpha} \\ &\textstyle\int b_t'(\theta,b)\, d\lambda_t(\theta,b) = B \\ &\textstyle\int \theta n_t(\theta,b)\, d\lambda_t(\theta,b) = N_t \end{aligned} \tag{12}$$

(v) The distribution of households λ_t and aggregate shocks evolve according to their laws of motion. Formally, denote $\Theta \times \mathcal{B}$ the sigma-algebra associated with the Cartesian product of the discrete set of productivity levels and the compact set of bond holdings, and $(\tilde{\Theta}, \tilde{\mathcal{B}})$ a subset of the sigma-algebra (in fact, those depend on t because of countercyclical income volatility, i.e. the set of productivity levels depends on z_t). Then,

$$\begin{aligned} &\lambda_{t+1}(\tilde{\Theta},\tilde{\mathcal{B}}) = \textstyle\int_{\Theta\times\mathcal{B}} Q_{\overline{\phi}_t,z_t}\left((\theta,b),(\tilde{\Theta},\tilde{\mathcal{B}})\right) d\lambda_t(\theta,b) \\ &\text{where}\quad Q_{\overline{\phi}_t,z_t}\left((\theta,b),(\tilde{\Theta},\tilde{\mathcal{B}})\right) = \mathbf{1}\left\{b_t'(\theta,b)\in\tilde{\mathcal{B}}\right\} \textstyle\sum_{\theta'\in\tilde{\Theta}} \Pi_\theta(\theta'|\theta) \end{aligned} \tag{13}$$

The transition function $Q_{\overline{\phi},z}$ depends on individual productivity, bonds, TFP and aggregate credit conditions. The latter two shift the distribution of bonds over time.

The market clearing risk-free rate and the wage are endogenous.The former is solved for numerically, but the wage can be solved for analytically using the firms' optimality condition and labor market clearing:

$$w_t = \alpha z_t N_t^{\alpha-1} = \alpha z_t \left(\int \theta n\left(\theta, b\right) d\lambda_t\left(\theta, b\right) \right)^{\alpha-1} \tag{14}$$

It is directly affected by the TFP shock z_t, and indirectly by the credit shock $\overline{\phi_t}$ through its effect on the distribution λ_t.

3 Decomposition of Precautionary Savings Motives

The model is solved numerically. This section presents a new solution for asset pricing models with heterogeneous agents, incomplete markets, and aggregate risk, which generates a decomposition of shocks' contributions to precautionary motives and of the dynamics of bond prices.

3.1 Decomposition

There are three main steps. The first two steps, with variants, are common to perturbation-based solutions of heterogeneous agents models with aggregate risk. First, I approximate the infinite-dimensional equilibrium variables by finite-dimensional objects: policy functions with linear splines, pricing functions, cross-sectional distribution with histogram weights. I obtain a finite parametrization that I refer to as the discrete model. Second, compute the stationary steady state of the discrete model. The solution of the steady state is global, and fully nonlinear with respect to idiosyncratic income risk, but abstracts from aggregate shocks. Third, perturb the discrete model around its steady state and compute the rational expectations solution of the perturbed system. The Jacobian and Hessian of the equilibrium system of equations, used as input to the perturbation method, are computed exactly with automatic differentiation.[8]

Then, I compare the results of a first- to a second-order perturbation. In the first-order solution, equilibrium variables depend linearly on the lagged values of the states, and the economy features certainty equivalence. That is, the volatility of aggregate shocks is irrelevant for households' decisions and aggregates. It only scales the impact of current shocks. In the second-order solution, variables depend nonlinearly on the lagged values, and they depend on the volatility of aggregate shocks, i.e. on aggregate risk.

[8] See Revels, Lubin and Papamarkou (2016).

First step Equilibrium conditions are stacked in a multivariate, vector-valued function $\mathcal{F}(.)$ that represents the nonlinear system of equations defining the equilibrium:

$$\mathbb{E}_t \left[\mathcal{F} \left(\mathbf{y_t}, \mathbf{y_{t+1}}, \mathbf{x_t}, \mathbf{x_{t+1}}, \epsilon^{\phi}_{t+1}, \epsilon^{z}_{t+1} \right) \right] = 0 \tag{15}$$

The vector of non-predetermined variables $\mathbf{y}$ contains the policy function coefficients (for labor and next period bonds; consumption is backed out from the budget constraint), the bond price, the wage, aggregate consumption and employment. The vector of predetermined variables $\mathbf{x}$ contains the histogram weights, aggregate TFP and credit shocks.

Second step Solving for the deterministic steady state boils down to solving a large nonlinear system of equations,

$$\mathcal{F}(\mathbf{y}, \mathbf{y}, \mathbf{x}, \mathbf{x}, 0, 0) = 0. \tag{16}$$

In theory, it could be solved directly using a nonlinear equation solver. In practice, there is no guarantee that numerical equation solvers will converge when we use projection methods to approximate policy functions. In addition to solving the households' consumption problem, the difficulty comes from having endogenous labor supply, endogenous government taxes, and solving for two equilibrium prices (wage and interest rate). I also solve for the value of the disutility of labor ψ that normalizes steady state output Y to 1.

Therefore, to make the problem more stable, I use the following variant of policy time iteration. First, given a guess for $\mathbf{x}$ and $\mathbf{y}$,[9] compute government taxes for all agents. Given taxes and the guess, solve for households' labor supply policy. Given that policy, solve then for households' savings policy. Using the policy functions, compute the implied stationary distribution (using an eigenvector method), and the new taxes. The process is repeated until policy functions converge. I use Broyden's method every time a numerical solver is needed, and automatic differentiation to compute exact derivatives. Since the convergence of the numerical solver is not guaranteed under any initial guess and parameter combination, I calibrate the steady state of the model with a homotopy method. That is, I slowly change parameters until the target is reached, starting from a combination under which the model steady state is easily computed. If needed, I modify the state space boundaries over that process.

[9] A good guess is obtained by using the endogenous grid method of Carroll (2006) to iterate backwards on the household's optimality conditions, starting from any feasible guess.

Third step Then, denote η the perturbation parameter scaling the amount of aggregate uncertainty in the economy. The solution to the equilibrium expectational difference equation $\mathbb{E}_t\left[\mathcal{F}(.)\right]=0$ is of the form:

$$\begin{aligned}\mathbf{x_{t+1}} &= \mathbf{h}\left(\mathbf{x_t},\eta\right)+\eta\begin{pmatrix}\mathbf{0}\\ \epsilon^{\phi}_{t+1}\\ \epsilon^{z}_{t+1}\end{pmatrix}\\ \mathbf{y_t} &= \mathbf{g}\left(\mathbf{x_t},\eta\right).\end{aligned} \tag{17}$$

The first-order approximation of the solution, in steady state deviations, is

$$\begin{aligned}\widehat{\mathbf{x_{t+1}}} &= \mathbf{h_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}+\eta\begin{pmatrix}\mathbf{0}\\ \epsilon^{\phi}_{t+1}\\ \epsilon^{z}_{t+1}\end{pmatrix}\\ \widehat{\mathbf{y_t}} &= \mathbf{g_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}.\end{aligned} \tag{18}$$

There are various ways to solve for the coefficients $\mathbf{h_x}\left(\mathbf{x},0\right)$ and $\mathbf{g_x}\left(\mathbf{x},0\right)$. I use the `gensys` algorithm of Sims (2001) This step involves computing the Jacobian of $\mathcal{F}(.)$.

Then, the second-order approximation of the solution writes:

$$\begin{aligned}\widehat{\mathbf{x_{t+1}}} &= \mathbf{h_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}+\tfrac{1}{2}\mathbf{h_{xx}}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}^2+\tfrac{1}{2}\mathbf{h}_{\eta\eta}\left(\mathbf{x},0\right)\eta^2+\eta\begin{pmatrix}\mathbf{0}\\ \epsilon^{\phi}_{t+1}\\ \epsilon^{z}_{t+1}\end{pmatrix}\\ \widehat{\mathbf{y_t}} &= \mathbf{g_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}+\underbrace{\frac{1}{2}\mathbf{g_{xx}}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}^2}_{\text{nonlinear term}}+\underbrace{\frac{1}{2}\mathbf{g}_{\eta\eta}\left(\mathbf{x},0\right)\eta^2}_{\text{non-certainty equivalence term}}.\end{aligned} \tag{19}$$

This step uses the Jacobian and the Hessian of $\mathcal{F}(.)$, which are usually sparse. It involves solving for second-order coefficients in a large linear system of equations involving the Jacobian and the Hessian of the system, and the first-order coefficients obtained earlier. Usual second-order solution methods for representative agents models (Schmitt-Grohe and Uribe (2008)) cannot be applied here, because they require constructing and inverting a matrix too large to be stored in most computers. To circumvent the problem, I use the `gensys2` algorithm of Kim et al. (2008). It applies a series of linear operations to the original system (Schur and singular value decomposition) to reduce its dimensionality and solve for the coefficients.

3.2 Departure From Certainty Equivalence

The bond pricing implications of the model come from departure from certainty equivalence. With certainty equivalence, only the current *level* of credit limits would affect households' savings. When departing from it, aggregate *fluctuations* in credit limits are priced into risk-free rates.

In the first-order approximation, deviations from the steady state are zero in the absence of aggregate shocks, so the steady state in the first-order approximation coincides with the deterministic steady state. In the second-order approximation, even in the absence of aggregate shocks, deviations from the deterministic steady state are nonzero and depend on the volatility of aggregate shocks, giving rise to a stochastic steady state. The latter can be interpreted as the average level of the economy when it is hit by a long-run simulation of aggregate shocks. In the second-order (and higher) approximation, the effects of aggregate shocks on equilibrium states and policy functions do not cancel out in the long-run, because of the deviation from certainty equivalence. To compute the deviations of the stochastic steady state from the deterministic one, I compute a fixed point of the pruned laws of motion of the economy.[10] The impulse response functions (IRF) to credit and TFP shocks are computed by feeding the laws of motion with nonzero innovations in the first period and iterating on them. I verify that market-clearing errors are close to zero over the simulated paths (Appendix A.4).

3.3 Interpretation

Relative to standard business cycle models with fixed borrowing constraints, which fail to match the comovements of risk-free rates, household debt, and consumption, this paper relaxes the assumption of zero-probability credit shocks. Many frameworks where borrowing constraints affect the price of risk-free bonds rely on transitional dynamics experiments: the tightening of households' borrowing constraints is a zero probability event, whose path is perfectly forecasted by households once it hits the economy (e.g., Guerrieri and Lorenzoni (2017)). Instead, in the economy that I study, households have rational expectations both in the stochastic steady state and along short-run transition paths. They know the stochastic process governing the credit shock and its current realization, but not its future realizations. This generates a precautionary motive, which

[10] I use pruned laws of motion to alleviate the well-known problem that iterating on second-order laws of motion gives rise to higher-order terms that do not increase the accuracy of the approximation and are likely to lead to explosive paths. Pruning essentially computes first-order projections of second-order terms, based on a first-order expansion of the conditional expectation of the system's deviation from steady state.

appears in the Euler equation of bondholders:

$$c_t^{-\gamma} = \beta(1 + r_t)\mathbb{E}_t \left[c_{t+1}^{-\gamma}\right] + \mu_t. \tag{20}$$

Given a consumption process, a high Lagrange multiplier μ_t on current credit limits implies a lower risk-free rate. Iterating the equation forward, the average and higher-order moments of future multipliers $\{\mu_s\}_{s\geq t}$ further lower the current rate.

In the solution of the model, prices and quantities are perturbed around their (deterministic) steady state values with fixed aggregate credit and TFP. They are decomposed as the sum of terms, which are linear and quadratic in the state variables' deviations from steady state, and a constant term that is linear in the variance of aggregate shocks. The constant terms permanently shift the model long-run (stochastic) steady state, hence the average level of the risk-free rate, and capture precautionary savings motive associated with aggregate shocks. These terms measure the respective contributions of credit supply versus TFP shocks to low risk-free rates. This decomposition is new relative to e.g. Favilukis et al. (2017), who also study aggregate shocks to borrowing constraints.[11]

Moreover, only linear term appears in the first-order perturbation of the model. Thus the nonlinearities associated with credit shocks can be estimated by computing the difference between first- and second-order terms once precautionary motives are measured (the fixed second order terms).[12] Below, I show that nonlinearities are key to explain the dynamics of risk-free rates in response to credit supply shocks.

4 Calibration and Model Fit

The model is calibrated at quarterly frequency to match annualized targets in the deterministic steady state of the economy. Table 1 summarizes the calibration. In order to deliver a realistic picture of households' use of liquid assets and short-term debt to smooth aggregate income and credit shocks over the cycle, I focus on matching level and cross-sectional moments for unsecured household credit, government taxes, transfers, and the real risk-free rate in the U.S. Average household income is normalized to 1.

Unsecured household credit For unsecured debt and liquid assets to GDP, I target values of 0.18 and 1.78, and obtain 0.22 and 1.73. Values are from the Flow of Funds of the

[11] The limitation of this solution is to be based on perturbation methods, which only makes it accurate in a neighborhood of the steady state. However, as I show below, the approximation remains accurate for large deviations, such as the deleveraging episode of the Great recession.

[12] This measure is not fully exact as it abstracts from higher-order terms.

Table 1: Baseline model parameters

Parameter	Explanation	Value	Target/source
β	Discount factor	0.9925	Risk-free rate $r = 2\%$ (FRB)
γ	Coefficient of relative risk aversion	5	–
η	Curvature of disutility of working	2	Frisch elasticity $= 1/2$
ψ	Disutility of working	11.5	Normalize $Y = 1$
$\overline{\phi}$	Average credit shock	2.6	Unsecured debt-to-GDP 0.18 (FRB)
$\phi(\theta)$	Credit limit function	$(1, 1.03, 1.06, 1.08, 2.33)$	Debt dispersion across incomes (SCF)
α	Cobb-Douglas parameter	2/3	Labor share of 2/3
$\tau_1(\theta)$	Tax function	$(0.05, 0.13, 0.17, 0.20, 0.28)$	Tax dispersion across incomes (CPS)
$T(\theta)$	Transfer function	$(1, 0.43, 0.24, 0.17, 0.13)$	Transfer dispersion across incomes (CPS)
B	Bond supply	6	Liquid assets-to-GDP 1.78 (FRB)
ρ_θ	Persistence of productivity shock	0.977	Persist. wage process (Kopecky and Suen (2010))
σ_θ	Std. dev. of productivity shock	0.12	Std. dev. wage process (Kopecky and Suen (2010))
ρ_ϕ	Persistence credit shock	0.99	Persist. risk-free rate 0.65 (FRB)
σ_ϕ	Std. dev. credit shock	0.025	Std. dev. risk-free rate 1.9% (FRB)
ρ_z	Persistence TFP shock	0.86	Persist. TFP
σ_z	Std. dev. TFP shock	0.0128	Std. dev. TFP
$\rho_{\phi z}$	Corr. credit and TFP shocks	0.5	Corr. debt-income 0.9 (FRB, BEA)

Notes: Quarterly frequency, targets are annualized. The top panel lists household-level, the middle-top panel firm-level, the middle-bottow panel government-level parameters, and the bottom panel aggregate shocks.

Federal Reserve Board (FRB) in 2006.[13] For that, I choose $\overline{\phi}$ to be equal to 2.4, and the sum of all households net bond positions, B, equal to 6 (1.5 quarterly).

To calibrate variables' distributions across income types, I map the income distribution in the model to the data by constructing the corresponding productivity quintiles. I then compute the empirical counterpart and distribution of model variables. In the steady state, 6.25% of households are in the first income bin, 25% in the second, 37.5% in the third, 25% in the fourth, and 6.25% in the fifth.[14]

In the 2007 Survey of Consumer Finances (SCF), I compute total unsecured credit as total household debt minus the total value of debt secured by primary residence (including mortgages and HELOC) and the total value of debt for other residential properties. This leaves other lines of credit, credit card balances, installment loans (including education and auto loans), and other debt. In the five income bins, average (median) unsecured debt in 2013 dollars is respectively: 130,190 (0); 17,150 (450); 25,340 (6,512); 141,190 (9,431); 302,920 (0). Replicating this distribution would require lower debt limits for θ_2 and θ_3 agents than for θ_1, at odds with the empirical evidence that debt *limits* increase

[13] Liquid assets are defined as the sum of all deposits and securities held directly by households. Data is from the Federal Reserve Board Flow of Funds (Z.1) table B.100, sum of lines 9 (inventory change), 16 (Treasury currency), 19 (checkable deposits and currency), 20 (time and savings deposits), 21 (money market fund shares), 24 (open market paper), and 25 (Treasury securities). Unsecured debt is from table B.100, line 34, which essentially corresponds to total household liabilities minus mortgage debt.

[14] Jappelli (1990) also computes access to credit conditional on household characteristics.

with income.[15] Therefore, I smooth the distribution by assuming that debt limits are uniformly distributed from θ_1 to θ_4, and thus increase with income. When normalized, they measure the relative debt limits faced by the various income types. Therefore, I use them to calibrate the idiosyncratic component of debt limits, $\phi(\theta) = (1, 1.03, 1.06, 1.08, 2.33)$. This allows to match the dispersion of average debt across income types. The dispersion of debt limits at the bottom of the income distribution, where more households are constrained, interacts with consumption smoothing, and determines the magnitude of precautionary savings motives.

In each income bin, average debt to average income is respectively: Inf, 3.13, 0.56, 0.78, 0.15.[16] In the model, household debt is positively related to their debt limits, and is endogenously chosen. Households occasionally borrow up to their debt limits. In particular, households with lower incomes are able to borrow a lower maximum amount than richer households, they borrow less on average when they do, and they are more frequently constrained. The fractions obtained are: 5.15, 1.09, 0.47, 0.05, 0.01. Overall, the model matches the decreasing profile of average debt to average income, and the fact that low income households hold on average disproportionately more debt relative to their income.

Government taxes and transfers I choose the tax and the transfer functions to replicate the empirical distribution of taxes and transfers across incomes. In Congressional Budget Office data for 2006[17], average total transfers to non-elderly households for the 5 quintiles of pretax market income (productivity in the model) are respectively (2006 dollars per households): 15,200; 6,600; 3,700; 2,600; 2,000. Normalizing by θ_1, it delivers a transfer function $T(\theta) = (1, 0.43, 0.24, 0.17, 0.13)$. Transfers represent 6.9% of average market income. To match that share in the model, I use a multiplicative factor identically applying to all transfers. Taxes by income quintiles are (2006 dollars per households) 2,600; 6,500; 11,800; 19,700; 68,100. This delivers a tax function $\tau_1(\theta) = (0.05, 0.13, 0.17, 0.20, 0.28)$.

Real risk-free rate: level The risk-free rate clears the bond market, and is directly impacted by credit limits because of a first-order (deleveraging) and a second-order order effect (precautionary savings). The discount factor $\beta = 0.9925$ is chosen to match the an-

[15]The reason for this pattern is likely due to facts not captured by the model, which lead both very low and very high income agents to accumulate a lot of debt. The model could be extended to include mechanisms such as large liquidity shocks (e.g. health expenditures, cost of college education).

[16]Average income is zero in the first bin because wage income is zero for unemployed agents.

[17]See "The Distribution of Federal Spending and Taxes in 2006", Exhibit 18 in Congressional Budget Office (2006).

nual Treasury Inflation Indexed long-term average yield of about 2.5% in 2009Q3 (based on the unweighted average bid yields for all TIPS with remaining terms to maturity of more than 10 years; source: FRB, H.15 Selected Interest Rates). Pflueger et al. (2020) report a value of 2.4% for the period 1973Q1-2014Q4, using the one-year Treasury bill rate net of one-year survey expectations of the inflation (GDP deflator) from the Survey of Professional Forecasters, expressed in percentage terms and linearly detrended. The value obtained using the lending rate adjusted for inflation using the GDP deflator is 2.5% in 2009 (source: IMF International Financial Statistics for the lending rate, and World Bank for the GDP deflator.).

When all households have the same β, the model generates a lower fraction of constrained θ_1 agents than of constrained θ_2 and θ_3 agents, in part due to government transfers being much larger θ_1 agents than for θ_2 and θ_3 agents. This is at odds with evidence that low income agents are on average more financially constrained. Therefore, I assume that agent with the lowest income θ_1 are also more impatient, with a 20% lower discount factor than the rest of agents.

Real risk-free rate: volatility and persistence The annual persistence and volatility of the risk-free rate in the data are 0.65 and between 1.50% (FRB) and 1.90% (Pflueger et al. (2020)). To match those, I structurally estimate the quarterly autocorrelation and volatility of the credit shock process, respectively 0.99 and 2.5%. The model counterpart for the risk-free rate persistence and volatility are computed in a long simulation ($T = 10,000$ period) of the linearized version of the model with random TFP and credit shocks, and then annualized.[18] I perform sensitivity analyses and show that these estimates are well-identified (Appendix Figure 8). In theory, an alternative would be to use unsecured credit changes from the data to estimate the parameters of the $\overline{\phi_t}$ process. In practice, however, contractions of debt limits themselves for a given $(household, loan)$ observation are less frequent than changes in credit standards, which continuously happen.

Real risk-free rate and household credit: procyclicality Though the model endogenously generates the positive correlation between the risk-free rate, household debt, and output, it is quantitatively not large enough to match the procyclicality of the real rate in

[18]I do not use the second-order approximation because the variable paths may diverge in long-run simulations even with pruning, a well-known problem with high-order approximations. The second-order approximation matters more for the average level than for the autocorrelation and standard deviation of the risk-free rate. Thus even if computing the stochastic steady state based on a long-run simulation was possible, the estimated values would likely be very similar to those of the first-order simulation, which correspond to the deterministic steady state.

U.S. data. I calibrate the correlation between TFP and credit shocks $\rho_{\phi z} = 0.50$ to match the empirical correlation between unsecured household debt and household income. Using various linearly detrended measures for consumption (personal consumption expenditures) or income (personal income) from the Bureau of Economic Analysis (BEA), and outstanding total consumer credit owned and securitized from the FRB (G.19 Consumer Credit), this correlation is about 0.90.

Aggregate productivity risk The TFP persistence $\rho_z = 0.86$ and volatility $\sigma_z = 0.0128$ are standard parameters in the business cycle literature (Fernald (2014)).

Risk aversion The coefficient of relative risk aversion γ determines the strength of the precautionary savings motive. I set it equal to 5, slightly lower than Favilukis et al. (2017) (8), the only other asset pricing paper with aggregate credit supply shocks. I experiment with different values. Larger values induce a lower risk-free rate and less debt in the stochastic steady state, and more amplification in the economy's response to shocks. For some parameter combinations, values below 4 imply an *increase* in aggregate consumption together with a decrease in debt to GDP following a *tightening* of borrowing constraint, a possibility so far undocumented in models of credit crises (e.g. Guerrieri and Lorenzoni (2017)). The reason is that if a low number of agents are constrained, and all agents have low precautionary savings motives, then the decrease in consumption by (close to) constrained households who mechanically deleverage may be more than offset by the corresponding increase in consumption by richer households, who decrease their net bond positions in order for the bonds market to clear when poorer households demand more bonds (this is facilitated by the sharp decrease in the risk-free rate), and thus save less and consume more.[19]

Idiosyncratic income risk The persistence and variance of the discretized productivity process are chosen to match those of the wage process in Kopecky and Suen (2010), 0.977 and 0.12. The model does not have exogenous unemployment risk, so the volatility of income is larger to proxy for the otherwise missing income volatility.[20] To calibrate the countercyclicality of income volatility, I use the estimate of Storesletten et al. (2004) with the standard deviation of individual income increasing by 0.09 from peak to trough. In

[19]That effect would be dampened if government bonds outstanding B varied over time through new issuances, instead of taxes τ_{0t}.

[20]It is computationally hard to add an unemployment state when solving for households' labor supply in the system of equilibrium conditions, because the system becomes not differentiable at that point. In the model, zero labor supply can be interpreted as unemployment, but there is no involuntary unemployment.

the model, a one standard deviation negative shock to TFP induces a -0.5% deviation from steady state output. For an average trough of -1.5% deviation from steady state, a three standard deviation TFP shock is needed. Thus I assume that a one standard deviation negative TFP shock raises the income volatility by 0.09/3=0.03 (credit shocks do not have any effect). Countercyclical income risk does not affect the deterministic steady state where TFP shocks are zero, but it slightly amplifies the economy's response to shocks in both the first- and the second-order solution of the model. The reason it affects the first-order solution of the model despite only changing moments of order 2 is that it increases the dispersion of individual income, hence the probability that households hit their borrowing constraints and are prevented to smooth consumption, an effect accounted for in the first-order model.

The curvature of the disutility from labor is chosen to obtain a Frisch elasticity of labor supply of 1/2, a standard value in macroeconomics.

Net worth distribution The shape of the bond distribution determines how borrowers responds to credit shocks. Unlike other aggregate shocks that affect it indirectly, credit shocks directly shift the distribution: to the right when borrowing limits are tightened, to the left when they are relaxed. Table 2 displays summary statistics for the unconditional distribution $\lambda(\mathbf{b})$ of net bond holdings. By construction, the model matches both aggregate assets and debt to income. It generates a realistic distribution, with a mass of constrained households, and a decreasing fraction of households with larger wealth. The fraction of constrained households lies between two of its empirical counterparts, the fraction of "poor hand-to-mouth" households (between 3.5% and 21% in 2001; Kaplan and Violante (2014)) and the fraction of households reporting to be without savings (about 33%; The Pew Charitable Trusts (2015)). It also generates realistic wealth inequality as measured by the mean to median ratio of net worth.

Table 2: Cross-sectional distribution of bonds

Statistic:	Data	Model
aggregate liquid assets/aggregate income	1.78	1.73
aggregate liquid debt/aggregate income	0.18	0.23
mean/median	4.60	4.90
share of Hand-to-Mouth households	0.33	0.35
P50 liquid net worth/aggregate income	0.15	0.30

Notes: Income is in annual terms. Source: Boar, Gorea and Midrigan (2021).

5 Macro-Finance Moments

This section shows that TFP and credit supply shocks are key to explain bond pricing, household balance sheet, and macroeconomic moments in a unified framework. First, it shows that the precautionary motive associated with aggregate shocks helps explain the levels of risk-free rates, household debt, consumption, and employment. Second, it shows that the amplification mechanism (or nonlinearities) associated with credit shocks helps explain the short-run comovements of those variables. Both are a challenge for standard business cycle models with fixed credit limits, or where credit contractions are unexpected.

5.1 Mechanism

The model has static and dynamic implications for how an economy with stochastic borrowing constraints differs from one with fixed credit supply. Because of the departure from certainty equivalence, households (imperfectly) anticipate future credit shocks and insure every period against future binding borrowing constraints. The economy's stochastic steady state shifts in response to increased precautionary motives: on average, households accumulate less debt and the risk-free rate is lower. For low magnitudes of the credit shocks, the large negative financial response only mildly affects real variables, because prices are flexible and adjust to clear markets.

When credit shocks are large, however, financial conditions affect real variables because of a composition effect that induces less productive households to work more and more productive ones to work less, and because of a wealth effect on labor supply. This gives rise to an economy with persistently low output, employment, debt and risk-free rate. Therefore, in terms of long-run aggregate consumption, credit changes lead to substantially larger costs of business cycles than TFP shocks.

Credit shocks also affect the economy dynamically, from the time they hit households' borrowing constraints to the time they revert back to their steady state values. Nonlinearities in the second-order solution of the model amplify the economy's response to a tightening of borrowing constraints, by capturing the different responses of households depending on their debt levels. Importantly, nonlinearities are close to zero for TFP shocks, which affect all households identically, but not for credit shocks. Thus over the business cycle, aggregate prices and quantities do not respond linearly to aggregate shocks, as they would in a representative agent economy.[21] As a result, the entire distribution of

[21] Krusell and Smith (1998) call this property "near-aggregation".

households' debt matters for the pricing of risk-free bonds.

5.2 Impact of Credit Supply Risk

Stochastic steady state By taking the difference between the second- and the first-order solution of the model, I obtain a numerical measure of nonlinearities and deviations from certainty equivalence due to aggregate risk. Table 3 displays the values of macroeconomic aggregates in the deterministic steady state and in the stochastic steady state (a long simulation of the economy). While other variables remain identical, the risk-free rate and household debt decrease in the stochastic steady state. This effect is small for the 1973-2005 period because the volatility of aggregate shocks is small. In addition, households have a moderate risk-aversion, and the impatience to consume of the income-poorest households – who are more constrained – conflicts with their precautionary savings motives that would further lower debt and the risk-free rate.

The strength of the precautionary motive is increasing in households' risk aversion, in the persistence and volatility of the credit shock, in the dispersion of debt limits across income types and in the countercyclicality of income risk. As the volatility of credit shocks increases, the risk-free rate required by agents to hold bonds decreases and so does aggregate debt, generating an environment like the post-Great Recession period. As implied by the distributions of net worth conditional on income in the previous section, debt decreases more for low-income and households close to the constraint.

Table 3: Financial and macroeconomic impact of credit supply risk

Variable	Deterministic steady state	$\sigma_\phi = 0.025$ (1973-2005)	$\sigma_\phi = 0.05$	$\sigma_\phi = 0.075$	$\sigma_\phi = 0.10$ (post-2005)
Interest rate	2.397%	-0.1%	-1.4%	-7.4%	-25.4%
Wage	1.491	0%	+0.07%	+0.3%	+0.9%
Profits	0.333	0%	0%	-0.6%	-1.5%
Employment	0.447	0%	-0.2%	-0.7%	-2.5%
Output	1.000	0%	0%	-0.5%	-1.6%
Consumption	1.000	0%	0%	-0.5%	-1.6%
Debt/GDP	0.229	-0.4%	-2.7%	-12.5%	-45%

Notes: Comparative statics analysis, holding other parameters fixed. Steady state summary statistics: risk-free rate, wage, profits, employment, output, consumption, aggregate net bond positions, debt/GDP, assets/GDP. Risk-free rate, debt/GDP and asset/GDP in annual terms. Columns 3-6 are percentage deviations from steady state values (column 2).

Comparison with productivity shocks The same comparative statics with the volatility of TFP shocks, shows that the differences between the deterministic and the stochastic steady states are close to zero, even for large TFP volatility (results available upon request). For instance, increasing $\sigma_z = 0.0128$ to $\sigma_z = 0.03$ does not affect variables compared to the deterministic steady state. Increasing it by a factor of five, to $\sigma_z = 0.06$, slightly decreases the risk-free rate by 0.10% (to 2.394%), while other variables remain unchanged. Thus credit volatility is a more important source of precautionary savings than TFP fluctuations.[22]

Three precautionary motives There are three types of precautionary savings motives in the model: a microeconomic, a macroeconomic, and a financial one. First, without aggregate risk, idiosyncratic income risk at the micro level increases the demand for insurance directed towards safe assets. One reason this motive arises is "prudence" (Kimball (1990)): $u'''(c) > 0$, so the volatility of income increases future expected marginal utility because of Jensen's inequality, which implies a decrease in current consumption and an increase in savings in households' Euler equation. Another reason is the presence of borrowing constraints: households fear that a sequence of negative income shocks will lead them to hit their borrowing constraints, preventing consumption smoothing; this effect is stronger for households at or near their constraints. Absent this first motive, the steady state risk-free rate would be equal to 8.33% (annualized percentage terms).[23] With this motive, it is equal to 2.40%, that is 71% lower.

Second, with aggregate risk, a motive due to TFP shocks arises at the macro level. Its mechanism is the same as the first motive, since TFP shocks result in labor income changes. The difference is that TFP changes across periods are smaller and less persistent than changes in idiosyncratic productivity, and they impact households uniformly. As a result, the associated decrease in the risk-free rate and aggregate consumption is close to zero. The quasi-absence of an effect on long-run consumption is consistent with the very low estimates of the cost of macroeconomic fluctuations in Lucas (2003).

Third, there is a new *financial* motive due to aggregate credit shocks. They contribute to lowering the steady state risk-free rate by an extra 0.13%, to 2.39% (1973-2005 period). In an economy similar to the post-Great Recession period ($\sigma_\phi = 0.10$), they lower the

[22] The lesser importance of TFP volatility for precautionary savings is consistent with the literature on high-order approximations of *representative agent* RBC models. In those models, the steady state and the dynamics of first- and second-order approximations of the economy are nearly identical (Fernandez-Villaverde, Rubio-Ramirez and Schorheide (2016)).

[23] The benchmark discount factor is $\beta = 0.9925$. A fraction 0.0625 of agents have a 20% lower discount factor, so the average discount factor is equal to $0.0625 \times 0.80 \times 0.9925 + (1 - 0.0625) \times 0.9925 \approx 0.979$.

steady state risk-free rate by an extra 25%, equivalent to a third of the micro motive. Furthermore, credit supply risk is not neutral in the long-run. It substantially increases the cost of macroeconomic fluctuations, by decreasing long-run aggregate consumption by 1.6%. This is a significant departure from results on the low cost of business cycle fluctuations.

Policy functions: the role of "middle-class" households To understand how credit risk affects aggregate savings, consumption, and employment in the long-run, I plot policy functions for the median income household when $\sigma_\phi = 0.10$ (Figure 1). The blue line is for the deterministic steady state, with fixed credit limits, and the orange line is for the stochastic steady state, where households anticipate credit shocks. In the latter case, households consume less, save and work more to achieve higher precautionary savings.

A comparison of policy functions across income levels (available upon request) reveals that this effect is stronger for more patient, low-income households ("middle-class" θ_2 and θ_3 agents). Those benefit less than the lowest income households (θ_1) from the progressivity of taxes and transfers, and they are less rich than more productive households (θ_4 and θ_5), so their precautionary motive dominates their impatience to consume. Finally, unlike other households, the most productive ones (θ_5) consume slightly more when anticipating credit shocks. Accordingly, they also save less and work less. This is due to the lower risk-free rate, which creates incentives for them to dissave and consume more of the consumption good and of leisure. With a fixed supply of bonds B, the lower risk-free rate clears the bond market when the demand for bonds from poorer households is higher, and that from richer households is lower.

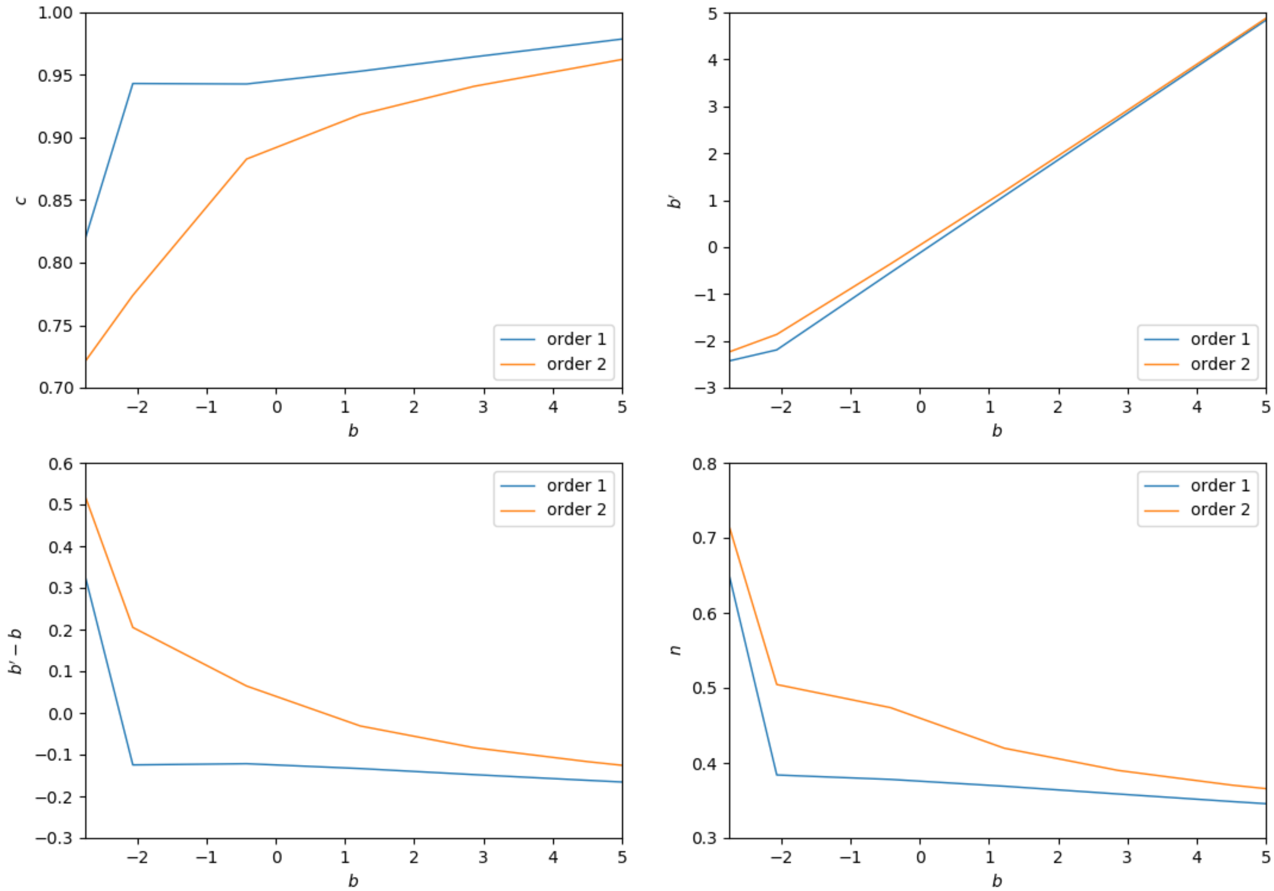
5.3 Business Cycle Volatility

Based on these results, I now investigate how changes in household credit contribute to financial and macroeconomic volatility in long run simulations of the economy. I implement a variance decomposition exercise using the economy's nonlinear laws of motion, to quantify the contribution of credit and TFP shocks. Overall, credit supply shocks are responsible for a larger share of volatility than TFP shocks, especially for bond prices and household debt.[24]

As Table 4 shows, the model assigns about or more than half of the variation of macroeconomic aggregates to credit shocks – except for wages, which scale one by one with TFP. This happens even in the absence of price rigidities, because of the adverse re-

[24] Appendix A.5 details the computations, which use the nonlinear laws of motion of the economy.

Figure 1: Consumption, savings, and employment policy functions



Notes: Policy function of the median income household by net bond level, starting at the corresponding steady state credit limit $\overline{\phi}\phi\left(\theta\right)$ for: consumption, next period bonds, savings and labor supply. Order 1 (blue) vs order 2 (orange).

action of employment analyzed in the previous section. This effect is reinforced by the volatility and persistence of credit shocks, which are much higher than their TFP counterparts.

5.4 Short-Run Comovements

I turn to comparing the economy's impulse response functions to credit and TFP shocks, in the first- and in the second-order approximations of the model. The amplification of the responses of financial and real variables to credit shocks is key to replicate their short-run comovements, especially around the Great Recession. Variables stay longer at lower values following a tightening of households' borrowing constraints. In the first-order approximation, households respond to the levels of current and expected future shocks,

Table 4: Credit and TFP shocks' contributions to financial and macroeconomic volatility

Variable:	Credit supply	TFP
Interest rate	59%	41%
Employment	52%	48%
Wage	21%	79%
Profits	59%	41%
Output	59%	41%

Notes: Variance decomposition: shares of the variance of variables in the first column accounted for by credit (second column) and TFP shocks (third column): bond price, employment, wages, profits, output, taxes. Variance shares are computed by bootstrap, as the Monte-Carlo average of the variance decompositions of generalized forecast errors at a large forecasting horizon ($H = 1000$ periods). Computations use $N = 500$ simulations.

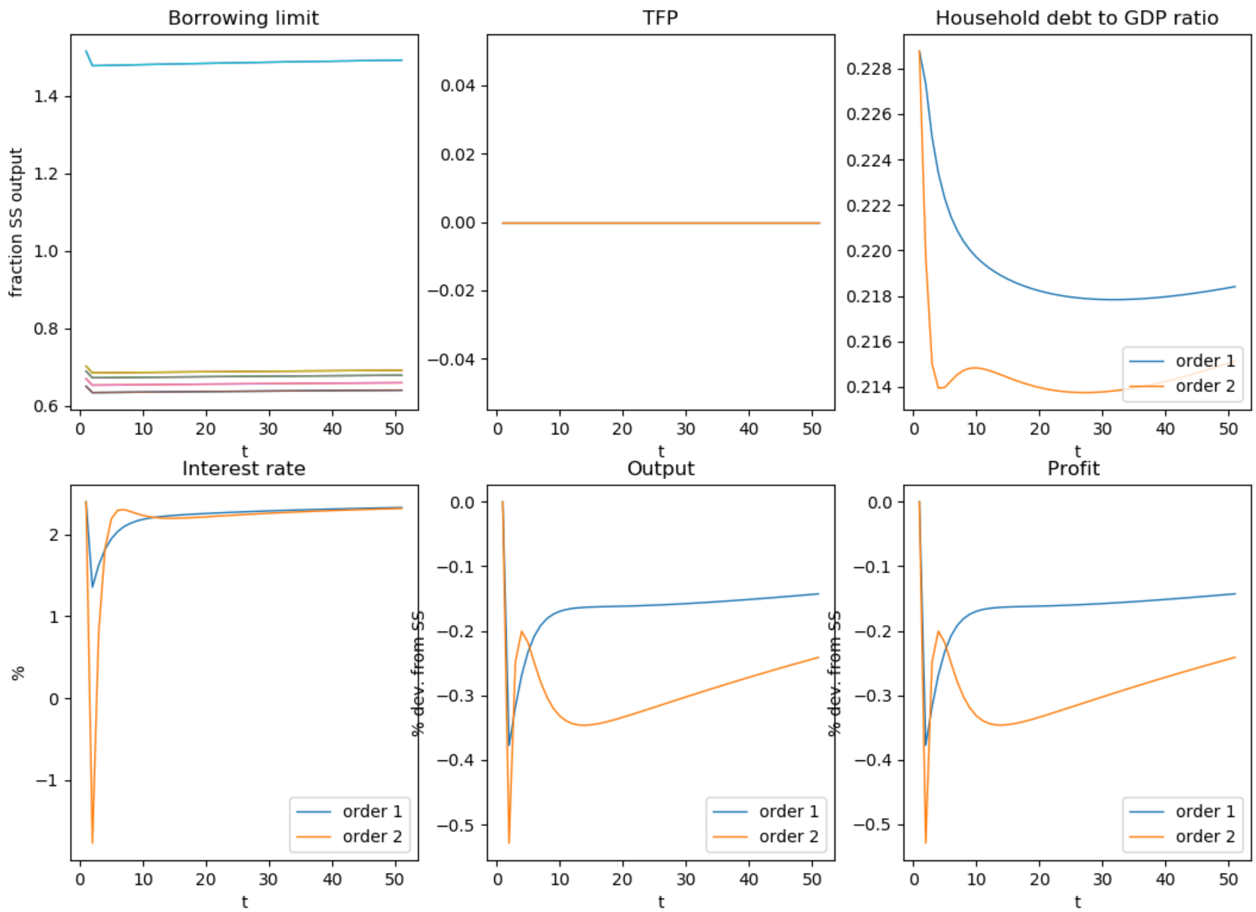
but not to their volatility, because of certainty equivalence. Policy functions, the cross-sectional distribution and prices respond linearly to shocks, and are linear functions of their lagged values. In the second-order approximation, households anticipate aggregate shocks, whose volatility enter linearly in the IRF. In addition, the economy evolves nonlinearly with respect to the level of shocks, with variables being linear and quadratic functions of their lagged values.

Amplification Figure 2 plots the economy's response to a one standard deviation credit shock, under the linear dynamics with certainty equivalence (order 1), and the nonlinear dynamics with households anticipating credit shocks (order 2). Aggregates are computed using the time-varying path of individual policy functions and of histogram weights. Deviations are from the deterministic steady state.[25]

All households' debt limits fall, but lower income households are able to borrow less than richer ones, reflecting idiosyncratic differences in their ability to borrow. As a result, constrained households are forced to reduce their debt and increase their net bond positions, thereby decreasing their consumption of goods and leisure. They trade-off working more to smooth consumption against the disutility of labor. Aggregate debt to GDP decreases, and stays persistently low, mostly because of the large persistence of the credit shock. The decrease in total consumption results from the composition of low-income constrained households decreasing their spending, and richer unconstrained households increasing theirs because they earn a lower return on their bonds. The decrease in the risk-free rate allows to balance a larger bond demand from the former with a lower demand from the latter, and to clear the bond market. Aggregate employment decreases,

[25]Results are similar if I compute deviations from the stochastic steady state.

Figure 2: Response to a credit supply shock: procyclical financial variables



Notes: Impulse response functions to a one standard deviation credit shock. Credit constraints (upper left panel) as a fraction of annual steady state output (upper left panel) are for θ_1 (lowest line) to θ_5 households (highest line). Other panels plot IRF in the 1st versus the 2nd order approximation of the model. Initial period: deterministic steady state.

causing a decline in output (see below). As implied by the economy's resource constraint, consumption falls with output.

Amplification is largest for the risk-free rate and household debt. Accounting for non-linearities significantly amplifies the response of aggregates to a credit shock. This result contrasts with recent findings, which focus on TFP shocks (e.g. Winberry (2018)). The initial response is amplified by a factor of 5 for debt to GDP, of 4 for the risk-free rate, of 1.5 for consumption, output and profit, and of 1.4 for the wage. While the risk-free rate decrease (in response to a one-time shock) is short-lived, other variables stay persistently low. The sharp decline in the rate causes consumption and employment to rebound (simultaneously, profits slightly increase and the wage slightly decreases). However, the rebound is short-lived, and the large persistence of the credit shock that induces borrowing constraints to stay persistently low, further decreases consumption and employment. Debt to GDP stays persistently low and barely rebounds. The price adjustment (the decrease in the risk-free rate) cannot offset the quantity restriction imposed by tighter borrowing constraints, which mechanically force constrained households to hold less debt. At the household level, these changes can be decomposed into changes in the cross-sectional distribution and in policy functions.

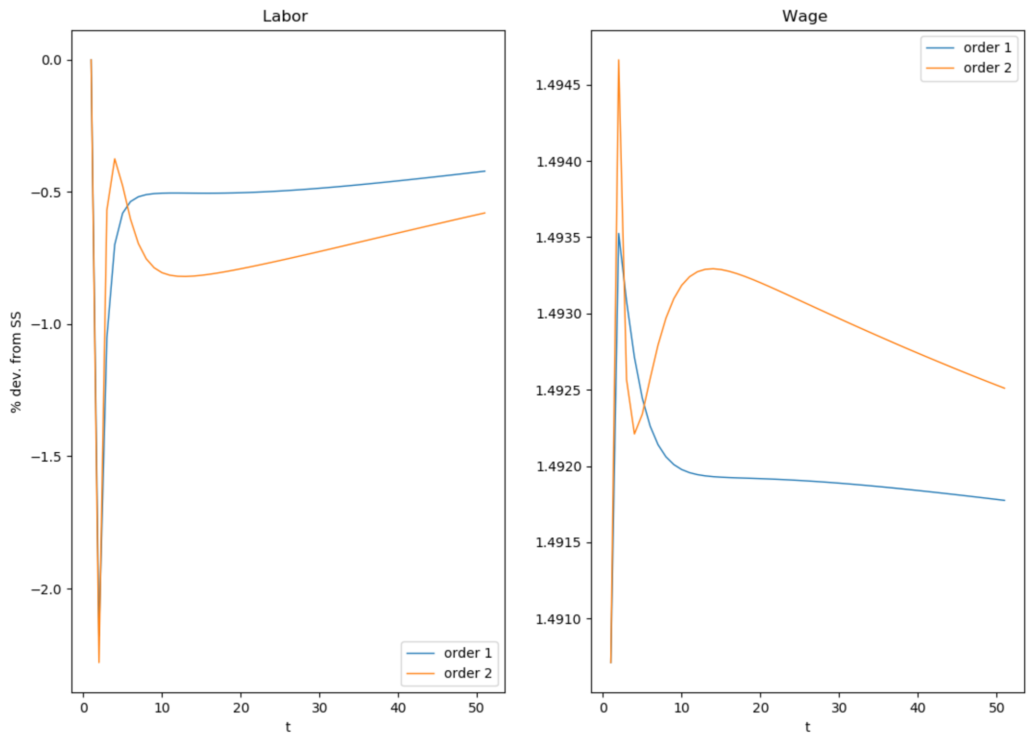
Credit shocks and employment In addition to risk-free bonds, households also insure against shocks by adjusting their labor supply. The employment response in Figure 3 results from the composition of less productive, constrained households who increase their labor supply to smooth consumption when they are forced to deleverage, and of more productive, unconstrained households who consume more leisure as they decrease their savings (wealth effect). In addition, the sharp decline in the risk-free rate creates an intertemporal substitution effect, which induces all households to consume more leisure in the current period.

The sign of the labor response depends on which effects dominates. Here output declines, mainly because more productive agents work less, despite less productive agents working more. This result differs from economies with fixed credit limits, where employment increases after a credit shock (e.g. Guerrieri and Lorenzoni (2017)). The difference between employment responses can be explained as follows. Here, the credit shock is mean-reverting. In economies with fixed credit limits and permanent shock unfolding over multiple periods, which are perfectly foreseen by agents, constrained households and those expecting to be constrained in the near future choose to work more. This effect is dampened in my model, because agents expect the credit shock to mean-revert. Thus even with flexible prices, small mean-reverting credit shocks can generate larger negative employment responses than unanticipated, large and permanent deleveraging episodes.

Without that effect, hours worked would increase after a tightening of credit supply, resulting in precautionary savings increasing less, and risk-free rates decreasing less. Thus it is important for the dynamics of bond prices. Importantly, it requires heterogeneous agents. With a representative agent, employment increases even after a credit supply shock, because this is the only way the agent can save more (e.g. Jones, Midrigan and Philippon (2020)).

Response to productivity shocks With only a TFP shock, borrowing constraints stay fixed. Because the economy becomes less productive, a decrease in TFP induces a fall in output, hence in profits. Because it induces a fall in the wage (the marginal productivity

Figure 3: Response to a credit supply shock: labor market



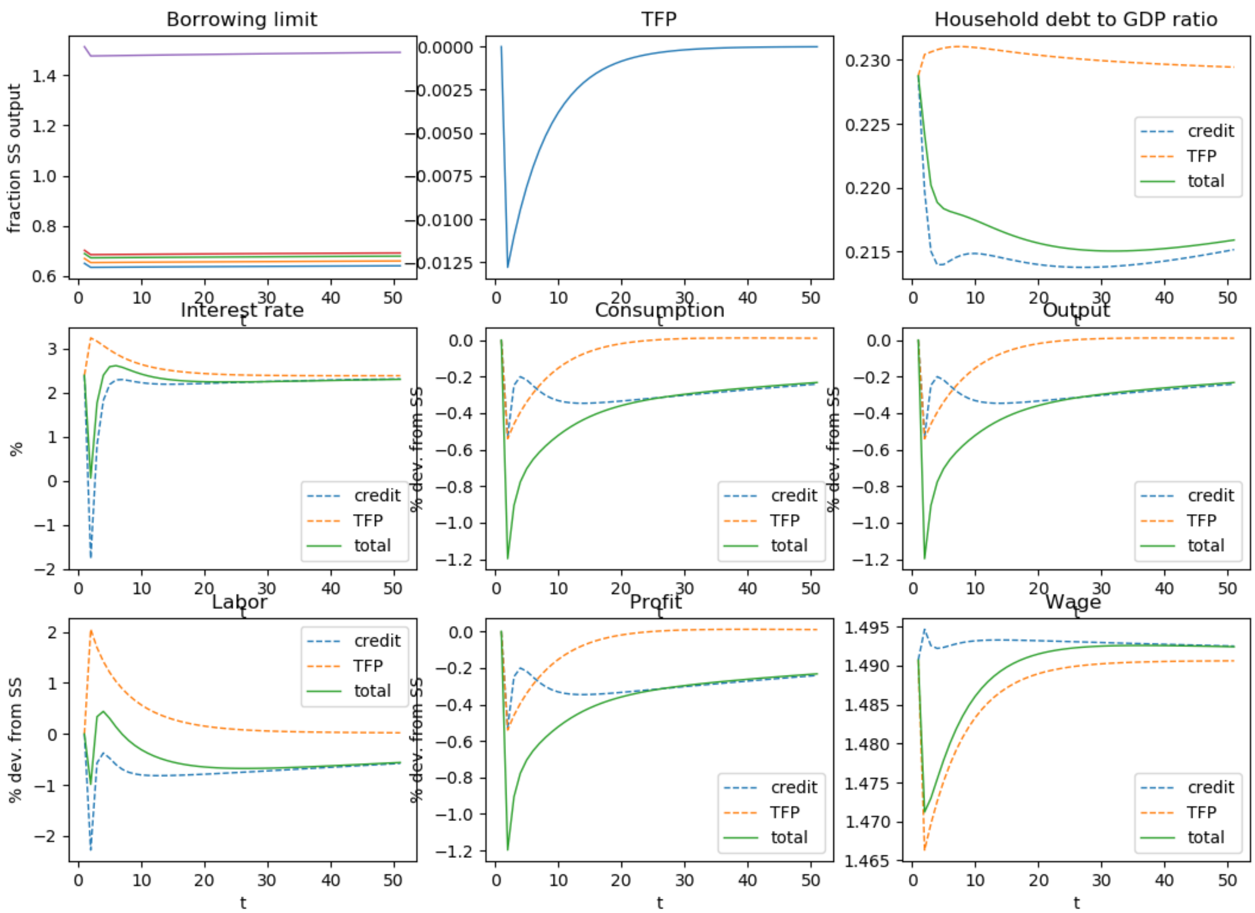
Notes: Impulse response functions to a one standard deviation credit shock for aggregate employment (left panel) and wages (right panel), in order 1 vs order 2. Initial period: deterministic steady state.

of labor decreases), households' income decreases and they supply more labor to compensate for the decrease. The wage decline induces a higher labor demand from firms. Summing the two effects, aggregate employment increases. To smooth their consumption when their incomes decline, lower income unconstrained households issue more debt, while richer households increase their savings. This increases wealth inequality. The risk-free rate increases to clear the bond market, with a larger demand for debt by low-income households and a larger demand for safe assets by richer ones. Together with the decline in output, this induces debt to GDP to increase, at odds with its procyclical pattern in the data.

Response to credit supply and productivity shocks As Figure 4 shows, the combination of credit and TFP shocks generate a realistic business cycle dynamics, which I exploit in the next section to estimate the sequence of structural shocks that explain the Great Recession period. Both households' borrowing constraints and TFP fall. The effects of the TFP shock dampen those of the credit shock. Debt to GDP decreases, but less than implied by the single credit shock, since low-income, unconstrained households resort to debt to smooth their consumption when their labor income declines. This partly offsets the precautionary savings effect implied by tighter borrowing constraints. Therefore, the risk-free rate falls less. Thus combining credit and TFP shocks makes the dynamics of the risk-free rate and household debt less extreme than with credit shocks only.

Employment also decreases, but less than with credit shocks. This is due to the decrease in the wage resulting from lower TFP, which makes hiring workers more attractive to firms. In contrast, consumption (hence output) is negatively affected twice – by lower TFP and tighter borrowing constraints. Firm profits also fall, tracking the decline in output.

Figure 4: Response to credit supply and TFP shocks: decomposition



Notes: Impulse response functions to a one standard deviation credit and TFP shock. Decomposition of the total response (green line) between credit shocks (dotted blue) and TFP shocks (dotted orange): order 2. The upper left panel plots the response of borrowing constraints to output for all income types (θ_1 for the lowest line, θ_5 for the highest)

6 Post-Credit Crisis Episodes

After the end of the Great Recession in 2009, aggregate consumption recovered but real risk-free rates remained low. This constitutes a "risk-free rate puzzle" for standard CCAPM with productivity risk. In those economies, a growing and stable consumption path should imply higher a risk-free rate, as households seek to borrow to smooth consumption over time (intertemporal substitution motive). In this section, I show that introducing both TFP and credit supply shocks in a heterogeneous agents economy allows to solve this puzzle, and to match the paths of financial and macroeconomic aggregates in the last 15 years.

I apply a particle filter to estimate the path of structural credit and TFP shocks that generated the observed paths for the risk-free rate and aggregate consumption from 2006 to 2017. The nonlinear solution of the model allows to match the data even in times of high volatility. The model also matches the path of household debt and employment, which were not targeted. Estimates show that: (i) TFP only fell during the Great Recession itself (2008-2009) and quickly reverted to its pre-crisis level; (ii) The implied measure of aggregate credit supply (inversely related to borrowing constraint tightness) kept decreasing until 2014, and stayed persistently low throughout the post-Great Recession period.

6.1 Model Fit

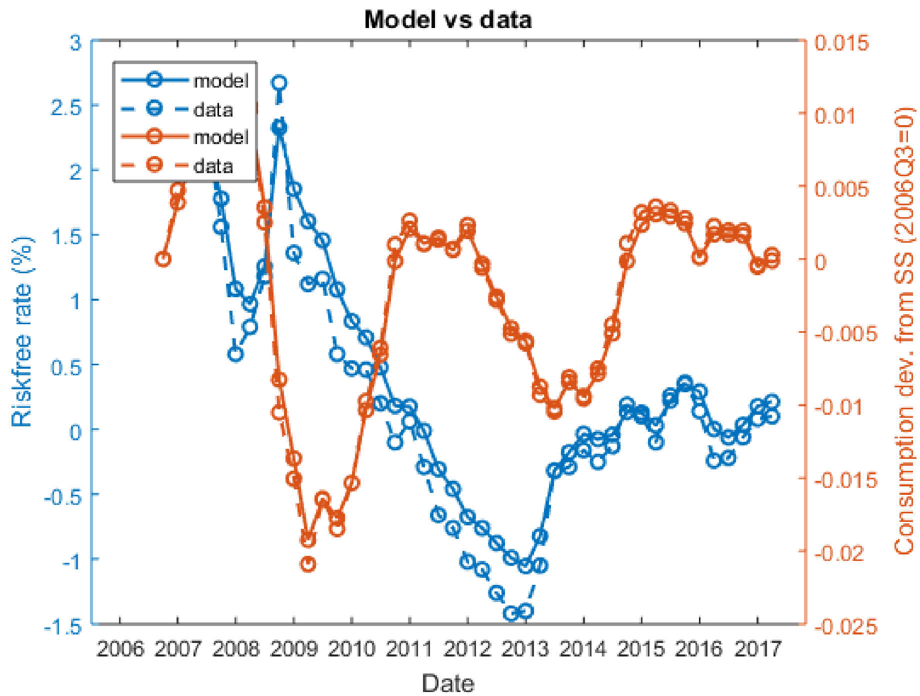
I use the model to recover measures of structural credit and TFP shocks implied by data on the risk-free rate and aggregate consumption from 2006Q3 to 2017Q2. This period of high volatility is a good test of the ability of the nonlinear model with household credit to match large changes in financial (risk-free rate, debt) and macroeconomic variables (output, employment).

To measure the real risk-free rate, I use the rate on 5-Year Treasury Inflation-Indexed Securities (Constant Maturity rate; source: Board of Governors of the Federal Reserve System).[26] To measure aggregate consumption, I use Real Personal Consumption Expenditures (source: U.S. Bureau of Economic Analysis). Because the deviation of consumption from its initial value in 2006Q3 is nonstationary over this period, I detrend it using a Hodrick-Prescott filter, and subtract the resulting initial value to normalize the detrended deviation to zero in the first period of the sample. Because the model is nonlinear, I use a particle filter to recover the sequence of structural shocks leading to the dynamics of the risk-free rate and consumption over this period (Appendix A.6.2 details the estimation).

[26] Results are similar for bonds of different maturities, see Appendix Figure 10.

As Figure 5 shows, the model matches bond and consumption data very well over the sample. It slightly misses the periods when the risk-free rate takes extremely low values, and in general the model forecast is slightly higher than its empirical counterpart. In addition, the tight link between consumption and TFP through the goods market clearing condition allows to exactly match the path of consumption deviations.

Figure 5: "Risk-free rate puzzle" after the Great Recession



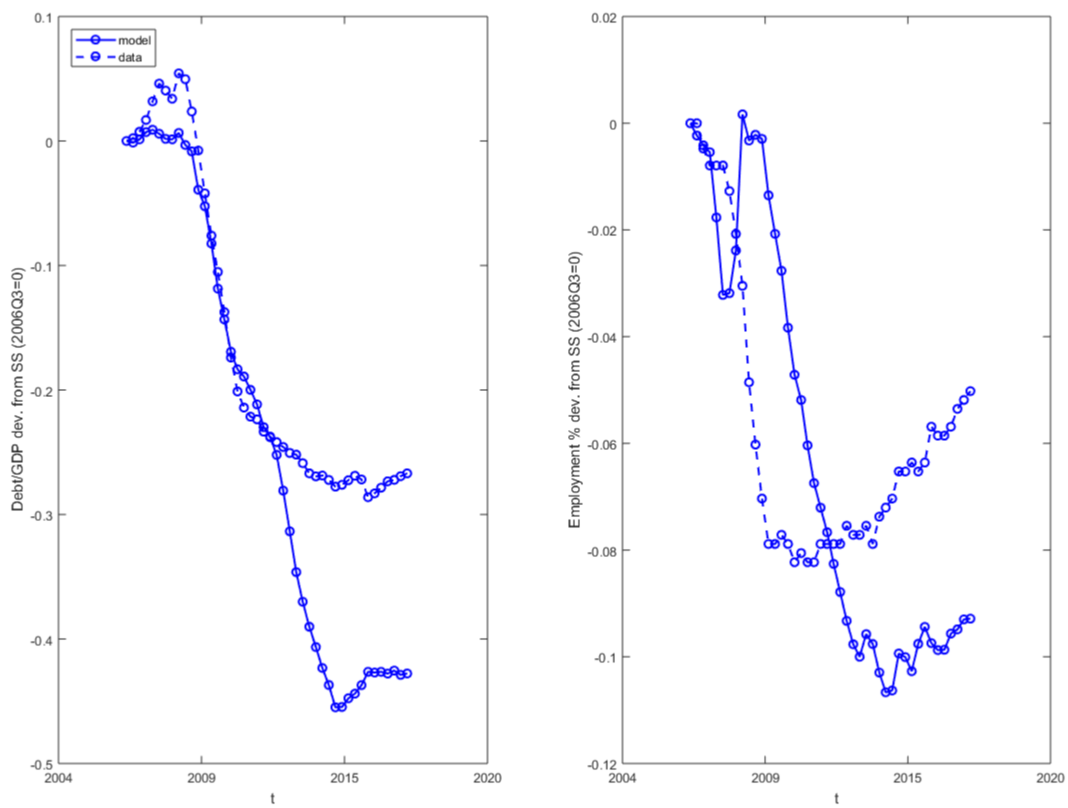
Notes: Risk-free rate (annualized, percentage) and consumption deviation from 2006Q3 value, predicted by particle filtering in the nonlinear version of the model (solid line) vs the data (dashed line). $N = 20,000$ particles simulated. The risk-free rate (left axis, blue) is measured as the 5-Year Treasury Inflation-Indexed Security, Constant Maturity rate, not seasonally adjusted (source: Board of Governors of the Federal Reserve System). Consumption (right axis, orange) is measured as Real Personal Consumption Expenditures, Billions of Chained 2009 Dollars, quarterly, seasonally adjusted (source: U.S. Bureau of Economic Analysis). Quarterly sample, 2006Q3-2017Q2.

The model-implied paths for debt-to-GDP and employment, which were not targeted by the particle filter, also closely match the data, which demonstrates the good fit of the model (Figure 6). The measure for household debt is Total Revolving Credit Owned and Securitized (source: Board of Governors of the Federal Reserve System). It is Personal Consumption Expenditures (source: U.S. Bureau of Economic Analysis) for GDP (in the model, output and consumption are equal), and Civilian Employment-Population Ratio (source: U.S. Bureau of Labor Statistics) for employment (the model features a continuum of measure 1 of households, so N is the ratio of employed to the entire population).[27]

[27] Results are similar with direct measures of aggregate consumption and hours worked. I did not use

The model replicates the hump-shaped pattern of household credit/GDP, with the run-up to the crisis until 2008, the decline and then the increase in credit around 2015. It fails to match the data in the last part of the sample, by overstating the decline in debt/GDP and in employment, for which the large persistence of credit shocks in the benchmark calibration and the wealth effect on labor supply may be responsible.

Figure 6: Model predictions: household debt and employment



Notes: Debt/GDP (left panel) and employment (right panel) implied by risk-free rate and consumption data, recovered by particle filtering. Model (solid line) versus data (dashed line). Results for $N = 20,000$ particles simulated. Variables are in log-deviations from their 2006Q3 values. Quarterly sample, 2006Q3-2017Q2.

6.2 Structural Estimation of Credit Supply and Productivity Shocks

Lastly, I plot the estimated paths for structural credit supply and TFP shocks (Figure 7). The model-based measure of credit shocks is inversely related to households' financial precautionary motive. The fall in the real risk-free rate over the sample, from 2.5% to -1.5% (annually), results from a large tightening of households' borrowing constraints. Compared to its 2006Q3 value, households' common component in the availability of

them for the estimation because they are less precisely measured.

credit decreases by more than 15%, and consistently stays 10% lower throughout the sample. The tightening prevents households from using debt to smooth consumption fluctuations. The precautionary motive is exacerbated by the (short-lived) drop in TFP, which induces constrained households to deleverage and save quickly. To clear the bond market, the risk-free rate collapses and stays persistently low, as long as borrowing constraints remain tight.

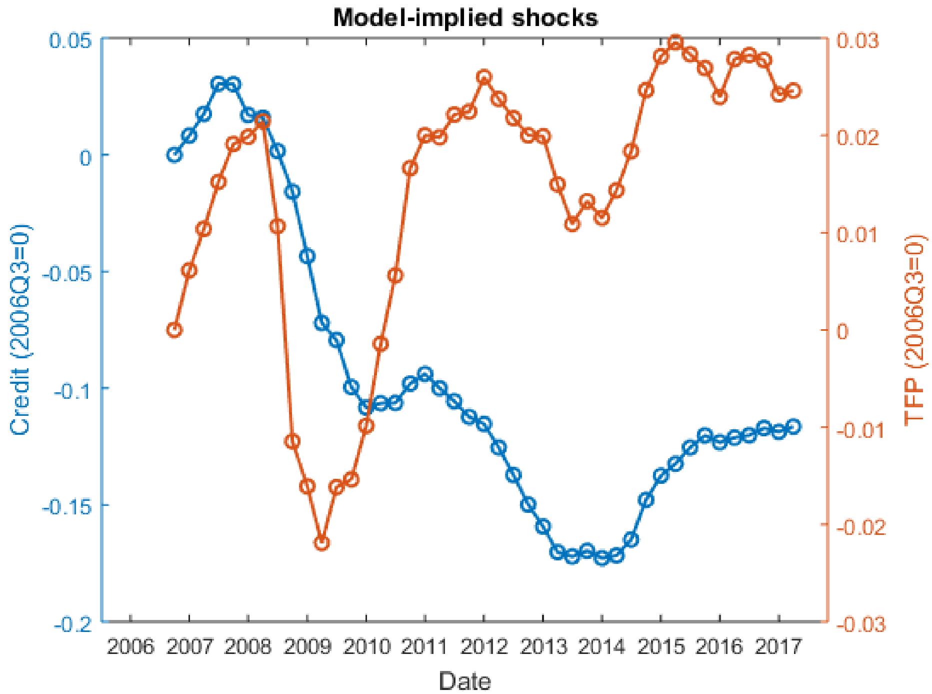
Credit shocks are a slow-moving variable, hence the lower-frequency fluctuations in debt and the risk-free rate. In contrast, TFP changes are more frequent, and track the higher frequency fluctuations in consumption. The 2% decline in TFP happens around 2008, to help match the fall in aggregate consumption. Then it reverts to its pre-crisis level within less than two years, and is more than 2% higher at the end of the sample. The model-based measure of TFP strongly correlates with empirical measures, providing external validation for these estimates (Appendix Figure 11). The same holds for credit supply shocks (Appendix Figure 12). Their pattern closely tracks lending standards measured from the Senior Loan Officer Survey on Bank Lending Practices until 2010 (Board of Governors of the Federal Reserve System). However, they then imply tighter credit constraints than captured by that measure alone. This is because bank lending standards measured in the survey only reflect credit tightness for new loans. Instead, my estimates of credit supply shocks capture the constraints applying to the total stock of unsecured credit at a given time, and the resulting decrease in real interest rates.

7 Conclusion

This paper uses a structural model of unsecured household credit to estimate the effect of a novel *financial* precautionary motive due to aggregate credit supply risk, which increases households' demand for risk-free bonds, and leads to lower real interest rates. Accounting for this motive, by introducing both credit supply and TFP shocks in an otherwise standard business cycle model, helps explain key properties of bond pricing and macroeconomic moments, which are puzzling in economies with fixed borrowing constraints. In particular, structural estimates show that persistently tight credit constraints since the Great Recession induced risk-free rates and household debt to remain low, despite consumption recovering.

Lastly, credit standards can be hard to assess empirically, as many measures rely on lenders' surveys, which are potentially hard to interpret. By providing a structural estimate of the common component of households' borrowing constraints, my approach

Figure 7: Structural estimates of credit supply and TFP shocks



Notes: Structural credit (left axis, blue) and TFP shocks (right axis, orange) estimated by particle filtering. Variables are in log-deviations from their 2006Q3 values. Quarterly sample, 2006Q3-2017Q2.

complements existing measures, with potential applications beyond the pricing of risk-free bonds.

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Appendix

A Computational appendix

A.1 Algorithm

1. (a) Variables are index by time t to denote their dependence on aggregate states $\left(\overline{\phi_t}, z_t, \lambda_t\right)$. The distribution of households over $\Theta \times \mathcal{B}$ is approximated as a histogram by a finite number of mass points on the Cartesian product of $\Theta = \{\theta_i\}_{i=1}^{N_\theta}$ and a fine bond grid $\{b_j\}_{j=1}^{N_b^f}$. $\Phi_t(\theta_i, b_j)$ denotes the fraction of households with productivity θ_i and bonds b_j. Its evolution is implied by policy functions according to:

$$\Phi_{t+1}\left(\theta_{i'}, b_{j'}\right) = \sum_{i=1}^{N_\theta} \sum_{j=1}^{N_b^f} \Pi_\theta\left(\theta_{i'}|\theta_i\right) \omega_{i,j,j',t} \times \Phi_t\left(\theta_i, b_j\right)$$
$$\text{where} \quad \omega_{i,j,j',t} = \begin{cases} \frac{b'-b_{j'-1}}{b_{j'}-b_{j'-1}} & \text{if } b'_t\left(\theta_i, b_j\right) \in \left[b_{j'-1}, b_j\right] \\ \frac{b_{j'+1}-b'}{b_{j'+1}-b_{j'}} & \text{if } b'_t\left(\theta_i, b_j\right) \in \left[b_{j'}, b_{j'+1}\right] \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

where $b_{j'-1}, b_{j'}, b_{j'+1}$ are bond points on the fine grid that bracket the value of next period bonds implied by the policy function. ω depend on t because policy functions depend on the aggregate state, i.e. $b'_t\left(\theta_i, b_j\right) = b'\left(\theta_i, b_j; \overline{\phi_t}, z_t, \lambda_t\right)$. For instance, if credit shocks $\overline{\phi_t}$ are low, tightening borrowing constraints, this distorts and shifts upwards the function $b'(.)$ because households are forced to save more, which through its impact on ω results in less mass on low bond values.

(b) Household saving and labor supply policy functions are interpolated using linear splines with respectively N_b and N_n knots. Households' saving function $b'(.)$ is characterized by a critical level of bonds χ_θ at which their borrowing constraints start binding, which depends on productivity. For every $\theta \in \Theta$, let $b_{\theta,j} = \chi_\theta + x_j$, with $0 = x_1 < ... < x_{N^b}$ denote the splines' knots for b' at which households' Euler equations hold with equality. For $b \leq \xi_\theta$, savings $b'(\theta, b) = -\overline{\phi}\phi(\theta) h(\theta)$ are determined by the borrowing limit ($\overline{\phi_t} = \overline{\phi}$ in the deterministic steady state). It defines the collocation nodes at which we force households' optimality conditions to hold to solve for policy functions. For a given aggregate state $\left(\overline{\phi}, z, \Phi\right)$, the saving function is finitely represented by $N_\theta \times (N_b + 1)$ coefficients giving the value of savings at the knots and the threshold below which households are constrained. So is the labor supply function, with $N_\theta \times N_n$ values at the knots for labor (which may differ from the knots for savings). The consumption function at the saving knots is backed out from the budget constraint:

$$c_t\left(\theta, b_{\theta,j}\right) = b_{\theta,j} + \left(1 - \tau_1\left(\theta\right)\right) w_t \theta n_t\left(\theta, b_{\theta,j}\right) + T\left(\theta\right) + \pi_t - \tau_{0t} - \frac{b'_t\left(\theta, b_{\theta,j}\right)}{1 + r_t} \tag{22}$$

(c) Equilibrium conditions for the discrete model are listed below. The first set of equations and the following two involve predetermined variables: the histogram weights (because weights should sum to 1, we keep only track of the number of weights minus 1), the credit and TFP shocks. The next sets of equations involve jump variables: the bond price, aggregate labor demand, the wage, profits, aggregate output, aggregate consumption, and the (discretized versions of) policy functions for labor and savings (including values of coefficients at knot points and borrowing constraint thresholds). The inclusion of some variables among jump variables, whose dynamics we want to solve for, is not strictly speaking necessary (it is the case for aggregate labor demand, the wage, profits, aggregate output and aggregate consumption). Their equation counterparts are definitional, and their values can be backed out from the other jump variables without including them explicitly in the equilibrium system of equation. However, including them makes the system dynamics better behaved numerically, because it provides more information to the code when taking derivatives with automatic differentiation.
In words, these equations are: the laws of motion for the distribution, credit and TFP; the market clearing conditions for bonds and labor; the definitions of aggregate output,[28] consumption, the wage and profits; the intratemporal optimality condition for households' labor supply, and the intertemporal optimality condition for savings/consumption (Euler equation). In the Euler equations, the t-conditional expectation is about $t+1$ values of aggregate shocks (next period borrowing constraints and wage influence current decisions), and

[28] Given the goods market clearing condition implied by the remaining equilibrium conditions and Walras law, aggregate output should equal aggregate consumption. During simulations, I recompute aggregate output fully nonlinearly using the policy functions and distributions implied by the perturbed solution, as $Y_t = z_t K^{1-\alpha}\left[\sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f}\theta_i n_t\left(\theta_i, b_j\right)\Phi_t\left(\theta_i, b_j\right)\right]^\alpha$. I check that the deviation from goods market clearing is close to 0.

is taken with respect to their values at t.

$$
\begin{aligned}
&\Phi_{t+1}\left(\theta_{i'}, b_{j'}\right) - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f}\Pi_\theta\left(\theta_{i'}|\theta_i\right)\left(\omega_{i,j,j',t}\Phi_t\left(\theta_i, b_j\right)\right) = 0,\;\; i' \in [|1, N_\theta|], j' \in \left[|1, N_b^f|\right] \\
&\log\overline{\phi}_{t+1} - \log\overline{\phi} - \rho_\phi(\log\overline{\phi}_t - \log\overline{\phi}) - \epsilon_{t+1}^q = 0 \\
&\log z_{t+1} - \rho_z \log z_t - \epsilon_{t+1}^z = 0 \\
&\begin{pmatrix}\epsilon^\phi \\ \epsilon^z\end{pmatrix} \overset{iid}{\sim} \mathcal{N}\left(\mathbf{0}, \begin{pmatrix}\sigma_\phi^2 & \sigma_\phi\sigma_z\rho_{\phi z} \\ \sigma_\phi\sigma_z\rho_{\phi z} & \sigma_z^2\end{pmatrix}\right) \\
&B - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f} b_j\Phi_{t+1}\left(\theta_i, b_j\right) = 0 \\
&N_t - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f}\theta_i n_t\left(\theta_i, b_j\right)\Phi_t\left(\theta_i, b_j\right) \\
&Y_t - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f} c_t\left(\theta_i, b_j\right)\Phi_t\left(\theta_i, b_j\right) \\
&C_t - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f} c_t\left(\theta_i, b_j\right)\Phi_t\left(\theta_i, b_j\right) \\
&w_t = \alpha z_t\left(\frac{1}{N_t}\right)^{1-\alpha} \\
&\pi_t = (1-\alpha) z_t K^{1-\alpha} N_t^\alpha \\
&\left(1 - \tau_1\left(\theta_i\right)\right) w_t\theta_i c_t\left(\theta_i, b_j\right)^{-\gamma} - \psi n_t\left(\theta_i, b_j\right)^\eta = 0,\;\; i \in [|1, N_\theta|], j \in [|1, N_b|] \\
&c_t\left(\theta_i, b_j\right)^{-\gamma} - \beta(1+r_t)\mathbb{E}_t\left\{\sum_{i'=1}^{N_\theta} c_{t+1}\left(\theta_{i'}, b'\left(\theta_i, b_j\right)\right)^{-\gamma}\right\} = 0,\;\; i \in [|1, N_\theta|], j \in [|1, N_b|]
\end{aligned} \tag{23}
$$

Denote as $\mathbf{y_t}$ the $6 + N_\theta \times (N_n + N_b + 1)$ vector of current jump (control) variables. Denote as $\mathbf{x_t}$ the $N_\theta \times N_b^f - 1 + 2$ vector of current state (predetermined) variables. Equilibrium conditions are stacked in a multivariate, vector-valued function $\mathcal{F}(.)$ that represents the nonlinear system of equations that defines the equilibrium:

$$\mathbb{E}_t\left[\mathcal{F}\left(\mathbf{y_t}, \mathbf{y_{t+1}}, \mathbf{x_t}, \mathbf{x_{t+1}}, \epsilon_{t+1}^q, \epsilon_{t+1}^z\right)\right] = 0 \tag{24}$$

2. Solving for the deterministic steady state of the economy (without aggregate shocks) amounts to finding $\mathbf{y}, \mathbf{x}$ that solve the following system of equation, which has as many unknowns as equations:

$$\mathcal{F}\left(\mathbf{y}, \mathbf{y}, \mathbf{x}, \mathbf{x}, 0, 0\right) = 0 \tag{25}$$

In theory, it could be solved directly using a nonlinear equation solver. In practice, there is no guarantee that numerical equation solvers will converge when we use projection methods to approximate policy functions. In addition to solving the households' consumption problem, the difficulty comes from having in addition a labor choice, endogenous government taxes, and solving for two prices in equilibrium (the wage and the interest rate). I also solve for the value of the disutility of labor ψ that normalizes steady state output Y to 1. I therefore use the following algorithm to make the problem more stable.

(a) Start with a guess for the bond price and labor demand $\left(p^{(0)}, N^{(0)}, \psi^{(0)}\right)$, for policy function values $\left(\mathbf{b}'^{(0)}(.), \mathbf{n}^{(0)}(.)\right)$, and the cross-sectional distribution $\mathbf{\Phi}^{(0)}(.)$ (it is only needed to compute the first iterate of government taxes). It is easier to solve for the bond price and labor demand demand, and back out the interest rate $1/p-1$ and the wage (from the firm's optimal labor choice) than solving directly for the latter. Thus having $\left(p^{(0)}, N^{(0)}\right)$ is equivalent to having $\left(r^{(0)}, w^{(0)}\right)$.

(b) Given those, use the endogenous grid method of Carroll (2006) to iterate backwards on the household's optimality conditions (the Euler and the labor intratemporal equations), and obtain a new guess for policy functions that will be supplied to the nonlinear policy solver solving the household's problem, $\left(\mathbf{b}'^{(1)}(.), \mathbf{n}^{(1)}(.)\right)$. This requires computing endogenous government taxes (fixed every period because we are at the steady state), which is why we need a guess for the cross-sectional distribution.

(c) The guess for prices is supplied to a second nonlinear solver wrapped around the policy solver, which solves for the prices clearing the bond and the labor market, and for the disutility of labor normalizing steady state output to 1. Within the price solver, I ensure that prices and labor disutility are positive ($p^{(n)}, N^{(n)}, \psi^{(n)} > 0$), and the stability condition $\beta/p^{(n)} \leq 1$ holds at every iteration n. The following steps occur within the price solver, and their iterates start at $n=1$.

(d) Given the exogenous law of motion for idiosyncratic income and the policy functions, compute the associated stationary distribution of households $\mathbf{\Phi}^{(1)}(.)$ (I use the eigenvector method). Also compute the wage and profits from the firm's optimality condition: $w^{(1)} = \alpha\left(\frac{1}{N^{(1)}}\right)^{1-\alpha}$, and $\pi^{(1)} = (1-\alpha)\left(N^{(1)}\right)^{\alpha}$. Then, given prices, policy functions and the distribution, compute endogenous government taxes $\tau_0^{(1)}$.

(e) Given prices, profits, taxes, and savings policies $\mathbf{b}'^{(1)}(.)$, solve the household's labor supply equation (using $\mathbf{n}^{(1)}(.)$ as a guess), and denote $\mathbf{n}^{(2)}(.)$ the new labor supply policy. It should always be non-negative. Here I use a nonlinear equation solver with Broyden's method, and supplies it with the Jacobian of the system of intratemporal equations. Here and later, derivatives are computed exactly with automatic differentiation, implemented with Julia's `ForwardDiff` package.

(f) Back out the associated consumption function from the budget constraint. If it has a non-positive entry at a point in the state space, adjust $\mathbf{n}^{(2)}(.)$ at that point such that the household consumes $c_{min} = 0.001$. This step helps with convergence of the solver when solving for savings in the next step.

(g) Given prices, profits, taxes and the new labor policy $\mathbf{n}^{(2)}(.)$, solve the household's Euler equation (using $\mathbf{b}'^{(1)}(.)$ as a guess), and denote $\mathbf{b}'^{(2)}(.)$ the new savings policy. Use the same solver as for labor.

(h) This completes one iterate in the loop solving for policy functions given prices. If the new policy functions $\left(\mathbf{n}^{(2)}(.)\,\mathbf{b}'^{(2)}(.)\right)$ are close enough to the previous ones $\left(\mathbf{n}^{(2)}(.)\,\mathbf{b}'^{(2)}(.)\right)$, then stop and we have solved the household's problem given prices $\left(p^{(0)}, N^{(0)}, \psi^{(0)}\right)$. Otherwise, iterate on steps (d)-(g). That is, given $\left(p^{(0)}, N^{(0)}, \psi^{(0)}\right)$ (hence the same wages and profits), compute new government taxes $\tau_0^{(n+1)}$. Then solve for new policy functions $\left(\mathbf{n}^{(n+1)}(.)\,\mathbf{b}'^{(n+1)}(.)\right)$, compare them to the previous ones $\left(\mathbf{n}^{(n)}(.)\,\mathbf{b}'^{(n)}(.)\right)$, and stop when they are close enough. This completes the solution of the household's problem given prices.

(i) Using the law of motion of the exogenous income shock and the optimal savings function, compute the stationary distribution $\mathbf{\Phi}^{(2)}$. Use it with policy functions to compute aggregate values for savings, labor supply and output. The price solver then chooses new values for prices and disutility of labor, $\left(p^{(1)}, N^{(1)}, \psi^{(1)}\right)$, to solve the following three equations:

$$
\begin{array}{l}
B - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f} b_j \Phi_{t+1}^{(2)}\left(\theta_i, b_j\right) = 0 \\
N^{(1)} - \sum_{i=1}^{N_\theta}\sum_{j=1}^{N_b^f} \theta_i n\left(\theta_i, b_j\right) \Phi^{(2)}\left(\theta_i, b_j\right) \\
Y^{(1)} - 1 = 0 \Leftrightarrow \left(N^{(1)}\right)^{\alpha} - 1 = 0
\end{array}
\tag{26}
$$

(j) Then go back to step (a) with the new prices, and iterate until convergence, i.e. policy functions and the stationary distribution have converged, and the three equations are satisfied. We then obtain prices, policy functions and a distribution that solve the model in the deterministic steady state.

3. Do a first- and a second-order perturbation of the discrete model around its steady

state. The solutions to the equilibrium expectational difference equation $\mathbb{E}_t\left[\mathcal{F}\left(.\right)\right] = 0$ are of the following form (Schmitt-Grohe and Uribe (2008)):

$$\begin{aligned} \mathbf{x_{t+1}} &= \mathbf{h}\left(\mathbf{x_t}, \eta\right) + \eta \begin{pmatrix} \mathbf{0} \\ \epsilon^q_{t+1} \\ \epsilon^z_{t+1} \end{pmatrix} \\ \mathbf{y_t} &= \mathbf{g}\left(\mathbf{x_t}, \eta\right) \end{aligned} \tag{27}$$

where η is the perturbation parameter (there is only one such parameter) scaling the amount of aggregate uncertainty in the economy. The goal is to solve for approximations of the functions $\mathbf{h}, \mathbf{g}$.

(a) For the first-order approximation of the model, several methods can be used. I check existence and uniqueness, and verify that I obtain identical results using Sims' `gensys` (Sims (2001)) and Klein's methods (Klein (2000)), commonly used in the macro literature. I briefly describe the input and the output of Klein's method because it has a clear interpretation in terms of jump and predetermined variables. We solve for a first-order approximation of $\mathbf{g}, \mathbf{h}$. Writing variables in deviations from their steady state values (denoted as $\widehat{x}, \widehat{y}$) and linearizing equilibrium conditions around 0 (where variables equal their steady state values), we obtain

$$\mathcal{F}_{\mathbf{y_t}}\widehat{\mathbf{y}}_\mathbf{t} + \mathcal{F}_{\mathbf{y_{t+1}}}\mathbb{E}_t\left[\widehat{\mathbf{y_{t+1}}}\right] + \mathcal{F}_{\mathbf{x_t}}\widehat{\mathbf{x}}_\mathbf{t} + \mathcal{F}_{\mathbf{x_{t+1}}}\mathbb{E}_t\left[\widehat{\mathbf{x_{t+1}}}\right] + \mathcal{F}_{\epsilon^q_{t+1}}\mathbb{E}_t\left[\widehat{\epsilon_{t+1}}^q\right] + \mathcal{F}_{\epsilon^z_{t+1}}\mathbb{E}_t\left[\widehat{\epsilon_{t+1}}^z\right] = 0 \tag{28}$$

where the derivatives of $\mathcal{F}$ are evaluated at the steady state. They are submatrices of the Jacobian of $\mathcal{F}$, computed exactly with automatic differentiation. $\widehat{\mathbf{y}}, \widehat{\mathbf{x}}$ terms are vectors, so their (matrix) products with the derivative matrices of $\mathcal{F}$ are vectors. The Jacobian is a matrix of dimension

$$\begin{aligned} &\left\{\left[N_\theta \times N_b^f - 1 + 2\right] + \left[6 + N_\theta \times \left(N_n + N_b + 1\right)\right]\right\} \\ &\times \left\{2 \times \left[N_\theta \times N_b^f - 1 + 2\right] + 2 \times \left[6 + N_\theta \times \left(N_n + N_b + 1\right)\right] + 2\right\} \end{aligned}$$

First-order approximations of the solution have the following form:

$$
\begin{aligned}
\widehat{\mathbf{x_{t+1}}} &= \mathbf{h_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}} + \eta \begin{pmatrix} \mathbf{0} \\ \epsilon^q_{t+1} \\ \epsilon^z_{t+1} \end{pmatrix} \\
\widehat{\mathbf{y_t}} &= \mathbf{g_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}
\end{aligned}
\tag{29}
$$

(b) For the second-order approximation of the model, I do a second-order approximation of equilibrium conditions around the steady state. It involves the Hessian of $\mathcal{F}$, a large three-dimensional array computed by automatic differentiation, of dimension:

$$
\begin{aligned}
&\left\{\left[N_\theta \times N_b^f - 1 + 2\right] + \left[6 + N_\theta \times (N_n + N_b + 1)\right]\right\} \\
&\times \left\{2 \times \left[N_\theta \times N_b^f - 1 + 2\right] + 2 \times \left[6 + N_\theta \times (N_n + N_b + 1)\right] + 2\right\}^2
\end{aligned}
$$

The second-order approximation of the solution has the form:

$$
\begin{aligned}
\widehat{\mathbf{x_{t+1}}} &= \mathbf{h_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}} + \tfrac{1}{2}\mathbf{h_{xx}}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}^2 + \tfrac{1}{2}\mathbf{h}_{\eta\eta}\left(\mathbf{x},0\right)\eta^2 + \eta \begin{pmatrix} \mathbf{0} \\ \epsilon^q_{t+1} \\ \epsilon^z_{t+1} \end{pmatrix} \\
\widehat{\mathbf{y_{t+1}}} &= \mathbf{g_x}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}} + \tfrac{1}{2}\mathbf{g_{xx}}\left(\mathbf{x},0\right)\widehat{\mathbf{x_t}}^2 + \tfrac{1}{2}\mathbf{g}_{\eta\eta}\left(\mathbf{x},0\right)\eta^2
\end{aligned}
\tag{30}
$$

where the terms equal to zero (in $h_\eta, g_\eta, h_{x\eta}, h_{\eta x}, g_{x\eta}, g_{\eta x}$) were canceled. $\widehat{\mathbf{x}}, \widehat{\mathbf{y}}$ terms are vectors, $\mathbf{g_x}, \mathbf{h_x}$ terms are matrices, $\mathbf{h_{xx}}, \mathbf{g_{xx}}$ are 3-dimensional arrays, and $\mathbf{h}_{\eta\eta}, \mathbf{g}_{\eta\eta}$ are vectors. Thus products of $\widehat{\mathbf{x}}, \widehat{\mathbf{y}}$ vectors with first-order derivative matrices are matrix products, those with second-order arrays are tensor products, and those with η are simple constant times vectors products. I use Kim et al. (2008)'s `gensys2` method to solve for the unknown coefficients. Schmitt-Grohe and Uribe (2008) propose instead to solve for the second-order coefficients in a linear system of equations involving the Jacobian and the Hessian of $\mathcal{F}$, and the first-order coefficients. While most papers with representative agent models use this method, it is not tractable in a setting with heterogeneous agents where the cross-sectional distribution is discretized as a histogram, since it involves constructing and inverting a matrix whose dimensions increases exponentially with the number of state variables. `gensys2` allows to reduce the dimensionality of the system of equation to solve by applying a sequence of linear operations to the original system (Schur and singular value decompositions).

A.2 Steady state in the second-order approximation of the model

Pruning essentially computes first-order projections of second-order terms, based on a first-order expansion of the conditional expectation of the system's deviation from steady state, according to the following steps.

First, gensys2 solves a linearly transformed system, where original variables $\begin{pmatrix}\widehat{\mathbf{x}} & \widehat{\mathbf{y}}\end{pmatrix}'$ that solve $\mathbb{E}_t\left[\mathcal{F}(.)\right] = 0$ are replaced by $Z'\begin{pmatrix}\widehat{\mathbf{x}} & \widehat{\mathbf{y}}\end{pmatrix}'$, where Z is a square, non-singular matrix. To simplify notation, denote the transformed variables as $\begin{pmatrix}\widehat{\mathbf{x}} & \widehat{\mathbf{y}}\end{pmatrix}'$ too. The second-order solution to the transformed system has the form (see the paper for details):

$$\begin{aligned}
&\widehat{\mathbf{x_{t+1}}} = F_1\widehat{\mathbf{x_t}} + F_2\eta\,\mathbf{ffl_{t+1}} + F_3\eta^2 + \tfrac{1}{2}F_{11}\widehat{\mathbf{x_t}}^2 + F_{12}\widehat{\mathbf{x_t}}\mathbf{ffl_{t+1}}\eta + \tfrac{1}{2}F_{22}\eta^2\mathbf{ffl_{t+1}}^2 \\
&\widehat{\mathbf{y_t}} = \tfrac{1}{2}M_{11}\widehat{\mathbf{x_t}}^2 + M_2\eta^2
\end{aligned} \tag{31}$$

The presence of cross-derivative terms in the transformed solution does not contradict their absence in the original solution, since they can be canceled by Z. Then, it implies that for $s > 0$:

$$\begin{aligned}
&\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s}}}\right] = F_1\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s-1}}}\right] + F_3\eta^2 + \tfrac{1}{2}F_{11}\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s-1}}}^2\right] + \tfrac{1}{2}F_{22}\eta^2\Omega_s \\
&= F_1\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s-1}}}\right] + F_3\eta^2 + \tfrac{1}{2}F_{11}\left(\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s-1}}}\right]^2 + \Sigma_{s-1}\right) + \tfrac{1}{2}F_{22}\eta^2\Omega_s \\
&\mathbb{E}_t\left[\widehat{\mathbf{y_{t+s}}}\right] = \tfrac{1}{2}M_{11}\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s}}}^2\right] + M_2\eta^2 \\
&= \tfrac{1}{2}M_{11}\left(\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s}}}\right]^2 + \Sigma_s\right) + M_2\eta^2 \\
&\Sigma_{s+1} = \eta^2F_2\Omega_tF_2 + F_1\Sigma_sF_1
\end{aligned} \tag{32}$$

where Ω_s is the t-conditional variance-covariance matrix of $\mathbf{ffl_{t+s}}$, and Σ_s is the t-conditional variance-covariance matrix of $\widehat{\mathbf{x_{t+s}}}$, defined recursively by a discrete Lyapunov equation (from the law of motion of $\widehat{\mathbf{x_{t+1}}}$).

Then, projecting $\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s-1}}}\right]$ terms on their first-order counterparts, denoted $\mathbb{E}_t^1\left[\widehat{\mathbf{x_{t+s-1}}}\right]$, we obtained the pruned law of motion of the transformed solution:

$$\begin{aligned}
&\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s}}}\right] = F_1\mathbb{E}_t\left[\widehat{\mathbf{x_{t+s-1}}}\right] + F_3\eta^2 + \tfrac{1}{2}F_{11}\left(\mathbb{E}_t^1\left[\widehat{\mathbf{x_{t+s-1}}}\right]^2 + \Sigma_{s-1}\right) + \tfrac{1}{2}F_{22}\eta^2\Omega_s \\
&\mathbb{E}_t\left[\widehat{\mathbf{y_{t+s}}}\right] = \tfrac{1}{2}M_{11}\left(\mathbb{E}_t^1\left[\widehat{\mathbf{x_{t+s}}}\right]^2 + \Sigma_s\right) + M_2\eta^2 \\
&\mathbb{E}_t^1\left[\widehat{\mathbf{x_{t+s}}}\right] = F_1\mathbb{E}_t^1\left[\widehat{\mathbf{x_{t+s-1}}}\right] \\
&\Sigma_{s+1} = \eta^2F_2\Omega_tF_2 + F_1\Sigma_sF_1
\end{aligned} \tag{33}$$

To compute the steady state of the second-order solution to the original system, we first compute the steady state of the transformed system using its laws of motion. In particular, we solve for the steady state value of expected deviations of transformed variables from their steady state (set $\eta = 1$):

$$\begin{aligned}
&\mathbb{E}\left[\widehat{\mathbf{x}}\right] = \left(I - F_1\right)^{-1}\left(F_3 + \tfrac{1}{2}F_{22}\Omega + \tfrac{1}{2}F_{11}\Sigma\right) \\
&\mathbb{E}\left[\widehat{\mathbf{y}}\right] = \tfrac{1}{2}M_{11}\Sigma + M_2 \\
&\text{where} \quad \Sigma = F_2\Omega_t F_2 + F_1\Sigma F_1
\end{aligned} \tag{34}$$

Finally, we back out the steady state values of original variables as $Z'^{-1}\begin{pmatrix}\mathbb{E}\left[\widehat{\mathbf{x}}\right] & \mathbb{E}\left[\widehat{\mathbf{y}}\right]\end{pmatrix}'$.

A.3 Impulse response functions in the second-order approximation of the model

To compute the economy's impulse response functions, we use the pruned version of the law of motion for transformed variables (for $\eta = 1$), for $t \geq 0$:

$$\begin{aligned}
&\widehat{\mathbf{x_{t+1}}} = F_1\widehat{\mathbf{x_t}} + F_2\mathbf{ffl_{t+1}} + F_3 + \tfrac{1}{2}F_{11}\widehat{\mathbf{x^1_t}}^2 + F_{12}\widehat{\mathbf{x^1_t}}\mathbf{ffl_{t+1}} + \tfrac{1}{2}F_{22}\mathbf{ffl_{t+1}}^2 \\
&\widehat{\mathbf{y_t}} = \tfrac{1}{2}M_{11}\widehat{\mathbf{x^1_t}}^2 + M_2 \\
&\widehat{\mathbf{x^1_{t+1}}} = F_1\widehat{\mathbf{x^1_t}} + F_2\mathbf{ffl_{t+1}}
\end{aligned} \tag{35}$$

We then back out the path of original variables as $\left\{Z'^{-1}\begin{pmatrix}\widehat{\mathbf{x_t}} & \widehat{\mathbf{y_t}}\end{pmatrix}'\right\}_t$.

A.4 Market-clearing errors

I measure the accuracy of the first- and second-order approximations by computing the residuals of equilibrium conditions, in particular market clearing conditions for bonds, consumption and labor. They are small in the first-order approximation of the model, and further decrease towards zero in the second-order approximation, proving the good fit of the model (Table 5).

A.5 Variance decomposition

A.5.1 First-Order

The vector $Y = \begin{pmatrix}x & y\end{pmatrix}$ of equilibrium objects contains the predetermined and the jump variables. It is in deviation from steady state, but it doesn't matter for this exercise because

Table 5: Solution accuracy

Market:	Bonds	Good	Labor
order 1	0.01% (0.03%)	0.04% (0.04%)	0.01% (0.01%)
order 2	0.00% (0.02%)	0.00% (0.00%)	0.00% (0.00%)

Notes: Average market clearing errors for IRF (sup norm in parentheses), computed as percentage differences normalized by the steady state value of the variable or by the initial value of the series.

we can just add the steady state vector, which will cancel out when taking variances. The output from `gensys` is a law of motion for Y, consisting of an AR(1) matrix Φ and an impact matrix Z:

$$(I - \Phi L)\, Y_{t+1} = Z\epsilon_{t+1} \tag{36}$$

where $\epsilon_{t+1} = \begin{pmatrix} \epsilon^{\phi}_{t+1} & \epsilon^{z}_{t+1} \end{pmatrix}'$ is the vector of the two shocks, with covariance matrix $\tilde{\Sigma}_{\epsilon} = \begin{pmatrix} 1 & \rho_{\phi,z} \\ \rho_{\phi,z} & 1 \end{pmatrix}$, and where the rows of Z corresponding to ϵ^{ϕ}_{t+1} and ϵ^{z}_{t+1} are $\begin{pmatrix} \sigma_{\phi} & 0 \\ 0 & \sigma_{z} \end{pmatrix}$. Thus $\text{Var}\left(\begin{pmatrix} \sigma_{\phi} & 0 \\ 0 & \sigma_{z} \end{pmatrix} \tilde{\Sigma}_{\epsilon} \right) = \begin{pmatrix} \sigma^2_{\phi} & \rho_{\phi,z}\sigma_{\phi}\sigma_{z} \\ \rho_{\phi,z}\sigma_{\phi}\sigma_{z} & \sigma^2_{z} \end{pmatrix} = \Sigma_{\epsilon}$.

First, we transform the shocks with covariance $\tilde{\Sigma_{\epsilon}}$ so that they are orthogonal, i.e. their covariance matrix is the identity matrix. This is done by Cholesky factorization. The new orthogonal shocks are defined as $\nu_t = Q\epsilon_t$, with Q such that $\mathbb{E}\left[\nu_t \nu_t'\right] = I$. Denoting $S = Q^{-1}$, $\epsilon_t = S\nu_t$ and $SS' = \tilde{\Sigma_{\epsilon}}$. S is a lower triangular matrix given by the Cholesky factorization of $\tilde{\Sigma}_{\epsilon}$.

Then, we transform the economy's law of motion from an AR(1) to an MA(∞) representation, using the fact that the eigenvalues of Φ are within the unit circle (we denote L the lag operator). We also substitute for $\epsilon_{t+1} = S\nu_{t+1}$.

$$\begin{aligned}
&(I - \Phi L)\, Y_{t+1} = Z\epsilon_{t+1} \\
&\Rightarrow Y_{t+1} = (I - \Phi L)^{-1} ZS\nu_{t+1} \\
&Y_{t+1} = \textstyle\sum_{k=0}^{\infty} \Phi^k L^k ZS\nu_{t+1} \\
&Y_{t+1} = \textstyle\sum_{k=0}^{\infty} \Phi^k ZS\nu_{t+1-k} \\
&\Rightarrow Y_{t+h} = \textstyle\sum_{k=0}^{\infty} \tilde{\Phi}^{(k)} \nu_{t+h-k}
\end{aligned} \tag{37}$$

for any forecasting horizon $h > 0$, and where $\tilde{\Phi}^{(k)} = \Phi^k ZS$ is a matrix of dimension (number of variables,number of shocks). Here we consider N variables and 2 shocks.

Then, forecast errors at horizon $h > 0$ are:

$$\begin{aligned} e_{t+h} &= Y_{t+h} - \mathbb{E}_t\left[Y_{t+h}\right] \\ &= \tilde{\Phi}^{(0)}\nu_{t+h} + \tilde{\Phi}^{(1)}\nu_{t+h-1} + \tilde{\Phi}^{(2)}\nu_{t+h-2} + ... + \tilde{\Phi}^{(h-1)} \\ &= \textstyle\sum_{i=1}^{h} \tilde{\Phi}^{(h-i)}\nu_{t+i} \end{aligned} \tag{38}$$

For variable $Y_j, j \in \{1, ...N\}$,

$$\begin{aligned} e_{j,t+h} &= \textstyle\sum_{i=1}^{h} \tilde{\Phi}_{j,\cdot}^{(h-i)}\nu_{t+i} \\ &= \textstyle\sum_{i=1}^{h}\left(\tilde{\Phi}_{j,1}^{(h-i)}\nu_{1,t+i} + \tilde{\Phi}_{j,2}^{(h-i)}\nu_{2,t+i}\right) \end{aligned} \tag{39}$$

So the total forecast error variance at horizon $h > 0$ for variable Y_j is, using the fact that ν's are mutually independent, identically distributed and serially uncorrelated:

$$\text{Var}\left(e_{j,t+h}\right) = \sum_{i=1}^{h}\left(\left(\tilde{\Phi}_{j,1}^{(h-i)}\right)^2 + \left(\tilde{\Phi}_{j,2}^{(h-i)}\right)^2\right) \tag{40}$$

Finally, the share of the forecast error variance of variable Y_j at horizon $h > 0$ accounted for by ν^1 and ν^2 (transformed versions of the original shocks ϵ^{ψ} and ϵ^{z}) are respectively:

$$\frac{\sum_{i=1}^{h}\left(\tilde{\Phi}_{j,1}^{(h-i)}\right)^2}{\sum_{i=1}^{h}\left(\left(\tilde{\Phi}_{j,1}^{(h-i)}\right)^2 + \left(\tilde{\Phi}_{j,2}^{(h-i)}\right)^2\right)} \text{ and } \frac{\sum_{i=1}^{h}\left(\tilde{\Phi}_{j,2}^{(h-i)}\right)^2}{\sum_{i=1}^{h}\left(\left(\tilde{\Phi}_{j,1}^{(h-i)}\right)^2 + \left(\tilde{\Phi}_{j,2}^{(h-i)}\right)^2\right)} \tag{41}$$

Results are sensitive to whether the matrix obtained from the Cholesky factorization is lower or upper triangular. A lower triangular S implies that ν^2 has no effect on nu^1. Note that because of the factorization, the ν shocks are not clearly interpretable as credit and TFP shocks. To have that interpretation for the results in the main text, I resolve the model with $\rho_{\phi,z} = 0$, and omit the factorization step. Results are displayed in Table 6, and are similar.

A.5.2 Second-Order

I use a generalized forecast error variance decomposition for nonlinear models (Lanne and Nyberg (2016)). The starting point is the nonlinear (quadratic) model given by `gensys2`, which can be written as

$$Y_{t+1} = f\left(Y_t, \epsilon_{t+1}\right) \tag{42}$$

Table 6: Credit and TFP shocks' contributions to financial and macroeconomic volatility

Variable:	Credit shock	TFP shock
$1/(1+r)$	51%	49%
N	56%	44%
w	6%	94%
π	70%	30%
Y	70%	30%

Notes: Variance decomposition, order 1: share of variance in left handside variables accounted for by credit and TFP shocks. Variance computed as the variance of forecast errors at a large forecasting horizon ($H = 1000$). See appendix for computations.

where G is a nonlinear function of the equilibrium vector and of innovations. As above, the interpretation of shocks is clearer when $\rho_{\phi,z} = 0$.
The generalized impulse-response function (GIRF) at horizon $i > 0$ (i.e. at date $t + i$) of variable Y_j, with respect to a credit shock (or TFP shock) of magnitude $\delta_{\phi,t+1}$ (or $\delta_{z,t+1}$) hitting at date $t + 1$, conditional on history of states $\omega_t = y_t$, is defined as:

$$\begin{aligned} &GI_j\left(i, \delta_{\phi,t+1}, \omega_t\right) = \mathbb{E}_t\left[Y_{j,t+i}|\epsilon^{\phi}_{t+1} = \delta_{\phi,t+1}, \omega_t\right] - \mathbb{E}_t\left[Y_{j,t+i}|\omega_t\right] \\ &\text{and } \; GI_j\left(i, \delta_{z,t+1}, \omega_t\right) = \mathbb{E}_t\left[Y_{j,t+i}|\epsilon^{z}_{t+1} = \delta_{z,t+1}, \omega_t\right] - \mathbb{E}_t\left[Y_{j,t+i}|\omega_t\right] \end{aligned} \tag{43}$$

Then, the generalized forecast error variance decomposition (GFEVD) of variable Y_j at horizon $h > 0$, is between the fraction of variance explained by credit shocks, and that explained by TFP shocks, respectively:

$$\begin{aligned} GFEVD_j(h, \delta_{\phi,t}) &= \frac{\sum_{i=0}^{h} GI_j\left(i,\delta_{\phi,t+1},\omega_t\right)^2}{\sum_{i=0}^{h} GI_j\left(i,\delta_{\phi,t+1},\omega_t\right)^2 + \sum_{i=0}^{h} GI_j(i,\delta_{z,t+1},\omega_t)^2} \\ GFEVD_j(h, \delta_{z,t}) &= \frac{\sum_{i=0}^{h} GI_j(i,\delta_{z,t+1},\omega_t)^2}{\sum_{i=0}^{h} GI_j\left(i,\delta_{\phi,t+1},\omega_t\right)^2 + \sum_{i=0}^{h} GI_j(i,\delta_{z,t+1},\omega_t)^2} \end{aligned} \tag{44}$$

Because GIRF are nonlinear, GFEVD depend on the sign and size of the innovations δ. I therefore compute average GFEVD using bootstrap. First, because the solution of the model is based on perturbations around the steady state, we can get rid of the history dependence in ω. Then, I simulate a history of credit and TFP innovations of length $T = 1000$, $\left\{\epsilon^{\phi}_t, \epsilon^{z}_t\right\}_{t=0}^{T} = \left\{\delta_{\phi,t}, \delta_{z,t}\right\}_{t=0}^{T}$ using $\begin{pmatrix}\epsilon^{\phi}_t \\ \epsilon^{z}\end{pmatrix} \overset{iid}{\sim} \mathcal{N}\left(0, I_2\right)$ (with gensys2 the innovation variances σ^2_{ϕ} and σ^2_z are incorporated in the GIRF matrices). For each innovation $\delta_{\phi,t}$, I compute the associated $GFEVD_j(h, \delta_{\phi,t})$ for variable Y_j at horizon h. Finally, the average GFEVD is obtained by averaging over individual $GFEVD_j(h, \delta_{\phi,t})$'s by using the probability associated to each $\delta_{\phi,t}$ by the standard normal p.d.f. (Because $\mathcal{N}(0,1)$ is symmetric, we should get something like an average of the GFEVD for a shock $\delta = -1$ and a shock $\delta = +1$.) Computations are parallelized over the N dimension. It takes about 17 hours to run the case $N = 500, H = 1000$ using 28 cores.

A.6 Estimation of structural shocks

A.6.1 Kalman filter (order 1)

A linear state space representation of the model is obtained from gensys. Using the above notation, the transition and the measurement equations are respectively:

$$\begin{aligned} Y_{t+1} &= \Phi Y_t + Z\epsilon_{t+1}, \;\; \epsilon_{t+1} \overset{iid}{\sim} \mathcal{N}(0,Q) \\ Y^{obs}_{t+1} &= H' Y_{t+1} + v_t, \;\; v_{t+1} \overset{iid}{\sim} \mathcal{N}(0,R) \end{aligned} \tag{45}$$

Φ and Z are readily obtained from gensys and $Q = I_2$ (variance-covariance terms are in Z by design). H is a selection matrix filled everywhere with zeros, and with ones for the entries corresponding to the observable variables in Y_{t+1} (bond price and consumption). There is no noise in the measurement equation, i.e. $R = 0_{2\times 2}$: the bond price and consumption are perfectly observed.
Using standard notation, denote $Y_{t|t-1} = \mathbb{E}\left[Y_t | Y^{obs,t-1}\right]$ (best linear predictor of Y_t given the history of observables Y^{obs} until $t-1$), $Y^{obs}_{t|t-1} = \mathbb{E}\left[Y^{obs}_t | Y^{obs,t-1}\right]$, and $Y_{t|t} = \mathbb{E}\left[Y_t | Y^{obs,t}\right]$. Also denote $\Sigma_{t|t-1} = \mathbb{E}\left[\left(Y_t - Y_{t|t-1}\right)\left(Y_t - Y_{t|t-1}\right)' | Y^{obs,t-1}\right]$ (predicting error variance-covariance matrix of Y_t given the history of observables until $t-1$),

$$\Omega_{t|t-1} = \mathbb{E}\left[\left(Y^{obs}_t - Y^{obs}_{t|t-1}\right)\left(Y^{obs}_t - Y^{obs}_{t|t-1}\right)' | Y^{obs,t-1}\right],$$

$$\Sigma_{t|t} = \mathbb{E}\left[\left(Y_t - Y_{t|t}\right)\left(Y_t - Y_{t|t}\right)' | Y^{obs,t}\right].$$

The goal of the Kalman filter here is to back out the sequences of forecasted observable variables and underlying states $\left\{Y^{obs}_{t|t-1}, Y_{t|t}\right\}$ implied by the model, given a sequence of observable variables $\left\{Y^{obs}_t\right\}$ taken from the data. The algorithm proceeds as follows:

1. At $t = 1$, initial conditions $Y_{1|0}, \Sigma_{1|0}$ are set equal to their (deterministic) steady state values. That is, $Y_{1|0} = 0$ (the initial system of equations was written in log deviations from steady state), and $\Sigma_{1|0}$ is the solution to the Riccati equation $\Sigma_{1|0} = \Phi\Sigma_{1|0}\Phi' + ZI_2Z'$, which is solved by iterating on a symmetric, positive definite

guess $\Sigma_{1|0}^{(0)}$ (using the stability of the system). I verify that the solution $\Sigma_{1|0}^{(\infty)} = \Sigma_{1|0}$ is symmetric and positive definite too. Following steps are for $t \geq 1$.

2. Given $\Sigma_{t|t-1}, Y_t^{obs}, Y_{t|t-1}^{obs}$, compute $\Omega_{t|t-1} = H'\Sigma_{t|t-1}H + R = H'\Sigma_{t|t-1}H$.

3. Compute $\text{Cov}_{t-1}\left(Y_t^{obs}, Y_t\right) = \mathbb{E}\left[\left(Y_t^{obs} - Y_{t|t-1}^{obs}\right)\left(Y_t - Y_{t|t-1}\right)' | Y^{obs,t-1}\right] = H'\Sigma_{t|t-1}$.

4. Compute the Kalman gain $K_t = \Sigma_{t|t-1}H\left(H'\Sigma_{t|t-1}H + R\right)^{-1} = \Sigma_{t|t-1}H\Omega_{t|t-1}^{-1}$.

5. Compute $Y_{t|t} = Y_{t|t-1} + K_t\left(Y_t^{obs} - H'Y_{t|t-1}\right)$ ("nowcast" of the state).

6. Compute $\Sigma_{t|t} = \Sigma_{t|t-1} - K_tH'\Sigma_{t|t-1}$ (variance-covariance matrix associated with the "nowcast" error).

7. Compute $\Sigma_{t+1|t} = \Phi\Sigma_{t|t}\Phi' + ZQZ' = \Phi\Sigma_{t|t}\Phi' + ZZ'$ (next period forecast error variance-covariance matrix).

8. Finally, compute $Y_{t+1|t} = \Phi Y_{t|t}$ and $Y_{t+1|t}^{obs} = H'Y_{t+1|t}$ (next period implied state, and next period forecasted observables).

A.6.2 Particle filter (order 2)

A nonlinear state space representation of the model is obtained from `gensys2`. Using the above notation, the transition and the measurement equations are respectively:

$$\begin{aligned} Y_{t+1} &= f\left(Y_t, W_{t+1}\right), \;\; W_{t+1} \overset{iid}{\sim} \mathcal{N}(0, Q) \\ Y_{t+1}^{obs} &= H'Y_{t+1} + v_t, \;\; v_{t+1} \overset{iid}{\sim} \mathcal{N}(0, R) \end{aligned} \tag{46}$$

f is the quadratic mapping (from `gensys2`) used to compute impulse responses in the second-order solution of the model (see above). $Q = I_2$ (variance-covariance terms are in the matrices part of f by design), and H is a selection matrix filled everywhere with zeros, and with ones for the entries corresponding to the observable variables in Y_{t+1} (bond price and consumption). I assume that there is some but very little noise in the measurement equation, i.e. $R = 10^{-6} \times I_2$: the bond price and consumption are close to perfectly observed. This is because the joint density of measurement errors is needed in the algorithm, so R cannot be zero.
Particles are i.i.d. draws $\left\{Y_{t-1}^i, W_{t-1}^i\right\}_{i=1}^N$ from the joint density $p\left(W_{t-1}, Y_{t-1}|Y_{t-1}^{obs}\right)$. Proposed particles are i.i.d. draws $\left\{Y_{t|t-1}^i, W_{t|t-1}^i\right\}_{i=1}^N$ from the joint density $p\left(W_t, Y_{t-1}|Y_{t-1}^{obs}\right)$.

There are N of each of them. Here, the structural innovations W are independent of the vector of predetermined and jump variables Y. Therefore, drawing from the proposed joint density boils down to drawing from the innovations' density, and then applying the nonlinear mapping f to the previous proposed Y and the new innovations w, to get the new proposed particle Y. As before, the sequence of observable variables $\left\{Y_t^{obs}\right\}_{t=0}^{T}$ is taken from the data, with $Y_0^{obs} = 0$. That is, I assume w.l.o.g. that the beginning of the sample represents the deterministic steady state (hence log-deviations are zero). The algorithm proceeds as follows.

1. At $t = 1$, set the initial condition $Y_{0|0}^i = Y_0^i = W_0^i = 0$ for all $i = 1, ...N$, i.e. the log-deviation from the deterministic steady state is assumed to be zero at $t = 0$.

2. Generate N i.i.d. draws of proposed particles $\left\{Y_{t|t-1}^i, W_{t|t-1}^i\right\}_{i=1}^{N}$ from $p\left(W_t, Y_{t-1}|Y_{t-1}^{obs}\right)$. That is, draw $w_{t|t-1}^i$ innovations from $\mathcal{N}\left(0, I_2\right)$ and obtain the associated $Y_{t|t-1}^i$ from f.

3. Evaluate the conditional density $p\left(Y_t^{obs}|w_{t|t-1}^i, Y_{t-1}^{obs}, Y_{t|t-1}^i\right)$ using the measurement equation and the distribution of measurement errors v. That is,

$$p\left(Y_t^{obs}|w_{t|t-1}^i, Y_{t-1}^{obs}, Y_{t|t-1}^i\right) = \phi\left(Y_t^{obs} - H'Y_t|w_{t|t-1}^i, Y_{t-1}^{obs}, Y_{t|t-1}^i\right)$$

   where ϕ is the (conditional) density of the multivariate standard normal distribution.

4. Evaluate the relative weights $q_t^i = \frac{p\left(Y_t^{obs}|w_{t|t-1}^i, Y_{t-1}^{obs}, Y_{t|t-1}^i\right)}{\sum_{j=1}^{N} p\left(Y_t^{obs}|w_{t|t-1}^j, Y_{t-1}^{obs}, Y_{t|t-1}^j\right)}$, normalized to be probabilities.

5. Re-sample, with replacement, N values $\left\{Y_{t|t-1}^i, W_{t|t-1}^i\right\}_{i=1}^{N}$ from the sample we had so far, now using the $\left\{q_t^i\right\}_{i=1}^{N}$ as probabilities. These new values are the particles, denoted $\left\{Y_t^i, W_t^i\right\}_{i=1}^{N}$.

6. Go back to step 2 for $t + 1$, generate new innovations and use the new swarm of particles $\left\{Y_t^i, W_t^i\right\}_{i=1}^{N}$ to generate a new swarm of proposed particles $\left\{Y_{t+1|t}^i, W_{t+1|t}^i\right\}_{i=1}^{N}$. Then iterate until reaching the end of the sample $t = T$.

Thus we obtain a sequence of swarms of particles $\left\{\left\{Y_t^i, W_t^i\right\}_{i=1}^{N}\right\}_{t=0}^{T}$, which represent empirical conditional densities at every point in time for the state Y, which are implied by the model, given the sequence of observables $\left\{Y_t^{obs}\right\}_{t=0}^{T}$ from the data. In the main text, I plot the sample averages of these empirical conditional densities at $t = 0, ...T$. This paper is to my knowledge the first paper to apply nonlinear filtering to the perturbation-based solution of a heterogeneous agents model with aggregate shocks. Computations are parallelized over the N dimension. It takes about 12 hours to run the case $N = 20,000, T = 44$ using 28 cores.

B Calibration

B.1 Computation Parameters

Table 7: Computation parameters

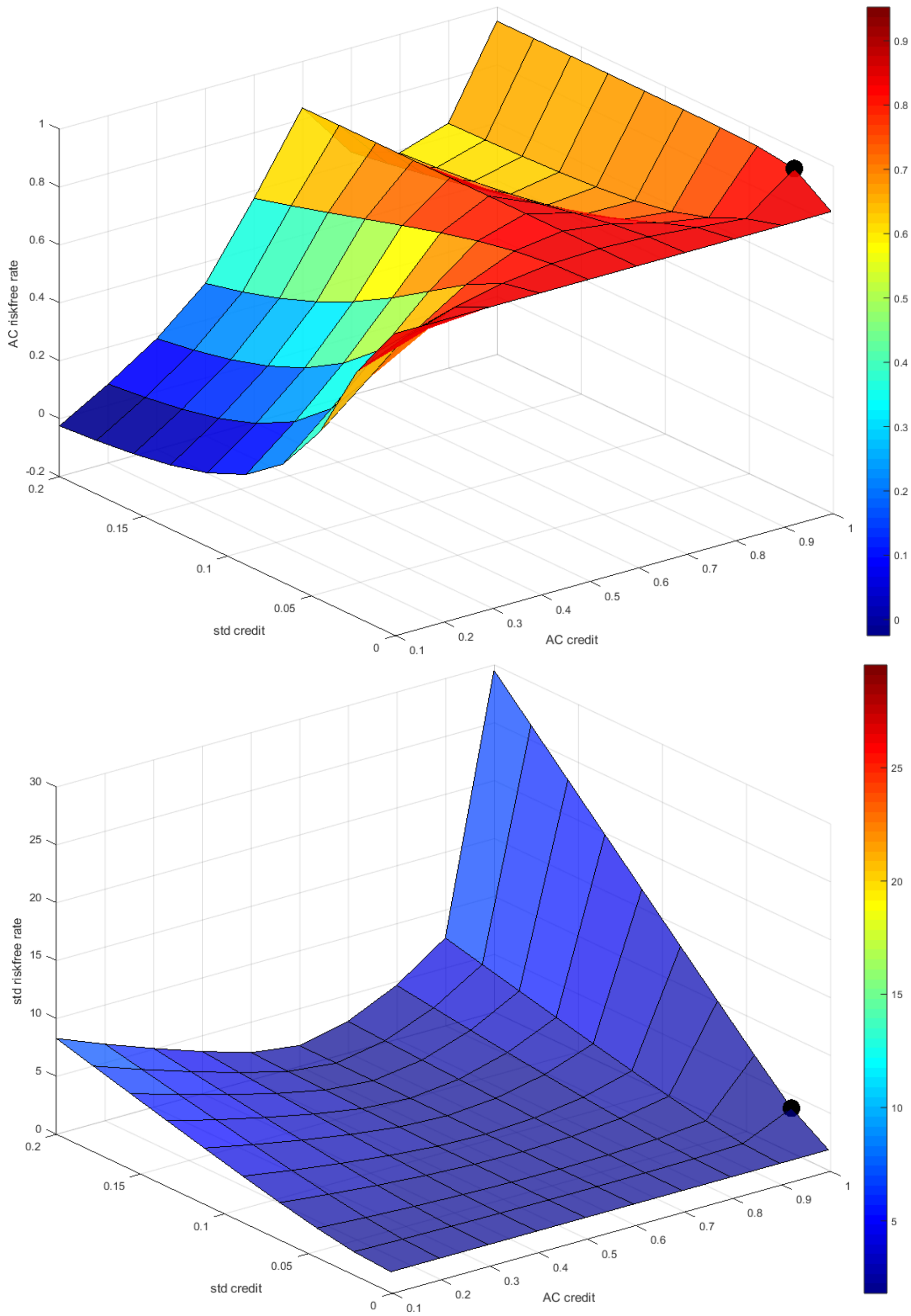
Parameter	Explanation	Value
N_θ	Nb. idiosyncratic income states	5
N_b^f	Length bond grid for distribution	60
N_b	Length bond grid for savings	20
N_n	Length bond grid for labor supply	20
$\bar{b}$	Max. bond grid	90
x_1	Min. x added to χ	0.001
c_{min}	Min. consumption	0.001
–	Nb. iterations endogenous grid for initial guess	150
–	Solver tolerance for policy functions	10^{-6}
–	Solver tolerance for prices and ψ	10^{-6}

On a 3.4 GHz Intel Core i5-7500 desktop with 8 GB of RAM, it takes 55s to solve for the model steady state, 7s and 821s to compute the Jacobian and the Hessian using automatic differentiation (in Julia), 8s and 170s to call `gensys` and `gensys2` (in MATLAB). Overall, the model is solved in 15-20min.

B.2 Identification of credit shock volatility and persistence

In Figure 8, I separately plot surfaces for the risk-free rate autocorrelation and volatility, as functions of the credit shock autocorrelation and volatility, to show that the latter are well-identified.

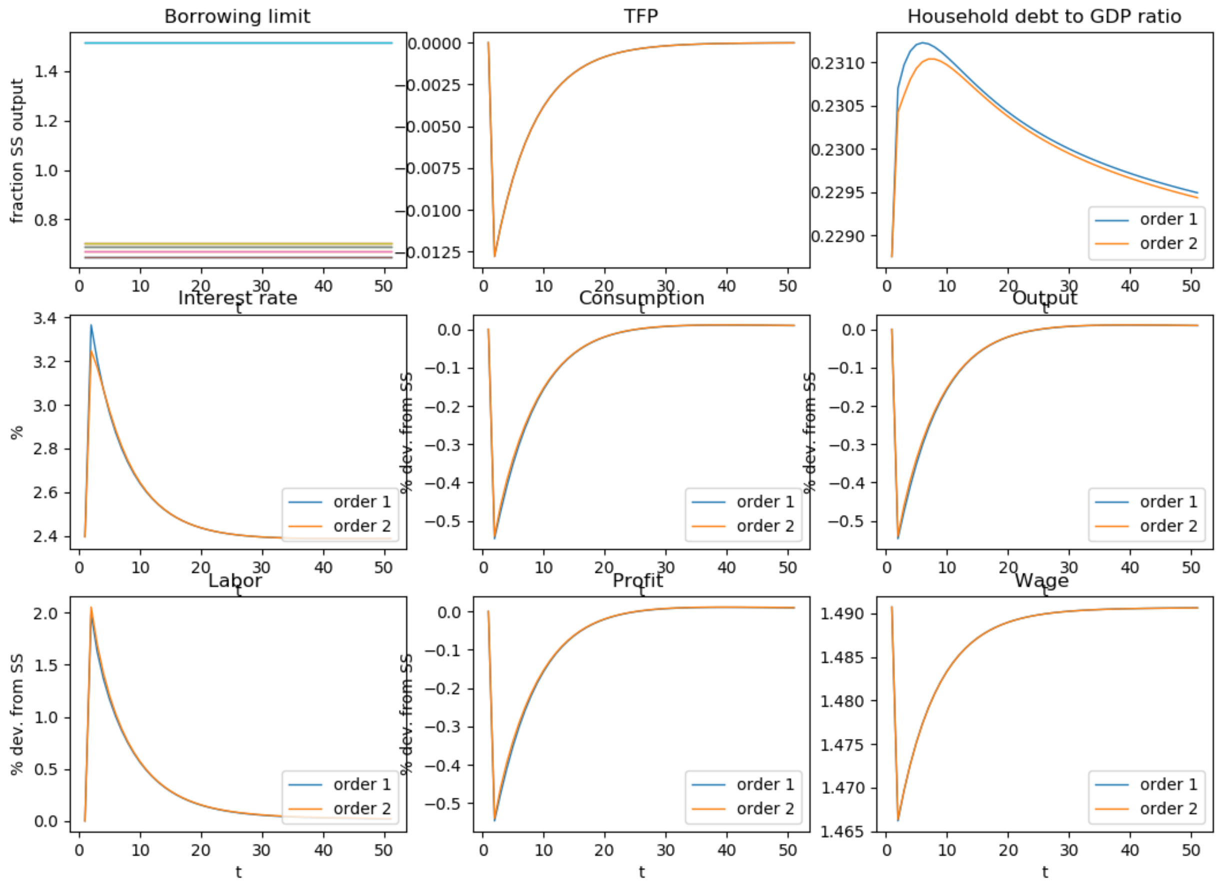
Figure 8: Identification of credit shock volatility and persistence with risk-rate data



Notes: Risk-free rate autocorrelation (upper panel) and annual % volatility (lower panel), as functions of the credit shock autocorrelation and volatility, estimated in a simulation of the linearized model with $T = 10,000$ periods. In each graph, the black dot represents the model calibration for the credit shock process. It is identified as it lies in non-flat areas of the (ρ, σ) surfaces.

C Response to productivity shocks

Figure 9: Response to TFP shock

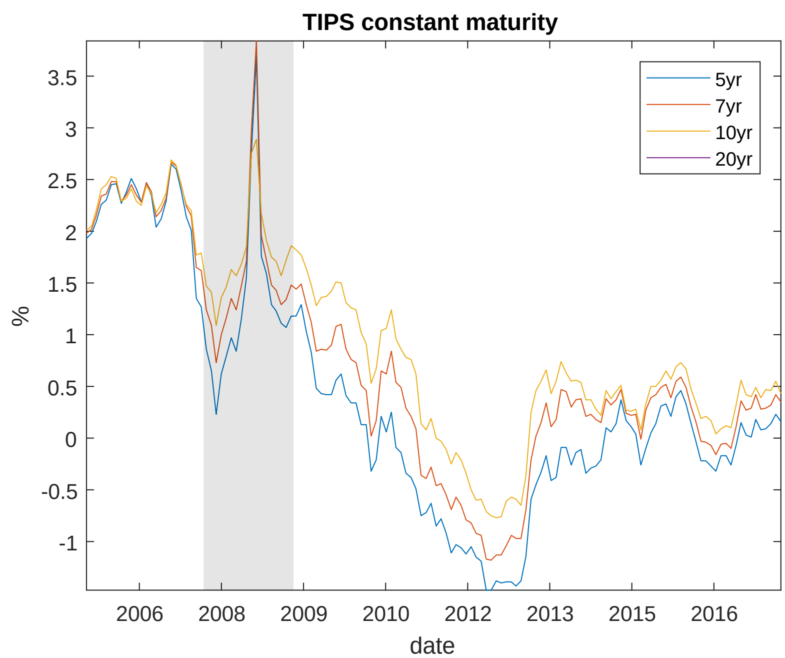


Notes: Impulse response function to a one standard deviation TFP shock: order 1 vs 2. The upper left panel plots the response of borrowing constraints to output for all income types (θ_1 for the lowest line, θ_5 for the highest), here zero. Initial period: deterministic steady state.

D Data

D.1 Real risk-free rate

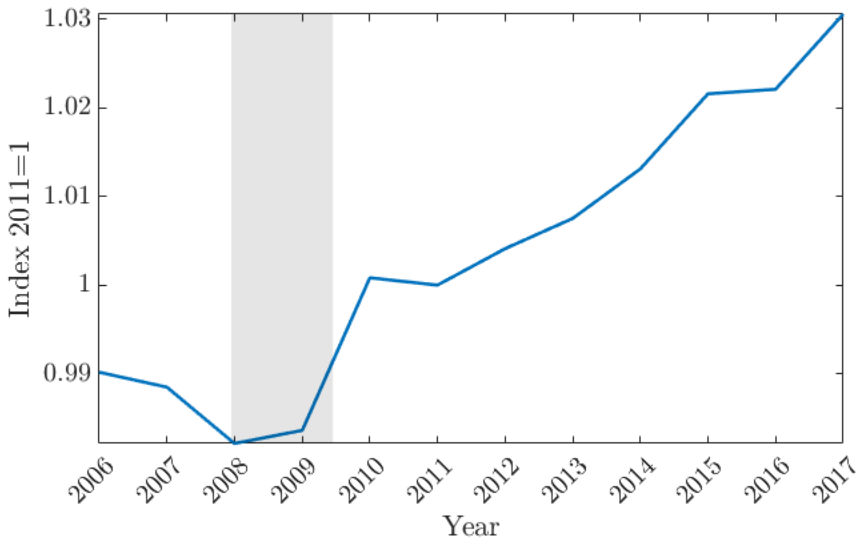
Figure 10: Real risk-free rate, various maturities



Notes: Treasury Inflation-Indexed Security, Constant Maturity, Percent, Monthly, Not Seasonally Adjusted. Maturity from 5 to 20 year. Source: FRB, H.15 Selected Interest Rates. Shaded areas represent NBER recession.

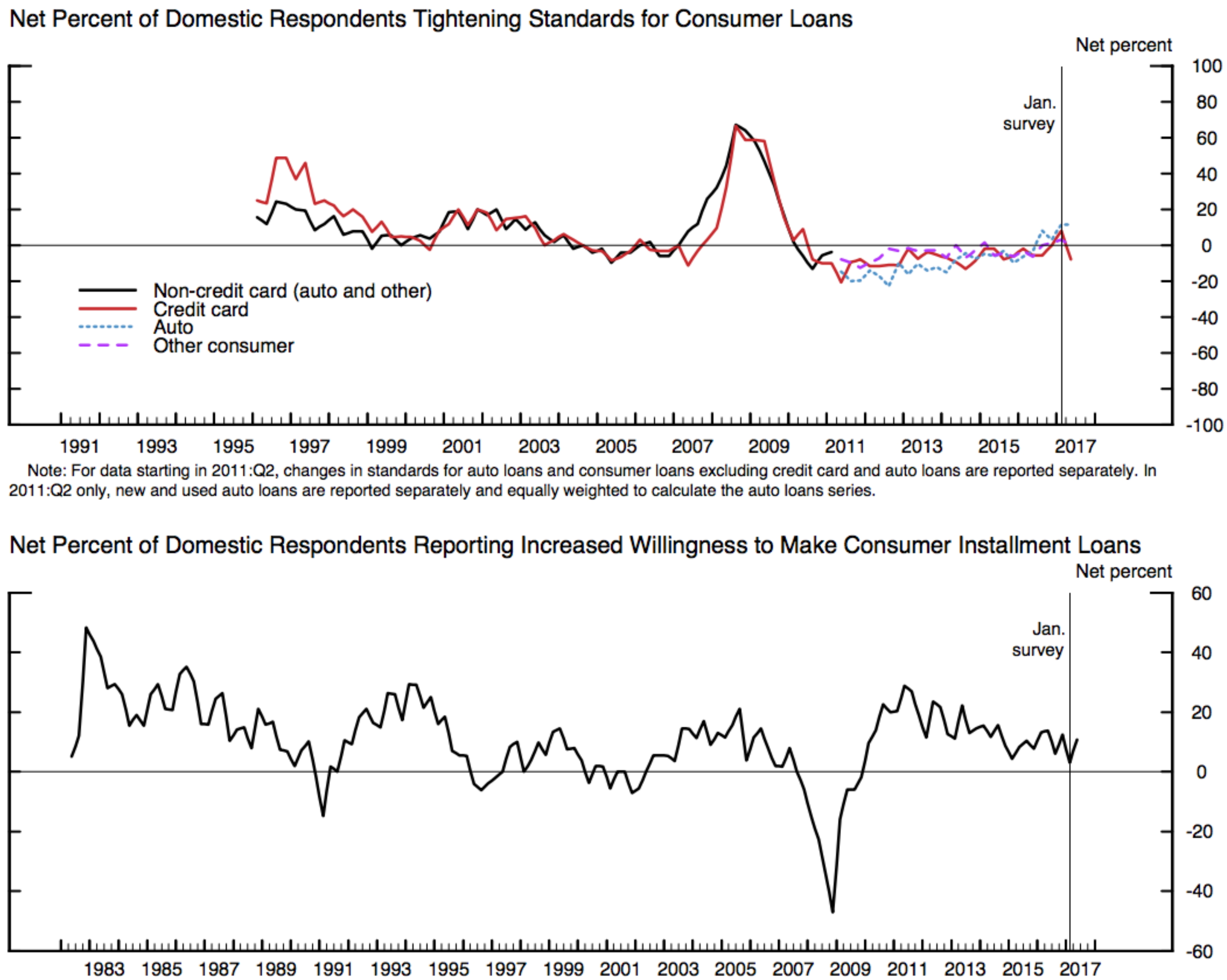
D.2 Productivity and lending standard measures

Figure 11: Total Factor Productivity



Notes: Total Factor Productivity at Constant National Prices for United States. Source: Penn World Table 9.0. Shaded area represents NBER recession.

Figure 12: Lending standards, unsecured household credit



*Source:*Ferederal Reserve Board, April 2017 Senior Loan Officer Opinion Survey on Bank Lending Practices. Quarterly frequency.