

# Modal analysis of resonant dielectric nano-structures

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## 1 Introduction

$$\begin{cases} \Delta u + \omega^2(1 + n(x))u = 0 & \text{in } \mathbb{R}^2 \setminus \partial D, \\ u^s := u - u^i \text{ satisfies the Sommerfeld radiation condition.} \end{cases} \quad (1.1)$$

with

$$n(x) = \eta \chi(\bar{D}). \quad (1.2)$$

### 1.1 Integral formulation

Denote by  $\Gamma^\omega$  the Green's function associated with Helmholtz' equation in the free space in the medium:

$$(\Delta + \omega^2) \Gamma(x, y) = \delta_y(x), \quad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d$$

**Proposition 1.1.** *Lippman-Schwinger equation*

$$u - u^i(x) = -\omega^2 \eta \int_D u(y) \Gamma^\omega(x, y) dy$$

**Definition 1.1.**

$$\begin{aligned} L^2(D) &\longrightarrow L^2(D) \\ \mathcal{K} \quad f &\longmapsto - \int_D f(y) \Gamma^\omega(\cdot, y) \cdot dy \end{aligned}$$

**Proposition 1.2.**

$$(I - \omega^2 \eta \mathcal{K}) [u|_D] = u^i \quad \text{in } D \quad (1.3)$$

### 1.2 Spectral analysis of $\mathcal{K}_D$

The following lemmas are from [1].

**Lemma 1.1.** *The operator  $\mathcal{K}$  is compact from  $L^2(D)$  to  $L^2(D)$ . In fact,  $\mathcal{K}$  is bounded from  $L^2(D)$  to  $H^2(D)$ . Moreover,  $\mathcal{K}$  is a Hilbert-Schmidt operator.*

**Lemma 1.2.** *Let  $\sigma(\mathcal{K})$  be the spectrum of  $\mathcal{K}$ :*

1.  $\sigma(\mathcal{K}) = \{0, \lambda_1, \dots, \lambda_n, \dots\}$  where  $|\lambda_1| \geq |\lambda_2| \geq \dots$  and  $\lambda_n \rightarrow 0$ .
2.  $\sigma(\mathcal{K}) \setminus \sigma_p(\mathcal{K}) = \{0\}$ ,  $\sigma_p$  being the point spectrum of  $\mathcal{K}$ .

**Lemma 1.3.** *Let  $H_j$  denote the generalized eigenspaces of the operator  $\mathcal{K}$ . Then the following decomposition holds*

$$L^2(D) = \overline{\bigcup_{j=1}^{\infty} H_j}$$

**Lemma 1.4.** *There exists a basis  $\{u_{j,l,k}\}$ ,  $1 \leq l \leq m_j$ ,  $1 \leq k \leq n_{j,l}$  for  $H_j$  such that*

$$\mathcal{K}[u_{j,l,k}] = \lambda_j u_{j,l,k} + u_{j,l,k-1} \quad \text{for all } j, l, k \quad (u_{j,l,k} = 0 \text{ for } k \leq 0).$$

$m_j$  is the geometric multiplicity of  $\lambda_j$ , given by the dimension of  $N(\lambda_j I - \mathcal{K})$  and  $\sum_l n_{j,l} = N_j$  is the algebraic multiplicity of  $\lambda_j$ . In  $H_j$ , in the basis  $\{u_{j,l,k}\}$ ,  $\mathcal{K}$  has the Jordan block representation:

$$K_D = \begin{pmatrix} J_{j,1} & & \\ & \ddots & \\ & & J_{j,m_j} \end{pmatrix} \quad \text{with} \quad J_{j,l} = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{pmatrix}$$

### 1.3 Non Normality, Exceptional points

The operator  $\mathcal{K}$  is not normal in  $L^2(D)$ . This can have two consequences:

- the existence of generalized eigenspaces that are not eigenspaces (degenerate eigenvalues  $\lambda_j$  such that  $n_{j,l} \geq 2$  for some  $1 \leq l \leq m_j$ )
- the lack of orthogonality between generalized eigenspaces

**Remark 1.1.** *In the first situation we are in presence of an exceptional point. This is a particular situation, that happens only for specific values of  $\omega$ . This situation is not stable under a small perturbation of the frequency. Therefore, at first we will assume that there are no exceptional points. This specific case will be studied later.*

**Conjecture 1.1.** *The set of  $\omega$  such that  $\mathcal{K}$  has an exceptional point is discrete.*

**Conjecture 1.2.** *Exceptional points are of order 2 or 4 ?*

## 2 Modal decomposition in the non exceptional case

In this section, we make the following assumption:

**Assumption 2.1.**  *$\omega$  is such that  $K$  has no exceptional eigenvalues, i.e.  $n_{j,l} = 1$  for every  $j \in \mathbb{N}, 1 \leq l \leq m_j$ .*

Denote  $\Gamma := \{(j, l, k) \in \mathbb{N}^3, 1 \leq l \leq m_j, 1 \leq k \leq n_{j,l}\}$  the set of indices for the basis functions. Equip  $\Gamma$  with the lexicographical order  $\preceq$ .

**Lemma 2.1.** *There exists an orthonormal basis  $\{e_\gamma : \gamma \in \Gamma\}$  for  $L^2(D)$  such that*

$$e_\gamma = \sum_{\gamma' \preceq \gamma} a_{\gamma, \gamma'} u_{\gamma'}$$

## 2.1 Modal expansion, resonance condition

**Proposition 2.1.**

$$u|_D(x) = \frac{1}{\tau\omega^2\varepsilon_c} \sum_{\gamma \in \Gamma} \frac{1}{\frac{1}{\tau\varepsilon_c} - \omega^2\lambda_j(\omega)} e_\gamma^i u_\gamma(x)$$

with  $e_\gamma^i = \int_D u^i(x) e_\gamma(x) dx$

## 3 Temporal modal analysis

Time dependent model, transverse electric case:

$$\varepsilon_D(x) \partial_t^2 u(x, t) - \Delta u(x, t) = 0$$

with

$$\begin{aligned} u(x, t) &= u^s(x, t) + u^i(x, t) \\ u^i(x, t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} u^i(x, \omega) d\omega \end{aligned}$$

Using proposition 2.1 we have:

$$u(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} \sum_{\gamma} e_\gamma^i(\omega) (I - \tau\varepsilon_c \omega^2 K_D(\omega)) [e_\gamma(\omega)] d\omega$$

this is a test

## References

- [1] Habib Ammari and Hai Zhang. Super-resolution in high-contrast media. In *Proc. R. Soc. A*, volume 471, page 20140946. The Royal Society, 2015.