Modal analysis of resonant dielectric nano-structures

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1 Introduction

$$\begin{cases} \Delta u + \omega^2(1+n(x))u = 0 \quad \text{in } \mathbb{R}^2 \backslash \partial D, \\ u^s := u - u^i \ \text{ satisfies the Sommerfeld radiation condition.} \end{cases} \tag{1.1}$$

with

$$n(x) = \eta \chi(\bar{D}). \tag{1.2}$$

1.1 Integral formulation

Denote by Γ^ω the Green's function associated with Helmholtz' equation in the free space in the medium:

$$\left(\Delta + \omega^2\right) \Gamma(x,y) = \delta_y(x), \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$$

Proposition 1.1. *Lippman-Schwinger equation*

$$u - u^i(x) = -\omega^2 \eta \int_D u(y) \Gamma^\omega(x,y) \mathrm{d}y$$

Definition 1.1.

$$\mathcal{K} \quad \begin{aligned} L^2(D) &\longrightarrow L^2(D) \\ f &\longmapsto -\int_D f(y) \Gamma^\omega(\cdot, y) \cdot \mathrm{d}y \end{aligned}$$

Proposition 1.2.

$$\left(I - \omega^2 \eta \mathcal{K}\right) \left[u\big|_D\right] = u^i \qquad \text{in } D \tag{1.3}$$

1.2 Spectral analysis of $\mathcal{K}_D$

The following lemmas are from [1].

Lemma 1.1. *The operator $\mathcal{K}$ is compact from $L^2(D)$ to $L^2(D)$. In fact, $\mathcal{K}$ is bounded from $L^2(D)$ to $H^2(D)$. Moreover, $\mathcal{K}$ is a Hilbert-Schmidt operator.*

Lemma 1.2. *Let $\sigma(\mathcal{K})$ be the spectrum of $\mathcal{K}$:*

1. $\sigma(\mathcal{K}) = \{0, \lambda_1, \dots, \lambda_n, \dots\}$ *where* $|\lambda_1| \geq |\lambda_2| \geq \cdots$ *and* $\lambda_n \to 0$.

2. $\sigma(\mathcal{K}) \setminus \sigma_p(\mathcal{K}) = \{0\}$, σ_p *being the point spectrum of* $\mathcal{K}$.

Lemma 1.3. *Let H_j denote the generalized eigenspaces of the operator $\mathcal{K}$. Then the following decomposition holds*

$$L^2(D) = \overline{\bigcup_{j=1}^{\infty} H_j}$$

Lemma 1.4. *There exists a basis $\{u_j, l, k\}$, $1 \leq l \leq m_j$, $1 \leq k \leq n_{j,l}$ for H_j such that*

$$\mathcal{K}[u_{j,l,k}] = \lambda_j u_{j,l,k} + u_{j,l,k-1} \quad \textit{for all } j,l,k \quad (u_{j,l,k} = 0 \textit{ for } k \leq 0).$$

m_j is the geometric multiplicity of λ_j, given by the dimension of $N(\lambda_j I - \mathcal{K})$ and $\sum_l n_{j,l} = N_j$ is the algebraic multiplicity of λ_j. In H_j, in the basis $\{u_j, l, k\}$, $\mathcal{K}$ has the Jordan block representation:

$$K_D = \begin{pmatrix} J_{j,1} & & \\ & \ddots & \\ & & J_{j,m_j} \end{pmatrix} \qquad \textit{with} \qquad J_{j,l} = \begin{pmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_j & 1 \\ & & & \lambda_j \end{pmatrix}$$

1.3 Non Normality, Exceptional points

The operator $\mathcal{K}$ is not normal in $L^2(D)$. This can have two consequences:

- the existence of generalized eigenspaces that are not eigenspaces (degenerate eigenvalues λ_j such that $n_{j,l} \geq 2$ for some $1 \leq l \leq m_j$)
- the lack of orthogonality between generalized eigenspaces

Remark 1.1. *In the first situation we are in presence of an* exceptional point*. This is a particular situation, that happens only for specific values of ω. This situation is not stable under a small perturbation of the frequency. Therefore, at first we will assume that there are no exceptional points. This specific case will be studied later.*

Conjecture 1.1. *The set of ω such that $\mathcal{K}$ has an exceptional point is discrete.*

Conjecture 1.2. *Exceptional points are of order* 2 *or* 4 *?*

2 Modal decomposition in the non exceptional case

In this section, we make the following assumption:

Assumption 2.1. *ω is such that K has no exceptional eigenvalues, i.e. $n_{j,l} = 1$ for every $j \in \mathbb{N}, 1 \leq l \leq m_j$.*

Denote $\Gamma := \{(j,l,k) \in \mathbb{N}^3, 1 \leq l \leq m_j,\ 1 \leq k \leq n_{j,l}\}$ the set of indices for the basis functions. Equip Γ with the lexicographical order $\preceq$.

Lemma 2.1. *There exists an orthonormal basis* $\{e_\gamma : \gamma \in \Gamma\}$ *for* $L^2(D)$ *such that*

$$e_\gamma = \sum_{\gamma' \preceq \gamma} a_{\gamma,\gamma'} u_{\gamma'}$$

2.1 Modal expansion, resonance condition

Proposition 2.1.

$$u\big|_D(x) = \frac{1}{\tau\omega^2\varepsilon_c} \sum_{\gamma\in\Gamma} \frac{1}{\frac{1}{\tau\varepsilon_c} - \omega^2\lambda_j(\omega)} e^i_\gamma u_\gamma(x)$$

with $e^i_\gamma = \int_D u^i(x) e_\gamma(x) \mathrm{d}x$

3 Temporal modal analysis

Time dependent model, transverse electric case:

$$\varepsilon_D(x)\partial_t^2 u(x,t) - \Delta u(x,t) = 0$$

with

$$u(x,t) = u^s(x,t) + u^i(x,t)$$
$$u^i(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t} u^i(x,\omega)\mathrm{d}\omega$$

Using proposition 2.1 we have:

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega t} \sum_\gamma e^i_\gamma(\omega) \left(I - \tau\varepsilon_c\omega^2 K_D(\omega)\right)[e_\gamma(\omega)]\mathrm{d}\omega$$

References

[1] Habib Ammari and Hai Zhang. Super-resolution in high-contrast media. In *Proc. R. Soc. A*, volume 471, page 20140946. The Royal Society, 2015.