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Non-Backtracking Random Walks and Cogrowth of Graphs

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Abstract. Let X be a locally finite, connected graph without vertices of degree 1. Non-backtracking random walk moves at each step with equal probability to one of the "forward" neighbours of the actual state, *i.e.,* it does not go back along the preceding edge to the preceding state. This is not a Markov chain, but can be turned into a Markov chain whose state space is the set of oriented edges of X. Thus we obtain for infinite X that the n-step non-backtracking transition probabilities tend to zero, and we can also compute their limit when X is finite. This provides a short proof of an old result concerning cogrowth of groups, and makes the extension of that result to arbitrary regular graphs rigorous. Even when X is non-regular, but *small cycles are dense in* X, we show that the graph X is non-amenable if and only if the non-backtracking n-step transition probabilities decay exponentially fast. This is a partial generalization of the cogrowth criterion for regular graphs which comprises the original cogrowth criterion for finitely generated groups of Grigorchuk and Cohen.

1 Introduction and Results

Let X be the vertex set of a locally finite, connected graph, possibly with multiple edges and loops. We write $e(x, y)$ for the number of edges between the vertices x and y, if $y \neq x$, while $e(x, x)$ is *twice* the number of loops at x (see §2 for a discussion). The degree of a vertex $x \in X$ is $\deg(x) = \sum_y e(x, y)$. We assume that $\deg(x) \geq 2$ for all $x \in X$. *Non-backtracking (simple) random walk* (NBRW) is the following random process: at the beginning, the walker starts at some vertex x and chooses with equal probability one of the incident edges. He steps to the other end of that edge. At the later steps, the rule is the same, but the walker selects with equal probability only among those incident edges that are different from the one transversed at the previous step.

We write $q^{(n)}(x, y)$ for the probability that the random walker, starting at vertex x, is at vertex y at the n-th step. Note that NBRW is *not* a Markov chain on X. The defining property of a Markov chain, that "the future depends only on the actual state and not on the past", is violated, since the walker has to remember the edge along which he reached the actual state before moving on.

However, it is easy to turn NBRW into a Markov chain by changing the state space: with each edge, we associate two oppositely oriented edges $e, \check{e}$ (with $\check{\check{e}} = e$). We write e^- and e^+ for the initial and terminal vertex of the edge e, so that $(\check{e})^- = e^+$ and $(\check{e})^+ = e^-$. (Note in particular, that for each *a priori* unoriented loop we get two oriented ones!) We now consider NBRW as a *Markov* process whose new state space

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is the set $E = E(X)$ of oriented edges, with transition matrix $Q_E = (q_E(e, f))_{e,f\in E}$ given by

$$q_E(e, f) = \begin{cases} \dfrac{1}{\deg(e^+) - 1} & \text{if } e \to f, \text{ that is, } f^- = e^+ \text{ and } f \neq \check{e}, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is the following link between edge-NBRW and vertex-NBRW.

Lemma 1.1 *For arbitrary vertices $x, y \in X$,*

$$q^{(n)}(x, y) = \frac{1}{\deg(x)} \sum_{\substack{e,f\in E:\\ e^+=x, f^+=y}} q_E^{(n)}(e, f),$$

where $q_E^{(n)}$ denotes the n-step transition probabilities, i.e., *the elements of the matrix power Q_E^n, with $Q_E^0 = I_E$, the identity matrix over E.*

(Note that $q^{(n)}(x, y)$ is *not* the (x, y)-element of an n-th matrix power over X!)

The following result is then a consequence of basic Markov chain theory.

Theorem 1.2

(i) *If X is finite, connected, with minimum degree* 2*, then for all $x, y \in X$,*

$$\lim_{n\to\infty} \frac{1}{n}\Big(q^{(1)}(x, y) + q^{(2)}(x, y) + \cdots + q^{(n)}(x, y)\Big) = \frac{\deg(y)}{|E(X)|}.$$

(ii) *If in addition to the assumptions of* (i) *X has minimum degree* 3*, then for all $x, y \in X$,*

$$\lim_{n\to\infty} q^{(n)}(x, y) = \frac{\deg(y)}{|E(X)|}, \quad \textit{if X is not bipartite,}$$

$$\lim_{n\to\infty} q^{(2n+\delta)}(x, y) = \frac{2\deg(y)}{|E(X)|}, \quad \textit{if X is bipartite,}$$

where $\delta = 0$ or $\delta = 1$ according to whether x and y are at even or odd distance.

(iii) *If X is infinite and connected, with minimum degree* 2*, then for all $x, y \in X$,*

$$\lim_{n\to\infty} q^{(n)}(x, y) = 0.$$

In statements (i) and (ii), note that $|E(X)|$ is twice the number of non-oriented edges.

As usual, the *distance* $d(x, y)$ between two vertices $x, y \in X$ is the minimum length of a path connecting the two. The *ball* of radius R centred at x is the subgraph $B(x, R) = \{y \in X : d(y, x) \le R\}$ of X. Recall that a *cycle* of length n in X consists of a sequence $e_n = e_0, \ldots, e_{n-1}$ of distinct edges whose initial vertices are all distinct, such that $e_{k-1} \to e_k$ for all $k = 1, \ldots, n$.

Definition 1.3 We say that *small cycles are dense in X*, if there is $R > 0$ such that every ball $B(x, R)$ in X contains a cycle.

Every finite, connected graph with minimum degree 2 satisfies this condition.

The *automorphism group* of X consists of all bijections $g: X \to X$ which satisfy $e(gx, gy) = e(x, y)$ for all $x, y \in X$. A graph is called *transitive*, (resp., *almost transitive*) if the automorphism group acts with one orbit, (resp., finitely many orbits) on X. Obviously, an infinite, almost transitive graph with minimum degree 2 has dense small cycles unless it is a tree. (To be precise, we require of a tree that it does not have multiple edges.)

Lemma 1.4 *If small cycles are dense in X, then $\rho(Q) = \limsup_{n\to\infty} q^{(n)}(x, y)^{1/n}$ is independent of $x, y \in X$, and $0 < \rho(Q) \leq 1$. (If X is finite, then $\rho(Q) = 1$.)*

The following strengthens Theorem 1.2(iii) for almost transitive graphs.

Theorem 1.5 *If X is infinite, connected, with minimum degree 2, and almost transitive, then for all $x, y \in X$, $\lim_{n\to\infty} q^{(n)}(x, y)/\rho(Q)^n = 0$.*

The *isoperimetric constant* $\iota(X)$ of a connected, locally finite graph X is

$$\iota(X) = \inf\left\{ \frac{\mathrm{Area}(F)}{\mathrm{Vol}(F)} : F \subset X \text{ finite} \right\},$$

where $\mathrm{Vol}(F) = \sum_{x\in F} \deg(x)$ and $\mathrm{Area}(F)$ is the number of edges with one endpoint in F and the other in $X \setminus F$. The graph is called *amenable* if $\iota(X) = 0$. Non-amenable graphs are also called (infinite) *expanders*.

Consider the Hilbert space $\ell^2(E)$ of all functions $F: E \to \mathbb{R}$ with $\langle F, F\rangle < \infty$, with the ordinary inner product

$$\langle F, G\rangle = \sum_{e\in E} F(e)G(e).$$

Then Q_E acts on this space by $Q_E F(e) = \sum_{f\in E} q_E(e, f)F(f)$. We denote by $\|Q_E\|$ the corresponding operator norm, and by $\rho_2(Q_E) = \lim_n \|Q_E^n\|^{1/n}$ its spectral radius. Note that $\rho(Q) \leq \rho_2(Q_E) \leq \|Q_E\|$ in general.

Proposition 1.6

(i) *One always has $\|Q_E\| = 1$.*
(ii) *If small cycles are dense in X, then $\rho(Q) = \rho_2(Q_E)$.*

Theorem 1.7 *Suppose that X is connected, that small cycles are dense, and that there is $M < \infty$ such that $2 \leq \deg(x) \leq M$ for all $x \in X$. Then X is amenable if and only if $\rho(Q) = 1$.*

With these results and their proofs we aim principally at extending and explaining previous material regarding *cogrowth* of graphs and groups and at shedding new light on cogrowth by studying it in terms of NBRW on the oriented edges. We also think that NBRW on the (oriented) edge set of an arbitrary graph is an interesting random process in its own right.

In §2, we first recall (ordinary) simple random walk on a graph and some of its basic properties in order to put our results on NBRW in the right perspective. We then consider cogrowth of graphs, which is best understood in terms of universal covering trees, and explain how Theorems 1.2, 1.5 and 1.7 apply. In §2 we also give various references; §3 is dedicated to the proofs of the results stated here. Some additional remarks and observations can be found in §4.

2 Simple Random Walk and Cogrowth of Graphs

Simple Random Walk

A simple random walk (SRW) is mostly considered on graphs without multiple edges, and loops are usually counted only once for the degree of a vertex. Here, multiple edges are admitted, and we count each loop twice. SRW is the Markov chain on the (vertex set of the) graph X with transition matrix $P = (p(x, y))_{x,y\in X}$ given by

$$p(x, y) = \frac{e(x, y)}{\deg(x)}.$$

Thus, contrary to NBRW, the walker does not remember whence he came at the previous step, and chooses at random any one among the outgoing edges at the actual vertex. A possible interpretation for counting each loop twice is that topologically, the walker standing at a vertex x sees two "ends" of each loop at x between which he may choose. We write $p^{(n)}(x, y)$ for the n-step transition probability from x to y.

The transition matrix P acts by $Pg(x) = \sum_y p(x, y)g(y)$ on the Hilbert space $\ell^2(X, \deg)$ of all functions $g\colon X \to \mathbb{R}$ with $\langle g, g\rangle < \infty$, where the inner product is

$$\langle g, h\rangle = \sum_{x\in X} g(x)h(x)\deg(x).$$

We denote by $\|P\|$ the norm of this operator.

Here is a list of well-known properties of SRW. (Recall once more that $E = E(X)$ is the set of oriented edges as in §1, so that $|E(X)|$ is twice the number of "ordinary" non-oriented edges.)

Proposition 2.1

(i) *If X is finite and not bipartite, then for all $x, y \in X$,*

$$\lim_{n\to\infty} p^{(n)}(x, y) = \frac{\deg(x)}{|E(X)|}.$$

If X is finite and bipartite, then for all $x, y \in X$, with $\delta \in \{0, 1\}$ such that $d(x, y) \equiv \delta \mod 2$,

$$\lim_{n\to\infty} p^{(2n+\delta)}(x, y) = 2\frac{\deg(x)}{|E(X)|}.$$

(ii) *If X is infinite, then for all $x, y \in X$,*

$$\lim_{n\to\infty} p^{(n)}(x, y) = 0.$$

(iii) *The* spectral radius

$$\rho(P) = \limsup_{n\to\infty} p^{(n)}(x, y)^{1/n}$$

is independent of $x, y \in X$, and $\|P\| = \rho(P)$.

(iv) *If X is infinite and almost transitive then*

$$\lim_{n\to\infty} p^{(n)}(x, y)/\rho(P)^n = 0.$$

(v) *X is amenable if and only if $\rho(P) = 1$.*

Statements (i) and (ii) follow from basic Markov chain theory, see Chung [2] or Seneta [14]: the Markov chain given by P is *irreducible* ($\forall\, x, y \in X$ there exists $n = n(x, y) \geq 0$ such that $p^{(n)}(x, y) > 0$). Its *period* $\mathfrak{d}(P) = \gcd\{n : p^{(n)}(x, x) > 0\}$ is equal to 2 when X is bipartite, and equal to 1, otherwise. Finally, $\mu(x) = \deg(x)$ defines an *invariant* measure. If X is finite, then $\mu(X) = |E(X)|$, and $\mu_0(x) = \mu(x)/|E(X)|$ is an invariant probability measure. Therefore, (i) follows from the basic convergence theorem, see [2, §1.6, Theorem 1] or [14, Theorem 4.2]. If X is infinite, then $\mu(X) = \infty$, whence the random walk cannot be *positive recurrent*, and (ii) must hold. We shall encounter these notions in more detail in §3.

For statement (iii), see Woess [17, §10]. In particular, the fact that $\rho(P) = \rho_2(P)$, the ℓ^2-spectral radius of P, follows from self-adjointness of P on $\ell^2(X, \deg)$.

Regarding statement (iv), this is immediate when $\sum_n p^{(n)}(x, y)/\rho(P)^n < \infty$. If the series diverges, then it follows from [17, Theorem 7.8] (which is basically due to Guivarc'h [7]) that $\rho(P) = 1$, and we can apply (ii).

Statement (v) has a long history, going back to Kesten's amenability criterion for finitely generated groups [10]. The version stated here is due to Dodziuk and Kendall [5] based on a previous paper by Dodziuk [4].

Cogrowth

Cogrowth is a notion of asymptotic density of a graph. It is best understood in terms of the *universal cover* of the graph X. This is a (unique) *tree* T together with a surjective mapping $\pi\colon T \to X$ which is a local homeomorphism, *i.e.,* if $\tilde{x}, \tilde{y}$ are neighbours in T, then so are $\pi(\tilde{x}), \pi(\tilde{y})$ in X, and $\deg_T(\tilde{x}) = \deg_X(\pi(\tilde{x}))$ for every vertex $\tilde{x} \in T$.

The covering tree can be constructed as follows: a non-backtracking walk of length $n \geq 0$ in X is a sequence $e_1, \ldots, e_n$ of edges such that $e_{k-1} \to e_k$ for $k = 2, \ldots, n$. Its initial and terminal vertices are e_1^- and e_n^+, respectively. If $n = 0$, we have an empty path, for which we have to specify its initial and terminal vertex. We now choose a root (reference vertex) $o \in X$, and define T as the set of all non-backtracking paths $\tilde{x}$ starting at o, including the empty path. Two such paths are defined to be neighbours in T if one of them extends the other by a single edge. The mapping π assigns to each $\tilde{x} \in T$ its terminal vertex $x \in X$.

Now let $x, y \in X$, and choose $\tilde{x} \in T$ such that $\pi(\tilde{x}) = x$. Write $T(y) = \{y \tilde{\in} T : \pi(\tilde{y}) = y\}$, and consider the sphere $S(\tilde{x}, n) = \{\tilde{v} \in T : d_T(\tilde{v}, \tilde{x}) = n\}$, where $d_T(\,\cdot\,,\,\cdot\,)$ is the distance in T. Then *(ordinary) cogrowth* at $x, y \in X$ is the sequence

$$\text{(2.1)} \qquad \operatorname{cog}_n(x, y) = \frac{|S(\tilde{x}, n) \cap T(y)|}{|S(\tilde{x}, n)|}, \quad n \geq 0.$$

The graph X being "small" corresponds to $(\operatorname{cog}_n(x, y))_n$ being "large". Besides finiteness, amenability is also a "smallness" condition, whence it is natural to look for a link between cogrowth and amenability.

Cogrowth was initially introduced by Grigorchuk [6] and later Cohen [3] for finitely generated *groups*. If Γ is such a group, then we can represent it as a factor $\mathbb{F}_s/N$, where $\mathbb{F}_s$ is the *free group* on s free generators $\tilde{a}_1, \ldots, \tilde{a}_s$, and N is a normal subgroup of $\mathbb{F}_s$. Let $\pi\colon \mathbb{F}_s \to \Gamma$ be the factor map. We write $\tilde{a}_{-i} = \tilde{a}_i^{-1}$ and set $\tilde{S} = \{\tilde{a}_i : i = \pm 1, \ldots, \pm s\}$. Then the Cayley graph of $\mathbb{F}_s$ with respect to $\tilde{S}$ is the $2s$-regular tree, which is the covering tree of the Cayley graph of Γ with respect to the generators $a_i = \pi(\tilde{a}_i)$. It is best to consider immediately the oriented edges of that Cayley graph: every $x \in \Gamma$ is the initial point of an edge of type $\tilde{a}_i$, whose endpoint is xa_i; the associated "inverse" edge goes from xa_i to x and has type $\tilde{a}_{-i}$ $(i = \pm 1, \ldots, \pm s)$. Every pair of this type corresponds to one unoriented edge. Note that generators with $a_i = a_{-i} \neq id$ give rise to multiple edges, and when $a_i = a_{-i} = id$, we get loops. This also explains why loops should be counted twice for the degrees. Thus, the factor map π becomes the covering map from the tree onto the Cayley graph.

Note that for groups, $\operatorname{cog}_n(x, x)$ is the same for all x. Amenability of a finitely generated group Γ is equivalent with amenability of any of its (locally finite) Cayley graphs. The main result of [6, 3], restated in our notation, was that

$$\text{(2.2)} \qquad \Gamma \text{ amenable} \iff \limsup_{n\to\infty} \operatorname{cog}_n(x, x)^{1/n} = 1.$$

This has been generalized to regular graphs by Northshield [11], who was also the first to explain cogrowth in terms of covering trees. One of the basic tools for studying cogrowth of regular graphs is a functional equation between the generating functions $C(x, y|t) = \sum_n \operatorname{cog}_n(x, y)t^n$ of the cogrowth sequence and $G(x, y|z) = \sum_n p^{(n)}(x, y)z^n$ of the transition probabilities of SRW: if X is d-regular, then with our notation and normalizations,

$$\text{(2.3)} \quad C(x, y|t) = \frac{1}{d}\delta_x(y) + \frac{(d-1)^2 - t^2}{d(d-1+t^2)} G(x, y|z(t)), \quad \text{where } z(t) = \frac{dt}{d-1+t^2}.$$

A first version of (2.3) is contained in Grigorchuk's Ph.D. thesis. Various proofs of that formula have appeared: Woess [16], Szwarc [15] (both for groups), Northshield [11] (shortest), Bartholdi [1] (more general). In spite of [1], there is no satisfactory version of that formula for non-regular graphs. Nevertheless, Northshield [12] proves a clever extension of (2.2) to *quasi-regular* graphs (non-regular graphs satisfying a certain uniform growth condition), and studies cogrowth of arbitrary graphs under certain restrictions [13].

More generally, we can consider a sequence $\nu = (\nu_{\tilde{x},n})_{\tilde{x}\in T, n\geq 0}$, where each $\nu_{\tilde{x},n}$ is a probability measure concentrated on the sphere $S(\tilde{x}, n)$ of radius n centred at $\tilde{x}$ in the covering tree T of X, with $\pi(\tilde{x}) = x$. Note that there is a natural bijection between $S(\tilde{x}, n)$ and $S(\tilde{x}', n)$, when $\pi(\tilde{x}) = \pi(\tilde{x}')$. We require that in this case, $\nu_{\tilde{x}',n}$ is the image of $\nu_{\tilde{x},n}$ under that bijection. Then we can define

$$\mathrm{cog}_n^\nu(x, y) = \nu_{\tilde{x},n}(T(y)), \quad x, y \in X,\ \pi(\tilde{x}) = x. \tag{2.4}$$

When each $\nu_{\tilde{x},n}$ is equidistribution on $S(\tilde{x}, n)$, this is ordinary cogrowth.

Another choice is to define

$$\nu_{\tilde{x},n}(\tilde{y}) = \frac{1}{\deg(\tilde{x})}\frac{1}{\deg(\tilde{x}_1) - 1}\cdots\frac{1}{\deg(\tilde{x}_{n-1}) - 1},$$

where $\tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_{n-1}, \tilde{y}$ are the consecutive vertices on the unique path in T from $\tilde{x}$ to $\tilde{y} \in S(\tilde{x}, n)$. Cogrowth with respect to this choice of ν is the same as NBRW:

$$\mathrm{cog}_n^\nu(x, y) = q^{(n)}(x, y). \tag{2.5}$$

In the specific case of regular graphs, the two concepts coincide. Thus, besides ordinary cogrowth, non-backtracking random walk is another way to extend cogrowth from regular to arbitrary graphs.

3 Proofs

In this section, we always use the basic assumption that X is a locally finite, connected graph with minimum degree 2. The following is straightforward.

Proof of Lemma 1.1 We have

$$\begin{aligned} q^{(n)}(x, y) &= \sum_{\substack{e_1\in E:\\ e_1^- = x}} \frac{1}{\deg(x)} \sum_{\substack{e_2,\ldots,e_n = f\in E:\\ e_{i-1}\to e_i,\, e_n^+ = y}} \frac{1}{\deg(e_1^+) - 1}\cdots\frac{1}{\deg(e_{n-1}^+) - 1} \\ &= \frac{1}{\deg(x)} \sum_{\substack{e_1,\, f\in E:\\ e_1^- = x,\, f^+ = y}} q_E^{(n-1)}(e_1, f) \end{aligned}$$

$$= \frac{1}{\deg(x)} \sum_{\substack{e_1,\, f \in E: \\ e_1^- = x,\, f^+ = y}} \frac{1}{\deg(x) - 1} \sum_{\substack{e \in E: \\ e \to e_1}} q_E^{(n-1)}(e_1, f)$$

$$= \frac{1}{\deg(x)} \sum_{\substack{e, f \in E: \\ e^+ = x,\, f^+ = y}} q_E^{(n)}(e, f),$$

since for any $e_1 \in E$ with $e_1^- = x$ there are $\deg(x) - 1$ edges e with $e \to e_1$. ■

It may be best to think of edge-NBRW as simple random walk on the *oriented line graph* (OLG) of X. This is the digraph whose vertex set is $E = E(X)$, and there is an oriented (second order) edge from e to f ($e, f \in E$) if $e \to f$. Our Markov chain with transition matrix Q_E is not symmetric, nor reversible like SRW on an unoriented graph. However, the counting measure λ, given by $\lambda(e) = 1$, is an invariant measure for Q_E, that is,

$$\sum_{e \in E} \lambda(e) q_E(e, f) = \lambda(f) \quad \forall\, f \in E. \tag{3.1}$$

We now recall a few basic Markov chain notions. We write $e \xrightarrow{*} f$ if there is $n \geq 0$ such that $q_E^{(n)}(e, f) > 0$ (*i.e.*, there is an oriented path from e to f in the OLG, a transitive relation), and $e \xleftrightarrow{*} f$ if $e \xrightarrow{*} f$ and $f \xrightarrow{*} e$. The equivalence classes with respect to the relation $\xleftrightarrow{*}$ are called *irreducible classes*. An *essential class* V is an irreducible class with the property that $e \in V$ and $e \xrightarrow{*} f$ imply $f \in V$. Its elements are also called essential. The Markov chain and its transition matrix Q_E are called irreducible if the state space E forms a single irreducible class. (In graph theoretic terminology, this means that the OLG is *strongly connected*.)

Lemma 3.1 *If X is finite, then Q_E is irreducible, unless X is a cycle.*

Proof Assume that X is not a cycle. Since X is connected, for any pair of edges e, f, at least one of $e \xrightarrow{*} f$, $e \xrightarrow{*} \check{f}$, $\check{e} \xrightarrow{*} f$, or $\check{e} \xrightarrow{*} \check{f}$ must hold. Therefore it is sufficient to show that $e \xrightarrow{*} \check{e}$ for every $e \in E$.

Let us first assume that e is not contained in any cycle of X. As $\deg(x) \geq 2$ for all x, we can find inductively a sequence $e = e_0, e_1, e_2, \dots$ of edges such that $e_{k-1} \to e_k$. By finiteness of X, there must be a minimal index m such that $e_m^+ = e_i^-$ for some $i \in \{1, \dots, m-1\}$. The edges $e_i, \dots, e_m$ form a cycle C_1, so that

$$e = e_0 \xrightarrow{*} e_m \to \check{e}_{i-1} \xrightarrow{*} \check{e}_0 = \check{e}.$$

Now assume that e is contained in a cycle C_1 formed by edges $e = e_0, \dots, e_m$. Since we are assuming that X is not a cycle, there is a vertex $e_i^- =: x$ in C_1 with $\deg(x) \geq 3$. Thus, there is an edge f with $f^- = x$ such that $f \notin \{\check{e}_{i-1}, e_i\}$ (for $i = 0$ we intend $e_{-1} = e_m$). If f does not lie on any cycle in X, we have already seen that $f \xrightarrow{*} \check{f}$, whence

$$e = e_0 \xrightarrow{*} e_{i-1} \to f \xrightarrow{*} \check{f} \to \check{e}_{i-1} \xrightarrow{*} \check{e}_0 = \check{e}.$$

On the other hand, assume that f is contained in a cycle C_2 formed by edges $f = f_0, \dots, f_\ell$. Then there must be another edge f_k ($k > 0$) incident with some vertex in C_1. Let j be the minimal index $\in \{0, \dots, m\}$ with $e_j^+ = f_k^+$ for some $k \in \{1, \dots, \ell\}$. Then

$$e \xrightarrow{*} e_{i-1} \to f = f_0 \xrightarrow{*} f_k \to \check{e}_j \xrightarrow{*} \check{e}. \qquad \blacksquare$$

If X is a finite cycle, then the OLG consists of two disjoint, oriented cycles of the same length, each of which constitutes an essential class of Q_E, on which NBRW moves "forward" deterministically.

Lemma 3.2 *If X is infinite, then for any edge $e \in E$ there are infinitely many edges $f \in E$ with $e \xrightarrow{*} f$.*

Proof Let $e \in E$ and X' be the graph that results from X by removing e and $\check{e}$. If X' is connected then by infiniteness, $e \xrightarrow{*} f$ for infinitely many $f \in E$. The same holds if e is directed towards an infinite component. Thus, let us assume that e is directed towards a finite component X_1' of X'. By infiniteness of X, $\check{e}$ is directed to the other infinite component, so that $\check{e} \xrightarrow{*} f$ for infinitely many $f \in E$. Applying the method of proof of Lemma 3.1 to X_1', we have $g \xrightarrow{*} \check{g}$ for some edge g with $e \to g$ in X (remember that we assumed that $\deg(e^+) \geq 2$). It follows that $e \to g \xrightarrow{*} \check{g} \to \check{e}$ and hence $e \xrightarrow{*} f$ for infinitely many $f \in E$. $\blacksquare$

In general, if Q_E is irreducible, then we can define its *period* by

$$\mathfrak{d} = \mathfrak{d}(Q_E) = \gcd\{n : q_E^{(n)}(e,e) > 0\},$$

which is independent of $e \in E$.

Lemma 3.3 *Let X be a finite, connected graph with $\deg(x) \geq 3$ for all $x \in X$. Then the period of the associated edge-NBRW is either 2 or 1, depending on whether X is bipartite or not (respectively).*

Proof First we shall show that $\mathfrak{d}(Q_E) \in \{1, 2\}$. Let e, f, g be three distinct edges with $e^- = f^- = g^- =: x$. By assumption, $\deg(e^+) \geq 3$ and there are two distinct edges $e_1, e_2 \neq \check{e}$ with $e^+ = e_1^- = e_2^-$. By Lemma 3.1 we have $e_1 \xrightarrow{*} \check{f}$ and $e_2 \xrightarrow{*} \check{g}$. Then there are non-backtracking closed paths such that $f \xrightarrow{*} \check{e}_1 \to \check{e} \to f$ in (say) m steps and $g \xrightarrow{*} \check{e}_2 \to \check{e} \to g$ in (say) n steps. But we also have $f \xrightarrow{*} f$ in $n + m - 2$ steps, via $f \xrightarrow{*} \check{e}_1 \to e_2 \xrightarrow{*} \check{g} \to f$. Therefore, $\mathfrak{d}(Q_E)$ must be a factor of n, m and $n + m - 2$, whence $\mathfrak{d}(Q_E) \in \{1, 2\}$.

It is now clear that we must have $\mathfrak{d}(Q_E) = 2$, if X is bipartite. Otherwise, X contains an odd cycle, so that $q_E^{(k)}(e,e) > 0$ for some odd k. Thus, we cannot have $\mathfrak{d}(Q_E) = 2$, that is, $\mathfrak{d}(Q_E) = 1$. $\blacksquare$

Proof of Theorem 1.2 (i) and (ii) If X is finite, but not a cycle, then we can use Lemma 3.1. Let $e, f \in E$ and $r \geq 0$ such that $q_E^{(r)}(e, f) > 0$. Then $q_E^{(n)}(e, f) > 0$ if and only if $n \equiv r \mod \mathfrak{d}$ and n is sufficiently large (see [14, Theorem 1.3]). The fundamental convergence theorem (see [2, §I, Theorem 1] or [14, Theorem 4.2]) implies that

$$\lim_{n\to\infty} q_E^{(n\mathfrak{d}+r)}(e, f) = \mathfrak{d}\,\lambda_0(f) = \frac{\mathfrak{d}}{|E(X)|}, \tag{3.2}$$

where λ_0 is the unique invariant probability measure, that is, $\lambda_0(f) = \frac{1}{|E|}$. In view of Lemma 3.3, this together with Lemma 1.1 yields statement (ii), when $\deg(x) \geq 3$ for all $x \in X$.

Otherwise,

$$\lim_{n\to\infty} \frac{1}{n}\left(q_E^{(1)}(e, f) + \cdots + q_E^{(n)}(e, f)\right) = \frac{1}{|E(X)|},$$

and combining this with Lemma 1.1, we obtain the limit proposed in statement (i) of Theorem 1.2.

In the case where X is a cycle, the $q^{(n)}(x, y)$ can be calculated explicitly, whence the claim of the Theorem follows. This is left as a simple exercise to the reader.

(iii) We distinguish two cases. First, if the edge-NBRW starting at $e \in E$ is *transient*, that is, the probability of returning to e is < 1, then $\sum_n q_E^{(n)}(e, f) < \infty$ for every $f \in E$, see [2, §I.6, Theorem 4]. Therefore, $q_E^{(n)}(e, f) \to 0$.

If the random walk starting at e is *recurrent*, *i.e.*, it returns to e with probability 1, then e must be an essential state, see [2, §I.4 Theorem 4] or [14, Lemma 5.2]. Now by Lemma 3.2, there are infinitely many $f \in E$ such that $e \xrightarrow{*} f$. Therefore, the (essential) irreducible class V of e is infinite. Since the random walk starting at e does not leave V, we can consider the restriction of Q_E to V. It defines an irreducible, recurrent Markov chain with invariant measure λ, the counting measure. Recurrence yields that this is the unique invariant measure up to normalization. It has total mass $\lambda(V) = \infty$; the chain is *null recurrent*, see [2, §I.6] or [14, §§5.2–5.3]. Therefore the convergence theorem for recurrent Markov chains yields that $q_E^{(n)}(e, f) \to 0$ for all $f \in V$. If $f \notin V$, then $q_E^{(n)}(e, f) = 0$ for all n. Since X is by assumption locally finite, Lemma 1.1 yields the result stated in (iii). ∎

Uniformly Irreducible Random Walks and Amenability

We now make a small detour regarding more general random walks on graphs, recalling and improving upon the material in [17, §10.B].

Let X be a locally finite, connected graph with graph metric $d(\cdot\,,\cdot)$, and consider the transition matrix $P = (p(x, y))_{x,y\in X}$ of an arbitrary random walk (Markov chain) on the set X. Then P is called *uniformly irreducible* if there are constants $K, \varepsilon_0 > 0$ such that for any pair of neighbours x, y there is some $k \leq K$ such that $p^{(k)}(x, y) \geq \varepsilon_0$. Furthermore, P is said to have *bounded range,* if there is $R > 0$ such that $p(x, y) > 0$ only if $d(x, y) \leq R$. These two are conditions of adaptedness of P to the graph structure.

If P has an *invariant measure* ν, then it acts on the Hilbert space $\ell^2(X,\nu)$ of all $F\colon X \to \mathbb{R}$ with $\langle F,F\rangle < \infty$, where $\langle F,G\rangle = \sum_x F(x)G(x)\,\nu(x)$. The operator norm satisfies $\|P\| \le 1$, and its ℓ^2-spectral radius is $\rho_2(P) = \lim_n \|P^n\|^{1/n}$. Note that for $\rho(P) = \limsup_n p^{(n)}(x,y)^{1/n}$ (independent of x, y by irreducibility) one has $\rho(P) \le \rho_2(P)$, and equality does not hold in general. The adjoint (more precisely, ν-adjoint) P^* of P on $\ell^2(X,\nu)$ has the stochastic kernel $p^*(x,y) = \nu(y)p(y,x)/\nu(x)$.

Theorem 3.4 *Suppose that X is connected with bounded vertex degrees, and that P is uniformly irreducible with bounded range and has an invariant measure ν satisfying $C^{-1} \le \nu(\cdot) \le C$ for some $C \ge 1$. Then $\rho_2(P) = 1$ if and only if the graph X is amenable.*

Outline of Proof Theorem 10.6 in [17] states that under the given assumptions, $\rho(P) = 1$ implies amenability of X. After the proof of that theorem, it is explained that the condition $\rho(P) = 1$ may be replaced with $\rho_2(P) = 1$.

Conversely, [17, Theorem 10.8] states that amenability of X implies $\|P\| = 1$. Now, let I be the identity operator (or matrix), and fix $n \ge 1$. Set $\bar{P} = \frac{1}{2}(I + P)$. Then $\bar{P}^n$ is uniformly irreducible, has bounded range and invariant measure ν. If X is amenable, then we get that $\|\bar{P}^n\| = 1$. This is true for every n. Consequently, $\rho_2(\bar{P}) = 1$. By basic spectral theory, also $\rho_2(P) = 1$. ■

More generally, the bounded range assumption can be replaced with tightness of the step length distributions of P and P^* as in [17, Theorem 10.8].

We want to apply Theorem 3.4, not to random walks on our "original" graph X, but to edge-NBRW on the OLG. However, the latter is not a graph (with unoriented edges), but a digraph. Therefore, we symmetrize it by "removing the arrows" from its edges. (Recall that the latter are "second order" edges, connecting edges of the original graph X). The resulting symmetrized oriented line graph (SOLG) still has as its vertex set the set E of *oriented* edges of the original graph X, but neighbourhood in the SOLG is given by $e \sim f$, if $e \to f$ or $f \to e$. We observe that in the SOLG, $q_E(e,f) > 0$ implies $e \sim f$, but *not conversely*.

Lemma 3.5 *If $2 \le \deg(x) \le M$ for all $x \in X$, and small cycles are dense in X, then there is $L > 0$ such that for each $e \in E$, we have $e \xrightarrow{*} \check{e}$ in at most L steps of edge-NBRW. In particular, Q_E is uniformly irreducible on the symmetrized OLG.*

Proof We may suppose that X is infinite. Observe that the first statement of the lemma implies uniform irreducibility. Indeed, let f be a neighbour of e in the SOLG. Then either $e^+ = f^-$, in which case $q_E(e,f) \ge 1/(M-1)$, or $f^+ = e^-$, in which case $e \xrightarrow{*} \check{e} \to \check{f} \xrightarrow{*} f$ in $k \le 2L+1$ steps with probability $\ge 1/(M-1)^{2L+1}$.

Now let $R > 0$ be such that $B(x,R)$ contains a cycle for every $x \in X$. Since the vertex degree in X is bounded by M, the number of (oriented) edges in each $B(x,R)$ cannot exceed a certain constant $K = K(M,R)$. Given $e \in E$, by Lemma 3.2 there are infinitely many edges f with $e \xrightarrow{*} f$. Thus, there is a non-backtracking path in X whose first edge is e and whose last edge f is the first one not lying in $B(e^-,R)$.

We may assume that this path has no repeated edges, so that its length (number of edges) is at most $K + 1$. Thus, $e \xrightarrow{*} f$ in at most K steps of the edge-NBRW. By assumption $B(f^+, R)$ contains a cycle C_1 formed by edges $e_1, \dots, e_m$ ($m \le K$). Since $d(e^-, f^+) > R$, neither e nor $\check{e}$ are edges inside the ball $B(f^+, R)$ in X, and consequently neither of the two is among the edges $e_1, \dots, e_m$ of C_1. Now, either $f \xrightarrow{*} e_i$ (case 1) or $\check{f} \xrightarrow{*} e_i$ (case 2) for some $i \in \{1, \dots, m\}$ in at most R steps. If f or $\check{f} = e_i$ for $i \in \{1, \dots, m\}$, then

$$\begin{aligned} & e \xrightarrow{*} f = e_i \to e_{i+1} \to \cdots \to e_m \to e_1 \to \cdots \to e_{i-1} \xrightarrow{*} \check{e} \\ \text{or} \qquad & e \xrightarrow{*} f = \check{e}_i \to \check{e}_{i+1} \to \cdots \to \check{e}_m \to \check{e}_1 \to \cdots \to \check{e}_{i-1} \xrightarrow{*} \check{e}, \end{aligned}$$

respectively, in $\le K + K + K = 3K$ steps. Now let us assume that $f, \check{f} \ne e_i$ for $i \in \{1, \dots, m\}$. Then we have in case 1,

$$e \xrightarrow{*} f \xrightarrow{*} e_i \to e_{i+1} \to \cdots \to e_m \to e_1 \to \cdots \to e_{i-1} \xrightarrow{*} \check{f} \xrightarrow{*} \check{e}$$

in $\le K + R + K + R + K = 2R + 3K$ steps. In case 2 we have to turn off on the way to f to arrive at the cycle C_1. More exactly, let $e = f_0, \dots, f_n = f$ be a walk from e to f in $n \le K$ steps. Now consider a walk from $\check{f} = \check{f}_n$ to e_i in $\le R$ steps. It contains at least one of the edges $\check{f}_n, \check{f}_{n-1}, \dots, \check{f}_1$. Let ℓ be the minimal index such that $\check{f}_\ell$ is not contained in the walk. Then we have

$$e \xrightarrow{*} f_\ell \xrightarrow{*} e_i \to e_{i+1} \to \cdots \to e_m \to e_1 \to \cdots \to e_{i-1} \xrightarrow{*} \check{f}_\ell \xrightarrow{*} \check{e},$$

again in $\le K + R + K + R + K = 2R + 3K$ steps. Thus setting $L = 2R + 3K$ we have $e \xrightarrow{*} \check{e}$ in $\le L$ steps. ■

Proof of Lemma 1.4 If X is not a cycle, then by Lemmas 3.1 and 3.5, Q_E is irreducible and a standard argument (see [14, Theorem 6.1] or [17, §1.B]) yields that

$$\rho(Q_E) = \limsup_n q_E^{(n)}(e, f)^{1/n} \tag{3.3}$$

is independent of $e, f \in E$. Since $1/\rho(Q_E)$ is the radius of convergence of each of the power series with non-negative coefficients $\sum_n q_E^{(n)}(e, f)\, z^n$ ($z \in \mathbb{C}$), where $e, f \in E$, Lemma 1.1 implies

$$\limsup_{n\to\infty} q^{(n)}(x, y)^{1/n} = \rho(Q_E)$$

for all $x, y \in X$, and Lemma 1.4 follows.

If X is a cycle of length k, then (without n-th roots)

$$\limsup_n q^{(n)}(x, y) = \begin{cases} 1 & \text{if } d(x, y) = 0 \text{ or } = k/2, \\ 1/2 & \text{otherwise,} \end{cases}$$

whence $\rho(Q) = 1$ is independent of x, y as well. ■

The fact that $\rho(Q) = \rho(Q_E)$, as stated in (3.3), is immediate from Lemma 1.1 and will be tacitly used several times.

Proof of Theorem 1.5 If X is a tree then for each pair $e, f \in E$ there is at most one n such that $q_E^{(n)}(e, f) > 0$.

Otherwise, X has a cycle, and since it is almost transitive, small cycles are dense in X. By Lemma 1.4, Q_E is irreducible, and the OLG of X is connected. Therefore the series $\sum_n q_E^{(n)}(e, f)/\rho(Q)^n$ either converge for all e, f or diverge for all $e, f \in E$, see [17, §1.B].

In the convergent case, $q_E^{(n)}(e, f)/\rho(Q)^n \to 0$.

In the divergent case, edge-NBRW is *ρ-recurrent*. The automorphism group Γ of X also acts with finitely many orbits on the OLG. Therefore we can apply an adaptation of a result of Guivarc'h [7], see [17, Theorem 7.8 and proof]: it yields that there is a positive function H on E such that $Q_E H = \rho(Q) \cdot H$, and

$$q_H(e, f) = \frac{q_E(e, f)H(f)}{\rho(Q)H(e)}$$

defines a new random walk which is Γ-invariant and recurrent. By [17, Theorem 3.26 and Lemma 3.25], Q_H has an invariant measure which is constant on each Γ-orbit, and consequently has infinite total mass. Therefore, Q_H is null recurrent, and $q_H^{(n)}(e, f) \to 0$ for all e, f. Since

$$q_H^{(n)}(e, f) = \frac{q_E^{(n)}(e, f)H(f)}{\rho(Q)^n H(e)},$$

we find that $q_E^{(n)}(e, f)/\rho(Q)^n \to 0$. ■

A *rough isometry* between two metric spaces (X, d), (X', d') is a mapping $\varphi\colon X \to X'$ with the following properties.

$$\begin{aligned} A^{-1}d(x, y) - A^{-1}B \le d'(\varphi x, \varphi y) \le A\, d(x, y) + B \quad \forall\, x, y \in X,\\ d'(x', \varphi X) \le B \quad \forall\, x' \in X', \end{aligned} \tag{3.4}$$

where $A \ge 1$ and $B \ge 0$. In this case we say that the two spaces are *roughly isometric*.

Proposition 3.6 *If X is a connected graph with $2 \le \deg(x) \le M$ which is not a cycle and has dense small cycles, then it is roughly isometric with its symmetrized oriented line graph.*

Proof Two finite connected graphs are always roughly isometric. Let us assume that X is infinite, with edge set E. Throughout this proof, we write $d_X(\cdot\,,\,\cdot)$ for the graph distance in X, and $d_E(\cdot\,,\,\cdot)$ for the graph distance in the SOLG of X. Define the mapping $\varphi\colon E \to X$ by $\varphi e = e^-$. Evidently, φ is surjective and hence

$$d_X(x, \varphi(E)) = 0 \quad \text{for all } x \in X. \tag{3.5}$$

Now given two vertices x, y in X with $d_X(x, y) = d$, it is clear that two arbitrary edges e, f starting in x and y, respectively, have distance at least d in the SOLG of X. It follows that

$$d_X(\varphi e, \varphi f) \le d_E(e, f). \tag{3.6}$$

On the other hand, we also obtain an upper bound for $d_E(e, f)$. Clearly, if e, f are oriented the "right way" we have $e \overset{*}{\to} f$ in $d_X(e^-, f^-)$ steps. If one of them is oriented the other way, by Lemma 3.5 it takes at most L steps to turn around, *i.e.,* to reach $\check{e}$ from e. Thus we have $e \overset{*}{\to} f$ in at most $2L + d_X(e^-, f^-)$ steps, so that

$$d_E(e, f) - 2L \le d_X(\varphi e, \varphi f). \tag{3.7}$$

Now, setting $A = 1$ and $B = 2L$ and combining (3.5)–(3.7) yields (3.4). ■

Proof of Proposition 1.6 (i) We have $\|Q_E\| = \|Q_E^* Q_E\|^{1/2}$, where the adjoint operator Q_E^* has kernel $q_E^*(e, f) = q_E(f, e)$. Let $F\colon E \to \mathbb{R}$, and let $e \in E$. Then

$$Q_E^* Q_E F(e) = \sum_{f \in E} \sum_{g \in E} q_E(g, e) q_E(g, f)\, F(f).$$

Thus, $Q_E^* Q_E$ is a symmetric stochastic operator which takes a weighted average of all values of F on each of the finite sets $\{f \in E : f^- = e^-\}$, where $e \in E$. Consequently, it has norm 1.

(ii) Instead of Q_E we shall use the new transition operator $\bar{Q}_E = \frac{1}{2}(I_E + Q_E)$, where I_E is the identity operator. Of course, its invariant measure is again the counting measure on E, and $\bar{Q}_E^* = \frac{1}{2}(I_E + Q_E^*)$. If we fix n, then $\bar{Q}_E^{*\,n} \bar{Q}_E^n$ is again doubly stochastic, has finite range, and all its matrix elements are bounded below by those of $c_n Q_E$, where $c_n = n/4^n$. Since Q_E is (uniformly) irreducible by Lemma 3.5, the same holds for $\bar{Q}_E^{*\,n} \bar{Q}_E^n$.

We shall now use the obvious, but crucial relation

$$q_E^{(n)}(e, f) = q_E^{(n)}(\check{f}, \check{e}), \tag{3.8}$$

which also holds for $\bar{Q}_E^n$ in the place of Q_E^n. Lemma 3.5 implies that for every $e \in E$,

$$\bar{q}_E^{(L)}(e, \check{e}) \ge 1/C, \quad \text{where } C = (2M)^L.$$

(M is the upper bound on the vertex degrees.) Therefore, using (3.8),

$$\begin{aligned} \bar{q}_E^{*\,(n)}(e, f) = \bar{q}_E^{(n)}(f, e) = \bar{q}_E^{(n)}(\check{e}, \check{f}) &\le C^2\, \bar{q}_E^{(L)}(e, \check{e})\, \bar{q}_E^{(n)}(\check{e}, \check{f})\, \bar{q}_E^{(L)}(\check{f}, f) \\ &\le C^2 \bar{q}_E^{(n+2L)}(e, f). \end{aligned}$$

In particular, we obtain that $\bar{Q}_E^{*\,n} \bar{Q}_E^n \le C^2\, \bar{Q}_E^{2n+2L}$ matrix-elementwise.

Now, since $\bar{Q}_E^{*\,n} \bar{Q}_E^n$ is symmetric (self-adjoint) and irreducible, Lemma 10.1 in [17] implies that its norm satisfies $\|\bar{Q}_E^{*\,n} \bar{Q}_E^n\| = \rho(\bar{Q}_E^{*\,n} \bar{Q}_E^n)$, the latter number being defined in the same way as in (3.3), but for the powers of $\bar{Q}_E^{*\,n}\bar{Q}_E^n$. Thus, if we take $e \in E$, then

$$\begin{aligned}\rho(\bar{Q}_E^{*\,n}\bar{Q}_E^n) &= \lim_{m\to\infty} \left\langle \left(\bar{Q}_E^{*\,n}\bar{Q}_E^n\right)^m \delta_e, \delta_e \right\rangle^{1/m} \\ &\le \lim_{m\to\infty} C^2 \left\langle \bar{Q}_E^{(2n+2L)m}\delta_e, \delta_e \right\rangle^{1/m} = \lim_{m\to\infty} C^2\, \bar{q}_E^{((2n+2L)m)}(e,e)^{1/m} \\ &\le C^2\, \rho(\bar{Q}_E)^{2n+2L},\end{aligned}$$

since $\bar{q}_E^{(k)}(e,e) \le \rho(\bar{Q}_E)^k$ for all $k \ge 0$ and $e \in E$, a well-known fact, see [14, §6.1] or [17, Lemma 1.9]. We infer that

$$\rho_2(\bar{Q}_E) = \lim_{n\to\infty} \|\bar{Q}_E^{*\,n}\bar{Q}_E^n\|^{1/2n} \le \lim_{n\to\infty} \left(C^2 \rho(\bar{Q}_E)^{2n+2L}\right)^{1/2n} = \rho(\bar{Q}_E).$$

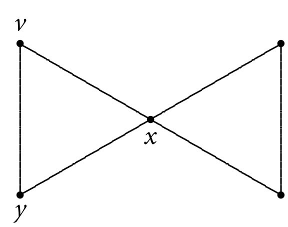
Since $\rho(\bar{Q}_E) = \frac{1}{2}(1+\rho(Q_E))$ and $\rho_2(\bar{Q}_E) = \frac{1}{2}(1+\rho_2(Q_E))$, we conclude that $\rho_2(Q_E) \le \rho(Q_E)$. The reversed inequality is obvious. ■

Proof of Theorem 1.7 It is by now a well-established fact that for connected graphs with bounded vertex degrees, amenability is rough-isometry-invariant. See [17, Theorem 4.7] (the isoperimetric inequality IS_∞ referred to there is the condition $\iota(X) > 0$, *i.e.,* nonamenability), or also the book by de la Harpe [8]. Thus, in view of Proposition 3.6, under the assumptions of Theorem 1.7 the graph X is amenable if and only if its SOLG is amenable. By (3.1), edge-NBRW has the counting measure λ on E as an invariant measure, and by Lemma 3.5, it is uniformly irreducible. Therefore, we can apply Theorem 3.4 to the SOLG, and Proposition 1.6(ii) allows us to replace the ℓ^2-spectral radius with $\rho(Q)$. ■

4 Final Remarks and Observations

Remark 4.1 Regarding Theorem 1.2(i) and (ii), the condition $\deg(x) \ge 3$ in Lemma 3.3 is necessary for the stronger convergence result of (ii), as the following example shows. Thus, if there are vertices of degree ≤ 2 it is in general not true that for vertex-NBRW, one has convergence of $q^{(2n+\delta)}(x,y)$ $(\delta \in \{0,1\})$ or $q^{(n)}(x,y)$ according to whether X is bipartite or not (respectively).

Example 4.1 Consider



Clearly, edge-NBRW has period $\mathfrak{d} = 3$. Write e for the edge from y to x and f for the edge from v to y. We have

$$q^{(3n)}(x,x) = 1 \quad \text{and} \quad q^{(3n+1)}(x,x) = q^{(3n+2)}(x,x) = 0 \quad \forall\, n.$$

For the edges terminating at y, we have $q_E^{(n)}(\check{e}, f) > 0$ only if $n \equiv 1 \mod 3$ and $q_E^{(n)}(f, \check{e}) > 0$ only if $n \equiv 2 \mod 3$, while $q_E^{(n)}(\check{e}, \check{e})$ and $q_E^{(n)}(f, f)$ are > 0 only if $n \equiv 0 \mod 3$. Therefore, using Lemma 1.1 and (3.2),

$$\begin{aligned} q^{(3n)}(y, y) &= \tfrac{1}{2}\left(q_E^{(3n)}(\check{e}, \check{e}) + q_E^{(3n)}(f, f)\right) \to \tfrac{1}{4}, \\ q^{(3n+1)}(y, y) &= \tfrac{1}{2} q_E^{(3n+1)}(\check{e}, f) \to \tfrac{1}{8}, \quad \text{and} \\ q^{(3n+2)}(y, y) &= \tfrac{1}{2} q_E^{(3n+2)}(f, \check{e}) \to \tfrac{1}{8}, \quad \text{as } n \to \infty\,. \end{aligned}$$

Remark 4.2 For regular, almost transitive graphs, Bartholdi [1, Lemma 3.9] states our Proposition 2.1(i)(ii)(iv) and Theorems 1.2 and 1.5. (We remark that in Lemma 3.9 of [1], the identity "$\limsup_n \frac{g_n}{\beta^n} = \limsup_n \frac{f_n}{\alpha^n} = \cdots$" should instead read "$\limsup_n \frac{g_n}{\beta^n} = \frac{d}{d-1} \limsup_n \frac{f_n}{\alpha^n} = \cdots$".) In [1], a proof for SRW is suggested where one starts with the finite case, while for an infinite graph, one takes the sequence of balls $B(o, r)$ around a "root" vertex, applies the "finite" result to each ball, and lets the radius tend to infinity, thereby exchanging two limits. Then [1] suggests to use the same argument for cogrowth. This argument has also found its way into a recent paper of Kapovich et al. [9], which states an extension to arbitrary regular graphs. However, the argument is problematic because it is by no means clear *a priori* that the two limits (for $n, r \to \infty$) may be exchanged.

As a matter of fact, this was the starting point for the present note, since several colleagues asked us how the mentioned argument can be made rigorous. When applied to regular graphs, our method provides a simple and rigorous proof of those statements for infinite graphs.

Remark 4.3 Theorems 1.2 and 1.5 extend the corresponding results of [16] from Cayley graphs to arbitrary graphs. At the same time, the functional equation (2.3) is no longer needed. The extension of the amenability criterion (Theorem 1.7) required more work, since the functional equation (2.3) can be used only in the regular case. Also, in the regular case, that criterion does not require denseness of small circles. However, our result *is* a full generalization of that amenability criterion for (Cayley graphs of) finitely generated groups. Indeed, according to our definition of the Cayley graph, small circles will always be dense in the latter unless the group is freely generated by the generating set that defines the Cayley graph. (Remember that when one of the generators satisfies $a_i = a_i^{-1} \neq id$, it leads to double edges. But double edges give rise to circles of length 2 according to our definition!)

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