Recursion relations for the metric form factors

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This file gives the recursion relations between the form factors $\chi_1(n)$, ..., $\chi_8(n)$ at n-1-loop in D=d+1 dimensions

The initial conditions are the tree-level n=1 and one-loop n=2 coefficients

```
In[ ]:= Quit

In[ ]:= SetDirectory[NotebookDirectory[]]
Out[ ]=
/Users/vanhove/Git/Metric

In[ ]:= Get["Recursion.txt"];

In[ ]:= Recχ1[n_] := χRecursion[n]〚1〛;

In[ ]:= Recχ2[n_] := χRecursion[n]〚2〛;

In[ ]:= Recχ3[n_] := χRecursion[n]〚3〛;

In[ ]:= Recχ4[n_] := χRecursion[n]〚4〛;

In[ ]:= Recχ5[n_] := χRecursion[n]〚5〛;

In[ ]:= Recχ6[n_] := χRecursion[n]〚6〛;
```

In[∘]:=
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Recχ7[n_] := χRecursion[n]⟦7⟧;
```

In[∘]:=
```
Recχ8[n_] := χRecursion[n]⟦8⟧;
```

Some examples of the coefficients of the recursion.
The matrices for recursion have block form with upper triangular matrices

In[223]:=
```
Get["Matrices-recursion.txt"];
```

In[235]:=
```
Style[Mat[1] // MatrixForm, 5]
```

Out[235]=

$$\begin{pmatrix} \frac{(-3+2d)(-1+n)n(2+(-2+d)n)+(-2+d)m^2(5+2(-2+d)n)-(-2+d)mn(5+2(-2+d)n)}{2(-1+d)(-1+n)n(2+(-2+d)n)} & -\frac{(-3+d)(-1+n)n(2+(-2+d)n)+m^2\left(-8+d(2-5n)+6n+d^2n\right)-m\left(4+4(-4+d)n+\left(10-7d+d^2\right)n^2\right)}{2(-1+d)(-1+n)n(2+(-2+d)n)} & \frac{-\left((-2+d)m^3(2+(-2+d)n)\right)+m\left(-4-2\left(4-6d+d^2\right)n-\left(-28+30d-10d^2+d^3\right)n^2+(-4+d)(-2+d)^2n^3\right)+(-1+n)n\left(4(-1+n)n+d^2(-1+n)n+d\left(-2+4n-4n^2\right)\right)+m^2\left(-d^3(-1+n)n+4\left(3-7n+4n^2\right)-4 \right.}{2(-1+d)(-1+n)n(2+(-2+d)n)(2(m-n)+d(-1-m+n))} \\ 0 & -\frac{2m(m-n)}{(-1+d)(-1+n)n(2+(-2+d)n)} & -\frac{m\left(d^2\left(m^2+n-mn-n^2\right)+2\left(-3+m^2+m(3-5n)+3n+n^2\right)+d\left(2-3m^2-5n+n^2+m(-2+7n)\right)\right)}{2(-1+d)(-1+n)n(2+(-2+d)n)(2(m-n)+d(-1-m+n))} \\ 0 & 0 & \frac{(-2+d)^2dm^4-2(-2+d)^2dm^3n+(-1+n)n\left(4(-1+n)n+d^2(-1+n)n+d\left(-2+4n-4n^2\right)\right)+m^2\left(14-34n+d^3n+20n^2+d\left(-19+43n-20n^2\right)+d^2\left(4-15n+5n^2\right)\right)+mn\left(-14+34n+d^3(-1+n)n-20n^2+d^2\right.(-}{2(-1+d)(d(-1+m)-2m)(-1+n)n(-2m+d(1+m-n)+2n)(2+(-2+d)n)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In[237]:=
```
Style[Mat[2] // MatrixForm, 4]
```

Out[237]=

$$\begin{pmatrix} \frac{(-2+d)\left((-1+n)n(2+(-2+d)n)+m^2(4+(-2+d)n)-mn(4+(-2+d)n)\right)}{4(-1+d)(-1+n)n(2+(-2+d)n)} & \frac{(m-n)(2m-(-1+n)(2+(-2+d)n))}{2(-1+d)(-1+n)n(2+(-2+d)n)} & \frac{-\left((-2+d)m^3(2+(-2+d)n)\right)-2m\left(2+(2-3d)n-\left(8-6d+d^2\right)n^2+(-2+d)^2n^3\right)+(-1+n)n\left(4(-1+n)n+d^2(-1+n)n+d\left(-2+4n-4n^2\right)\right)+m^2\left(8(-1+n)^2+d^2n(-1+2n)-2d\left(3-5n+4n^2\right)\right)}{4(-1+d)(-1+n)n(2+(-2+d)n)(2(m-n)+d(-1-m+n))} & 0 & 0 \\ 0 & \frac{-\left((-1+d)(-1+n)n(2+(-2+d)n)\right)-m^2\left(-2+d(2-4n)+4n+d^2n\right)+mn\left(-2+d(2-4n)+4n+d^2n\right)}{(-1+d)(-1+n)n(2+(-2+d)n)} & \frac{\left(2-3d+d^2\right)m^3(1+(-2+d)n)-(-1+d)(-1+n)n\left(4(-1+n)n+d^2(-1+n)n+d\left(-2+4n-4n^2\right)\right)+m^2\left(d^3(2-3n)n+d\left(-8+27n-28n^2\right)+2\left(1-7n+8n^2\right)+2d^2\left(2-7n+8n^2\right)\right)+m\left(6-18n+26n^2-16n^3+d^3n\left(1-4n+3n^2\right)+d^2\left(2-13n+25n^2-16n^3\right)+d\left(-6+29n-47n^2+28n^3\right)\right)}{2(-1+d)(-1+n)n(2+(-2+d)n)(2(m-n)+d(-1-m+n))} & 0 & 0 \\ 0 & 0 & -\frac{(-2+d)^3dm^4n-2(-2+d)^3dm^3n^2+(-3+2d)(-1+n)n\left(4(-1+n)n+d^2(-1+n)n+d\left(-2+4n-4n^2\right)\right)-(-2+d)mn\left(16-38n+22n^2+d^3(-1+n)n^2-2d^2n\left(4-5n+2n^2\right)+d\left(-16+39n-27n^2+4n^3\right)\right)+(-2+d)m^2\left(16-38n+22n^2+d^3n^2(-1+2n)-2d^2n\left(4-5n+4n^2\right)+d\left(-16+39n-27n^2+8n^3\right)\right)}{4(-1+d)(d(-1+m)-2m)(-1+n)n(-2m+d(1+m-n)+2n)(2+(-2+d)n)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{(-2+d)(-1+d)m(m}{d(-1+n)(2+(-2+d)} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[230]:=

```
Mat[3] // MatrixForm
```

Out[230]//MatrixForm=

$$
\begin{pmatrix}
\frac{m (m-n)}{3 n (1+n)} & \frac{m (m-n)}{3 n (1+n)} & -\frac{m (m-n) (-3+n)}{6 (2+m-n) n (1+n)} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-2 m^2+2 m n+3 n (1+n)}{3 n (1+n)} & \frac{3 m^3-2 m^2 (3+2 n)+3 n \left(2+n-n^2\right)+m \left(-3+6 n+4 n^2\right)}{6 (2+m-n) n (1+n)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{8 m^4-16 m^3 n+m n \left(-7+7 n-8 n^2\right)+3 n \left(-2-n+n^2\right)+m^2 \left(7-7 n+16 n^2\right)}{12 (-2+m) (2+m-n) n (1+n)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3 m (m-n)}{2 n (1+n)} & \frac{m (m-n)}{2 n (1+n)} & \frac{m (-m+n)}{n (1+n)} & \frac{m (-2+3 m-3 n) (m-n)}{2 (2+m-n) n (1+n)} \\
0 & 0 & 0 & 0 & 0 & \frac{3 m (m-n)}{2 n (1+n)} & \frac{m^2-m n}{n+n^2} & \frac{m (m-n)}{2 n (1+n)} \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{2 m (m-n)}{n (1+n)} & \frac{m (-2+m-n) (m-n)}{(2+m-n) n (1+n)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{m \left(3 m^3-2 n-6 m^2 n+m \left(2+3 n^2\right)\right)}{2 (-2+m) (2+m-n) n (1+n)}
\end{pmatrix}
$$

In[229]:=

```
Mat[4] // MatrixForm
```

Out[229]//MatrixForm=

$$
\begin{pmatrix}
\frac{3-2 d}{4-4 d} & \frac{2 m-3 n+d n}{4 n-4 d n} & \frac{-\left((-2+d) m^2 \left(-5+d (2-4 n)+4 n+d^2 n\right)\right)+m \left(4+\left(10-9 d+2 d^2\right) n+(-3+d) (-2+d)^2 n^2\right)+n \left(4 (-1+n) n+d^2 (-1+n) n+d \left(-2+4 n-4 n^2\right)\right)}{4 (-1+d) n (2+(-2+d) n) (2 (m-n)+d (-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{m \left(d (2-2 m-3 n)+d^2 n+2 (-3+2 m+n)\right)}{4 (-1+d) n (2+(-2+d) n) (2 (m-n)+d (-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-\left((-2+d) m^2 \left(-6+d (4-6 n)+8 n+d^2 n\right)\right)+(-2+d) m n \left(-6+d (4-6 n)+8 n+d^2 n\right)+n \left(4 (-1+n) n+d^2 (-1+n) n+d \left(-2+4 n-4 n^2\right)\right)}{4 (-1+d) (d (-1+m)-2 m) n (-2 m+d (1+m-n)+2 n) (2+(-2+d) n)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

In[228]:=

```
Mat[5] // MatrixForm
```

Out[228]//MatrixForm=

$$\begin{pmatrix} \frac{1}{4(-1+d)} & \frac{1}{4-4d} & -\frac{(-2+d)\,m\,(m-n)\,(1+(-2+d)\,n)}{4(-1+d)\,n\,(2+(-2+d)\,n)\,(2(m-n)+d(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2(-1+d)} & \frac{2(-2+d)\,m^2\,(1+(-2+d)\,n)+n\left(4(-1+n)\,n+d^2(-1+n)\,n+d\left(-2+4n-4n^2\right)\right)+m\left(d^2\left(n-3n^2\right)-2\left(1-5n+6n^2\right)+d\left(2-7n+12n^2\right)\right)}{4(-1+d)\,n\,(2+(-2+d)\,n)\,(2(m-n)+d(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(-2+d)\,m^2\left(2+\left(-4+d^2\right)n\right)-(-2+d)\,m\,n\left(2+\left(-4+d^2\right)n\right)+n\left(4(-1+n)\,n+d^2(-1+n)\,n+d\left(-2+4n-4n^2\right)\right)}{4(-1+d)\,(d(-1+m)-2m)\,n\,(-2m+d(1+m-n)+2n)\,(2+(-2+d)\,n)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[227]:=

```
Mat[6] // MatrixForm
```

Out[227]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{(-2+d)\,m\,(-m+n)}{4(-1+d)\,n\,(2+(-2+d)\,n)\,(2m-2n+d(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{m\left(-2-4m+d(2+2m-5n)+6n+d^2n\right)}{4(-1+d)\,n\,(2+(-2+d)\,n)\,(2(m-n)+d(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(-2+d)\,m\,(m-n)\left(-2-2dn+d^2n\right)}{4(-1+d)\,(d(-1+m)-2m)\,n\,(-2m+d(1+m-n)+2n)\,(2+(-2+d)\,n)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[226]:=

```
Mat[7] // MatrixForm
```

Out[226]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \frac{(-2+d)\,m\,(m-n)}{4\,(-1+d)\,n\,(2+(-2+d)\,n)\,(2\,(m-n)+d\,(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{m\,(-2-4\,m+2\,d\,(1+m)+6\,n+(-5+d)\,d\,n)}{4\,(-1+d)\,n\,(2+(-2+d)\,n)\,(2\,m-2\,n+d\,(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(-2+d)\,m\,(-m+n)\,(-2+(-2+d)\,d\,n)}{4\,(-1+d)\,(d\,(-1+m)-2\,m)\,n\,(2+(-2+d)\,n)\,(2\,m-2\,n+d\,(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[225]:=

```
Mat[8] // MatrixForm
```

Out[225]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \frac{(-2+d)\,m\,(d\,(-1+n)-2\,n)\,(m-n)}{4\,(-1+d)\,n\,(2+(-2+d)\,n)\,(2\,m-2\,n+d\,(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(d\,(-1+n)-2\,n)\,(-m+n)\,\left(-2-4\,m+d\,(2+2\,m-3\,n)+2\,n+d^2\,n\right)}{4\,(-1+d)\,n\,(2+(-2+d)\,n)\,(2\,(m-n)+d\,(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{(d\,(-1+n)-2\,n)\,\left(-((-1+d)\,n\,(2+(-2+d)\,n))-(-2+d)\,m^2\,(-2+(-2+d)\,d\,n)+(-2+d)\,m\,n\,(-2+(-2+d)\,d\,n)\right)}{4\,(-1+d)\,(d\,(-1+m)-2\,m)\,n\,(2+(-2+d)\,n)\,(2\,m-2\,n+d\,(-1-m+n))} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The initial condition is set the tree-level contribution

In[∘]:=

$$\chi_1[1] = 4;\ \chi_2[1] = 0;\ \chi_3[1] = 0;\ \chi_4[1] = 0;\ \chi_5[1] = 0;\ \chi_6[1] = 0;\ \chi_7[1] = 0;\ \chi_8[1] = 0;$$

At one-loop order one can directly extract the coefficients in D=d+1 dimensions

In[∘]:=
$$\chi_1[2] = \frac{2\,(6 + d\,(-9 + 4\,d))}{(-1 + d)^2};$$
$$\chi_2[2] = \frac{2\,(-2 + d)^2}{(-1 + d)^2};$$
$$\chi_3[2] = -\frac{2\,(-2 + d)^2}{(-1 + d)^2};$$
$$\chi_4[2] = 8 - \frac{4}{-1 + d};$$
$$\chi_5[2] = \frac{4}{-1 + d};$$
$$\chi_6[2] = -4;$$
$$\chi_7[2] = 0;\ \chi_8[2] = 0;$$

The recursion is valid from n>=3

In[∘]:=
```
RecχList[n_] := Table[χr[n] = Simplify[Recχr[n]], {r, 1, 8}];
```

In[∘]:=
```
RecχList[3]
```

Out[∘]=
$$\left\{\frac{16\,(-6 + d)}{3\,(-4 + d)\,(-1 + d)},\ 0,\ 0,\ -\frac{4\,\left(-176 + 656\,d - 874\,d^2 + 529\,d^3 - 146\,d^4 + 15\,d^5\right)}{3\,(-4 + d)\,(-1 + d)^3\,(-4 + 3\,d)},\ -\frac{4\,\left(112 - 216\,d + 138\,d^2 - 33\,d^3 + 3\,d^4\right)}{3\,(-4 + d)\,(-1 + d)^3\,(-4 + 3\,d)},\right.$$
$$\left.-\frac{4\,\left(48 - 104\,d + 66\,d^2 + 5\,d^3 - 16\,d^4 + 3\,d^5\right)}{3\,(-4 + d)\,(-1 + d)^3\,(-4 + 3\,d)},\ \frac{8\,(-2 + d)^3}{3\,(-4 + d)\,(-1 + d)^3\,(-4 + 3\,d)},\ \frac{16\,(-3 + d)\,(-2 + d)^3}{3\,(-4 + d)\,(-1 + d)^3\,(-4 + 3\,d)}\right\}$$

In[ ]:=
```
RecχList[4] // Simplify
```

Out[ ]=

$$\left\{-\frac{2\left(1200-3944\,d+5408\,d^2-3870\,d^3+1499\,d^4-299\,d^5+24\,d^6\right)}{3\,(-4+d)^2\,(-1+d)^4},\ -\frac{6\,(-3+d)\,(-2+d)^5}{(-4+d)^2\,(-1+d)^4},\ \frac{6\,(-3+d)\,(-2+d)^5}{(-4+d)^2\,(-1+d)^4},\right.$$

$$\frac{1056-1584\,d-372\,d^2+2068\,d^3-1551\,d^4+477\,d^5-52\,d^6}{3\,(-4+d)^2\,(-1+d)^4\,(-3+2\,d)},\ \frac{-96+1728\,d-3772\,d^2+3312\,d^3-1413\,d^4+295\,d^5-24\,d^6}{3\,(-4+d)^2\,(-1+d)^4\,(-3+2\,d)},$$

$$\frac{-2976+12\,432\,d-21\,964\,d^2+21\,076\,d^3-11\,769\,d^4+3811\,d^5-664\,d^6+48\,d^7}{3\,(-4+d)^2\,(-1+d)^4\,(-3+2\,d)},$$

$$\left.\frac{(-2+d)^4\left(-26+37\,d-15\,d^2+2\,d^3\right)}{(-4+d)^2\,(-1+d)^4\,(-3+2\,d)},\ \frac{(-2+d)^4\,(1+d)\,(-8+3\,d)}{(-4+d)^2\,(-1+d)^4\,(-3+2\,d)}\right\}$$

In[ ]:=
```
RecχList[5] // Simplify
```

Out[ ]=

$$\left\{-\frac{64\left(-1020+2192\,d-1723\,d^2+629\,d^3-105\,d^4+6\,d^5\right)}{15\,(-4+d)^2\,(-1+d)^3\,(-8+3\,d)},\ 0,\ 0,\right.$$

$$-\Big(\Big(4\left(-20\,736+197\,184\,d-632\,656\,d^2+1\,040\,216\,d^3-1\,007\,272\,d^4+604\,798\,d^5-226\,391\,d^6+50\,983\,d^7-6249\,d^8+315\,d^9\right)\Big)\Big/$$
$$\Big(15\,(-4+d)^2\,(-1+d)^5\,(-8+3\,d)\,(-8+5\,d)\Big)\Big),$$

$$\Big(4\left(59\,136-237\,056\,d+386\,192\,d^2-334\,440\,d^3+168\,680\,d^4-50\,930\,d^5+9099\,d^6-918\,d^7+45\,d^8\right)\Big)\Big/\Big(15\,(-4+d)^2\,(-1+d)^5\,(-8+3\,d)\,(-8+5\,d)\Big),$$

$$\Big(4\left(52\,992-242\,432\,d+450\,448\,d^2-429\,944\,d^3+210\,320\,d^4-32\,698\,d^5-16\,531\,d^6+9543\,d^7-1881\,d^8+135\,d^9\right)\Big)\Big/$$

$$\left.\Big(15\,(-4+d)^2\,(-1+d)^5\,(-8+3\,d)\,(-8+5\,d)\Big),\ -\frac{16\,(-2+d)^5\left(2-9\,d+3\,d^2\right)}{5\,(-4+d)^2\,(-1+d)^5\,(-8+3\,d)\,(-8+5\,d)},\ -\frac{32\,(-2+d)^5\,(-5+2\,d)\left(2-9\,d+3\,d^2\right)}{5\,(-4+d)^2\,(-1+d)^5\,(-8+3\,d)\,(-8+5\,d)}\right\}$$

Solution of the recursion relation in general dimension until n=90
The expressions are large

In[ ]:=
```
Get["listsolutions-90.txt"];
```

In[•]:=
```
(Table[χr[50], {r, 1, 8}] /. listsolution) /. d → 3
```
Out[•]=

$$\left\{8,\ 0,\ 0,\ \frac{70201}{1300},\ \frac{2601}{1300},\ -\frac{209}{52},\ \frac{1}{1300},\ \frac{47}{1300}\right\}$$

The partial general solution for n>=3 for d=3+ϵ

In[•]:=
```
SolPartχ1[n_] := 8 + O[ϵ];
SolPartχ2[n_] := (1 + (-1) ^n) (n - 1) (n - 3) / (2 n (-n + 2)) ϵ + O[ϵ] ^2;
SolPartχ3[n_] := - (1 + (-1) ^n) (n - 1) (n - 3) / (2 n (-n + 2)) ϵ + O[ϵ] ^2;
SolPartχ4[n_] := 4 + n (-1) ^n + (1 + 3 (-1) ^n) / (2 n (n + 2)) + O[ϵ];
SolPartχ5[n_] := 2 + (1 + 3 (-1) ^n) / (2 n (n + 2)) + O[ϵ];
SolPartχ6[n_] := -4 + 1 / n - (n + 1) (1 + 3 (-1) ^n) / (2 n (n + 2)) + O[ϵ];
SolPartχ7[n_] := (1 + 3 (-1) ^n) / (2 n (n + 2)) + O[ϵ];
SolPartχ8[n_] := (1 + 3 (-1) ^n) / (2 n (n + 2)) (n - 3) + O[ϵ];
```

In[•]:=
```
For[r = 1, r ≤ 8, r++, {Print[Table[((χr[n] - SolPartχr[n]) /. d → 3 + ϵ) + O[ϵ] ^2, {n, 3, 5}]]}]
```

$\{O[\epsilon]^1,\ O[\epsilon]^1,\ O[\epsilon]^1\}$

$\{O[\epsilon]^2,\ O[\epsilon]^2,\ O[\epsilon]^2\}$

$\{O[\epsilon]^2,\ O[\epsilon]^2,\ O[\epsilon]^2\}$

$\{O[\epsilon]^1,\ O[\epsilon]^1,\ O[\epsilon]^1\}$

$\{O[\epsilon]^1,\ O[\epsilon]^1,\ O[\epsilon]^1\}$

$\{O[\epsilon]^1,\ O[\epsilon]^1,\ O[\epsilon]^1\}$

$\{O[\epsilon]^1,\ O[\epsilon]^1,\ O[\epsilon]^1\}$

$\{O[\epsilon]^1,\ O[\epsilon]^1,\ O[\epsilon]^1\}$

The metric solution with ρ=G M/r and D=d+1

In[•]:=
```
Clear[h, d]
```

In[∘]:=
```
h[n_] := ρ^n {χ1[n] - χ2[n], χ2[n] (1 + (n - 1) (d + 1 - 3)) / (2 - (n - 1) (d + 1 - 3)), χ2[n] ((n) (d + 1 - 3)) / (2 - (n - 1) (d + 1 - 3))}
```

In[∘]:=
```
Table[h[n] /. listsolution /. d → 3, {n, 1, 15}]
```

Out[∘]=

$$\{\{4\rho, 0, 0\}, \{7\rho^2, \rho^2, \rho^2\}, \{8\rho^3, 0, 0\}, \{8\rho^4, 0, 0\}, \{8\rho^5, 0, 0\}, \{8\rho^6, 0, 0\}, \{8\rho^7, 0, 0\}, \{8\rho^8, 0, 0\}, \{8\rho^9, 0, 0\}, \{8\rho^{10}, 0, 0\}, \{8\rho^{11}, 0, 0\}, \{8\rho^{12}, 0, 0\}, \{8\rho^{13}, 0, 0\}, \{8\rho^{14}, 0, 0\}, \{8\rho^{15}, 0, 0\}\}$$

In[∘]:=
```
-1 + (1 + ρ) ^3 / (1 - ρ) + O[ρ] ^5
```

Out[∘]=

$$4\rho + 7\rho^2 + 8\rho^3 + 8\rho^4 + O[\rho]^5$$

In[∘]:=
```
Table[h[n] /. listsolution, {n, 1, 10}] // Simplify
```

Out[∘]=

$$\Big\{\{4\rho, 0, 0\}, \Big\{\frac{2(-2+3d)\rho^2}{-1+d}, -\frac{2(-2+d)^2\rho^2}{(-4+d)(-1+d)}, -\frac{4(-2+d)^3\rho^2}{(-4+d)(-1+d)^2}\Big\}, \Big\{\frac{16(-6+d)\rho^3}{3(-4+d)(-1+d)}, 0, 0\Big\},$$

$$\Big\{-\frac{2\left(-336+1160d-1368d^2+702d^3-167d^4+15d^5\right)\rho^4}{3(-4+d)^2(-1+d)^3}, \frac{6(-3+d)(-2+d)^5(-5+3d)\rho^4}{(-4+d)^2(-1+d)^4(-8+3d)}, \frac{24(-3+d)(-2+d)^6\rho^4}{(-4+d)^2(-1+d)^4(-8+3d)}\Big\},$$

$$\Big\{-\frac{64\left(-1020+2192d-1723d^2+629d^3-105d^4+6d^5\right)\rho^5}{15(-4+d)^2(-1+d)^3(-8+3d)}, 0, 0\Big\},$$

$$\Big\{\left(4\left(-368\,640+2\,044\,416d-4\,720\,160d^2+6\,017\,600d^3-4\,711\,992d^4+2\,370\,932d^5-772\,376d^6+158\,007d^7-18\,492d^8+945d^9\right)\rho^6\right) /$$
$$\left(45(-4+d)^3(-1+d)^5(-8+3d)\right), -\frac{20(-2+d)^7(-9+5d)\left(-108+147d-64d^2+9d^3\right)\rho^6}{3(-4+d)^3(-1+d)^6(-8+3d)(-12+5d)}, -\frac{40(-2+d)^8\left(-108+147d-64d^2+9d^3\right)\rho^6}{(-4+d)^3(-1+d)^6(-8+3d)(-12+5d)}\Big\},$$

$$\Big\{\left(128\left(-2\,479\,680+10\,489\,968d-19\,116\,744d^2+19\,710\,916d^3-12\,673\,390d^4+5\,260\,560d^5-1\,403\,017d^6+229\,501d^7-20\,454d^8+720d^9\right)\rho^7\right) /$$
$$\left(315(-4+d)^3(-1+d)^5(-8+3d)(-12+5d)\right), 0, 0\Big\},$$

$$\Big\{-\Big(\Big(2\Big(9\,839\,370\,240-79\,243\,794\,432d+286\,616\,733\,696d^2-619\,916\,946\,176d^3+899\,064\,385\,536d^4-927\,803\,860\,416d^5+$$
$$704\,491\,752\,256d^6-400\,800\,095\,616d^7+172\,043\,104\,176d^8-55\,549\,274\,996d^9+13\,301\,009\,436d^{10}-2\,293\,234\,359d^{11}+$$
$$269\,390\,745d^{12}-19\,314\,045d^{13}+637\,875d^{14}\Big)\rho^8\Big) / \left(315(8-3d)^2(-4+d)^4(-1+d)^7(-12+5d)\right)\Big),$$

$$\left(14(-2+d)^9(-13+7d)\left(76\,896-217\,776d+252\,658d^2-153\,881d^3+51\,955d^4-9231d^5+675d^6\right)\rho^8\right) /$$
$$\left(3(8-3d)^2(-4+d)^4(-1+d)^8(-12+5d)(-16+7d)\right),$$

$$\left(112\ (-2+d)^{10}\ \left(76\,896-217\,776\ d+252\,658\ d^2-153\,881\ d^3+51\,955\ d^4-9231\ d^5+675\ d^6\right)\ \rho^8\right)\Big/$$
$$\left(3\ (8-3\ d)^2\ (-4+d)^4\ (-1+d)^8\ (-12+5\ d)\ (-16+7\ d)\right)\Big\},$$
$$\Big\{-\left(\left(512\ \left(14\,537\,698\,560-97\,809\,486\,336\ d+300\,932\,220\,528\ d^2-561\,333\,392\,256\ d^3+709\,289\,388\,056\ d^4-642\,218\,241\,288\ d^5+\right.\right.\right.$$
$$429\,558\,942\,903\ d^6-215\,461\,241\,674\ d^7+81\,334\,357\,596\ d^8-22\,945\,306\,494\ d^9+4\,745\,695\,595\ d^{10}-693\,931\,632\ d^{11}+$$
$$\left.\left.67\,182\,552\ d^{12}-3\,789\,720\ d^{13}+90\,720\ d^{14}\right)\ \rho^9\right)\Big/\left(2835\ (8-3\ d)^2\ (-4+d)^4\ (-1+d)^7\ (-12+5\ d)\ (-16+7\ d)\right)\Big),\ 0,\ 0\Big\},$$
$$\Big\{\left(4\ \left(73\,251\,363\,225\,600-747\,595\,647\,221\,760\ d+3\,525\,719\,701\,635\,072\ d^2-10\,234\,667\,807\,698\,944\ d^3+20\,534\,142\,343\,258\,112\ d^4-\right.\right.$$
$$30\,289\,527\,194\,921\,472\ d^5+34\,093\,605\,377\,577\,216\ d^6-29\,980\,916\,901\,846\,912\ d^7+20\,904\,867\,345\,334\,080\ d^8-$$
$$11\,656\,711\,176\,148\,608\ d^9+5\,215\,267\,705\,005\,504\ d^{10}-1\,869\,360\,948\,530\,664\ d^{11}+533\,242\,880\,747\,556\ d^{12}-119\,492\,170\,871\,546\ d^{13}+$$
$$\left.\left.20\,584\,255\,829\,505\ d^{14}-2\,631\,598\,861\,284\ d^{15}+235\,217\,751\,930\ d^{16}-13\,120\,737\,210\ d^{17}+343\,814\,625\ d^{18}\right)\ \rho^{10}\right)\Big/$$
$$\left(14\,175\ (8-3\ d)^2\ (-4+d)^5\ (-1+d)^9\ (-12+5\ d)\ (-16+7\ d)\right),\ -\Big(\Big(36\ (-2+d)^{11}\ (-17+9\ d)$$
$$\left(4\,233\,600-16\,769\,760\ d+28\,696\,376\ d^2-27\,720\,170\ d^3+16\,541\,097\ d^4-6\,246\,684\ d^5+1\,458\,702\ d^6-192\,666\ d^7+11\,025\ d^8\right)\ \rho^{10}\Big)\Big/$$
$$\left(5\ (8-3\ d)^2\ (-4+d)^5\ (-1+d)^{10}\ (-12+5\ d)\ (-16+7\ d)\ (-20+9\ d)\right)\Big),$$
$$-\Big(\Big(72\ (-2+d)^{12}\ \left(4\,233\,600-16\,769\,760\ d+28\,696\,376\ d^2-27\,720\,170\ d^3+16\,541\,097\ d^4-6\,246\,684\ d^5+1\,458\,702\ d^6-192\,666\ d^7+11\,025\ d^8\right)$$
$$\rho^{10}\Big)\Big/\left((8-3\ d)^2\ (-4+d)^5\ (-1+d)^{10}\ (-12+5\ d)\ (-16+7\ d)\ (-20+9\ d)\right)\Big)\Big\}\Big\}$$

In[ ]:=
```
Sum[ρ^n Simplify[(χ₁[n] - χ₂[n]) /. listsolution], {n, 1, 10}]
```

Out[ ]=

$$4\,\rho+\frac{(-4+6\,d)\,\rho^2}{-1+d}+\frac{16\,(-6+d)\,\rho^3}{3\,(-4+d)\,(-1+d)}-$$

$$\frac{2\,\left(-336+1160\,d-1368\,d^2+702\,d^3-167\,d^4+15\,d^5\right)\rho^4}{3\,(-4+d)^2\,(-1+d)^3}-\frac{64\,\left(-1020+2192\,d-1723\,d^2+629\,d^3-105\,d^4+6\,d^5\right)\rho^5}{15\,(-4+d)^2\,(-1+d)^3\,(-8+3\,d)}+$$

$$\left(4\,\left(-368\,640+2\,044\,416\,d-4\,720\,160\,d^2+6\,017\,600\,d^3-4\,711\,992\,d^4+2\,370\,932\,d^5-772\,376\,d^6+158\,007\,d^7-18\,492\,d^8+945\,d^9\right)\rho^6\right)\Big/$$
$$\left(45\,(-4+d)^3\,(-1+d)^5\,(-8+3\,d)\right)+$$

$$\left(128\,\left(-2\,479\,680+10\,489\,968\,d-19\,116\,744\,d^2+19\,710\,916\,d^3-12\,673\,390\,d^4+5\,260\,560\,d^5-1\,403\,017\,d^6+229\,501\,d^7-20\,454\,d^8+720\,d^9\right)\rho^7\right)\Big/$$
$$\left(315\,(-4+d)^3\,(-1+d)^5\,(-8+3\,d)\,(-12+5\,d)\right)-$$

$$\left(2\,\left(9\,839\,370\,240-79\,243\,794\,432\,d+286\,616\,733\,696\,d^2-619\,916\,946\,176\,d^3+899\,064\,385\,536\,d^4-927\,803\,860\,416\,d^5+\right.\right.$$
$$704\,491\,752\,256\,d^6-400\,800\,095\,616\,d^7+172\,043\,104\,176\,d^8-55\,549\,274\,996\,d^9+13\,301\,009\,436\,d^{10}-2\,293\,234\,359\,d^{11}+$$
$$\left.\left.269\,390\,745\,d^{12}-19\,314\,045\,d^{13}+637\,875\,d^{14}\right)\rho^8\right)\Big/\left(315\,(8-3\,d)^2\,(-4+d)^4\,(-1+d)^7\,(-12+5\,d)\right)-$$

$$\left(512\,\left(14\,537\,698\,560-97\,809\,486\,336\,d+300\,932\,220\,528\,d^2-561\,333\,392\,256\,d^3+709\,289\,388\,056\,d^4-642\,218\,241\,288\,d^5+429\,558\,942\,903\,d^6-\right.\right.$$
$$\left.215\,461\,241\,674\,d^7+81\,334\,357\,596\,d^8-22\,945\,306\,494\,d^9+4\,745\,695\,595\,d^{10}-693\,931\,632\,d^{11}+67\,182\,552\,d^{12}-3\,789\,720\,d^{13}+90\,720\,d^{14}\right)$$
$$\left.\rho^9\right)\Big/\left(2835\,(8-3\,d)^2\,(-4+d)^4\,(-1+d)^7\,(-12+5\,d)\,(-16+7\,d)\right)+$$

$$\left(4\,\left(73\,251\,363\,225\,600-747\,595\,647\,221\,760\,d+3\,525\,719\,701\,635\,072\,d^2-10\,234\,667\,807\,698\,944\,d^3+20\,534\,142\,343\,258\,112\,d^4-\right.\right.$$
$$30\,289\,527\,194\,921\,472\,d^5+34\,093\,605\,377\,577\,216\,d^6-29\,980\,916\,901\,846\,912\,d^7+20\,904\,867\,345\,334\,080\,d^8-$$
$$11\,656\,711\,176\,148\,608\,d^9+5\,215\,267\,705\,005\,504\,d^{10}-1\,869\,360\,948\,530\,664\,d^{11}+533\,242\,880\,747\,556\,d^{12}-119\,492\,170\,871\,546\,d^{13}+$$
$$\left.\left.20\,584\,255\,829\,505\,d^{14}-2\,631\,598\,861\,284\,d^{15}+235\,217\,751\,930\,d^{16}-13\,120\,737\,210\,d^{17}+343\,814\,625\,d^{18}\right)\rho^{10}\right)\Big/$$
$$\left(14\,175\,(8-3\,d)^2\,(-4+d)^5\,(-1+d)^9\,(-12+5\,d)\,(-16+7\,d)\right)$$