Picard-Fuchs Equations of Twisted Differential forms associated to Feynman Integrals

Pierre Vanhove

ABSTRACT. Dimensionally or analytically regulated Feynman integrals lead to relative twisted period integrals. We present a recent extension of the Griffiths-Dwork pole reduction algorithm for deriving the D-module of differential operators acting on the twisted differential forms from Feynman integrals.

We illustrate the application of this algorithm by providing twisted Picard-Fuchs operators for hypergeometric, elliptic and Calabi-Yau differential motives arising from families of Feynman integrals.

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Contents

Part 1. Feynman integrals in Physics

Feynman integrals are a cornerstone in understanding fundamental interactions and the elementary building blocks of nature. Scattering amplitudes are used in particle physics to compare theoretical predictions with experimental measurements in particle colliders (see [1–3] for instance), more recently to gravitational wave physics [4–6], or the evaluation of the correlation functions of quantum fields at the end of inflation as they provide tools for analysing the formation of structure in the Universe [7,8]. Their accurate calculation, whether analytically or numerically, is needed for precision physics, but this remains a significant hurdle.

There are growing evidences [9–25] that Feynman integrals needed for precision physics correspond to (relative) period integrals of (singular) Calabi–Yau geometries.

Given a family of Feynman integrals attached to a Feynman graph, it is desirable to answer the questions:

(1) What is the class of functions to which belongs a given Feynman integral.
(2) Determine the complete set of partial differential operators acting on a given family Feynman integrals.

Feynman integrals satisfy several remarkable important properties:

(1) They are D-finite functions, that is they satisfy a differentiable module of partial differential equations with respect to their parameters [26, 27].
(2) Feynman integrals are relative period integrals of a variation of mixed Hodge structures [9, 28].

These two properties set the question of analysing the nature of Feynman integral in a clear Hodge theoretical framework. This approach has been developed in [9–11, 17, 28–36].

We recall, in section 2, how Feynman integrals and motives are attached to a graph, and we make explicit the appearance of twisted differential in section. In section 3, we explain the extension of the Griffths-Dwork reduction for the specific twisted Feynman differentials detailled in [37]. In section 4, we make explicit twisted differential operators from Feynman integrals: hypergeometric differential operator in section 4.1, hyperelliptic and elliptic differential operators in section 4.2, and Calabi–Yau differential operators in section 4.3.

Part 2. Hodge structures of Feynman integrals

2.1. Feynman Graph polynomials

Definition 2.1. A *Feynman* graph Γ is a finite collection of vertices $V(\Gamma)$, edges $E(\Gamma)$, and half-edges $H(\Gamma)$ satisfying the usual definitions; edges are adjacent to two vertices, and half-edges are adjacent to a single vertex, and allowing multiple edges between pairs of vertices.

We let $e(\Gamma) = |E(\Gamma)|$ the number of edges. To each edge of Γ we attach a mass variable $m_e \in \mathbb{R}$ and to each half-edge we attach a momentum vector $p_h \in \mathbb{R}^{1,D-1}$ in the D-dimensional Minkowski space equipped with a metric of signature $(1, D-1)$. To each half edge of Γ attach a vector $p_h \in \mathbb{C}^D$ subject to the so-called momentum conservation relation

$$\sum_{h \in H(\Gamma)} p_h = 0. \tag{2.1}$$

We assume that each vertex of Γ has a single outgoing half-edge. Therefore, one may view Γ as a graph in the usual sense, allowing multiple edges between vertices. For physical processes these vectors belong to of the D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$. To simplify notation, we view momenta as being attached to vertices, and write p_v instead of p_h. Furthermore, we consider only the completely massive case with $m_e^2 > 0$ and all external vectors are of non-zero norm $p_v \cdot p_v \neq 0$. We take m_e, p_v as having complex values. The analytic properties of the Feynman integrals are studied by using analytic continuation in the multi-dimensional complex plane spanned by the number of independent scalar products $p_i \cdot p_j$ with $1 \leq i, j \leq |H(\Gamma)|$ and the masses m_i^2 with $1 \leq i \leq e(\Gamma)$.

We associate to the graph Γ two polynomials which are defined as follows [38, 39]. Let $\{x_e \mid e \in e(\Gamma)\}$ be variables attached to all edges of Γ. A spanning tree of Γ is a subgraph T of Γ which contains all vertices of Γ, and so that $b_1(\mathsf{T}) = 0$ and $b_0(\mathsf{T}) = 1$. For each spanning tree T of Γ we attach the monomial $x^{\mathsf{T}} = \prod_{e \notin \mathsf{T}} x_e$. The *first Symanzik polynomial* is the polynomial

$$\mathscr{U} = \sum_{\substack{\text{Spanning}\\ \text{trees of } \Gamma}} x^{\mathsf{T}} . \tag{2.2}$$

A spanning k-forest of Γ is a subgraph F of Γ containing all vertices of Γ and so that $h_1(\mathsf{F}) = 0$ and $h_0(\mathsf{F}) = k$. We attach the polynomial $x^{\mathsf{F}} = \prod_{e \notin \mathsf{F}} x_i$ to each spanning 2-forest. A 2-forest is a disjoint union of two sub-trees $\mathsf{F} = \mathsf{T}_1 \cup \mathsf{T}_2$, and we define $s_{\mathsf{F}} = \sum_{(v_1,v_2) \in \mathsf{F} = \mathsf{T}_1 \cup \mathsf{T}_2} p_{v_1} \cdot p_{v_2}$ Where the $\cdot$-product is the scalar product on $\mathbb{C}^D$. Then

$$\mathscr{V}(\vec{s}, \vec{m}; D) = \sum_{\substack{\text{Spanning}\\ \text{2-forests of } \Gamma}} s_{\mathsf{F}} x^{\mathsf{F}}, \quad \mathscr{F}(\vec{s}, \vec{m}; D) = \mathscr{U} \times \left(\sum_{e \in E(\Gamma)} m_e^2 x_e \right) - \mathscr{V}(\vec{s}, \vec{m}; D) . \tag{2.3}$$

The polynomial $\mathscr{F}(\vec{s}, \vec{m}; D)$ is called the *second Symanzik polynomial* of Γ, depends the mass parameters and kinematic invariants, respectively:

$$\vec{m} := \left\{ m_1^2, \ldots, m_{e(\Gamma)}^2 \right\} \in \mathbb{R}_{>0}^{e(\Gamma)}, \qquad \vec{s} = \{p_i \cdot p_j, i, j \in v(\Gamma)\}. \tag{2.4}$$

When $|v(\Gamma)| > D$, not all the scalar products are independent, and the number of independent variables satisfies certain Gram determinant conditions [40]. The discriminant locus of $\mathscr{F}$ depends on the linear relations between the scalar products in $\vec{s}$. This polynomial is a homogeneous polynomial of degree $L+1$ in the variables x_e for $e \in e(\Gamma)$, where $L = b_1(\Gamma)$. This L is often called the *loop order* of Γ. Henceforward, we will write instead $\mathscr{F}$ to simplify our notation.

2.2. Feynman integrals in parametric representation

Using the two polynomials associated to a graph Γ, we define a family of Feynman integrals [39, 41]

$$I_\Gamma(\underline{z}; D, \underline{\nu}) = \int_{[0,+\infty[^{e(\Gamma)}} \frac{\mathscr{U}^{\nu-(l+1)\frac{D}{2}}}{\mathscr{F}^{\nu-l\frac{D}{2}}} \, \delta\left(\sum_{i=1}^{e(\Gamma)} x_i - 1 \right) \prod_{i=1}^{e(\Gamma)} x_i^{\nu_i - 1} dx_i \, . \tag{2.5}$$

where we have set $\underline{\nu} := (\nu_1, \ldots, \nu_{e(\Gamma)})$ and collected the kinematic factors (the internal masses m_i and the independent scalar products between the external momenta) into $\underline{z} = (\vec{s}, \vec{m})$.

Since the coordinate scaling $(x_1, \ldots, x_{e(\Gamma)}) \to \lambda(x_1, \cdots, x_{e(\Gamma)})$ leaves invariant the integrand and the domain of integration, we can rewrite this integral as

$$I_\Gamma(\underline{z}; D, \underline{\nu}) = \int_{\Delta_{e(\Gamma)}} \Omega_\Gamma^{D,\underline{\nu}} \tag{2.6}$$

with

$$\Omega_\Gamma^{D,\underline{\nu}} := \frac{\mathscr{U}^{\nu-(l+1)\frac{D}{2}}}{\mathscr{F}^{\nu-l\frac{D}{2}}} \prod_{e\in e(\Gamma)} x_e^{\nu_e-1}\,\Omega_0 \tag{2.7}$$

with the differential $e(\Gamma) - 1$-form

$$\Omega_0 := \sum_{j=1}^{e(\Gamma)} (-1)^{j-1} x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_{e(\Gamma)}, \tag{2.8}$$

where $\widehat{dx_j}$ means that dx_j is omitted. The domain of integration is defined as

$$\Delta_{e(\Gamma)} := \left\{ [x_1, \ldots, x_{e(\Gamma)}] \in \mathbb{P}^{e(\Gamma)-1} | x_i \in \mathbb{R}, x_i \geq 0 \right\}. \tag{2.9}$$

2.3. Mixed Hodge structures for Feynman graph integrals

We define the vanishing loci for the Symanzik polynomials attached to the graph Γ:

$$X_{\Gamma;D} = \{\mathscr{F}(\vec{s}, \vec{m}; D) = 0 | x_i \in \mathbb{P}^{e(\Gamma)-1}(\mathbb{R})\}; \qquad Y_\Gamma = \{\mathscr{U} = 0 | x_i \in \mathbb{P}^{e(\Gamma)-1}(\mathbb{R})\}. \tag{2.10}$$

Notice that the vanishing locus for $\mathscr{F}$ depends on the space-time dimension D through the linear relations between the external momenta.

The integrand of the Feynman integral (2.6) is a differential form representing a class of $\mathrm{H}^{e(\Gamma)-1}(\mathbb{P}^{e(\Gamma)-1} - Z_{\Gamma;D})$ where $Z_{\Gamma;D}$ is the singular locus of the integrand. We see that if $e(\Gamma) - \frac{(L+1)D}{2} < 0$, $Z_{\Gamma;D} = Y_\Gamma$ and that if and $e(\Gamma) - \frac{LD}{2} > 0$ then $Z_{\Gamma;D} = X_{\Gamma;D}$. If neither of these inequalities is satisfied, then $Z_{\Gamma;D} = X_{\Gamma;D} \cup Y_\Gamma$.

Although the integrand $\Omega_\Gamma^{D,\underline{\nu}}$ is a closed form such that $\eta \in \mathrm{H}^{e(\Gamma)-1}(\mathbb{P}^{e(\Gamma)-1} - Z_{\Gamma;D})$, in general the domain $\Delta_{e(\Gamma)}$ has a boundary and therefore its homology class is not in $\mathrm{H}_{e(\Gamma)-1}(\mathbb{P}^{e(\Gamma)-1} - Z_{\Gamma;D})$. This difficulty is resolved by considering the relative cohomology [28, 32].

We need to consider a blow-up in $\mathbb{P}^{e(\Gamma)-1}$ of linear space $f : \mathscr{P} \to \mathbb{P}^{e(\Gamma)-1}$, such that all the vertices of $\Delta_{e(\Gamma)}$ lie in $\mathscr{P}\backslash\mathscr{X}$ where $\mathscr{X}$ is the strict transform of $Z_{\Gamma;D}$. Let $\mathscr{B}$ be the total inverse image of the coordinate simplex $\{x_1x_2\cdots x_{e(\Gamma)} = 0 | [x_1, \ldots, x_{e(\Gamma)}] \in \mathbb{P}^{e(\Gamma)-1}\}$.

As been explained by Bloch, Esnault and Kreimer in [28] all of this lead to the mixed Hodge structure associated to the Feynman graph

$$M(\Gamma) := H^{e(\Gamma)-1}(\mathscr{P}\backslash\mathscr{X}, \mathscr{B}\backslash\mathscr{B} \cap \mathscr{X}; \mathbb{Q}). \tag{2.11}$$

2.4. Regulated Feynman integrals and twisted differential forms

The Feynman integral defined in (2.6) is a function of the parameters D and $\underline{\nu}$. For integer values of D and the powers ν_i the integral can be divergent. There are the ultaviolet or infrared divergences which have a special meaning in quantum field theory. We refer to e.g. [42] for a physics based discussion.

One can prove that the Feynman integral is a meromorphic function of $(D, \underline{\nu})$ in $\mathbb{C}^{e(\Gamma)+1}$, with simple poles located on affine hyperplanes defined by linear equations $c_0 D + \sum_{r=1}^{e(\Gamma)} c_r \nu_r$ with integer coefficients $(c_0, c_1, \ldots, c_{e(\Gamma)}) \in \mathbb{Z}^{e(\Gamma)+1}$. One can show that there is an open subset of $(D, \nu_1, \ldots, \nu_{e(\Gamma)}) \in \mathbb{C}^{e(\Gamma)+1}$ where the integral converges. The (unique) value of the Feynman integral is defined by analytic continuation. We refer to [43] for a throughouly discussion.

Using these properties there are two commonly used regularisation in physics:

(1) The dimensional regularisation where the analytic continuation is done in the spacetime D with fixed values for the indices $\underline{\nu}$ taken to be integers.
(2) The analytic continuation with a fixed integer value for the spacetime dimension D, but the analytic continuation is done with respect to (a subset) of the indices $\underline{\nu}$.

In the following we combine both of these regulators, and we introduce the following notations

$$I_\Gamma^{\epsilon,\kappa}(\underline{z}) := I_\Gamma(\underline{z}; 2\delta - 2\epsilon, \mathsf{v}_1 + \mu_1\kappa, \ldots, \mathsf{v}_{e(\Gamma)} + \mu_{e(\Gamma)}\kappa) \tag{2.12}$$

with δ an integer spacetime dimension of interest (say for instance $\delta = 2$), and $(\mathsf{v}_1, \ldots, \mathsf{v}_{e(\Gamma)}, \mu_1, \ldots, \mu_{e(\Gamma)}) \in \mathbb{Z}^{2e(\Gamma)}$, and ϵ and κ are positive real numbers. We set

$$I_\Gamma^{\epsilon,\kappa}(\underline{z}) = \int_{\Delta_{e(\Gamma)}} \Omega_\Gamma^{\epsilon,\kappa}, \tag{2.13}$$

with

$$\Omega_\Gamma^{\epsilon,\kappa} = \omega_\Gamma^{\mathrm{Rat}} \times \left(\frac{\mathscr{U}^{L+1}}{\mathscr{F}^L}\right)^{\epsilon} \prod_{i=1}^{e(\Gamma)} \left(\frac{x_i \mathscr{U}}{\mathscr{F}}\right)^{\mu_i\kappa} dx_1 \cdots dx_{e(\Gamma)}, \tag{2.14}$$

with the rational function

$$\omega_\Gamma^{\mathrm{Rat}} = \frac{\mathscr{U}^{\mathsf{v}_1 + \cdots + \mathsf{v}_{e(\Gamma)} - (L+1)\delta}}{\mathscr{F}^{\mathsf{v}_1 + \cdots + \mathsf{v}_{e(\Gamma)} - L\delta}} \prod_{i=1}^{e(\Gamma)} x_i^{\mathsf{v}_i - 1} \tag{2.15}$$

We make a few remarks:

(1) When $\epsilon = \kappa = 0$ we have that $\Omega_\Gamma^{0,0} = \omega_\Gamma^{\mathrm{Rat}}$ is a rational differential form.
(2) We remark that $\mathscr{U}^{L+1}/\mathscr{F}^L$ and $x_i\mathscr{U}/\mathscr{F}$ are degree zero rational functions in $\mathbb{P}^{e(\Gamma)-1}$, therefore when $\epsilon \neq 0$ or $\kappa \neq 0$ we have well-defined twisted differential forms.
(3) The twists in (2.14) does not introduce new poles. Therefore, the twisted differential has the *same* singular locus as the rational differential form (2.15).

Feynman integrals naturally lead to twisted differential forms of the kind studied in [44–47], which has been considered in recent applications, e.g. [48–50], for reducing the families of Feynman integrals attached to a given graph onto a basis of integrals.

In the present text will discuss a different approach aimed to derive the differential operators acting on the Feynman integrals [37]. Our approach uses that the twist for Feynman integral is a rational function build from the graph polynomials.

Part 3. D-modules for twisted differential forms

An important property of Feynman integral is that they are holonomic functions, which means that they satisfy finite order differential equation when differentiating with respect to their physical parameters $\underline{z} := \{\vec{s}, \vec{m}\}$.

Let us consider r parameters from the set of internal masses and independent kinematics, $\underline{z} := \{z_1, \dots, z_r\} \in \vec{m} \cup \vec{s}$. We seek differential operators annihilating the twisted differential form $\Omega_\Gamma^{\epsilon,\kappa}$ in (2.14) in cohomology

$$\left(\sum_{a_1=0}^{o_1} \cdots \sum_{a_r=0}^{o_r} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}\right) \Omega_\Gamma^{\epsilon,\kappa} = d\beta_\Gamma^{\epsilon,\kappa}, \tag{3.1}$$

where $c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa)$ are rational functions of the physical parameters, but they are independent of the edge variables $x_1, \dots, x_{e(\Gamma)}$, and $o_1, \dots, o_r$ are some positive integers. The inhomogeneous term $d\beta_\Gamma^{\epsilon,\kappa}$ is a total derivative in x_i's where the only allowed poles are those already present in $\Omega_\Gamma^{\epsilon,\kappa}$ [51]. Because the domain of integration (2.9) of the Feynman integral does not depend on the physical parameters, we then deduce

$$\left(\sum_{a_1=0}^{o_1} \cdots \sum_{a_r=0}^{o_r} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}\right) I_\Gamma^{\epsilon,\kappa} = \mathscr{S}_\Gamma^{\epsilon,\kappa}\,, \tag{3.2}$$

where $\mathscr{S}_\Gamma^{\epsilon,\kappa}$ is an inhomogeneous term obtained by integrating $d\beta_\Gamma^{\epsilon,\kappa}$ over the boundary of the positive orthant (2.9). This is a non-trivial task because one needs to blow-up the intersections between the graph hypersurface and the domain of integration, so the integral is well-defined [9, 11, 28, 52]. For instance, section 3.2 of [11] gives a detailed derivation of the inhomogeneous term for the two-loop sunset integral along these lines.

If the integration is done over a cycle $\mathcal{C}$, like the one defined by the torus $\mathcal{C}_{\max} := \{|x_1| = \cdots = |x_{e(\Gamma)}| = 1\}$, the resulting integral is annihilated by the action of the differential operator [53]

$$\left(\sum_{a_1=0}^{o_1} \cdots \sum_{a_r=0}^{o_r} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}\right) \int_{\mathcal{C}_{\max}} \Omega_\Gamma^{\epsilon,\kappa} = 0\,. \tag{3.3}$$

The ideal generated by these differential operators is a differential module (or D-module). Thus, the differential equations we are seeking can be obtained by deriving annihilators of $\Omega_\Gamma^{\epsilon,\kappa}$, i.e., partial differential operators that annihilate the integrand by acting on the physical parameter and the edge variables. An example of a system of partial differential equations for Feynman integrals is the Gel'fand-Kapranov-Zelevinskiĭ (GKZ) system, which provides a D-module of differential operators acting on the toric generalisation of the Feynman integral [15, 53–60]. Because the graph polynomials in the expression for $\Omega_\Gamma^{\epsilon,\kappa}$ in (2.14) are not generic polynomials the differential module acting on a given Feynman integral is obtained after restricting the GKZ D-module which is a higher non-trivial task [54, 61–63] and its systematic implementation is still an open problem. In the following sections we present an algorithmic procedure to derive the differential equations based on an extension of the Griffiths-Dwork reduction for twisted differential forms.

A motivation is to have an algorithm that applies to a large class of analytic regularised Feynman integrals which is missing for the commonly used programs in theoretical physics. This way we can analyse how the twists parameters ϵ, from space-time dimension, and κ, from the analytic regulator, deform the minimal order of the differential operators.

3.1. Variation of mixed Hodge structure and ODEs

If $\mathcal{H}_{\mathbb{Q}}$ is the local system underlying a variation of mixed Hodge structure over a 1-dimensional base M, and $\boldsymbol{s}$ is a meromorphic section of $\mathcal{H}_{\mathbb{Q}} \otimes \mathcal{O}_M$ then there is a minimal differential equation $\mathscr{L}_{\boldsymbol{s}}$ annihilating the period functions attached to $\boldsymbol{s}$.

Selecting a parameter t amongst the physical parameters $\vec{m} \cup \vec{s}$ we consider a pencil of graph $\mathscr{F}_\Gamma(t)$ and the differential form

$$\Omega_\Gamma^{D,\underline{\nu}}(t) = \frac{\mathscr{U}^{\sum_{i=1}^{e(\Gamma)} \nu_i - (L+1)D/2}}{(\mathscr{F}_\Gamma(t))^{\sum_{i=1}^{e(\Gamma)} \nu_i - LD/2}} \prod_{i=1}^{e(\Gamma)} x_i^{\nu_i - 1}\, \Omega_0, \tag{3.4}$$

determines a section of $\mathcal{H}_{\Gamma;D} \otimes \mathcal{O}_M$ where $\mathcal{H}_{\Gamma;D}$ is a variation of mixed Hodge structure over an open subset M of $\mathbb{A}_t^1$.

Definition 3.1. Let $\mathscr{L}_\Gamma^D$ denote the minimal differential operator in $\mathbb{C}[M]\langle \partial_t \rangle$ which annihilates the form $\Omega_\Gamma^{D,\underline{\nu}}(t)$ in $\mathcal{H}_{\Gamma;D} \otimes \mathcal{O}_M$.

We recall how an ordinary differential equation is associated with a variation of mixed Hodge structure along with a holomorphic section of the underlying local system.

Definition 3.2. A ($\mathbb{Q}$-)variation of mixed Hodge structure of weight n consists of several pieces of data

(1) A $\mathbb{Q}$-local system $\mathcal{H}_{\mathbb{Q}}$ over a complex manifold M,
(2) An increasing weight filtration by $\mathbb{Q}$-local systems $\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \cdots \subseteq \mathcal{W}_{2n} = \mathcal{H}_{\mathbb{Q}}$,
(3) A decreasing Hodge filtration $\mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \cdots \subseteq \mathcal{F}^0 = \mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{Q}} \otimes_{\underline{\mathbb{Q}}_M} \underline{\mathbb{C}}_M$,
(4) A flat connection $\nabla : \mathcal{H}_{\mathbb{C}} \otimes \mathcal{O}_M \to \mathcal{H}_{\mathbb{C}} \otimes \Omega_M^1$ so that $\nabla(\mathcal{F}^i) \subseteq \mathcal{F}^{i-1}$,

so that on each fibre $\mathcal{H}_{\mathbb{Q}}$, the data $(\mathcal{H}_{\mathbb{Q}}, \mathcal{F}_t^\bullet, \mathcal{W}_\bullet)$ is a mixed Hodge structure.

Given a local section $\boldsymbol{s}$ of $\mathcal{H}_{\mathbb{C}} \otimes \mathcal{O}_M$, and a local parameter t on M, we can construct local (or multivalued) period functions

$$\boldsymbol{\pi}_{\boldsymbol{s}}(t) = \langle \boldsymbol{s}, \gamma_t \rangle \tag{3.5}$$

for a flat section γ_t of $\mathcal{H}_{\mathbb{Q}}^\vee$. We will often take $\boldsymbol{s} = \Omega_\Gamma^{D,\underline{\nu}}(t)$ and let $\mathcal{H}_{\mathbb{Q}}^\vee$ is the homology bundle underlying the family of varieties $\mathbb{P}^{e(\Gamma)-1} - X_{\Gamma;D}(t)$, in which case the pairing is integration.

Given a variation of mixed Hodge structure, $(\mathcal{H}_{\mathbb{Q}}, \mathcal{W}_\bullet, \mathcal{F}^\bullet)$ over $M \subseteq \mathbb{A}^1$ with Gauss–Manin connection ∇, we have differential operators

$$\nabla_{\partial_t} : \mathcal{H} \otimes \mathcal{O}_M \to \mathcal{H} \otimes \mathcal{O}_M, [\omega] \mapsto \nabla([\omega])(\partial_t) \tag{3.6}$$

where ∂_t denotes the vector field corresponding to a choice of variable t. The pairing satisfies

$$\frac{d}{dt}\langle \boldsymbol{s}, \gamma_t\rangle = \langle \nabla_{\partial_t}(\omega), \gamma_t\rangle. \tag{3.7}$$

Consequently, there is a minimal collection of elements $\{f_0(t), \dots, f_n(t)\}$ in the $\mathbb{C}(t)$-vector space $\Gamma(\mathcal{H} \otimes \mathcal{O}_M) \otimes \mathbb{C}(t)$ so that

$$\left[f_n(t)\nabla^n_{\partial_t} + f_{n-1}(t)\nabla^{n-1}_{\partial_t} + \cdots + f_1(t)\nabla_{\partial_t} + f_0(t)\right] \boldsymbol{s} = 0 \tag{3.8}$$

and thus there is a linear differential operator

$$\mathscr{L}_{\boldsymbol{s}} = f_n(t)\frac{d^n}{dt^n} + f_{n-1}(t)\frac{d^{(n-1)}}{dt^{(n-1)}} + \cdots + f_1(t)\frac{d}{dt} + f_0(t) \tag{3.9}$$

whose solutions are the period functions $\boldsymbol{\pi}_{\boldsymbol{s}}(t)$. The local system $\mathcal{H}^\vee_{\mathbb{Q}}$ is equipped with a weight filtration $\mathcal{W}^*_\bullet$ dual to the weight filtration on $\mathcal{H}_{\Gamma;D}(t)$ determined by $\mathcal{W}^*_i = (\mathcal{W}_{-i-1})^\vee$. The pairing (3.5) induces a map from $\mathcal{H}^\vee_{\mathbb{Q}}$ to $\mathcal{O}_M$ whose image is $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$, the local system of solutions of $\mathscr{L}_{\boldsymbol{s}}$. Therefore, $\mathcal{W}^*_i$ induces a filtration on $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$.

Lemma 3.3 (section 3.1 of [36]). *The local system $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$ is a quotient of the dual local system $\mathcal{H}^\vee_{\mathbb{Q}}$ by a sub-local system $\mathbb{K}_{\boldsymbol{s}}$. If $\boldsymbol{s} \in \mathcal{W}_i \otimes \mathcal{O}_M$ then $\mathcal{W}^*_i \subseteq \mathbb{K}_{\boldsymbol{s}}$.*

We summarize the results of [36]

(1) The local systems $\mathcal{S}o\ell(\mathscr{L}_{\Gamma;D})$ are quotients of $\mathcal{H}^\vee_{\Gamma;D}$.
(2) The filtration induced by $\mathcal{W}^*_\bullet$ corresponds to a factorisation of $\mathscr{L}_{\Gamma;D}$, however there may be factorisations of $\mathscr{L}_{\Gamma;D}$ which do not correspond to $\mathcal{W}^*_\bullet$.
(3) The monodromy representation of $\mathcal{S}o\ell(\mathscr{L}_{\Gamma;D})$ is upper triangular with diagonal blocks equal to the monodromy representations of the factors of $\mathscr{L}_{\Gamma;D}$.

3.2. Griffiths-Dwork reduction for twisted differential forms

We present the Griffiths-Dwork reduction for twisted differential forms applied to the case of the differential form $\Omega^{\epsilon,\kappa}_\Gamma$ defined in (2.14).

Choosing r variables amongst the kinematic parameters $\underline{z} := \{z_1, \dots, z_r\} \in \vec{m} \cup \vec{s}$, the differentiation of $\Omega^{\epsilon,\kappa}_\Gamma$ leads to

$$\sum_{\substack{\mathsf{a}=a_1+\cdots+a_r\\ a_i\geq 0}} c_{\underline{a}}(\vec{m}, \vec{s}; \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r} \Omega^{\epsilon,\kappa}_\Gamma = \sum_{\substack{\mathsf{a}=a_1+\cdots+a_r\\ a_i\geq 0}} \frac{c_{\underline{a}}(\vec{m}, \vec{s}; \epsilon, \kappa) P^{\underline{a}}(\underline{x})}{\mathscr{F}^{\mathsf{a}}} \Omega^{\epsilon,\kappa}_\Gamma, \tag{3.10}$$

where $\underline{a} = (a_1, \dots, a_r) \in \mathbb{N}^r$ and $P^{\underline{a}}(\underline{x})$ is a homogeneous polynomial of degree $(L+1)(a_1 + \cdots + a_r)$ in the edge variables $\underline{x}$. The sum is over the differential operators of order $a_1 \geq 0, \dots, a_r \geq 0$ and fixed total order $\mathsf{a} := a_1 + \cdots + a_r$. The pole order in the second Symanzik polynomial $\mathscr{F}$ has increased by a.

We present an extension of the Griffths pole reduction [64, 65] adapted to include the twist factor in $\Omega^{\epsilon,\kappa}_\Gamma$:

Step 1: Reduction of polynomial $P^{\underline{a}}(\underline{x})$ in the Jacobian ideal of $\mathscr{F}$, $\mathrm{Jac}(\mathscr{F}_\Gamma) := \langle \vec{\nabla}\mathscr{F}(\underline{x})\rangle$

$$P^{\underline{a}}(\underline{x}) = \vec{C}^{\underline{a}}(\underline{x}) \cdot \vec{\nabla}\mathscr{F}\,, \tag{3.11}$$

with $\vec{\nabla}\mathscr{F} := \left(\partial_{x_1}\mathscr{F}(\underline{x}), \dots, \partial_{x_{e(\Gamma)}}\mathscr{F}(\underline{x})\right)$.

Step 2: Reduction of $\vec{C}^{\underline{a}}(\underline{x})$ in the Jacobian ideal of $\mathscr{U}$, $\mathrm{Jac}(\mathscr{F}_\Gamma) := \langle \vec{\nabla}\mathscr{U}(\underline{x})\rangle$

$$\vec{C}^{\underline{a}}(\underline{x}) \cdot \vec{\nabla}\mathscr{U} = c^{\underline{a}}(\underline{x})\,\mathscr{U}\,, \tag{3.12}$$

with $\vec{\nabla}\mathscr{U} := \left(\partial_{x_1}\mathscr{U}(\underline{x}), \dots, \partial_{x_{e(\Gamma)}}\mathscr{U}(\underline{x})\right)$.

Step 3: Thanks to steps 1 and 2 the differential form

$$\beta^{\underline{a}} = \sum_{1\leq i<j\leq e(\Gamma)} \frac{x_i C_j^{\underline{a}}(\underline{x}) - x_j C_i^{\underline{a}}(\underline{x})}{\mathscr{F}^{\mathsf{a}-1}}\,\Omega_\Gamma^{\epsilon,\kappa}\, dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_{e(\Gamma)}\,. \tag{3.13}$$

satisfies the property

$$d\beta_\Gamma^{\underline{a}} = (\mathsf{a}-1)\frac{P^{\underline{a}}(\underline{x})}{\mathscr{F}^{\mathsf{a}}}\Omega_\Gamma^{\epsilon,\kappa} + \frac{\vec{\nabla}\cdot\vec{C}^{\underline{a}}(\underline{x}) + \lambda_U\, c^{\underline{a}}(\underline{x})}{\mathscr{F}^{\mathsf{a}-1}}\,\Omega_\Gamma^{\epsilon,\kappa}\,. \tag{3.14}$$

where $\lambda_U = e(\Gamma) - (L+1)(\delta-\epsilon) + \kappa\sum_{i=1}^{e(\Gamma)}\mu_i$ is the power of $\mathscr{U}$ in $\Omega_\Gamma^{\epsilon,\kappa}$.

Therefore, for a given set of derivatives, we have performed the pole reduction

$$\sum_{\substack{\mathsf{a}=a_1+\cdots+a_r\\ a_i\geq 0}} c_{\underline{a}}(\vec{m},\vec{s};\epsilon,\kappa)\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\Omega_\Gamma^{\epsilon,\kappa}$$
$$= \sum_{\substack{\mathsf{a}=a_1+\cdots+a_r\\ a_i\geq 0}} \frac{\vec{\nabla}\cdot\vec{C}^{\underline{a}}(\underline{x}) + \lambda_U\, c^{\underline{a}}(\underline{x})}{(\mathsf{a}-1)\mathscr{F}^{\mathsf{a}-1}}\,\Omega_\Gamma^{\epsilon,\kappa} + d\beta_\Gamma^{\epsilon,\kappa}, \tag{3.15}$$

Iterating this procedure gives the partial differential equation (3.1). We refer to [37] for a proof of this reduction and details.

REMARK 3.4. We remark that this way of solving the linear system includes implicitly the freedom given by the syzygies of $\mathrm{Jac}(\mathscr{F})$ and $\mathrm{Jac}(\mathscr{U})$ since they belong to the kernel of the linear systems from (3.11) and (3.12) respectively. It was noticed in [51], that in the rational case, only the first order syzygies are needed to take into account the non-isolated singularities of Feynman integrals.

Part 4. Differential operators for various graphs

In this section we present some differential equations acting on the regulated Feynman integrals. The twist does not change the singular locus of the integrand. Consequently, the real singularities of the associated differential operator are the same as when there are no twist ($\epsilon = \kappa = 0$). We illustrate the effects of the twist one some classes of period integrals.

4.1. Hypergeometric differential operators: the massless box graph

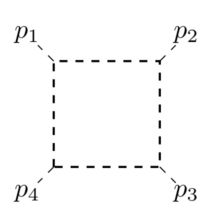

FIGURE 1. The box graph with massless external and internal states.

For the box graph in Fig. 1, we have four massless momenta p_i with $1 \leq i \leq 4$ such that $p_1 + \cdots + p_4 = 0$ and $p_1^2 = \cdots = p_4^2 = 0$. After scaling the integral and setting $X = \frac{p_1 \cdot p_4}{p_1 \cdot p_2}$ the graph polynomials are given by

$$\mathscr{U}_\square = x_1 + \cdots + x_4, \qquad \mathscr{F}_\square(X) = x_2 x_4 + X x_1 x_3 \,. \tag{4.1}$$

The dimensionally regulated Feynman integral in $D = 4 - 2\epsilon$ is given by the twisted differential

$$I_\square^{\epsilon,0}(X) = \int_{\Delta_4} \left(\frac{\mathscr{U}_\square^2}{\mathscr{F}_\square(X)} \right)^\epsilon \frac{\Omega_0}{\mathscr{F}_\square(X)^2} \,, \tag{4.2}$$

with $\Omega_0 = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3$, and $\Delta_4 = \{x_i \geq 0,\ 1 \leq i \leq 4\}$. The application of the algorithm described in section 3.2 gives the differential equation

$$\mathscr{L}_\square^\epsilon I_\square^{\epsilon,0}(X) = \frac{(\epsilon+1)\Gamma(-\epsilon-1)^2}{\Gamma(-2\epsilon)} \left(1 + X^{-\epsilon-1}\right) \,, \tag{4.3}$$

with the differential operator

$$\mathscr{L}_\square^\epsilon = (X+1)X \frac{d}{dX} + 1 + X + \epsilon \,. \tag{4.4}$$

The Feynman integral integrates to hypergeometric functions, with the ϵ expansion as polylogarithms because we have a variation of mixed Tate motives [66, 67]

$$\begin{aligned} I_\square^{\epsilon,0}(X) &= \frac{4}{\epsilon^2 X} - \frac{4 + 2\ln(X)}{X\epsilon} + \frac{2\ln(X) - \frac{5\pi^2}{3} + 4}{X} \\ &+ \frac{\epsilon}{X} \Big(2\,\mathrm{Li}_3(-X) - 2\ln(X)\,\mathrm{Li}_2(-X) - \ln(X)^2 \ln(X+1) + \frac{\ln(X)^3}{3} + \frac{4\ln(X)\pi^2}{3} \\ &\qquad - 2\ln(X) - 10\zeta(3) + \frac{5\pi^2}{3} - 4 - \pi^2 \ln(X+1) \Big) + O(\epsilon^2) \,, \end{aligned} \tag{4.5}$$

where $\mathrm{Li}_r(X) = \sum_{n \geq 1} X^n / n^r$ are the polylogarithms.

4.2. Hyperelliptic differential operators: Planar two-loop graphs

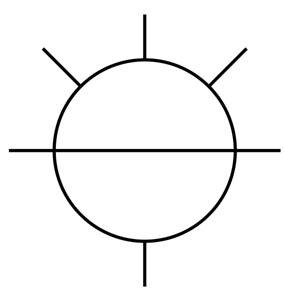

FIGURE 2. A two-loop graphs with $a = 4$, $b = 1$ and $c = 2$.

Two-loop graphs can be labelled by the number of edges (a, b, c) on each cycle, in figure 2 we have represented a graph with $a = 4$, $b = 1$, $c = 2$. Planar graphs are graph for which least one edge number is equal to 1, and their graph polynomials are given by

$$\mathscr{U}_{(a,1,c)} = \left(z + \sum_{i=1}^{c} x_i\right)\left(\sum_{i=1}^{a} y_i\right) + z\left(\sum_{i=1}^{c} x_i\right), \tag{4.6}$$
$$\mathscr{V}_{(a,1,c);D} = z\left(\sum_{i=1}^{c}\sum_{j=1}^{a} r_{ij}^2 x_i y_j\right) + \left(z + \sum_{i=1}^{a} y_i\right)\left(\sum_{1\leq i<j\leq c} p_{ij}^2 x_i x_j\right)$$
$$+ \left(z + \sum_{i=1}^{c} x_i\right)\left(\sum_{1\leq i<j\leq a} q_{ij}^2 y_i y_j\right),$$
$$\mathscr{F}_{(a,1,c);D} = \mathscr{U}_{(a,1,c)}\left(\sum_{i=1}^{c} m_{i+a}^2 x_i + \sum_{i=1}^{a} m_i^2 y_i + m_{a+c+1}^2 z\right) - \mathscr{V}_{(a,1,c);D}.$$

We define the class of mixed Hodge structures (MHS) than can arise from the two-loop Feynman integrals:

Definition 4.1 (MHS for planar two-loop graphs).
(1) Let $\mathbf{MHS}_{\mathbb{Q}}$ denote the Abelian category of $\mathbb{Q}$-mixed Hodge structures.
(2) The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(C;\mathbb{Q})$ for every hyperelliptic curve C is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$.
(3) The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(E;\mathbb{Q})$ for every elliptic curve E is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}$.

The main theorem of [36] states for generic values of $(a,1,c)$ and the spacetime dimension D the Feynman integral attached the planar two-loop graphs

$$I_{(a,1,c)}^{D,\underline{\nu}} = \int_{\Delta_{a+1+c}} \frac{\mathscr{U}^{\sum_{i=1}^{a+1+c}\nu_i - \frac{3D}{2}}}{\mathscr{F}^{\sum_{i=1}^{a+1+c}\nu_i - D}} \prod_{i=1}^{a} x_i^{\nu_i} \prod_{i=1}^{c} y_i^{\nu_{a+i}} z^{\nu_{a+c+1}} \Omega_0\,. \tag{4.7}$$

is a (relative) period integrals of $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$ because the singular locus is determined by the vanishing locus of $\mathscr{F}_{(a,1,c);D}$

Theorem 4.2 (DHV [36]). *For any values of a, c, the cohomology groups of* $X_{(a,1,c);D} = \{\mathscr{F}_{(a,1,c);D} = 0 | (x_i, y_i, z) \in \mathbb{P}^{a+c}\}$ *are contained in* $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$.

Depending on the value of parameters D, a, c and the kinematic invariants p_{ij}^2, q_{ij}^2, r_{ij}^2, and the internal masses m_i^2, the singularities of the integrand change and Feynman integral can be become a period integral of $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}$ or just $\mathbf{MHS}_{\mathbb{Q}}$.

4.2.1. The elliptic curve: the two-loop sunset graph. We present the case of the sunset integral attached to the two-loop graph with $a = b = c = 1$.

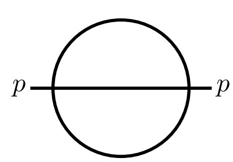

Figure 3. Two-loop sunset with $a = b = c = 1$

The graph polynomials are, in the notations introduced above,

$$\begin{aligned}\mathscr{U}_{(1,1,1)} &= x_1 z + x_1 y_1 + y_1 z, \\ \mathscr{V}_{(1,1,1);D} &= p^2 x_1 y_1 z, \\ \mathscr{F}_{(1,1,1);D} &= \mathscr{U}_{(1,1,1)} \times \left(m_1^2 x_1 + m_2^2 y_2 + m_3^2 z\right) - \mathscr{V}_{(1,1,1);D}.\end{aligned} \tag{4.8}$$

and the Feynman integral for $D = 2 - 2\epsilon$ and $\nu_1 = \nu_2 = \nu_3 = 1$ reads

$$I^{\epsilon}_{(1,1,1)} = \int_{\Delta_3} \left(\frac{\mathscr{U}^3_{(1,1,1)}}{\mathscr{F}^2_{(1,1,1)}}\right)^{\epsilon} \frac{z dx_1 \wedge dy_1 - y_1 dx_1 \wedge dz + x_1 dy_1 \wedge dz}{\mathscr{F}_{(1,1,1)}}. \tag{4.9}$$

For $\epsilon = 0$ it is shown in [11, 30] that the integral is a regulator period integral of the $\mathbf{MHS}^{\mathrm{ell}}_{\mathbb{Q}}$ associated to the elliptic curve defined by $\mathscr{F}_{(1,1,1)} = 0$ in $\mathbb{P}^2$.

The equal-mass case: When all the mass parameters are the same $m_1 = m_2 = m_3$ and setting $t = p^2/m_1^2$, the sunset Feynman satisfies the differential equation

$$\left(\mathscr{L}^{(2)}_{(1,1,1)} + \epsilon \mathscr{L}^{(1)}_{(1,1,1)} + \epsilon^2 \mathscr{L}^{(0)}_{(1,1,1)}\right) I^{\epsilon}_{(1,1,1)} = -6 \frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)} \tag{4.10}$$

with the differential operators $\mathscr{L}^{(r)}_{\ominus(3)}$ of order r

$$\begin{aligned}\mathscr{L}^{(2)}_{(1,1,1)} &= \frac{d}{dt}\left(t(t-1)(t-9)\frac{d}{dt}\right) + (t-3), \\ \mathscr{L}^{(1)}_{(1,1,1)} &= (3t^2 - 10t - 9)\frac{d}{dt} + 3t - 5, \\ \mathscr{L}^{(0)}_{(1,1,1)} &= 2(t+1).\end{aligned} \tag{4.11}$$

The $\epsilon = 0$ piece is the Picard-Fuchs operator associated to the modular curve $X_1(6)$ defined by $(x_1 y_1 + x_1 y + y_1 z)(x_1 + y_1 + z) = t x_1 y_1 z$ [30]. The twist induces the differential $\mathscr{L}^1_{(1,1,1)}$ and $\mathscr{L}^0_{(1,1,1)}$ without affecting the real singularities of the total differential operator.

The different mass case: For the non-equal-mass case $m_1 \neq m_2 \neq m_3$ the order of the differential equation is four with the following ϵ expansion [37, 68, 69]

$$\underbrace{\left(\mathscr{L}^{(4)}_0 + \sum_{r=0}^{4} \epsilon^{1+r} \hat{\mathscr{L}}^{(4-r)}_{r+1}\right)}_{=:\mathscr{L}^{\epsilon}_{(1,1,1)}} I^{\epsilon}_{(1,1,1)} = \mathscr{S}^{\epsilon}_{(1,1,1)}\,. \tag{4.12}$$

where the differential operators $\mathscr{L}^{(r)}_s$ are of order r. The $\epsilon = 0$ piece is a fourth order differential operator that factorizes

$$\mathscr{L}^{(4)}_0 = \mathscr{L}^{(1)}_a \circ \mathscr{L}^{(1)}_b \circ \mathscr{L}^{3-\mathrm{mass}}_{\ominus(3)} \tag{4.13}$$

where $\mathscr{L}^{(1)}_a$ and $\mathscr{L}^{(1)}_b$ are order one differential operators. The differential operator $\mathscr{L}^{3-\mathrm{mass}}_{\ominus(3)}$ is the Picard-Fuchs operator for the three masses two-loop sunset integral in two dimensions determined by the sunset elliptic curve $(x_1 y_1 + x_1 z + y_1 z)(m_1^2 x_1 + m_2^2 y_1 + m_3^2 z) = p^2 x_1 y_1 z$ [11].

The differential operators $\hat{\mathscr{L}}^{(4-r)}_{r+1}$ are irreducible differential operators of order $4 - r$, therefore the differential operator (4.12) is irreducible for generic values of ϵ.

The factorisation of the differential operator (4.13) is understood from the fact that the Feynman integral are (relative) periods of the motive $\mathbf{MHS}^{\rm ell}_{\mathbb{Q}}$ attached to the sunset elliptic curve. As shown in [36] such factorisation appears for the $(a,1,c)$ graphs:

THEOREM 4.3 (Factorisation of differential operators). *For any a,c, the operator $\mathscr{L}_{(a,1,c);D}$ admits a factorisation*

$$\mathscr{L}_1\mathscr{L}_2\dots\mathscr{L}_k$$

where $\mathcal{S}o\ell(\mathscr{L}_i)$ is either:

(a) a local system with finite order monodromy or
(b) a subquotient of the local system underlying a family of hyperelliptic curves over a Zariski open subset of $\mathbb{A}^1$.

In particular, if a or c is ≤ 2 then the monodromy representation of $\mathcal{S}o\ell(\mathscr{L}_i)$ is either

(a) finite, or
(b) a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

The twist in the differential form (4.9) induces an ϵ deformation of the Picard-Fuchs operator $\mathscr{L}^{\epsilon}_{(1,1,1)}$. This does not affect the real singularities of the differential operator because the ϵ factor in (4.9) does not change the nature of the singular locus which is still given by the same elliptic curve as in the $\epsilon=0$ case. Therefore, the ϵ deformation only affects the local monodromies and the apparent singularities of the differential operator, as can be seen from the coefficient of the highest order term

$$\mathscr{L}^{\epsilon}_{(1,1,1)}\Big|_{(d/dt)^4} = (p^2)^3\prod_{i=1}^4(p^2-\mu_i^2)\Big(-(2\epsilon+5)\,(p^2)^2-2\left(m_1^2+m_2^2+m_3^2\right)(1+2\epsilon)\,p^2 + (7+6\epsilon)\prod_{i=1}^4\mu_i\Big)\,, \quad (4.14)$$

where $\mu_i=\{m_1+m_2+m_3, -m_1+m_2+m_3, m_1-m_2+m_3, m_1+m_2-m_3\}$ are the thresholds.

The action of $\mathscr{L}^{\epsilon}_{(1,1,1)}$ leads to the inhomogeneous differential equation (4.12), with an inhomogeneous term given by

$$\mathscr{S}_{\ominus}(\vec m,t,\epsilon)=\frac{c_{23}(t,\epsilon)\Gamma(\epsilon+1)^2}{(m_2m_3)^{2\epsilon}\Gamma(1+2\epsilon)}+\frac{c_{13}(t,\epsilon)\Gamma(\epsilon+1)^2}{(m_1m_3)^{2\epsilon}\Gamma(1+2\epsilon)}+\frac{c_{12}(t,\epsilon)\Gamma(\epsilon+1)^2}{(m_1m_2)^{2\epsilon}\Gamma(1+2\epsilon)}\,, \quad (4.15)$$

obtained from the integration of the exact differential $d\beta^{\epsilon}_{(1,1,1)}$ in (3.1). The coefficients $c_{12}(t,\epsilon)$, $c_{13}(t,\epsilon)$ and $c_{23}(t,\epsilon)$ are polynomials of degree 4 in t and degree 2 in ϵ, respectively which expressions are given on the `SageMath` worksheet Sunset-Twoloop-3mass-Epsilon.ipynb. Expanding in powers of ϵ, we have

$$\mathscr{S}_{\ominus}(\vec m,t,\epsilon)=\mathscr{S}^0_{\ominus}(\vec m,t)+\left(c_0^{(1)}(\vec m)+\sum_{i=1}^3 c_i^{(1)}(\vec m)\log(m_i)\right)\epsilon+O(\epsilon^2) \quad (4.16)$$

with the leading term given by

$$\mathscr{S}^0_{\ominus}(\vec{m},t) = 60t^4 + 56\left(m_1^2+m_2^2+m_3^2\right)t^3 - 308\prod_{i=1}^{4}\mu_i. \tag{4.17}$$

For $\epsilon = 0$ the two-loop sunset integral satisfies the differential equation [11, 70]

$$\mathscr{L}^{3-\text{mass}}_{\ominus} f^{(0)}_{\ominus}(t) = s_0(\vec{m},t) + \sum_{i=1}^{3} s_i(\vec{m},t)\log(m_i^2)\,. \tag{4.18}$$

It can be checked that

$$\mathscr{S}^0_{\ominus}(\vec{m},t) = \mathscr{L}^{(1)}_1 \mathscr{L}^{(2)}_1 \mathscr{L}^{3-\text{mass}}_{\ominus} f^{(0)}_{\ominus}(t), \tag{4.19}$$

showing that the structure of the inhomogeneous term is compatible with the factorisation of the $\epsilon = 0$ piece of the differential operator in (4.12).

4.3. Calabi–Yau differential operators: sunset multiloop graphs

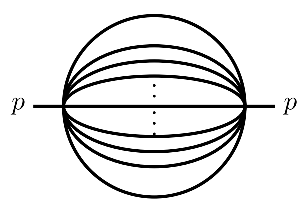

FIGURE 4. Multi-loop sunset with n edges

We now turn to the $n-1$-loop sunset integral in $D = 2-2\epsilon$ dimensions attached to the graph in fig. 4 which reads

$$I^{\epsilon}_{\ominus(n)}(p^2,\vec{m},t) = \int_{\Delta_n}\left(\frac{\mathscr{U}^n_{\ominus(n)}}{\mathscr{F}^{n-1}_{\ominus(n)}}\right)^{\epsilon} \frac{\sum_{i=1}^n(-1)^{i-1}\bigwedge_{\substack{j=1\\ j\neq i}}^n dx_j}{\mathscr{F}_{\ominus(n)}}, \tag{4.20}$$

with the domain of integration $\Delta_n = \{x_i \geq 0, 1 \leq i \leq n\}$ and the graph polynomials

$$\begin{aligned}
\mathscr{U}_{\ominus(n)} &= x_1\cdots x_n \sum_{i=1}^{n}\frac{1}{x_i}\,,\\
\mathscr{F}_{\ominus(n)} &= \mathscr{U}_{\ominus(n)} \times \sum_{i=1}^{n} m_i^2 x_i - p^2 x_1\cdots x_n\,.
\end{aligned} \tag{4.21}$$

Notice that $\mathscr{U}^n_{\ominus(n)}/\mathscr{F}^{n-1}_{\ominus(n)}$ is a homogeneous rational function of degree 0 in $(x_1,\dots,x_n)$. As usual the differential form is defined in the complement of the vanishing locus of the denominator in $X_n = \{\mathscr{F}_{\ominus(n)} = 0\}$. The Feynman integral being a (relative) period of a Calabi–Yau manifold of complex dimension $n-2$ defined by the equation $\mathscr{F}_{\ominus(n)} = 0$ [10, 11, 13, 16, 17, 19, 30, 71].

The equal-mass case: For the equal-mass case $m_1 = \cdots = m_n$ and $p^2 = t\, m_n^2$ in (4.21) the sunset Feynman integral satisfies the differential equation

$$\underbrace{\left(\sum_{r=0}^{n-1} \epsilon^r \mathscr{L}_{\ominus(n)}^{(n-1-r)}\right)}_{=:\mathscr{L}^{\epsilon}_{\ominus(n)}} I_{\ominus}^{\epsilon}(t) = -n! \frac{\Gamma(1+\epsilon)^{n-1}}{\Gamma(1+(n-1)\epsilon)}. \tag{4.22}$$

Like the case $n=3$ described above the term of order ϵ^r is a differential operator $\mathscr{L}_{\ominus(n)}^{(r)}$ of order r in t. The coefficient of ϵ^0 is the differential operator of order $n-1$ derived in [35] (see as well [16, 72–76]).

The ϵ deformation does not change the real singularities of the differential operators because the twist in (4.20) does not introduce new singularities.

This is seen as well on the form of the ϵ-deformed Picard-Fuchs operator for the $n=4$ case with $m_1 = m_2 = m_3 = m_4$

$$\begin{aligned} \mathscr{L}^{\epsilon}_{\ominus(4)} = &-(t-16)(t-4)t^2 \left(\frac{d}{dt}\right)^3 - 6(t^3-15t^2+32t)\left(\frac{d}{dt}\right)^2 - (7t^2-68t+64)\left(\frac{d}{dt}\right) - t + 4 \\ &+ \epsilon\left(-6(t-10)t^2\left(\frac{d}{dt}\right)^2 - 6(3t-20)t\left(\frac{d}{dt}\right) + 18 - 6t\right) \\ &+ \epsilon^2\left(-(11t^2-28t-64)\left(\frac{d}{dt}\right) - 11t + 14\right) + \epsilon^3\left(-6t-12\right), \end{aligned} \tag{4.23}$$

where the $\mathscr{L}^{\epsilon}_{\ominus(4)}|_{\epsilon=0}$ operator is the Picard-Fuchs operator for the $K3$ surface with Picard number 19 [10].

Different mass case: We represent the result for the four different masses $m_1 \neq m_2 \neq m_3 \neq m_4$ for the case $n=4$. In that case the singular locus is a $K3$ surface of Picard number 16 [51]. The ϵ-deformed differential operator has order 11 and has the ϵ expansion

$$\mathscr{L}^{\epsilon}_{\ominus(4)} = \sum_{r=0}^{16} \epsilon^r \mathscr{L}_r^{(11)} + \sum_{r=0}^{11} \epsilon^{16+r} \mathscr{L}_{16+r}^{(11-r)}, \tag{4.24}$$

where the differential operators $\mathscr{L}_s^{(r)}$ are of order r. The order 11 part of this deformed Picard-Fuchs operator has degree 16 in ϵ.

The order $\epsilon = 0$ operator factorises as

$$\mathscr{L}_0^{(11)} = \mathscr{L}_{a_1}^{(1)} \circ \cdots \circ \mathscr{L}_{a_5}^{(1)} \circ \mathscr{L}_{\ominus(4)}^{4-\text{mass}}, \tag{4.25}$$

where $\mathscr{L}_{a_1}^{(1)}, \dots, \mathscr{L}_{a_5}^{(1)}$ are first order operators and $\mathscr{L}_{\ominus(4)}^{4-\text{mass}}$ is the sixth order differential operator for the three-loop sunset integral with the all different mass configurations given in section 4.3 of [51].

The coefficient of the highest order term $(d/dt)^{11}$ is given by

$$\begin{aligned}\mathscr{L}^{\epsilon}_{\ominus(4)}\Big|_{(d/dt)^{11}} = t^{11}\left(t-(m_1+m_2-m_3-m_4)^2\right)\\ \times\left(t-(m_1-m_2+m_3-m_4)^2\right)\left(t-(m_1+m_2+m_3-m_4)^2\right)\\ \times\left(t-(m_1-m_2-m_3+m_4)^2\right)\left(t-(m_1+m_2-m_3+m_4)^2\right)\\ \times\left(t-(m_1-m_2+m_3+m_4)^2\right)\left(t-(-m_1+m_2+m_3+m_4)^2\right)\\ \times\left(t-(m_1+m_2+m_3+m_4)^2\right)q^{[1111]}(t,\epsilon). \quad (4.26)\end{aligned}$$

The ϵ dependence appears only in the apparent singularities determined by the polynomial $q^{[1111]}(t,\epsilon)$ of degree 17 in t and 16 in ϵ. The polynomial is given in the only worksheet Sunset-Threeloop-Epsilon.ipynb.

4.4. Discussion

We have presented a generalisation of the Griffiths-Dwork reduction for deriving differential operators acting on Feynman integrals in dimensional or analytic regularisation. The algorithm makes a special use of the fact that the twist from the regularisations is the power of a degree zero homogeneous rational function build from the graph polynomials.

The algorithm gives the minimal order (non-factorisable) D-module of differential operators acting on regulated Feynman integrals. At each derivative order the procedure consists of solving the linear systems from the reductions with the respect the Jacobian ideal of the graph polynomials $\mathscr{F}$ in (3.11) and $\mathscr{U}$ in (3.12) in order to determine the coefficients $c_{\underline{a}}(\underline{z})$ and the inhomogeneous term $\beta_\Gamma^{\underline{a}}$ in (3.1).

Because the twisted differential $\Omega_\Gamma^{\epsilon,\kappa}$ has the same singularities as $\Omega_\Gamma^{0,0}$, the regularisation parameters ϵ or κ do not affect the discriminant locus but only the local monodromies. This reflects on the fact that these parameters only affect the apparent singularities of the differential operators.

With this algorithm we can derive a Gröbner basis of partial differential operators in some multiple scale cases. The differential operators produced by the algorithm of this paper might arise as specialisation of the system of partial differential operators obtained by GKZ approach. The restriction of the GKZ D-module is a difficult open problem, which we leave for further investigations.

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Institut de Physique Théorique, Université Paris-Saclay, CEA, CNRS, F-91191 Gif-sur-Yvette Cedex, France