Differential Equations for Feynman Integrals

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ABSTRACT

Feynman integrals are notoriously difficult multidimensional integrals arising in many physical problems. We present various approaches for determining the differential equations satisfied by Feynman integrals. The approach is based on the analysis of the algebraic geometry associated with the Feynman integrals and the use of the creative telescoping algorithm.

CCS CONCEPTS

• **Applied computing → Physics**.

KEYWORDS

Feynman integrals, period integrals, Picard-Fuchs equations.

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1 DEFINITIONS

Feynman integrals enter the evaluation of many physical observable quantities in particle physics, gravitational physics, statistical physics, and solid-state physics. They are multidimensional integrals that cannot be evaluated with elementary methods. It is an important question to understand what kind of special functions they are of their physical parameters.

Feynman integrals come in families of integrals

$$I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D) := \frac{\prod_{i=1}^n \Gamma(\nu_i)}{\pi^{\frac{lD}{2}}\Gamma(\omega)} \int_{(\mathbb{R}^{1,D-1})^l} \frac{\prod_{i=1}^l d^D\ell_i}{\prod_{i=1}^n Q_i^{\nu_i}}, \qquad (1)$$

where D is the dimension of space-time and the integration is over the vectors $\ell_i := (\ell_i^1, \ldots, \ell_i^D)$ in the Minkowskian space $\mathbb{R}^{1,D-1}$ with the flat space measure of integration $d^D\ell_i := \prod_{r=1}^D d\ell_i^{(r)}$. The exponents ν_i are complex numbers, and we have set

$$\omega := \sum_{i=1}^n \nu_i - \frac{lD}{2}. \qquad (2)$$

The denominators Q_i are quadratic forms in the integration vectors ℓ_i

$$Q_i = \sum_{r,s=1}^l \alpha_{r,s}^{(i)} \ell_r \cdot \ell_s + \sum_{r=1}^l \sum_{s=1}^v \beta_{r,s}^{(i)} \ell_r \cdot p_s + \sum_{r,s=1}^v \gamma_{rs}^{(i)} p_r \cdot p_s - m_i^2 + i\varepsilon. \qquad (3)$$

In this expression we have

- The $\cdot$-product is the natural metric on $\mathbb{R}^{1,D-1}$: $u \cdot v = u^1v^1 - \sum_{r=2}^D u_r v_r$.
- The vectors $p_i \in \mathbb{R}^{1,D-1}$ subject to the condition $p_1 + \cdots + p_v = 0$, and v is a positive integer. The set of scalar products is denoted $\underline{s} := (p_i \cdot p_j)_{1\leq i,j\leq n}$.
- The coefficients m_i^2 with $1 \leq i \leq n$ are real positive numbers. They are the so-called mass parameters. The set of mass parameters is denoted $\underline{m}^2 := (m_1^2, \ldots, m_n^2)$.
- The coefficients $(\alpha_{rs}^{(i)})_{1\leq r,s\leq l}$, $(\beta_{rs}^{(i)})_{1\leq r\leq l; 1\leq s\leq v}$ and $(\gamma_{rs}^{(i)})_{1\leq r,s\leq v}$ are integer coefficients taking values in $\{-1, 0, 1\}$.
- The term $i\varepsilon$ with $\varepsilon > 0$, together with a very specific contour of integration, is Feynman's prescription for avoiding the poles of the integral when the denominator factors Q_i vanish (see [1] for reference).

We introduce the Schwinger parameters $(x_i)_{1\leq i\leq n}$, one for each denominator factor, such that

$$\frac{\Gamma(\nu_i)}{Q_i^{\nu_i}} = \int_0^\infty e^{-x_i Q_i} x_i^{\nu_i - 1} dx_i. \qquad (4)$$

We then collect the various contributions

$$\sum_{i=1}^n x_i Q_i =: \sum_{i,j=1}^l \Omega_{ij} \ell_i \cdot \ell_j + \sum_{i=1}^l \ell_i \cdot P_i - J, \qquad (5)$$

and we perform the standard Gaussian integration on the loop momenta ℓ_i

$$\int_{(\mathbb{R}^{1,D-1})^l} e^{-\sum_{i,j=1}^l \Omega_{ij}\ell_i\cdot\ell_j} \prod_{i=1}^l d^D\ell_i = \frac{\pi^{\frac{lD}{2}}}{\det(\Omega)^{\frac{D}{2}}} \qquad (6)$$

to obtain the parametric representation that we will use in this text (see [2] for details)

$$I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D) = \int_0^\infty \cdots \int_0^\infty \frac{\mathcal{U}^{\omega - \frac{D}{2}}}{\mathcal{F}^\omega} \prod_{i=1}^n x_i^{\nu_i - 1} dx_i. \qquad (7)$$

The integral involves:

- the so-called first Symanzik polynomial

$$\mathcal{U} = \det\left((\Omega_{ij})_{1\leq i,j\leq l}\right), \qquad (8)$$

  which is a homogeneous polynomial of degree l in the Schwinger parameters x_i. It is at most linear in each of the x_i variables. It does not depend on the physical parameters.

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- The so-called second Symanzik polynomial

$$\mathcal{F} := \left(\sum_{i,j=1}^{l} (\Omega^{-1})_{ij} P_i \cdot P_j - J\right)$$
$$= \mathcal{U}\left(\sum_{i=1}^{n} m_i^2 x_i\right) - \sum_{1\leq i\leq j\leq n} p_i \cdot p_j \mathcal{G}_{ij}(x_1,\dots,x_n) \qquad (9)$$

where $\mathcal{G}_{ij}(x_1,\dots,x_n)$ is a homogeneous polynomial of degree $l+1$. It is at most linear in each of the x_i variables. The polynomial $\mathcal{F}$ is homogeneous of degree $l+1$ in the variables $(x_1,\dots,x_n)$.

As a final comment, we mention that the first and second Symanzik polynomial have an interpretation in graph theory. To the family of Feynman integrals in (1) one can associate a graph Γ with l (homology) loops, n edges and v vertices. The $\mathcal{U}$ polynomial is the Kirchhoff polynomial of graph Γ, which is as well the determinant of the minor of the Laplacian of the graph with one row and column removed. The polynomial $\sum_{1\leq i\leq j\leq n} p_i \cdot p_j \mathcal{G}_{ij}(x_1,\dots,x_n)$ is given by the sum over all spanning 2-forests of the graph Γ. We refer to [3, 4] for an extensive discussion about the graph theory of Feynman integrals. This connection to graph theory will not be needed for the present discussion, so we do not develop this further.

2 ANALYTIC PROPERTIES

The Feynman integrals (1) can diverge from the region of large values of the loop momenta, i.e. when $|\ell_i| \to \infty$. They are called the ultraviolet divergences. Weinberg proved in [5] that when all the mass parameters are non-vanishing $m_i^2 > 0$ for $1 \leq i \leq n$, the Feynman integral is absolutely convergent when $\omega > 0$ in (2). The other source of divergence arises from the region of integration where some vectors $|\ell_r| \sim 0$ vanish and some of the mass parameters $m_i = 0$ in the denominator factors in (3). They are the called infrared divergences. The conditions for convergences is $\sum_i \nu_i - D/2h < 0$ where $0 \leq h \leq l$ and the sum is over a subset of exponents ν_i for which the mass parameters vanish (see [6, 7] for details).

To sum up, the conditions for the absolute convergence of the Feynman integrals is a set of hyperplanes on the variables $(\underline{\nu}, D)$ in $\mathbb{C}^{n+1}$. Because the polynomials $\mathcal{U}$ and $\mathcal{F}$ are independent of the dimension of space-time D and the exponents $\underline{\nu}$, the Feynman integral is a meromorphic function with singularities on linear hypersurfaces on $(\underline{\nu}, D)$ in $\mathbb{C}^{n+1}$ as shown by [8] and reproved by Panzer in [9].

The parametric representation in (7) allows to consider the Feynman integral for non-integer values of the dimension D. It is customary in physics to perform a Laurent expansion of the integral near a positive integer dimension $D_c = 2, 3, \cdots$

$$I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D) = \sum_{n\geq -2l} (D - D_c)^n I_\Gamma^{(n)}(\underline{s}, \underline{m}^2; \underline{\nu}). \qquad (10)$$

The coefficients of this expansion are finite integrals. One would like to calculate the various terms of this expansion. In the rest of this text we will be focusing on ultraviolet and infrared finite Feynman graph integrals. A discussion of divergent graphs can be found in [10] for instance.

Feynman integrals are highly transcendental functions of the kinematics coefficients $\underline{s} := (p_i \cdot p_j)_{1\leq i,j\leq v}$ and the internal masses $\underline{m}^2 = (m_1^2, \dots, m_n^2)$. They are multi-valued functions, with non-trivial monodromies. They present branch cuts (associated with particle production) and their analytic or numerical evaluations are challenging.

3 PERIOD INTEGRALS

Broadhurst and Kreimer [11, 12] remarked that the definition of the Feynman integrals in (7) resembles the definition of period integrals given by Kontsevich and Zagier in [13]: *a period is a complex number that can be expressed as an integral of an algebraic function over an algebraic domain*

$$\int_\Delta \frac{f(x_1,\dots,x_n)}{g(x_1,\dots,x_n)} \prod_{i=1}^{n} dx_i\,, \qquad (11)$$

where $f(x_1,\dots,x_n)$ and $g(x_1,\dots,x_n)$ belong to $\mathbb{Z}[x_1,\dots,x_n]$ and the domain of integration Δ is a domain in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

To understand the connection between Feynman integrals and period integrals, we need a more general definition of abstract periods given in [13]. Let's consider $X(\mathbb{C})$ a smooth algebraic variety of dimension n over $\mathbb{Q}$. Consider $D \subset X$ a divisor with normal crossings, which means that locally this is a union of coordinate hyperplanes of dimension $n-1$. Let $\eta \in \Omega^n(X)$, and let $\Delta \in H_n(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$ a singular n-chain on $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$. To the quadruple (X, D, ω, Δ), we can associate a complex number called the *period of the quadruple*

$$P(X, D, \omega, \Delta) = \int_\Delta \eta\,. \qquad (12)$$

For this definition to be compatible under change of variables, we define the space of *effective periods* as the $\mathbb{Q}$-vector space of equivalence classes modulo (a) linearity in η and Δ, (b) under change variables, (c) and integration by part of Stokes formula.

We can rewrite the Feynman integral (7) in this form according

$$I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D) = \int_{\Delta_n} \frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega} \prod_{i=1}^{n} x_i^{\nu_i - 1}\, \Omega_0, \qquad (13)$$

where Ω_0 is the natural differential $n-1$-form on the real projective space $\mathbb{P}^{n-1}$

$$\Omega_0 := \sum_{j=1}^{n} (-1)^{j-1} x_j\, dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n\,, \qquad (14)$$

where $\widehat{dx_j}$ means that dx_j is omitting in this sum. The domain of integration Δ_n is defined as

$$\Delta_n := \left\{[x_1,\dots,x_n] \in \mathbb{P}^{n-1} | x_i \in \mathbb{R}, x_i \geq 0\right\}. \qquad (15)$$

To a Feynman graph, one can associate two graph hypersurfaces. One defined from the first Symanzik polynomial in (8)

$$\mathbb{V}(\mathcal{U}) := \{\mathcal{U}(x_i) = \det \Omega = 0 | x_i \in \mathbb{P}^{n-1}(\mathbb{R})\}\,, \qquad (16)$$

and the vanishing locus of the second Symanzik polynomial $\mathcal{F}$ in (9)

$$\mathbb{V}(\mathcal{F}) := \{\mathcal{F}(x_i) = 0 | x_i \in \mathbb{P}^{n-1}(\mathbb{R})\}\,. \qquad (17)$$

The polar part X_Γ of the Feynman integral in (7) is the union of these two graph hyper-surfaces unless for $\omega = D/2$ when it is only given by $\mathbb{V}(\mathcal{F})$ or $\omega = 0$ when it is only given by $\mathbb{V}(\mathcal{F})$. Although the integrand is a closed-form

$$\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\,\Omega_0 \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_\Gamma), \tag{18}$$

the domain Δ_n has a boundary and therefore its homology class is not in $H_n(\mathbb{P}^{n-1}\backslash X_\Gamma)$. As well, the polar part X_Γ of the differential form entering the expression of the Feynman integral intersects the boundary of the domain of integration $\partial\Delta_n \cap X_\Gamma \neq \emptyset$. We need to consider a blow-up in $\mathbb{P}^{n-1}$ of linear space $f: \mathcal{P} \to \mathbb{P}^{n-1}$, such that all the vertices of Δ_n belong to $\mathcal{P}\backslash\mathcal{X}$ where $\mathcal{X}$ is the strict transform of X_Γ. Let $\mathcal{B}$ be the total inverse image of the coordinate simplex $\{x_1x_2\cdots x_n = 0 | [x_1,\dots,x_n] \in \mathbb{P}^{n-1}\}$.

Bloch, Esnault, and Kreimer showed in [10] that all of this lead to the mixed Hodge structure associated with the Feynman graph

$$\mathfrak{M}(\Gamma) := H^{n-1}(\mathcal{P}\backslash\mathcal{X}, \mathcal{B}\backslash\mathcal{B}\cap\mathcal{X}; \mathbb{Q}). \tag{19}$$

4 DIFFERENTIAL EQUATIONS

Following [14], we introduce the differential operators in the Weyl algebra

$$A^{(n)}[D] := \mathbb{C}[-D/2]\langle x_1,\dots,x_n,\partial_{x_1},\dots,\partial_{x_n} | [\partial_{x_i}, x_j] = \delta_{ij}\rangle \tag{20}$$

acting inside the Feynman integral, and the operators in the shift algebra

$$S^{(n)}[D] := \mathbb{C}[-D/2]\langle \mathbf{1}^+,\dots,\mathbf{n}^+,\mathbf{1}^-,\dots,\mathbf{n}^- | [-\mathbf{j}^-, \mathbf{i}^+] = \delta_{ij}\rangle. \tag{21}$$

The shift operators act on functions $F(\underline{\nu})$ of $\underline{\nu}$ as follows

$$(\mathbf{i}^- F)(\underline{\nu}) = F(\underline{\nu} - e_i), \qquad (\mathbf{i}^+ F)(\underline{\nu}) = \nu_i F(\underline{\nu} + e_i) \tag{22}$$

where $e_i = (0,\dots,0,1,0,\dots,0)$ with the 1 at position i. The insertion of the variable x_i inside the integral is the same as raising the index $\nu_i \to \nu_i + 1$, i.e. $x_i \to \mathbf{i}^+$, differentiating with respect to x_i lowers the index $\nu_i \to \nu_i - 1$, i.e. $\partial_{x_i} \to -\mathbf{i}^-$.

A result of Bernshtein [15] states that the module

$$A^{(n)}[-D/2]\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right) \tag{23}$$

is holonomic. Then by a theorem by Loeser and Sabbah [16] the dimension of the vector space

$$V_\Gamma := \sum_{\underline{\nu}\in\mathbb{Z}^n} \mathbb{C}(-D/2,\underline{\nu}) I_\Gamma(\underline{s},\underline{m}^2;\underline{\nu},D) \tag{24}$$

is given by the topological Euler characteristic of the complement of the graph hyper-surface [14]

$$\dim(V_\Gamma) = (-1)^{n+1}\chi\left((\mathbb{C}^*)^n\backslash\mathbb{V}(\mathcal{U})\cup\mathbb{V}(\mathcal{F})\right), \tag{25}$$

with the vanishing loci defined in (16) and (17).

Since this vector space is finite-dimensional one can expand integral in the family of Feynman integrals $I_\Gamma(\underline{s},\underline{m}^2;\underline{\nu},D)$ on a basis of, so-called master integrals, $M_\Gamma(\underline{s})$ with coefficients given by rational functions of the parameters $\underline{s}$,$\underline{m}^2$, $\underline{\nu}$ and the dimension D.

Differentiating with respect to the physical parameters $\underline{s}$, the master integrals $M_\Gamma(\underline{s})$ satisfy the first order this differential system of equations (see [17] for a physicist's review)

$$dM_\Gamma(\underline{s}) = A_\Gamma \wedge M_\Gamma(\underline{s}). \tag{26}$$

The matrix A_Γ is a flat connection satisfying

$$dA_\Gamma + A_\Gamma \wedge A_\Gamma = 0. \tag{27}$$

One can convert the first order system in (26) into a system of differential operators acting on the the Feynman integral $I_\Gamma(\underline{s},\underline{m}^2;\underline{\nu},D)$. It is enough to get a Gröbner basis of such operators.

For a subset of variables $\underline{z} := \{z_1,\dots,z_r\} \in \underline{s}$ of variables satisfies we define a differential operator annihilating the integrand

$$\mathcal{T}_{\underline{z}}\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right) = 0\,. \tag{28}$$

We decompose this operator

$$\mathcal{T}_{\underline{z}} := \mathcal{L}_{\underline{z}} + \sum_{i=1}^n \frac{\partial}{\partial x_i} Q_i(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}) \tag{29}$$

into a Picard-Fuchs differential operator $\mathcal{L}_{\underline{z}}$ acting only on the parameters $\underline{z}$ of the Feynman integrals

$$\mathcal{L}_{\underline{z}} = \sum_{\substack{0\leq a_i\leq o_i\\ 1\leq i\leq r}} p_{a_1,\dots,a_r}(\underline{s},\underline{m}^2)\prod_{i=1}^r\left(\frac{\partial}{\partial z_i}\right)^{a_i} \tag{30}$$

and a certificate part $Q_i(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x})$ which is a differential operator acting on the parameters $\underline{z}$ and the integration variables

$$Q_i(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}) = \sum_{\substack{0\leq a_i\leq o'_i\\ 1\leq i\leq r}}\sum_{\substack{0\leq b_j\leq \tilde{o}_j\\ 1\leq j\neq n}} q^{(i)}_{a_1,\dots,a_r}(\underline{s},\underline{m}^2;\underline{x}) \times\prod_{i=1}^r\left(\frac{\partial}{\partial z_i}\right)^{a_i}\prod_{j=1}^n\left(\frac{\partial}{\partial x_i}\right)^{b_j}. \tag{31}$$

In these expressions o_i, o'_i with $1 \leq i \leq r$ and $\tilde{o}_j$ and $1 \leq j \leq n$, are positive integers.

Integrating (28) over the domain σ we have

$$\begin{aligned}0 &= \int_\sigma \mathcal{T}_{\underline{z}}\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right)\prod_{i=1}^n dx_i\\ &= \mathcal{L}_{\underline{z}}\int_\sigma\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right)\prod_{i=1}^n dx_i + \int_\sigma d_X Q\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right)\end{aligned} \tag{32}$$

where we have set

$$d_X Q := \sum_{i=1}^n \frac{\partial}{\partial x_i} Q_i(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}) dx^i. \tag{33}$$

Integrating by part the last term we obtain

$$\int_\sigma d_X Q\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right) = -\int_{\partial\sigma} Q\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right). \tag{34}$$

For a cycle without a boundary $\partial\sigma = \emptyset$, this integral vanishes

$$\mathcal{L}_{\underline{z}}\int_\sigma\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right) = 0\,. \tag{35}$$

In the case of the Feynman integral I_Γ, this is no longer true because $\partial\Delta_n \neq \emptyset$, therefore

$$\mathcal{L}_{\underline{z}} I_\Gamma(\underline{s},\underline{m}^2;\underline{\nu},D) = \int_{\partial\Delta_n} Q\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{\nu_i-1}\right) \neq 0\,. \tag{36}$$

5 COMPUTATIONAL CHALLENGES

Deriving the system of differential equations in (26) is computationally challenging [17–20] because the dimension of the vector space V_Γ in (24) has a rapid growth with the loop order. On the other side the rank of the motive $\mathfrak{M}(\Gamma)$ in (19) has a much milder growth. For instance for the case of the so-called sunset integrals with the number of edges $n = l + 1$ simply related to the loop order l. The dimension $\dim(V_\Gamma)$ is given by $2^n - 1$ [14, 21] whereas the rank of the motive has a much milder linear growth with n the number of edges. This milder growth is due to the fact that the vanishing locus of the second Symanzik polynomial $\mathcal{F}$ is a degree n polynomial in $\mathbb{P}^{n-1}$, the motive in (19) is associated with a $n-2$-fold Calabi-Yau [22].

One approach for deriving such a system of differential operators is to make use of the creative telescoping algorithm [23, 24]. With this algorithm, one can derive a Gröbner basis of differential operators acting on the integrand of the Feynman integral, despite the presence of non-isolated singularities.

6 THE HYPERGEOMETRIC DIFFERENTIAL EQUATIONS

In a way to understand the system of differential equations satisfied by the family of Feynman integrals, we note that that the parametric representation in (7) is a particular case of generalised Euler integrals [25–27]

$$\int_\sigma \prod_{i=1}^r P_i(x_1,\dots,x_n)^{\alpha_i} \prod_{i=1}^n x_i^{\beta_i} dx_i \tag{37}$$

studied by Gel'fand, Kapranov and Zelevinski (GKZ) in [28, 29]. They have shown that this class of integrals satisfy a system of hypergeometric differential equations.

For applying this approach we consider the following generalisation of the first and second Symanzik polynomial

$$\mathrm{U}(u_1,\dots,u_r) := \sum_{\mathrm{a}=(a_1,\dots,a_{n-1})\in \mathrm{A}_{\mathcal{U}}} z_{\boldsymbol{a}} \prod_{i=1}^{n-1} x_i^{a_i}, \tag{38}$$

where $\mathrm{A}_{\mathcal{U}} = (\mathbf{a}_1,\dots,\mathbf{a}_r)$ is the finite subset of $\mathbb{Z}^n$ for the exponent of the monomial in the first Symanzik $\mathcal{U}$. By construction $\mathrm{U}(1,\dots,1) = \mathcal{U}$. Likewise

$$\mathrm{F}(f_1,\dots,f_s) := \sum_{\mathrm{a}=(a_1,\dots,a_{n-1})\in \mathrm{A}_{\mathcal{F}}} z_{\boldsymbol{a}} \prod_{i=1}^{n-1} x_i^{a_i}, \tag{39}$$

with $\mathrm{A}_{\mathcal{F}} = (\mathbf{a}_1,\dots,\mathbf{a}_r)$ is the finite subset of $\mathbb{Z}^n$ for the exponent of the monomial in the first Symanzik $\mathcal{F}$. By construction we recover the second Symanzik $\mathcal{F}$ when the complex numbers $f_1,\cdots,f_s$ are identified with the physical parameters in (9).

Setting $\underline{z} := \{u_1,\dots,u_r,f_1,\dots,f_s\}$, we consider the integral

$$\Pi_\Gamma(\underline{z}) := \frac{1}{(2i\pi)^{n-1}} \int_{|x_1|=\dots=|x_{n-1}|=1} \frac{\mathrm{U}(u_1,\dots,u_r)^{\omega-\frac{D}{2}}}{\mathrm{F}(f_1,\dots,f_s)^\omega} \prod_{i=1}^n x_i^{\nu_i-1} dx_i, \tag{40}$$

which differs from the Feynman integral in (7) with the replacement of the Symanzik polynomials by the ones in (38) and (39) and the domain of integration which is a torus.

The GKZ construction implies that this integral satisfies the following set of differential equations:

- For every vector $\boldsymbol{\ell} = (\ell_1,\dots,\ell_r) \in \mathbb{L}$ in the lattice

$$\mathbb{L} := \{(\ell_1,\dots,\ell_r)\in\mathbb{Z} | \sum_{i=1}^r \ell_i \mathbf{a}_i = 0, \ell_1+\dots+\ell_r = 0\}, \tag{41}$$

one associates a differential operator $\square_{\boldsymbol{\ell}}$ that annihilates the integral

$$\left(\square_{\boldsymbol{\ell}} := \prod_{\ell_i>0} \partial_{z_i}^{\ell_i} - \prod_{\ell_i<0} \partial_{z_i}^{-\ell_i}\right) \Pi_\Gamma(\underline{z}) = 0, \tag{42}$$

- n differential operators $\mathbf{E} := (E_1,\dots,E_{n-1})$

$$\mathbf{E} := \mathbf{a}_1 z_1 \frac{\partial}{\partial z_1} + \dots + \mathbf{a}_r z_r \frac{\partial}{\partial z_r}, \tag{43}$$

such that for $\mathbf{c} = (c_1,\dots,c_{n-1}) \in \mathbb{C}^{n-1}$ we have

$$(\mathbf{E}-\mathbf{c})\Pi_\Gamma(\underline{z}) = 0. \tag{44}$$

Notice that $E_1 = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i}$ is the Euler operator and c_1 is the degree of homogeneity of the integral.
These operators satisfy the commutation relations

$$\begin{aligned} \mathbf{z^u E} - \mathbf{E z^u} &= -(\mathbf{A}\cdot\mathbf{u})\, \mathbf{z^u}, \\ \partial_z^{\mathbf{u}} \mathbf{E} - \mathbf{E} \partial_z^{\mathbf{u}} &= (\mathbf{A}\cdot\mathbf{u})\, \partial_z^{\mathbf{u}}, \end{aligned} \tag{45}$$

with $\mathbf{z^u} := \prod_{i=1}^r z_r^{u_r}$ and $\partial_z^{\mathbf{u}} := \prod_{i=1}^r \partial_{z_r}^{u_r}$.

The set of differential equations need to be restricted to the slice for the physical parameters. It is particularly difficult to restrict the D-module of differential operators by imposing the relations on the $\underline{z}$ parameters with the physical parameters. As well, it is known that the GKZ construction does not provide all the differential operators for a minimal dimension space of differential operators acting on the periods [30]. On the other side, an application of the creative telescoping algorithm leads to the minimal dimension set of differential operators [22].

7 AN EXAMPLE IN TWO DIMENSIONS

In this section, we illustrate the application the GKZ formalism and the creative telescoping algorithm to the following Feynman integral with $\omega = D/2 = 1$

$$I_\circ(p^2, m_1^2, m_2^2) := \iint_0^\infty \frac{dx_1 dx_2}{\mathcal{F}_\circ(p^2, m_1^2, m_2^2; x_1, x_2)}, \tag{46}$$

with the graph polynomial

$$\mathcal{F}_\circ(x_1, x_2, t, m_1^2, m_2^2) = p^2 x_1 x_2 - (m_1^2 x_1 + m_2^2 x_2)(x_1 + x_2). \tag{47}$$

Since the integral in (46) is projective one can set $x_2 = 1$ and integrate only over x_1. The most general degree two polynomial in $\mathbb{P}^1$ with at most degree two monomial is given by

$$\mathrm{F}_\circ(x_1, x_2; z_1, z_2, z_3) = z_1 x_1^2 + z_2 x_2^2 + z_3 x_1 x_2. \tag{48}$$

This polynomial has three parameters which is exactly the number of independent physical parameters as in (47)

$$z_1 = -m_1^2, \qquad z_2 = -m_2^2, \qquad z_3 = p^2 - (m_1^2 + m_2^2). \tag{49}$$

Setting

$$\mathbf{A}_\circ = \begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{pmatrix} = \begin{pmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ 1 & 0 & 0\end{pmatrix}, \tag{50}$$

with $\mathbf{a}_i = (1, a_i^1, a_i^2)$ we have the lattice

$$\mathbb{L}_\circ := \{\boldsymbol{\ell} := (\ell_1, \ell_2, \ell_3) \in \mathbb{Z}^3 | \ell_1 \mathbf{a}_1 + \ell_2 \mathbf{a}_2 + \ell_3 \mathbf{a}_3 = \boldsymbol{\ell} \cdot \mathrm{A}_\circ = 0\}. \tag{51}$$

This means that the elements of $\mathbb{L}_\circ$ are in the kernel of $\mathbf{A}_\circ$. This lattice in $\mathbb{Z}^3$ has rank one $\mathbb{L}_\circ = (1, 1, -2)\mathbb{Z}$. Notice that all the elements automatically satisfy the condition $\ell_1 + \ell_2 + \ell_3 = 0$.

Because the rank is one the GKZ system of differential equations is generated by the following three operators

$$\begin{aligned} \Box_1 &:= \frac{\partial^2}{\partial z_1 \partial z_2} - \frac{\partial^2}{(\partial z_3)^2},\\ E_1 &:= \sum_{r=1}^3 z_r \frac{\partial}{\partial z_r}, \qquad E_2 := z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}. \end{aligned} \tag{52}$$

By construction for any $\alpha \in \mathbb{C}$

$$\begin{aligned} &\Box_1 (\mathrm{F}_\circ)^\alpha = 0,\\ &E_1 (\mathrm{F}_\circ)^\alpha = \alpha\, (\mathrm{F}_\circ)^\alpha,\\ &E_2 (\mathrm{F}_\circ)^\alpha = \frac{1}{2}\, \left(\partial_{x_1} \left(x_1 (\mathrm{F}_\circ)^\alpha\right) - \partial_{x_2} \left(x_2 (\mathrm{F}_\circ)^\alpha\right)\right), \end{aligned} \tag{53}$$

therefore the action of the derivative E_2 vanishes on the integral but not the integrand

$$E_2 \int_\sigma (\mathrm{F}_\circ)^\alpha = 0 \qquad \text{for} \qquad \partial\sigma = \emptyset. \tag{54}$$

GKZ [28, 29] have shown that this system of differential equations are solved by the hypergeometric series $\gamma_i \notin \mathbb{Z}$

$$\Phi^\circ_{\mathbb{L},\boldsymbol{\gamma}} = \sum_{\boldsymbol{\ell} \in \mathbb{L}_\circ} \prod_{i=1}^3 \frac{z_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)}, \tag{55}$$

in this sum, we have $\boldsymbol{\ell} = n(1, 1, -2)$ with $n \in \mathbb{Z}$, and the condition $\sum_{i=1}^3 \gamma_i \mathbf{a}_i = (0, 0, -1)$ which can be solved using $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = \gamma(1, 1, -2) + (0, 0, -1)$, leading to

$$\Phi^\circ_{\mathbb{L},\boldsymbol{\gamma}} = \frac{1}{z_3} \sum_{n \in \mathbb{Z}} \frac{u_1^n}{\Gamma(n + \gamma + 1)^2 \Gamma(-2n + \gamma)}, \tag{56}$$

where we have introduced the new toric coordinate

$$u_1 := \frac{z_1 z_2}{z_3^2} = \frac{m_1^2 m_2^2}{\left(p^2 - (m_1^2 + m_2^2)\right)^2}, \tag{57}$$

which is the natural coordinate dictated by the invariance of the integral under the transformation $(x_1, x_2) \longrightarrow (\lambda x_1, \lambda x_2)$ and $(z_1, z_2, z_3) \longrightarrow (z_1/\lambda, z_2/\lambda, z_3/\lambda)$.

This GKZ hypergeometric function is a combination of ${}_3F_2$ hypergeometric functions

$$\begin{aligned} \Phi^\circ_{\mathbb{L},\boldsymbol{\gamma}} = \frac{1}{z_3^{1-2\gamma}} \Bigg(&\frac{u_1^{\gamma-1}}{\Gamma(\gamma)\Gamma(\gamma+2)}\, {}_3F_2 \left(\begin{matrix} 1,\ 1-\gamma,\ 1-\gamma \\ 1+\frac{\gamma}{2},\ \frac{3}{2}+\frac{\gamma}{2} \end{matrix} \middle| \frac{1}{4u_1} \right) \\ &+ \frac{u_1^\gamma}{\Gamma(\gamma+1)^2}\, {}_3F_2 \left(\begin{matrix} 1,\ \frac{1}{2}-\frac{\gamma}{2},\ \frac{1}{2}-\frac{\gamma}{2} \\ 1+\gamma,\ 1+\gamma \end{matrix} \middle| 4u_1 \right) \Bigg). \end{aligned} \tag{58}$$

For $\gamma = 0$ the series is trivially zero as the system is resonant and needs to be regularised [30, 31] . The regularisation is to use the functional equation for the Γ-function $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$ to replace the pole term by

$$\lim_{\epsilon \to 0} \frac{\Gamma(\epsilon)}{\Gamma(-2n + \epsilon)} = \Gamma(1 + 2n), \qquad n \in \mathbb{Z}\backslash\{0\}, \tag{59}$$

and write the associated regulated sum as

$$\pi_\circ = \lim_{\epsilon \to 0} \frac{1}{z_3} \sum_{n \in \mathbb{N}} \frac{u_1^n \Gamma(\epsilon)}{\Gamma(n+1)^2 \Gamma(-2n + \epsilon)}, \tag{60}$$

which is easily shown to be

$$\begin{aligned} \pi_\circ(z_1, z_2, z_3) &= \frac{1}{z_3}\, {}_2F_1 \left(\begin{matrix} \frac{1}{2}\ 1 \\ 1 \end{matrix} \middle| 4u_1 \right) = \frac{1}{\sqrt{z_3^2 - 4 z_1 z_2}},\\ &= \frac{1}{\sqrt{(p^2 - (m_1 + m_2)^2)(p^2 - (m_1 - m_2)^2)}}. \end{aligned} \tag{61}$$

This expression of course matches the expression for the integral on the torus

$$\pi_\circ(p^2, m_1^2, m_2^2) = \frac{1}{(2i\pi)^2} \int_{|x_1| = |x_2| = 1} \frac{dx_1 dx_2}{\mathcal{F}_\circ(x_1, x_2)}. \tag{62}$$

7.1 Picard-Fuchs operators

The creative telescoping algorithm [23, 24] gives the following operators acting on the integrand of (46):

$$\mathcal{T}_z := \mathcal{L}_z - \frac{\partial}{\partial x_1} Q_z \tag{63}$$

with $z = p^2$, m_1^2 or m_2^2, such that

$$\mathcal{T}_z \left(\frac{1}{\mathcal{F}_\circ(p^2, m_1^2, m_2^2; x_1, x_2)} \right) = 0. \tag{64}$$

For $z = p^2$ we have the differential operator

$$\begin{aligned} \mathcal{L}_{p^2} &:= p^2 \frac{d}{dp^2} + \frac{p^2(p^2 - m_1^2 - m_2^2)}{(p^2 - (m_1 + m_2)^2)(p^2 - (m_1 - m_2)^2)}, \\ Q_{p^2} &:= \frac{p^2(2m_2^2 - (p^2 - (m_1^2 + m_2^2))x_1)}{(p^2 - (m_1 + m_2)^2)(p^2 - (m_1 - m_2)^2) \mathcal{F}_\circ(x_1, 1, p^2, \underline{\xi}^2)}. \end{aligned} \tag{65}$$

For $z = m_1^2$ we have the differential operator

$$\begin{aligned} \mathcal{L}_{m_1^2} &:= m_1^2 \frac{d}{dm_1^2} - \frac{m_1^2(p^2 - m_1^2 + m_2^2)}{(p^2 - (m_1 + m_2)^2)(p^2 - (m_1 - m_2)^2)}, \\ Q_{m_1^2} &:= \frac{((p^2 - m_2^2)^2 - m_1^2(p^2 + m_2^2))x_1 - m_2^2(p^2 + m_1^2 - m_2^2)}{(p^2 - (m_1 + m_2)^2)(p^2 - (m_1 - m_2)^2)\, \mathcal{F}_\circ(x_1, 1, p^2, \underline{\xi}^2)}, \end{aligned}$$

with of course a similar operators $\mathcal{L}_{m_2^2}$ and $Q_{m_2^2}$ obtained by the exchange of m_1 and m_2.

On the integral $\pi_\circ(p^2, m_1^2, m_2^2)$ in (61) we have

$$\mathcal{L}_z \pi_\circ(p^2, m_1^2, m_2^2) = 0 \qquad \text{for} \qquad z = p^2, m_1^2, m_2^2. \tag{66}$$

The action of these differential operators on the Feynman integral (46) is easily obtained by integrating the total derivative. The action of the Picard-Fuchs operators on the Feynman integral $I_\circ(p^2, m_1^2, m_2^2, 2)$ are given by

$$L^\circ_{PF,(1)}\, I_\circ(p^2, m_1^2, m_2^2, 2) = -\frac{2}{(p^2-(m_1+m_2)^2)(p^2-(m_1-m_2)^2)}\,, \tag{67}$$

and

$$L^\circ_{PF,(2)}\, I_\circ(p^2, m_1^2, m_2^2, 2) = \frac{m_1^2-m_2^2-p^2}{(p^2-(m_1+m_2)^2)(p^2-(m_1-m_2)^2)}\,. \tag{68}$$

These equations integrate to the following expression

$$\begin{aligned} I_\circ(p^2, m_1^2, m_2^2) &= \frac{1}{\sqrt{(p^2-(m_1+m_2)^2)(p^2-(m_1-m_2)^2)}} \\ \times\log&\left(\frac{p^2-(m_1^2+m_2^2)-\sqrt{(p^2-(m_1+m_2)^2)(p^2-(m_1-m_2)^2)}}{p^2-(m_1^2+m_2^2)+\sqrt{(p^2-(m_1+m_2)^2)(p^2-(m_1-m_2)^2)}}\right)\,. \end{aligned} \tag{69}$$

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8 BIOGRAPHY

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