Differential equations and Motives of Feynman integrals

Pierre Vanhove

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Abstract. In this text

- We review the arguments why Feynman integrals are relative motivic periods. We illustrate this by giving the Hodge structure of some of some Feynman graph
- We discuss the differential equations for Feynman integral: we present the GKZ approach, and an extension of the Griffith-Dwork reduction for twisted differential needed for regulated Feynman integrals. We give some example of differential equations.
- We give some analytic evaluations of some Feynman integrals.

These notes are based on the following work [1–6] and review articles [7, 8].

These lecture notes are dedicated to Pierre Cartier.

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Contents

Part 1. Feynman integral in Physics

Feynman integrals are key ingredients in various areas of physics, and their accurate calculation, whether analytically or numerically, remains a significant hurdle in advancing our understanding of physical phenomena.

They are the key objects used to understand fundamental interactions and the elementary constituents in Nature. It is well known that scattering amplitudes are used in particle physics to compare the theoretical predictions to experimental measurements in particle colliders (see [9–11] for instance). More recently the use of modern developments in scattering amplitudes have been extended to gravitational physics like unitarity methods to gravitational wave physics [12–14]. Feynman integrals enters as well in the evaluation of the correlation functions of quantum fields at the end of inflation are important in cosmology as they provide the seeds for the formation of structure in the Universe [15, 16].

Feynman integrals are function of the kinematics invariants $\underline{s} = \{s_{ij} = (p_i + p_j)^2, 1 \leq i, j \leq n\}$ where p_i are the incoming momenta of the external particle, and the internal masses $\underline{m}^2 = (m_1, \ldots, m_r)$. They are multivalued functions presenting branch cuts associated to particle production in the complex energy plane. They are given by multi-dimensional integrals (reviewed in section 2.1), which are difficult to analytically or numerically evaluate for the physical processes needed for data analysis. A lot of efforts in the physics community is devoted to precise evaluation of the Feynman integrals as function of their physical parameters. One important question is to understand the class of functions that appear in the analytic expression of Feynman integrals. And that is the optimal representation, bearing in mind the need of numerical evaluation for a wide range of physical parameter to compare with the experimental data from particle physics experiments or gravitational wave signals.

In particular, identifying the specific types of special functions required to evaluate Feynman integrals has been an ongoing challenge since the early days of quantum field theory [17, 18] and continues to be an active research field as recently reviewed, for instance in [19–24].

The answer to these questions depends very strongly on which properties one wants to display. An analytic representation suitable for an high precision numerical evaluation may not be the one that displays the mathematical nature of the integral.

For instance the two-loop sunset integral has received many different, but equivalent, analytical expressions: hypergeometric and Lauricella functions [25,26], Bessel integral representation [27–29], Elliptic integrals [30, 31], Elliptic polylogarithms [3, 32–37] and trilogarithms [3].

Feynman integral satisfy several remarkable important properties:

(1) They are D-finite function, that is they satisfy a (D-module) of partial differential equations with respect to its parameters [38, 39]. Physics have designed integrations by part algorithm to generate the set of differential operators acting on the Feynman integrals [40, 41]. There exists various implementation algorithms [42–45] but their efficiency drops for complicated Feynman graphs. So alternative approach are required. The evaluation of cosmological correlators [46] needs an analytical regularisation of the associated Feynman integrals which is not possible with the above cited integration-by-part algorithms. The set of differential

operators acting on a Feynman integral gives important information about its analytic nature. Moreover, the differential equation is important for evaluating physical observables by solving the system of differential equations associated with Feynman integrals, either analytically, in perturbation with respect to the kinematic parameters or numerically. For instance, the differential operator has real singularities at the position of thresholds and pseudo-thresholds, and the order of the differential operator is connected to the underlying algebraic geometry of the singular locus of the integrand [4]. Intriguingly, there is growing evidence suggesting that certain Feynman integrals correspond to relative period integrals of singular Calabi–Yau geometries, a connection explored in a number of studies, including [2, 3, 47–59]. We will present this approach in part 3.

(2) Feynman integrals are relative period integrals of a variation of mixed Hodge structure [47,60]. The relation between Feynman integrals and periods is described in 2.2. This connexion between Feynman integrals and periods makes more natural the appearance of Calabi–Yau geometries noticed by physicits. This gives a algebraic geometrical setup for identifying the class of functions appearing from Feynman integrals, and give some hope to make some complexity statements of the computations needed in physics.

Part 2. Feynman integrals and motivic periods

2.1. Feynman integrals

In these section we give different representations for Feynman integrals. We start with the propagator representation that naturally arises from the rules for the interactions and propagations of elementary particles. Then we give two derivations of the parametric representation which is better suited for the derivations of the algebraic geometry associated to a Feynman graph.

2.1.1. The Feynman integrals propagator representation. A connected Feynman graph Γ is determined by the number n of propagators (internal edges), the number l of loops, and the number v of vertices. The Euler characteristic of the graph relates these three numbers as $l = n - v + 1$, therefore only the number of loops l and the number n of propagators are needed.

In a momentum representation an l-loop with n propagators Feynman graph reads

(2.1.1)
$$I_\Gamma^D(p_i, m_i) := \frac{(\mu^2)^{\sum_{i=1}^n \nu_i - l\frac{D}{2}}}{\pi^{\frac{lD}{2}}} \frac{\prod_{i=1}^n \Gamma(\nu_i)}{\Gamma(\sum_{i=1}^n \nu_i - l\frac{D}{2})} \int_{(\mathbb{R}^{1,D-1})^l} \frac{\prod_{i=1}^l d^D\ell_i}{\prod_{i=1}^n (q_i^2 - m_i^2 + i\varepsilon)^{\nu_i}}$$

where μ^2 is a scale of dimension mass squared. Some of the vertices are connected to external momenta p_i with $i = 1, \dots, v_e$ with $0 \le v_e \le v$. The internal masses are positive $m_i \ge 0$ with $1 \le i \le n$. Finally $+i\varepsilon$ with $\varepsilon > 0$ is the Feynman prescription for the propagators for a space-time metric of signature $(+ - \cdots -)$, and D is the space-time dimension, and we set $\nu := \sum_{i=1}^n \nu_i$.

2.1.2. Parametric representation. Introducing the size l vector of loop momenta $L^\mu := (\ell_1^\mu, \dots, \ell_l^\mu)^T$ corresponding to the minimal set of linearly independent momenta flowing along the graph. We introduce as well the size v_e vector of external momenta $P^\mu = (p_1^\mu, \dots, p_{v_e}^\mu)^T$. Since we take the convention that all momenta are incoming momentum conservation implies that $\sum_{i=1}^{v_e} p_i = 0$.

Putting the momenta q_i flowing along the graph in a size n vector $q^\mu := (q_1^\mu, \dots, q_n^\mu)^T$. Momentum conservation at each vertices of the graph gives the relation

$$q^\mu = \rho \cdot L^\mu + \sigma \cdot P^\mu \,. \tag{2.1.2}$$

The matrix ρ of size $n \times l$ has entries taking values in $\{-1, 0, 1\}$, the signs depend on an orientation of the propagators. The matrix σ of size $n \times v_e$ has only entries taking values in $\{0, 1\}$ because have the convention that all external momenta are incoming.

We introduce the Schwinger proper-times α_i conjugated to each internal propagators using

$$\frac{1}{(d_i^2)^{\nu_i}} = \frac{1}{\Gamma(\nu_i)} \int_0^\infty e^{-\alpha d_i^2} \alpha^{\nu_i - 1} d\alpha_i, \tag{2.1.3}$$

to get

$$I_\Gamma^D(p_i, m_i) = \frac{(\mu^2)^{\nu - l\frac{D}{2}}}{\pi^{\frac{lD}{2}} \Gamma(\nu - l\frac{D}{2})} \int_{(\mathbb{R}^{1,D-1})^l} \int_{[0,+\infty[^n} e^{-\sum_{i=1}^n \alpha_i (q_i^2 - m_i^2 + i\varepsilon)} \prod_{i=1}^n \frac{d\alpha_i}{\alpha_i^{1-\nu_i}} \prod_{i=1}^l d^D \ell_i \,. \tag{2.1.4}$$

Setting $T = \sum_{i=1}^n \alpha_i$ and $\alpha_i = T\, x_i$ this integral becomes

$$I_\Gamma^D(p_i, m_i) = \frac{(\mu^2)^{\nu - l\frac{D}{2}}}{\pi^{\frac{lD}{2}} \Gamma(\nu - l\frac{D}{2})} \int_{(\mathbb{R}^{1,D-1})^l} \int_{[0,+\infty[^{n+1}} e^{-T\mathcal{Q}} \delta(\sum_{i=1}^n x_i - 1) \frac{dT}{T^{1-\nu}} \prod_{i=1}^n \frac{dx_i}{x_i^{1-\nu_i}} \prod_{i=1}^l d^D \ell_i \,, \tag{2.1.5}$$

where we have defined

$$\mathcal{Q} := \sum_{i=1}^n x_i (q_i^2 - m_i^2) \,. \tag{2.1.6}$$

Introducing the $n \times n$ diagonal matrix $X = \mathrm{diag}(x_1, \cdots, x_n)$, one rewrites this expression exhibiting the quadratic form in the loop momenta

$$\mathcal{Q} = (L^\mu + \Omega^{-1} Q^\mu)^T \cdot \Omega \cdot (L^\mu + \Omega^{-1} Q^\mu) - J - (Q^\mu)^T \cdot \Omega^{-1} \cdot Q^\mu \,, \tag{2.1.7}$$

where we have defined

$$\Omega := \rho^T X \rho, \quad Q^\mu := \rho^T X \sigma\, P^\mu, \quad J := (P^\mu)^T \sigma^T X \sigma P^\mu + \sum_{i=1}^n x_i (m_i^2 - i\varepsilon) \tag{2.1.8}$$

and we made use of the fact that the square $l \times l$ matrix Ω is symmetric and invertible. Performing the Gaussian integral over the loop momenta L^μ one gets

$$I_\Gamma^D(p_i, m_i) = \frac{(\mu^2)^{\nu - l\frac{D}{2}}}{\Gamma(\nu - l\frac{D}{2})} \int_{[0,+\infty[^{n+1}} e^{-T\mu^2 \mathcal{F}_\Gamma \mathcal{U}_\Gamma^{-1}} \frac{\delta(\sum_{i=1}^n x_i - 1)}{\mathcal{U}_\Gamma^{l\frac{D}{2}}} \prod_{i=1}^n \frac{dx_i}{x_i^{1-\nu_i}} \frac{dT}{T^{1-\nu+l\frac{D}{2}}} \,. \tag{2.1.9}$$

Introducing the notations for the first Symanzik polynomial

$$\mathcal{U}_\Gamma := \det(\Omega) \tag{2.1.10}$$

and using the adjugate matrix of $\mathrm{Adj}(\Omega) := \det \Omega\, \Omega^{-1}$, we define the second Symanzik polynomial

$$\mathcal{F}_\Gamma := \frac{-J\,\mathcal{U}_\Gamma + (Q^\mu)^T \cdot \mathrm{Adj}(\Omega) \cdot Q^\mu}{\mu^2} \,. \tag{2.1.11}$$

A modern approach to the derivation of these polynomials using graph theory is review in section 2.1.4.

Performing the integration over T, one arrives at the expression for a Feynman graph given in quantum field theory textbooks like [61]

$$I_\Gamma^D(p_i,m_i) = \int_{[0,+\infty[^n} \frac{\mathcal{U}_\Gamma^{\nu-(l+1)\frac{D}{2}}}{\mathcal{F}_\Gamma^{\nu-l\frac{D}{2}}}\,\delta(\sum_{i=1}^n x_i - 1)\prod_{i=1}^n x_i^{\nu_i-1}dx_i\,. \tag{2.1.12}$$

Since the coordinate scaling $(x_1,\dots,x_n)\to\lambda(x_1,\cdots,x_n)$ leaves invariant the integrand and the domain of integration, we can rewrite this integral as

$$I_\Gamma^D(p_i,m_i) = \int_\Delta \prod_{i=1}^n x_i^{\nu_i-1}\,\frac{\mathcal{U}_\Gamma^{\nu-(l+1)\frac{D}{2}}}{\mathcal{F}_\Gamma^{\nu-l\frac{D}{2}}}\,\omega \tag{2.1.13}$$

where ω is the differential $n-1$-form

$$\omega := \sum_{j=1}^n (-1)^{j-1}\,x_j\,dx_1\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n \tag{2.1.14}$$

where $\widehat{dx_j}$ means that dx_j is omitting in this sum. The domain of integration $\mathcal{D}$ is defined as

$$\Delta := \{[x_1,\cdots,x_n]\in\mathbb{P}^{n-1}|x_i\in\mathbb{R}, x_i\geq 0\}. \tag{2.1.15}$$

This representation as a differential form in $\mathbb{P}^{n-1}$ is at the heart of the cohomological description given in section 2.2

2.1.3. Maximal cut. Another closely related integral is the so-called maximal cut of a Feynman graph has a nice parametric representation. This integral is of huge interest because this is an explicit representation of a "fundamental" period associated with a Feynman graph.

The maximal cut is defined by

$$\pi_\Gamma(\underline{s},\underline{\xi}^2,D) := \frac{1}{\Gamma(\omega)(2i\pi)^n\pi^{\frac{lD}{2}}}\int_{(\mathbb{R}^{1,D-1})^L}\prod_{i=1}^l d^D\ell_i\prod_{i=1}^n\delta(q_i^2-m_i^2+i\varepsilon), \tag{2.1.16}$$

of the Feynman integral $I_\Gamma(\underline{s},\underline{\xi}^2,D)$ which is obtained from the Feynman integral in (2.1.1) by replacing all propagators by a delta-function

$$\frac{1}{d^2}\to\frac{1}{2i\pi}\delta(d^2). \tag{2.1.17}$$

Using the representation of the δ-function

$$\delta(x) = \int_{-\infty}^{+\infty} dwe^{iwx}, \tag{2.1.18}$$

we obtain that the integral is

$$\pi_\Gamma(\underline{s},\underline{m},D) := \frac{1}{\Gamma(\omega)(2i\pi)^n\pi^{\frac{lD}{2}}}\int_{\mathbb{R}^{(1,D-1)L}} e^{-i\sum_{i=1}^n x_i(\ell_i^2+m_i^2-i\varepsilon)}\prod_{i=1}^l d^D\ell_i\prod_{i=1}^n dx_i\,. \tag{2.1.19}$$

At this stage the integral is similar to the one leading to the parametric representation with the replacement $x_r \to ix_r$ with $x_r \in \mathbb{R}$. Setting $\tilde{x}_r = ix_r$ and performing the Gaussian integrals over the loop momenta, we get

$$\pi_n(\underline{s}, \underline{\xi}^2, D) := \frac{1}{(2i\pi)^n} \int_{i\mathbb{R}^n} \frac{\widetilde{\mathcal{U}_\Gamma}^{\omega-\frac{D}{2}}}{\widetilde{\mathcal{F}_\Gamma}^{\omega}} \prod_{i=1}^{n} \delta(1 - \sum_{i=1}^{n} \tilde{x}_i) d\tilde{x}_i \,. \tag{2.1.20}$$

using the projective nature of the integrand we have $\frac{\widetilde{\mathcal{U}_\Gamma}^{\omega-D/2}}{\widetilde{\mathcal{F}_\Gamma}^{\omega}} = i^{-n} \frac{\mathcal{U}_\Gamma^{\omega-D/2}}{\mathcal{F}_\Gamma^{\omega}}$ and the integral can be rewritten as the torus integral

$$\pi_\Gamma(\underline{s}, \underline{\xi}^2, D) := \frac{1}{(2i\pi)^n} \int_{|x_1]=\cdots=|x_{n-1}|=1} \frac{\mathcal{U}_\Gamma^{\omega-D/2}}{\mathcal{F}_\Gamma^{\omega}} \prod_{i=1}^{n-1} dx_i \,. \tag{2.1.21}$$

This integral shares the same integrand with the Feynman integral I_Γ in (2.1.13) but the cycle of integration differs since we are integrating over a n-torus. We show in section 3.4.1 that this maximal cut arises naturally from the toric formalism.

2.1.4. Graph polynomials.

Definition 2.1.1. A *Feynman* graph Γ is a finite collection of vertices $V(\Gamma)$, edges $E(\Gamma)$, and half-edges $H(\Gamma)$ satisfying the usual definitions; edges are adjacent to two vertices, and half-edges are adjacent to a single vertex, and allowing multiple edges between pairs of vertices. We let $e(\Gamma) = |E(\Gamma)|$. To each edge of Γ there is a mass variable $m_e \in \mathbb{R}$ and to each half-edge there is a momentum vector $p_h \in \mathbb{R}^{1,D-1}$ in the D-dimensional Minkowski space equipped with a metric of signature $(1, D-1)$. To each half edge of Γ attach a vector $p_h \in \mathbb{C}^D$ subject to the so-called momentum conservation relation

$$\sum_{h\in H(\Gamma)} p_h = 0. \tag{2.1.22}$$

For physical processes these vectors belong to of the D-dimensional Minkowski space $\mathbb{R}^{1,D-1}$. The analytic properties of the Feynman integrals are studied by using analytic continuation in the multi-dimensional complex plane.

Henceforward, this general setup is simplified by the assumption that each vertex of Γ has a single outgoing half-edge. Therefore, one may view Γ as a graph in the usual sense, allowing multiple edges between vertices. To simplify notation, view momenta as being attached to vertices, and write p_v instead of p_h. Furthermore, we consider only the completely massive case with $m_e^2 > 0$ and all external vectors are of non-zero norm $p_v \cdot p_v \neq 0$. Often, we will view m_e, p_v as having complex values instead of real values to simplify the algebro-geometric arguments in this paper.

We associate to the graph Γ two polynomials which are defined as follows [23, 62]. Let $\{x_e \mid e \in e(\Gamma)\}$ be variables attached to all edges of Γ. A spanning tree of Γ is a subgraph T of Γ which contains all vertices of Γ, and so that $b_1(\mathsf{T}) = 0$ and $b_0(\mathsf{T}) = 1$. For each spanning tree T of Γ we attach the monomial $x^{\mathsf{T}} = \prod_{e\notin \mathsf{T}} x_e$. The *first Symanzik polynomial* is the polynomial

$$\mathcal{U}_\Gamma = \sum_{\substack{\text{Spanning}\\ \text{trees of }\Gamma}} x^{\mathsf{T}} \,. \tag{2.1.23}$$

A spanning k-forest of Γ is a subgraph F of Γ containing all vertices of Γ and so that $h_1(\mathsf{F}) = 0$ and $h_0(\mathsf{F}) = k$. We attach the polynomial $x^{\mathsf{F}} = \prod_{e\notin \mathsf{F}} x_i$ to each spanning 2-forest. A 2-forest is a disjoint union of two sub-trees $\mathsf{F} = \mathsf{T}_1 \cup \mathsf{T}_2$, and we define $s_{\mathsf{F}} = \sum_{(v_1,v_2)\in \mathsf{F}=\mathsf{T}_1\cup\mathsf{T}_2} p_{v_1}\cdot p_{v_2}$ Where the $\cdot$-product is the scalar product on $\mathbb{C}^D$. Then

$$\mathcal{V}_\Gamma = \sum_{\substack{\text{Spanning}\\ \text{2-forests of }\Gamma}} s_{\mathsf{F}} x^{\mathsf{F}}, \qquad \mathcal{F}_\Gamma(\vec{s},\vec{m}) = \mathcal{U}_\Gamma \left(\sum_{e\in e(\Gamma)} m_e^2 x_e\right) - \mathcal{V}_\Gamma\,. \tag{2.1.24}$$

The polynomial $\mathcal{F}_\Gamma(\vec{s},\vec{m})$ is called the *second Symanzik polynomial* of Γ. This is a homogeneous polynomial of degree $L+1$ in the variables $x_e : e \in e(\Gamma)$, where $L = b_1(\Gamma)$. This L is often called the *loop order* of Γ. Henceforward, we will write instead $\mathcal{F}_\Gamma$ to simplify our notation.

2.1.5. The graph polynomials. The graph polynomials are special homogeneous polynomials because either their coefficients are all set to 1 in $\mathcal{U}_\Gamma$ or the different coefficients are related by the kinematical relations as in $\mathcal{F}_\Gamma$. The form of the Feynman integrals is very similar to a generalized Euler integral as considered by GKZ [63,64] but the fact that the graph polynomial are not generic makes these integrals very special.

- The first Symanzik polynomial $\mathcal{U}(x_1,\ldots,x_n)$ is an homogeneous polynomial of degree l in the Feynman parameters x_i and it is at most linear in each of the x_i variables. It does not depend on the physical parameters. This polynomial is also known as the Kirchhoff polynomial of graph Γ. Which is as well the determinant of the Laplacian of the graph see [65, eq (35)] for a definition.
- The polynomial $\mathcal{U}(x_1,\ldots,x_n)$ can be seen as the determinant of the period matrix Ω of the punctured Feynman graph [7], i.e. the graph with amputated external legs. Or equivalently it can be obtained by considering the degeneration limit of a genus l Riemann surfaces with n punctures. This connection plays an important in understanding the quantum field theory Feynman integrals as the $\alpha' \to 0$ limit of the corresponding string theory integrals [66,67].
- The graph polynomial $\mathcal{F}_\Gamma$ is homogeneous of degree $l+1$ in the variables $(x_1,\ldots,x_n)$. This polynomial depends on the internal masses ξ_i^2 and the kinematic invariants $s_{ij} = (p_i\cdot p_j)/\mu^2$. The polynomials $\mathcal{G}_{ij}$ are at most linear in all the variables x_i since this is given by the spanning 2-trees [65]. Therefore if all internal masses are vanishing then $\mathcal{F}_\Gamma$ is linear in the Feynman parameters x_i.
- The $\mathcal{U}_\Gamma$ are independent of the dimension of space-time. But this is not the case for the coefficient of $\mathcal{F}_\Gamma$ because of the linear relation between the external vectors which depend on the dimension of D. This is important as this affects the nature of the singular locus of the Feynman integral as we will discuss in section 2.2.3 for two-loop graphs.
- All the physical parameters, the internal masses ξ_i^2 and the kinematic variables $s_{ij} = (p_i\cdot p_j)/\mu^2$ (that includes the external masses) enter linearly. This will be important for the toric approach described in §3.4.

2.1.6. Ultraviolet and infrared divergences. The space-time dimension enters in the powers of $\mathcal{U}_\Gamma$ and $\mathcal{F}_\Gamma$ in the parametric representation for the Feynman graphs. One can prove that the Feynman integral as a meromorphic function of $(\underline{\nu}, D)$ in $\mathbb{C}^{1+n}$. The Feynman integral I_Γ may diverge for integer values of D and the exponents ν_i, but there is an open subset of $(D, \nu_1, \ldots, \nu_n) \in \mathbb{C}^{n+1}$ where the integral converges. The (unique) value of the Feynman integral is defined by analytic continuation We refer to the book by [68] for a throughouly discussion.

2.2. Motives

In the survey [69], Kontsevich and Zagier give the following definition of the ring $\mathcal{P}$ of periods: *a period is a complex number that can be expressed as an integral of an algebraic function over an algebraic domain.*

In more precise terms $z \in \mathcal{P}$ is a period if its real part $\Re\mathrm{e}(z)$ and imaginary part $\Im\mathrm{m}(z)$ are of the form

$$\int_\Delta \frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)} \prod_{i=1}^n dx_i \,, \tag{2.2.1}$$

where $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ belong to $\mathbb{Z}[x_1, \cdots, x_n]$ and the domain of integration Δ is a domain in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Since sums and products of periods remain periods, therefore the periods form a ring, and the periods form a sub $\bar{\mathbb{Q}}$-algebra of $\mathbb{C}$ (where $\bar{\mathbb{Q}}$ is the set of algebraic numbers).

Examples of periods represented by single integral

$$\sqrt{2} = \int_{2x^2 \leq 1} dx; \quad \log(2) = \int_{1 \leq x \leq 2} \frac{dx}{x} \tag{2.2.2}$$

or by a double integral

$$\zeta(2) = \frac{\pi^2}{6} = \int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_2}{t_2} \frac{dt_1}{1 - t_1} \,. \tag{2.2.3}$$

This example the value at 1 of the dilogarithm $\mathrm{Li}_2(1) = \zeta(2)$ where

$$\mathrm{Li}_2(x) = \sum_{n \geq 1} \frac{x^n}{n^2}; \qquad \text{for} \qquad 0 \leq x < 1 \,. \tag{2.2.4}$$

In particular, it is familiar to the quantum field theory practitioner that the finite part of one-loop amplitudes in four dimensions is expressed in terms of dilogarithms. At one-loop around four dimensions the structure of the integrals is now very well understood [70–75]. General formulas for all one-loop amplitudes can be found in [76, 77], some higher-loop recent considerations can be found in [78].

Under change of variables and integration a period can take a form given in (2.2.1) or not. One example is π which can be represented by the following two-dimensional or one-dimensional integrals

$$\pi = \int_{x^2+y^2 \leq 1} dx dy = 2 \int_0^{+\infty} \frac{dx}{1 + x^2} = \int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}} \tag{2.2.5}$$

or by the following contour integral

$$2i\pi = \oint \frac{dz}{z} \,. \tag{2.2.6}$$

At a first sight, the definition of a period given in (2.2.1) and the Feynman representation of the Feynman graph in (2.1.12) *look similar*. A relation between these two objects was remarked in the pioneer work of Broadhurst and Kreimer [79, 80].

We actually need a more general definition of abstract periods given in [69]. Let's consider $X(\mathbb{C})$ a smooth algebraic variety of dimension n over $\mathbb{Q}$. Consider $D \subset X$ a divisor with normal crossings, which means that locally this is a union of coordinate hyperplanes of dimension $n-1$. Let $\eta \in \Omega^n(X)$, and let $\Delta \in H_n(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q})$ a singular n-chain on $X(\mathbb{C})$ with boundary on the divisor $D(\mathbb{C})$. To the quadruple (X, D, ω, Δ) we can associate a complex number called the *period of the quadruple*

$$P(X, D, \omega, \Delta) = \int_\Delta \eta \,. \tag{2.2.7}$$

In order that this definition is compatible with the example of periods given previously, in particular the behaviour under change of variables one needs to introduce the notation of equivalence classes of quadruples for periods leading to the same period (2.2.7). To this end one defines the space $\mathbf{P}$ of *effective periods* as the $\mathbb{Q}$-vector space of equivalence classes modulo (a) linearity in η and Δ, (b) under change variables, (c) and integration by part of Stokes formula. The map from $\mathbf{P}$ to the space of periods $\mathcal{P}$ is clearly surjective, and it is conjectured to be injective providing an isomorphism. We refer to the review article [69] for more details.

2.2.1. Mixed Hodge structures for Feynman graph integrals. We define the vanishing loci for the Symanzik polynomials attached to the graph Γ:

$$X_{\Gamma;D} = \{\mathcal{F}_\Gamma(x_i) = 0 | x_i \in \mathbb{P}^{e(\Gamma)-1}(\mathbb{R})\}; \qquad Y_\Gamma = \{\mathcal{U}_\Gamma(x_i) = 0 | x_i \in \mathbb{P}^{e(\Gamma)-1}(\mathbb{R})\} \,. \tag{2.2.8}$$

Notice that the vanishing locus for $\mathcal{F}_\Gamma$ depends on the space-time dimension D through the linear relations between the external momenta.

We recall the expression for the Feynman integral (up to a normalising constant) in fixed dimension D to the graph Γ derived in previously

$$I_\Gamma(\vec{s}, \vec{m}) = \int_\Delta \omega_{\Gamma;D}, \quad \omega_{\Gamma;D} := \frac{\mathcal{U}_\Gamma^{e(\Gamma)-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma^{e(\Gamma)-\frac{LD}{2}}} \Omega_0, \quad \Omega_0 = \sum_{e\in E(\Gamma)} \bigwedge_{e'\neq e} dx_{e'}. \tag{2.2.9}$$

The integrand is a differential form representing a class of $\mathrm{H}^{e(\Gamma)-1}(\mathbb{P}^{e(\Gamma)-1} - Z_{\Gamma;D})$ where $Z_{\Gamma;D}$ is the singular locus of the integrand. We see that if $e(\Gamma) - \frac{(L+1)D}{2} < 0$, $Z_{\Gamma;D} = Y_\Gamma$ and that if and $e(\Gamma) - \frac{LD}{2} > 0$ then $Z_{\Gamma;D} = X_{\Gamma;D}$. If neither of these inequalities is satisfied, then $Z_{\Gamma;D} = X_{\Gamma;D} \cup Y_\Gamma$. The domain of integration is $\Delta := \{[x_0 : \cdots : x_N] \mid x_i \in \mathbb{R}_{\geq 0}\}$. The Feynman integral is a function of the mass parameters and kinematic invariants, respectively:

$$\vec{m} := \left\{m_1^2, \ldots, m_{e(\Gamma)}^2\right\} \in \mathbb{R}_{>0}^{e(\Gamma)}, \qquad \vec{s} = \{p_i \cdot p_j, i, j \in v(\Gamma)\}. \tag{2.2.10}$$

When $|v(\Gamma)| > D$, not all the scalar products are independent, and the number of independent variables satisfies certain Gram determinant conditions [81]. This affects the nature of the period integrals as explained in [4] and exemplified with the double box case result given in section 2.2.3.

Although the integrand η is a closed form such that $\eta \in \mathrm{H}^{e(\Gamma)-1}(\mathbb{P}^{e(\Gamma)-1} - Z_{\Gamma;D})$, in general the domain Δ has a boundary and therefore its homology class is not in $\mathrm{H}_{e(\Gamma)-1}(\mathbb{P}^{e(\Gamma)-1} - Z_{\Gamma;D})$. This difficulty will be resolved by considering the relative cohomology.

In general the polar part $Z_{\Gamma;D}$ of the differential form entering the expression of the Feynman integral in (2.1.13), intersects the boundary of the domain of integration $\partial\Delta \cap Z_{\Gamma;D\neq}\emptyset$. We need to consider a blow-up in $\mathbb{P}^{e(\Gamma)-1}$ of linear space $f : \mathcal{P} \to \mathbb{P}^{e(\Gamma)-1}$, such that all the vertices of Δ lie in $\mathcal{P}\backslash\mathcal{X}$ where $\mathcal{X}$ is the strict transform of $Z_{\Gamma;D}$. Let $\mathcal{B}$ be the total inverse image of the coordinate simplex $\{x_1x_2 \cdots x_{e(\Gamma)} = 0|[x_1, \ldots, x_{e(\Gamma)}] \in \mathbb{P}^{e(\Gamma)}\}$.

As been explained by Bloch, Esnault and Kreimer in [60] all of this lead to the mixed Hodge structure associated to the Feynman graph

$$M(\Gamma) := H^{e(\Gamma)-1}(\mathcal{P}\backslash\mathcal{X}, \mathcal{B}\backslash\mathcal{B} \cap \mathcal{X}; \mathbb{Q})\,. \tag{2.2.11}$$

In the second part of this text we will give a description of the motive for particular Feynman integral. A list of mixed Hodge structures associated with vacuum graphs can be found in [82, 83].

2.2.2. The massive one-loop triangle. As an illustration we consider the example of the mixed Hodge structure for the massive one-loop triangle following [84, section 13].

The Feynman integral is given by

$$I_{\triangleright}(p_1, p_2, p_3) = \frac{\mu^2}{\pi^2} \int \frac{d^4\ell}{\ell^2\,(\ell + p_1)^2\,(\ell - p_3)^2}\,, \tag{2.2.12}$$

with $p_i^2 = m_i^2 \neq 0$ non vanishing external masses and the momentum conservation constraint $p_1+p_2+p_3 = 0$. This integral is finite being free of ultraviolet divergences in four dimensions, and of infrared divergences for non vanishing masses.

A direct application of section 2.1.1, for the case of a graph at $l = 1$ loop, with $n = 3$ edges in $D = 4$ dimensions, leads to the Schwinger representation

$$I_{\triangleright}(p_1, p_2, p_3) = \mu^2 \int_{x_1,x_2,x_3 \geq 0} \frac{x_1 dx_2 \wedge dx_3 - x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2}{(x_1 + x_2 + x_3)(x_1x_2m_3^2 + x_1x_3m_2^2 + x_2x_3m_1^2)}\,. \tag{2.2.13}$$

The graph hyper-surfaces are the line $X_\Gamma^0 = \{x_1 + x_2 + x_3 = 0|x_i \in \mathbb{P}^2(\mathbb{R})\}$ and the conic $X_\triangleright := \{x_1x_2m_3^2 + x_1x_3m_2^2 + x_2x_3m_1^2 = 0|x_i \in \mathbb{P}^2(\mathbb{R})\}$. The polar part of the integrand is $Z_{\Gamma;D} = X_\Gamma^0 \cup X_\triangleright$. The domain of integration is the triangle $\Delta = \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$. The intersection of the graph polar part and the domain of integration is given by the three points $Z_{\Gamma;D} \cap \Delta = \{[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]\}$. Let $f : \mathcal{P} \to \mathbb{P}^2$ the blow-up of the three vertices $\{(x_1 = x_2 = 0), (x_1 = x_3 = 0), (x_2 = x_3 = 0)\}$. Let $E_i \subset \mathcal{P}$ the exceptional divisors such that E_i lies over the intersection $(x_j = x_k = 0)$, with (i, j, k) a permutation of $(1, 2, 3)$. Finally, let $F_i \subset \mathcal{P}$ be the strict transform of the locus $\{x_i = 0\}$. The blown up domain of integration is the hexagon $\mathcal{B} := f^*\Delta = \cup_{i=1}^3 E_i \cup_{i=1}^3 F_i$. If we denote by $\mathcal{X} = \mathcal{L} \cup \mathcal{C}$ the union of the strict transform $\mathcal{L}$ of the line $x_1 + x_2 + x_3 = 0$ and the strict transform $\mathcal{C}$ of the conic $X_\triangleright$. The mixed Hodge structure of the one-loop massive triangle graph in four dimension is given by (2.2.11).

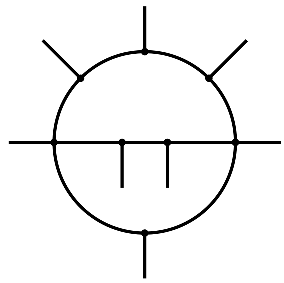


FIGURE 1. A two-loop graphs with $a = 4$, $b = 3$ and $c = 2$.

2.2.3. Motives for two-loop graphs. In this work, we focus our attention almost solely on two-loop Feynman graphs, particularly those of (a, b, c) type as defined below. An example of such a graph is depicted in Figure 1.

DEFINITION 2.2.1. Let (a, b, c) denote a graph with $a + b + c - 1$ vertices and $e(\Gamma) = a + b + c$ edges, so that two of these vertices are trivalent and connected by three chains of edges containing a, b, and c edges respectively.

In this case $L = 2$, if $a + b + c \leq D$ the vanishing locus of $\mathbf{U}_{(a,b,c)}$ is a quadric hypersurface in $\mathbb{P}^{a+b+c-1}$, and the periods of the mixed Hodge structure in (2.2.11) are rather simple, in the sense that the mixed Hodge structure is mixed Tate. In $D = 4$ dimensions this was shown by Brown in [**47**]. On the other hand, if $a+b+c > D$ then the denominator of the integrand in (2.2.9) is $\mathcal{F}_\Gamma$, which is a cubic hypersurface. The cohomology of a cubic hypersurface need not have mixed Tate cohomology, so from the perspective of Hodge theory, this is a more complicated situation. The first step toward understanding this is to study the mixed Hodge structure on $\mathrm{H}^*(X_{(a,b,c);D}; \mathbb{Q})$ when $a + b + c > D$. The main result of [4] is about the graphs of the type $(a, 1, c)$. The mixed Hodge structures of $(a, 1, c)$ graph hypersurfaces are simple, in the sense that they come from hyperelliptic curves.

DEFINITION 2.2.2. (1) Let $\mathbf{MHS}_\mathbb{Q}$ denote the abelian category of $\mathbb{Q}$-mixed Hodge structures.
(2) The largest extension-closed subcategory of $\mathbf{MHS}_\mathbb{Q}$ containing the Tate twists of $\mathrm{H}^1(C; \mathbb{Q})$ for every hyperelliptic curve C is called $\mathbf{MHS}_\mathbb{Q}^{\mathrm{hyp}}$.
(3) The largest extension-closed subcategory of $\mathbf{MHS}_\mathbb{Q}$ containing the Tate twists of $\mathrm{H}^1(E; \mathbb{Q})$ for every elliptic curve E is called $\mathbf{MHS}_\mathbb{Q}^{\mathrm{ell}}$.

THEOREM (Theorem [4]). *For any positive integers a, c and any space-time dimension D,*

$$\mathrm{H}^{a+c-1}(X_{(a,1,c);D}; \mathbb{Q}) \in \mathbf{MHS}_\mathbb{Q}^{\mathrm{hyp}}.$$

It is interesting to compare this to the results of Marcolli–Tabuada for the two-loop sunset [**85**].
In other words, the cohomology of $X_{(a,b,c);D}$ is obtained by iterated extension involving Tate twists of the cohomology of hyperelliptic curves, and the Tate Hodge structure.

THEOREM (Theorem [4]). *If $3D/2 \leq a + c$ then*

$$\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - {}^{\boldsymbol{b}}X_{(a,1,c);D}; B_\Gamma - (B_\Gamma \cap {}^{\boldsymbol{b}}X_{(a,1,c);D})) \in \mathbf{MHS}_\mathbb{Q}^{\mathrm{hyp}}.$$

In particular, this means that the Feynman integrals in all of these cases can be constructed from algebraic functions and periods of htyperelliptic curves.

In [4] we give a detailed analysis of the motive for the double box graph hypersurface, $X_{(3,1,3);D}$. Recall that for a projective algebraic variety X, $\mathrm{H}^*(X;\mathbb{Q})$ is equipped with a canonical mixed Hodge structure, and that $\mathrm{Gr}^W_j \mathrm{H}^i(X;\mathbb{Q}) \cong 0$ if $j > i$.

Theorem (Theorem [4]). *For arbitrary kinematic parameters, and arbitrary space-time dimension D, $W_4\mathrm{H}^5(X_{(3,1,3);D};\mathbb{Q})$ is mixed Tate.*

(1) *If $D \geq 6$ then $\mathrm{Gr}^W_5 \mathrm{H}^5(X_{(3,1,3);D};\mathbb{Q}) \cong \mathrm{H}^1(C;\mathbb{Q})(-2)$ for a curve C which has genus 2 for generic kinematic parameters.*
(2) *If $D = 4$ then $\mathrm{Gr}^W_5 \mathrm{H}^5(X_{(3,1,3);D};\mathbb{Q}) \cong \mathrm{H}^1(E;\mathbb{Q})(-2)$ for a curve E which is elliptic for generic kinematic parameters.*
(3) *If $D \leq 4$ then $\mathrm{H}^5(X_{(3,1,3);E};\mathbb{Q})$ is mixed Tate.*

We get sharper results for the motive of the vanishing locus of $\mathcal{F}_{(a,1,c);D}$ in the case where $c = 1, 2$ for graphs of type $(a, 1, 1)$ in Figure 1

Theorem (Theorems [4]). *Suppose $a \leq 2$ or $c \leq 2$. Then*

$$\mathrm{H}^{a+c-1}(X_{(a,1,c);D};\mathbb{Q}) \in \mathbf{MHS}^{\mathrm{ell}}_{\mathbb{Q}}.$$

Corollary 2.2.3. *If $3D/2 \leq a + c$ and either $a \leq 2$ or $c \leq 2$ then*

$$\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - {}^{\boldsymbol{b}}X_{(a,1,c);D}; B - (B_\Gamma \cap {}^{\boldsymbol{b}}X_{(a,1,c);D})) \in \mathbf{MHS}^{\mathrm{ell}}_{\mathbb{Q}}.$$

This means that the mixed Hodge structure $\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - {}^{\boldsymbol{b}}X_{(a,1,c);D}; B - (B \cap {}^{\boldsymbol{b}}X_{(a,1,c);D}))$ is constructed by taking iterated extensions of $\mathrm{H}^1(E;\mathbb{Q})(-a)^{r_1}$ and $\mathbb{Q}(-b)^{r_2}$ for different values of a, b, r_1, and r_2, and with various possibly different elliptic curves. Therefore the Feynman integrals in these cases are built from algebraic and elliptic functions.

Remark 2.2.4. All of the results are expressed in the category of mixed Hodge structures, rather than the category of motives. This is done in order to make closer contact with discussions of periods and Feynman integrals appearing in the physics literature. However, the geometric tools used to obtain these results are compatible with motivic constructions.

Part 3. Differential equations

In this work, we give an algorithmic procedure for deriving such differential equations and the inhomogeneous part without having to go through the integral reduction to master integrals and the construction of a reducible system of differential equations satisfied by the set of master integrals. Among the motivations for finding a shortcut to derive differential equations without relaying on master integrals reduction is that the integration-by-parts reduction leads to large system of master integrals that may obscure the algebraic geometry underlying the analytic structure of Feynman integrals. Another motivation is the application to cosmological correlators which give rise to analytic regularisation for which the commonly used integration by part algorithms are not developed out-of-the-box. Finding a system of partial differential equations (PDEs) is also useful for generalised Feynman integrals in the context of Gel'fand-Kapranov-Zelevinskiĭ (GKZ) systems, which gives a D-module of differential operators acting on the Feynman integral [8, 51, 86–92]. However, the transition of this D-module to the PDEs of Feynman integrals requires a restriction which is highly non-trivial and still an open problem [86, 93–95].

3.1. Feynman integrals are holonomic functions

The Feynman integrals have the representation in momentum space

$$I_\Gamma(\underline{s}, \underline{m}; \underline{\nu}, D) = \int_{\mathbb{R}^{1,D-1}} \frac{\prod_{r=1}^{L} d^D \ell_r}{\prod_{i=1}^{n} D_i^{\nu_i}} \tag{3.1.1}$$

with $(\nu_1, \dots, \nu_n, D) \in \mathbb{C}^{n+1}$ and D_i are the quadratic form involving the loop momenta as described in section 2.1.1.

We can consider the family of integral

$$V_\Gamma = \sum_{\underline{\nu}=\underline{\nu}_0+\mathbb{Z}^n} A(\underline{s}, \underline{m}, \underline{\nu}, D)\, I_\Gamma(\underline{s}, \underline{m}; \underline{\nu}, D) \tag{3.1.2}$$

where $\underline{\nu}_0 \in \mathbb{C}^n$ is non vanishing only in the case of analytic regularisation.

The Feynman integral satisfy a set of integration-by-part identities

$$\int \partial_{\ell^\mu_{r_0}} \frac{v^\mu}{\prod_{i=1}^{n} D_i^{\nu_i}} \prod_{r=1}^{L} d^D \ell_r = 0 \tag{3.1.3}$$

that imply that integral with indices $\underline{\nu}$ differing by integers are not independent and satisfy relations induced by (3.1.3).

One consequence of this is that the dimension of dimension of the vector space V_Γ is finite, a basis of this vector space is called in physics literature the master integrals. And that Feynman integrals are holonomic functions:

Definition A multivariate function $f(x_1, \dots, x_n)$ is **D-finite** over $\mathbb{F}(x_1, \dots, x_n)$ if for each $i \in \{1, \dots, n\}$ the function satisfies a linear partial differential equation

$$c_{i,r_i}(\underline{x}) \frac{\partial^{r_i}}{\partial x_i^{r_i}} f(\underline{x}) + c_{i,r_i-1}(\underline{x}) \frac{\partial^{r_i-1}}{\partial x_i^{r_i-1}} f(\underline{x}) + \cdots + c_{i,0}(\underline{x}) = 0\,.$$

where the coefficients $c_{i,j}(\underline{x}) \in \mathbb{F}[x_1, \dots, x_n]$.

3.1.1. Masters. Following [96], we introduce the differential operators in the Weyl algebra

$$A^{(n)}[D] := \mathbb{C}[-D/2]\, \langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \mid [\partial_{x_i}, x_j] = \delta_{ij} \rangle \tag{3.1.4}$$

acting inside the Feynman integral, and the operators in the shift algebra

$$S^{(n)}[D] := \mathbb{C}[-D/2]\, \left\langle \mathbf{1}^+, \dots, \mathbf{n}^+, \mathbf{1}^-, \dots, \mathbf{n}^- \mid \left[-\mathbf{j}^-, \mathbf{i}^+\right] = \delta_{ij} \right\rangle \tag{3.1.5}$$

The shift operators act on functions $F(\underline{v})$ of $\underline{v}$ as follows

$$\left(\mathbf{i}^- F\right)(\underline{v}) = F\left(\underline{v} - e_i\right), \quad \left(\mathbf{i}^+ F\right)(\underline{v}) = v_i F\left(\underline{v} + e_i\right) \tag{3.1.6}$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 at position i. The insertion of the variable x_i inside the integral is the same as raising the index $v_i \to v_i + 1$, i.e. $x_i \to \mathbf{i}^+$, differentiating with respect to x_i lowers the index $v_i \to v_i - 1$, i.e. $\partial_{x_i} \to -\mathbf{i}^-$.

A result of Bernshtein [97] states that the module

$$A^{(n)}[-D/2]\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\prod_{i=1}^n x_i^{v_i-1}\right) \tag{3.1.7}$$

is holonomic. Then by a theorem by Loeser and Sabbah [98] the dimension of the vector space

$$V_\Gamma := \sum_{\underline{v}\in\mathbb{Z}^n} \mathbb{C}(-D/2,\underline{v}) I_\Gamma\left(\underline{s},\underline{m}^2;\underline{v},D\right) \tag{3.1.8}$$

is given by the topological Euler characteristic of the complement of the graph hyper-surface [90, 96].

$$\dim\left(V_\Gamma\right) = (-1)^{n+1}\chi\left(\left(\mathbb{C}^*\right)^n \backslash \mathbb{V}(\mathcal{U}) \cup \mathbb{V}(\mathcal{F})\right) \tag{3.1.9}$$

with the vanishing loci defined in (2.2.8).

Since this vector space is finite-dimensional one can expand integral in the family of Feynman integrals $I_\Gamma\left(\underline{s},\underline{m}^2;\underline{v},D\right)$ on a basis of, so-called master integrals, $M_\Gamma(\underline{s})$ with coefficients given by rational functions of the parameters $\underline{s},\underline{m}^2,\underline{v}$ and the dimension D. Feynman integrals are holonomic functions of their physical parameters [38, 39, 96, 99]. This means that Feynman integrals satisfy systems of (inhomogeneous) partial differential equations of finite order with respect to their physical parameters $\vec{m}$ and $\vec{s}$.

Differentiating with respect to the physical parameters $\underline{s}$, the master integrals $M_\Gamma(s)$ satisfy the first order this differential system of equations

$$dM_\Gamma(\underline{s}) = A_\Gamma \wedge M_\Gamma(\underline{s}) \tag{3.1.10}$$

The matrix A_Γ is a flat connection satisfying

$$dA_\Gamma + A_\Gamma \wedge A_\Gamma = 0 \tag{3.1.11}$$

One can convert the first order system into a system of differential operators acting on the the Feynman integral $I_\Gamma\left(\underline{s},\underline{m}^2;\underline{v},D\right)$. It is enough to get a Gröbner basis of such operators.

3.1.2. The D-module of differential equations. A Feynman integral satisfies inhomogenous differential equations with respect to any set of variables $\underline{z} \in \{p_i\cdot p_j, m_1^2,\dots,m_n^2\}$

$$\mathcal{L}_\Gamma(\underline{z})\, I_\Gamma = \mathcal{S}_\Gamma \tag{3.1.12}$$

The aim of this talk is to present some new methods for deriving such system of differential equation and its underlying (algebraic) geometry

For a subset of variables $\underline{z} := \{z_1,\dots,z_r\} \in \underline{s}$ of variables satisfies we define a differential operator annihilating the integrand

$$\mathcal{T}_{\underline{z}}\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}\prod_{i=1}^{n}x_i^{v_i-1}\right)=0 \tag{3.1.13}$$

We decompose this operator

$$\mathcal{T}_{\underline{z}}:=\mathcal{L}_{\underline{z}}+\sum_{i=1}^{n}\frac{\partial}{\partial x_i}Q_i\left(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}\right) \tag{3.1.14}$$

into a Picard-Fuchs differential operator $\mathcal{L}_{\underline{z}}$ acting only on the parameters $\underline{z}$ of the Feynman integrals

$$\mathcal{L}_{\underline{z}}=\sum_{\substack{0\leq a_i\leq\leq_i\\1\leq i\leq r}}p_{a_1,\dots,a_r}\left(\underline{s},\underline{m}^2\right)\prod_{i=1}^{r}\left(\frac{\partial}{\partial z_i}\right)^{a_i} \tag{3.1.15}$$

and a certificate part $Q_i\left(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}\right)$ which is a differential operator acting on the parameters $\underline{z}$ and the integration variables

$$\begin{aligned}Q_i\left(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}\right)=&\sum_{\substack{0\leq a_i\leq o_i\\1\leq i\leq r}}\sum_{\substack{0\leq b_j\leq o_j\\1\leq j\neq n}}q^{(i)}_{a_1,\dots,a_r}\left(\underline{s},\underline{m}^2;\underline{x}\right)\\&\times\prod_{i=1}^{r}\left(\frac{\partial}{\partial z_i}\right)^{a_i}\prod_{j=1}^{n}\left(\frac{\partial}{\partial x_i}\right)^{b_j}.\end{aligned} \tag{3.1.16}$$

In these expressions o_i, o'_i with $1\leq i\leq r$ and $\tilde{o}_j$ and $1\leq j\leq n$, are positive integers.

Integrating (28) over the domain σ we have

$$\begin{aligned}0&=\int_\sigma\mathcal{T}_{\underline{z}}\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}\prod_{i=1}^{n}x_i^{v_i-1}\right)\prod_{i=1}^{n}dx_i\\&=\mathcal{L}_{\underline{z}}\int_\sigma\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}\prod_{i=1}^{n}x_i^{v_i-1}\right)\prod_{i=1}^{n}dx_i+\int_\sigma d_XQ\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}\prod_{i=1}^{n}x_i^{v_i-1}\right)\end{aligned} \tag{3.1.17}$$

where we have set

$$d_XQ:=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}Q_i\left(\underline{s},\underline{m}^2,\partial_{\underline{z}};\partial_{x_i},\underline{x}\right)dx^i \tag{3.1.18}$$

Integrating by part the last term we obtain

$$\int_\sigma d_XQ\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}\prod_{i=1}^{n}x_i^{v_i-1}\right)=-\int_{\partial\sigma}Q\left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^{\omega}}\prod_{i=1}^{n}x_i^{v_i-1}\right) \tag{3.1.19}$$

For a cycle without a boundary $\partial\sigma=\emptyset$, this integral vanishes

$$\mathcal{L}_{\underline{z}} \int_\sigma \left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega} \prod_{i=1}^n x_i^{v_i-1} \right) = 0 \tag{3.1.20}$$

In the case of the Feynman integral I_Γ, this is no longer true because $\partial\Delta_n \neq \emptyset$, therefore

$$\mathcal{L}_{\underline{z}} I_\Gamma \left(\underline{s}, \underline{m}^2; \underline{v}, D\right) = \int_{\partial\Delta_n} Q \left(\frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega} \prod_{i=1}^n x_i^{v_i-1} \right) \neq 0 \tag{3.1.21}$$

3.2. Variation of mixed Hodge structures

We need to introduce the notation of *variation of Hodge structure* needed to take into account that Feynman integrals lead to families of Hodge structures parametrized by the variation of the kinematics invariants. These concepts have been introduced by Griffiths in [100] and generalized to mixed Hodge modules over complex varieties by M. Saito in [101]. We refer to these works for details about this, but a particular case of variation of mixed Hodge structure for polylogarithms is discussed in the next section.

A *pure Hodge structure of weight* n is an algebraic structure generalizing the Hodge theory for compact complex manifold. For a compact complex manifold $\mathcal{M}$ the de Rham cohomology groups $H^n(\mathcal{M}) := H^n(\mathcal{M}, \mathbb{R}) \otimes \mathbb{C}$ can be decomposed

$$H^n(\mathcal{M}) = \bigoplus_{p+q=n} H^{p,q}(\mathcal{M}), \qquad \text{with} \qquad \overline{H^{p,q}(\mathcal{M})} = H^{q,p}(\mathcal{M})\,. \tag{3.2.1}$$

The Dolbeaut cohomology groups $H^{p,q}(\mathcal{M})$ are defined as the $\bar\partial$-closed (p,q)-forms modulo $\bar\partial A^{p,q-1}(\mathcal{M})$ (see [102] for a more detailed exposition). By definition $H^n(\mathcal{M})$ is pure Hodge structure of weight n.

When one does not have a complex structure one defines a pure Hodge structure from a Hodge filtration. Let consider a finite dimensional $\mathbb{Q}$-vector space $H = H_\mathbb{Q}$. Suppose given a decreasing filtration $F^\bullet H_\mathbb{C}$ on $H_\mathbb{C} := H_\mathbb{Q} \otimes \mathbb{C}$,

$$H_\mathbb{C} \supseteq \cdots \supseteq F^{p-1}H_\mathbb{C} \supseteq F^p H_\mathbb{C} \supseteq F^{p+1}H_\mathbb{C} \supseteq \cdots \supseteq (0)\,. \tag{3.2.2}$$

One says that $F^\bullet H_\mathbb{C}$ defines a *pure Hodge structure of weight* n

$$H^n_\mathbb{C} := \bigoplus_{p+q=n} H^{p,q}, \qquad \text{where} \qquad H^{p,q} := F^p H_\mathbb{C} \cap \overline{F^q H_\mathbb{C}} \tag{3.2.3}$$

where $\overline{F^q H_\mathbb{C}}$ is the complex conjugate of $F^p H_\mathbb{C}$.

Pure Hodge structures are defined for smooth compact manifolds $\mathcal{M}$, but Feynman graph integrals involve non-compact or non-smooth varieties which require using the generalizations provided by the mixed Hodge structures introduced by Deligne [103].

The only pure Hodge structure of dimension one is the *Tate Hodge structure* $\mathbb{Q}(n)$ with

$$F^p\mathbb{Q}(n)_\mathbb{C} = \begin{cases} 0 & \text{for } p > -n \\ \mathbb{Q}(n)_\mathbb{C} & \text{for } i \leq -n\,. \end{cases} \tag{3.2.4}$$

This means that $\mathbb{Q}(n)_{\mathbb{C}} = H^{-n,-n}(\mathbb{Q}(n)_{\mathbb{C}})$ and $\mathbb{Q}(n)$ has weight $-2n$. Notice that $\mathbb{Q}(n)\otimes\mathbb{Q}(m) = \mathbb{Q}(n+m)$, therefore $\mathbb{Q}(n) = \otimes^n\mathbb{Q}(1)$ for $n\in\mathbb{Z}$.

A *mixed Hodge structure* on H is a pair of (finite, separated, exhaustive) filtrations: (a) an increasing filtration $W_\bullet H_{\mathbb{Q}}$ called the *weight filtration*, (b) a decreasing filtration, the Hodge filtration $F^\bullet H_{\mathbb{C}}$ described earlier. The Hodge structure on H induces a filtration on the graded pieces for the weight filtration $gr_n^W H := W_nH/W_{n-1}H$. By definition for a mixed Hodge structure, the filtration $gr_n^W H$ should be a pure Hodge structure of weight n.

A mixed Hodge structure H is called *mixed Tate* if

$$gr_n^W H = \begin{cases} 0 & \text{for } n = 2m-1 \\ \bigoplus \mathbb{Q}(-m) & \text{for } n = 2m\,. \end{cases} \tag{3.2.5}$$

From mixed Hodge structures one can define a matrix of periods. For a mixed Tate Hodge structure the weight and the Hodge filtrations are opposite since $F^{p+1}H_{\mathbb{C}}\cap W_{2p}H_{\mathbb{C}} = (0)$, and $H_{\mathbb{C}} = \bigoplus_p F^pH_{\mathbb{C}}\cap W_{2p}H_{\mathbb{C}}$. We first make a choice of a basis $\{e_i^{p,p}\in F^pH_{\mathbb{C}}\cap W_{2p}H_{\mathbb{C}}\}$ of $H_{\mathbb{C}}$. Then expressing the basis elements $\{\varepsilon_i\}$ for $W_\bullet H$ in terms of the basis for $H_{\mathbb{C}}$ gives a *period matrix* with columns composed by the basis elements of $F^\bullet H$. With a proper choice of the basis for $W_\bullet H$ one can ensure that the period matrix is block lower triangular, with the block diagonal elements corresponding to $gr_n^W H$ given by $(2i\pi)^{-n}$. In the case of the polylogarithms such a period matrix is given in eq. (4.1.2).

3.2.1. Variation of mixed Hodge structure and ODEs. If $\mathcal{H}_{\mathbb{Q}}$ is the local system underlying a variation of mixed Hodge structure over a 1-dimensional base M, and $\boldsymbol{s}$ is a meromorphic section of $\mathcal{H}_{\mathbb{Q}}\otimes\mathcal{O}_M$ then there is a minimal differential equation $\mathscr{L}_{\boldsymbol{s}}$ annihilating the period functions attached to $\boldsymbol{s}$. We explain this in detail below and study the relationship between the solution sheaf of $\mathscr{L}_{\boldsymbol{s}}$ and the weight-graded pieces of the variation of the original mixed Hodge structure. To each Feynman graph, we attach an operator $\mathscr{L}_{\Gamma;D}$ and we determine the irreducible factors of $\mathscr{L}_{\Gamma;D}$ when Γ is a planar two-loop graph using the Hodge structure analysis of [4] and review in section 2.2.3.

We now give a brief description of how one may obtain an ODE starting with a variation of mixed Hodge structure along with a holomorphic section of the underlying local system.

Definition 3.2.1. A ($\mathbb{Q}$-)variation of mixed Hodge structure of weight n consists of several pieces of data

(1) A $\mathbb{Q}$-local system $\mathcal{H}_{\mathbb{Q}}$ over a complex manifold M,
(2) An increasing weight filtration by $\mathbb{Q}$-local systems $\mathcal{W}_0\subseteq\mathcal{W}_1\subseteq\cdots\subseteq\mathcal{W}_{2n} = \mathcal{H}_{\mathbb{Q}}$,
(3) A decreasing Hodge filtration $\mathcal{F}^n\subseteq\mathcal{F}^{n-1}\subseteq\cdots\subseteq\mathcal{F}^0 = \mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{Q}}\otimes_{\underline{\mathbb{Q}}_M}\underline{\mathbb{C}}_M$,
(4) A flat connection $\nabla:\mathcal{H}_{\mathbb{C}}\otimes\mathcal{O}_M\to\mathcal{H}_{\mathbb{C}}\otimes\Omega^1_M$ so that $\nabla(\mathcal{F}^i)\subseteq\mathcal{F}^{i-1}$,

so that on each fibre $\mathcal{H}_{\mathbb{Q}}$, the data $(\mathcal{H}_{\mathbb{Q}},\mathcal{F}_t^\bullet,\mathcal{W}_\bullet)$ is a mixed Hodge structure.

Given a local section $\boldsymbol{s}$ of $\mathcal{H}_{\mathbb{C}}\otimes\mathcal{O}_M$, and a local parameter t on M, we can construct local (or multivalued) period functions

$$\boldsymbol{\pi}_{\boldsymbol{s}}(t) = \langle\boldsymbol{s},\gamma_t\rangle \tag{3.2.6}$$

for a flat section γ_t of $\mathcal{H}_{\mathbb{Q}}^{\vee}$. For us, we will often take $\boldsymbol{s} = \omega_{\Gamma;D}(t)$ and let $\mathcal{H}_{\mathbb{Q}}^{\vee}$ is the homology bundle underlying the family of varieties $\mathbb{P}^{e(\Gamma)-1} - X_{\Gamma;D}(t)$, in which case the pairing is integration.

Given a variation of mixed Hodge structure, $(\mathcal{H}_{\mathbb{Q}}, \mathcal{W}_{\bullet}, \mathcal{F}^{\bullet})$ over $M \subseteq \mathbb{A}^1$ with Gauss–Manin connection ∇, we have differential operators $\nabla_{\partial_t} : \mathcal{H} \otimes \mathcal{O}_M \to \mathcal{H} \otimes \mathcal{O}_M, [\omega] \mapsto \nabla([\omega])(\partial_t)$ where ∂_t denotes the vector field corresponding to a choice of variable t. The pairing satisfies

$$\frac{d}{dt}\langle \boldsymbol{s}, \gamma_t \rangle = \langle \nabla_{\partial_t}(\omega), \gamma_t \rangle. \tag{3.2.7}$$

Consequently, there is a minimal collection of elements $\{f_0(t), \ldots, f_n(t)\}$ in the the $\mathbb{C}(t)$-vector space $\Gamma(\mathcal{H} \otimes \mathcal{O}_M) \otimes \mathbb{C}(t)$ so that

$$\left[f_n(t)\nabla_{\partial_t}^n + f_{n-1}(t)\nabla_{\partial_t}^{n-1} + \cdots + f_1(t)\nabla_{\partial_t} + f_0(t)\right] \boldsymbol{s} = 0 \tag{3.2.8}$$

and thus there is a linear differential operator

$$\mathscr{L}_{\boldsymbol{s}} = f_n(t)\frac{d^n}{dt^n} + f_{n-1}(t)\frac{d^{(n-1)}}{dt^{(n-1)}} + \cdots + f_1(t)\frac{d}{dt} + f_0(t) \tag{3.2.9}$$

whose solutions are the period functions $\boldsymbol{\pi}_{\boldsymbol{s}}(t)$. The local system $\mathcal{H}_{\mathbb{Q}}^{\vee}$ is equipped with a weight filtration $\mathcal{W}_{\bullet}^*$ dual to the weight filtration on $\mathcal{H}_{\Gamma;D}(t)$ determined by $\mathcal{W}_i^* = (\mathcal{W}_{-i-1})^{\vee}$. The pairing (3.2.6) induces a map from $\mathcal{H}_{\mathbb{Q}}^{\vee}$ to $\mathcal{O}_M$ whose image is $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$, the local system of solutions of $\mathscr{L}_{\boldsymbol{s}}$. Therefore, $\mathcal{W}_i^*$ induces a filtration on $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$.

Lemma 3.2.2. *The local system $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$ is a quotient of the dual local system $\mathcal{H}_{\mathbb{Q}}^{\vee}$ by a sub-local system $\mathbb{K}_{\boldsymbol{s}}$. If $\boldsymbol{s} \in \mathcal{W}_i \otimes \mathcal{O}_M$ then $\mathcal{W}_i^* \subseteq \mathbb{K}_{\boldsymbol{s}}$.*

Proof. The image of the induced map $\int_{(-)} \boldsymbol{s} : \mathcal{H}_{\mathbb{Q}}^{\vee} \to \mathcal{O}_M$ obtained by the pairing (3.2.6) is isomorphic to $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$. This proves the first result. The second follows by definition. $\square$

A mixed Hodge structure is mixed Tate if it is an iterated extension of the Tate Hodge structure. We will also say that a variation of mixed Hodge structure is mixed Tate if all fibres are mixed Tate.

3.2.2. Filtrations and ODEs. We now discuss the relationship between filtrations on the local system $\mathcal{S}o\ell(\mathscr{L})$ and factorisations of $\mathscr{L}$. The following Propositions are certainly well-known, but we offer proofs for the sake of convenience.

Proposition 3.2.3. *Let $\mathscr{L}$ be a differential operator on $\mathcal{O}_M$ for M an open subset of $\mathbb{A}^1$. There is a bijection between*

1. *Increasing filtrations on $\mathcal{S}o\ell(\mathscr{L})$*
2. *Factorisations of $\mathscr{L}$ in $\mathbb{C}[M]\langle\partial_t\rangle$.*

Proof. Given a factorization of $\mathscr{L} = \mathscr{L}_1 \ldots \mathscr{L}_k$ we obtain a filtration of $\mathcal{S}o\ell(\mathscr{L})$,

$$\mathcal{S}o\ell(\mathscr{L}_k) \subseteq \mathcal{S}o\ell(\mathscr{L}_{k-1}\mathscr{L}_k) \subseteq \cdots \subseteq \mathcal{S}o\ell(\mathscr{L}_1 \ldots \mathscr{L}_k).$$

Given a filtration $\mathcal{W}_0 \subseteq \cdots \subseteq \mathcal{W}_k$ of $\mathcal{S}o\ell(\mathscr{L})$, we obtain, for $\mathcal{W}_{k-1}$ a monodromy-invariant subspace $\mathscr{V}_{k-1} \subseteq \mathcal{S}o\ell(\mathscr{L})$. A monodromy invariant subspace of $\mathcal{S}o\ell(\mathscr{L})$ provides a factorisation $\mathscr{L} = \mathscr{L}_k\mathscr{L}'$ so that $\mathcal{S}o\ell(\mathscr{L}') = \mathscr{V}_k$. Iterating this we obtain the desired factorisation. $\square$

Proposition 3.2.4. *Suppose $\mathscr{L}$ is an ODE on a Zariski open subset $M \subseteq \mathbb{A}^1$ and $\mathscr{L} = \mathscr{L}_1 \dots \mathscr{L}_k$. The monodromy representation of $\mathscr{L}$ can be written in block upper-triangular form whose diagonal blocks are the monodromy representations of $\mathcal{S}o\ell(\mathscr{L}_i)$.*

Proof. We show that if $\mathscr{L} = \mathscr{L}_1\mathscr{L}_2$ then the monodromy representation is an extension of $\mathcal{S}o\ell(\mathscr{L}_1)$ by $\mathcal{S}o\ell(\mathscr{L}_2)$. As noted above, there is an injection $\mathcal{S}o\ell(\mathscr{L}_2) \subseteq \mathcal{S}o\ell(\mathscr{L}_1\mathscr{L}_2)$. Near a point $b \in M$, choose a basis of solutions $\{f_1(t), \dots, f_k(t)\}$ of $\mathscr{L}_1$, choose particular solutions $\{p_1(t), \dots, p_k(t)\}$ of $\mathscr{L}$, so that $\mathscr{L}_1 p_i(t) = f_i(t)$. For any $f(t) = a_1 f_1(t) + \dots + a_k f_k(t)$ let $p_f(t) = a_1 p_1(t) + \dots + a_k p_k(t)$. Then

$$\mathcal{S}o\ell(\mathscr{L})_b = \operatorname{span}\{p_f(t) \mid f \in \mathcal{S}o\ell(\mathscr{L}_1)_b\} + \mathcal{S}o\ell(\mathscr{L}_2)_b.$$

Given a loop $\gamma \in \pi_1(M, b)$ and multi-valued function h on M, let $\gamma \cdot h(t)$ denote the action of monodromy on $h(t)$. Then

$$\mathscr{L}_2\left(\gamma \cdot p_f(t)\right) = \gamma \cdot (\mathscr{L}_2 p_i(t)) = \gamma \cdot f_i(t)$$

Therefore $\gamma \cdot p_f(t) \equiv p_{\gamma \cdot f}(t) \bmod \mathcal{S}o\ell(\mathscr{L}_2)$. The claim follows by induction. ☐

The weight filtration on $\mathcal{H}_{\Gamma;D} \otimes \underline{\mathbb{C}}_M$ may be extended to a maximal filtration, $\mathcal{W}_\bullet^{\max}$. The graded pieces of this maximal filtration are local systems which we denote $\mathbb{L}_1, \dots, \mathbb{L}_n$. By the Jordan–Hölder theorem, these are independent of the choice of maximal filtration on $\mathcal{H}_{\Gamma;D}$. In particular, since $\mathcal{S}o\ell(\mathscr{L}_{\boldsymbol{s}})$ is isomorphic to a quotient of $\mathcal{H}_{\Gamma;D} \otimes \underline{\mathbb{C}}_M$, we obtain the following result.

Proposition 3.2.5. *Suppose $\mathscr{L}_{\boldsymbol{s}} = \mathscr{L}_1 \dots \mathscr{L}_k$ is a factorisation of $\mathscr{L}_{\boldsymbol{s}}$ so that each $\mathscr{L}_i$ is irreducible. Then for each i there is an index j_i so that $\mathcal{S}o\ell(\mathscr{L}_i) \cong \mathbb{L}_{j_i}$.*

Precisely, we look at many variations of mixed Hodge structure in this paper where the graded pieces $\mathrm{Gr}_i^{\mathcal{W}}$ are either isomorphic to polarized pure Tate variations of Hodge structure or variations of Hodge structure underlying families of elliptic curves.

Proposition 3.2.6. *(1) Suppose $(\mathcal{H}_\mathbb{Z}, \mathcal{F}^\bullet, \mathcal{W}_\bullet)$ is a polarized pure Tate variation of Hodge structure. Then the monodromy representation of the local system is finite.*

(2) Suppose $(\mathcal{H}_\mathbb{Z}, \mathcal{F}^\bullet, \mathcal{W}_\bullet)$ is a polarized variation of pure Hodge structure underlying a non-isotrivial family of elliptic curves. Then $\mathcal{H}$ is a simple local system.

(3) Suppose $(\mathcal{H}_\mathbb{Z}, \mathcal{F}^\bullet, \mathcal{W}_\bullet)$ is a polarized variation of pure Hodge structure underlying an isotrivial family of elliptic curves. Then $\mathcal{H}$ is has finite monodromy.

Proof. A pure Tate polarized variation of Hodge structure admits a flat, positive definite, symmetric bilinear pairing $(\bullet, \bullet) : \mathcal{H}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{H}_\mathbb{Z} \to \underline{\mathbb{Z}}_M$. Therefore, its monodromy representation is a subgroup of the orthogonal group of a positive definite lattice. Consequently, its monodromy group is finite.

If the monodromy representation of a non-isotrivial family of elliptic curves admits a nontrivial subrepresentation on $\mathbb{Z}^2$, the rank implies that the monodromy group is solvable. The monodromy representation of a non-isotrivial family of elliptic curves is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z})$ is not solvable no finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$ is solvable.

If a family of elliptic curves is isotrivial then it becomes trivial after a finite base-change and hence the monodromy is finite. ☐

3.2.3. Differential operators attached to pencils of graph hypersurfaces. At points throughout this paper, we have discussed how cohomology of the graph hypersurfaces relate to the exploratory and computational work done by in [5]. In that article, Lairez and the third author study particular pencils of graph hypersurfaces of the form

$$\mathbf{F}_\Gamma(t) = \mathcal{U}_\Gamma \left(\sum_e m_e x_e \right) - t\mathbf{V}_\Gamma \tag{3.2.10}$$

varying with a parameter t and sometimes randomly chosen kinematic and parameters. Recall from the introduction that, attached to this pencil, there is a variation of mixed Hodge structure over an open subset M of $\mathbb{A}^1_t$ which we denote $\mathcal{H}_{\Gamma;D}$, and that the differential form

$$\omega_{\Gamma,D}(t) = \frac{\mathcal{U}_\Gamma^{e(\Gamma)-(L+1)D/2}}{(\mathbf{F}_\Gamma(t))^{e(\Gamma)-LD/2}} \Omega_0, \tag{3.2.11}$$

determines a section of $\mathcal{H}_{\Gamma;D} \otimes \mathcal{O}_M$.

Definition 3.2.7. Let $\mathscr{L}_{\Gamma;D}$ denote the minimal differential operator in $\mathbb{C}[M]\langle\partial_t\rangle$ which annihilates the form $\omega_{\Gamma;D}(t)$ in $\mathcal{H}_{\Gamma;D} \otimes \mathcal{O}_M$.

Many of the operators $\mathscr{L}_{\Gamma;D}$ discovered in [5] admit factorisations, as computed by the factorisation algorithm of the Ore algebra package [104,105] in `SAGE` [106]. The discussion in Sections 3.2.2 and 3.2.1 allow us to interpret these factorisations. We summarize our discussion.

(1) The local systems $\mathcal{S}o\ell(\mathscr{L}_{\Gamma;D})$ are quotients of $\mathcal{H}^\vee_{\Gamma;D}$.
(2) The filtration induced by $\mathcal{W}^*_\bullet$ corresponds to a factorisation of $\mathscr{L}_{\Gamma;D}$, however there may be factorisations of $\mathscr{L}_{\Gamma;D}$ which do not correspond to $\mathcal{W}^*_\bullet$ (Proposition 3.2.3).
(3) The monodromy representation of $\mathcal{S}o\ell(\mathscr{L}_{\Gamma;D})$ is upper triangular with diagonal blocks equal to the monodromy representations of the factors of $\mathscr{L}_{\Gamma;D}$ (Proposition 3.2.4).

The results of [4] give the following result:

Theorem 3.2.8. *For any a, c, the operator $\mathscr{L}_{(a,1,c);D}$ admits a factorisation*

$$\mathscr{L}_1\mathscr{L}_2 \dots \mathscr{L}_k$$

where $\mathcal{S}o\ell(\mathscr{L}_i)$ is either:

(a) a local system with finite order monodromy or
(b) a subquotient of the local system underlying a family of hyperelliptic curves over a Zariski open subset of $\mathbb{A}^1$.

In particular, if a or c is ≤ 2 then the monodromy representation of $\mathcal{S}o\ell(\mathscr{L}_i)$ is either

(a) finite, or
(b) a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Remark 3.2.9. Starting with the definition of a Feynman integral given by [60] and [107] and explained in section 3, it need not be the case that the Feynman integral of a graph Γ satisfies an inhomogeneous differential equation $\mathscr{L}_{\Gamma;D} I_{\Gamma;D}(t) = h(t)$ where $h(t)$ is a collection of periods taken on the boundary B_Γ. The form β will exist on $\mathbb{P}^{e(\Gamma)-1}$, and its polar locus is contained $X_{\Gamma;D}$ but the polar locus of

$\boldsymbol{b}^*\beta$ may include the exceptional divisor E of $\boldsymbol{b}$. If β has no poles on E then it is true that we indeed have

$$\mathscr{L}_{\Gamma;D}\Omega_{\Gamma;D} = d\widetilde{\beta} \tag{3.2.12}$$

and therefore

$$\mathscr{L}_{\Gamma;\Delta}\int_{\widetilde{\Delta}}\boldsymbol{b}^*\omega_{\Gamma;D} = \int_{\widetilde{\Delta}}\mathscr{L}_{\Gamma;D}\boldsymbol{b}^*\omega_{\Gamma;D} = \sum_i\int_{(\partial\widetilde{\Delta})_i}\boldsymbol{b}^*\beta =: h(t) \tag{3.2.13}$$

where $(\partial\widetilde{\Delta})_i$ denote the components of the boundary of $\widetilde{\Delta}$. However, if β has poles along some component of E then the final integral is undefined. We suggest two possible solutions to this problem. The first is to check directly that the form $\beta \in \Omega^n_{\mathbb{P}^n}(*X_{\Gamma;D})$ computed in [5] does not acquire extra poles under the blow up map. The second is to use the methods developed by Lairez [108] directly on the toric variety $\mathbb{P}_\Gamma$.

3.3. The differential equations

In general a Feynman integral $I_\Gamma(\underline{s},\underline{\xi}^2,\underline{\nu},D)$ satisfies an inhomogeneous system of differential equations

$$\mathcal{L}_\Gamma I_\Gamma = \mathcal{S}_\Gamma, \tag{3.3.1}$$

where the inhomogeneous term $\mathcal{S}_\Gamma$ essentially arises from boundary terms corresponding to reduced graph topologies where internal edges have been contracted. Knowing the maximal cut integral allows to determine differential operators $\mathcal{L}_\Gamma$

$$\mathcal{L}_\Gamma\pi_\Gamma(\underline{s},\underline{\xi}^2,D) = 0, \tag{3.3.2}$$

This fact has been exploited in [109–112] to obtain the minimal order differential operator. The important remark in this construction is to use that the only difference between the Feynman integral I_Γ and the maximal cut π_Γ is the choice of cycle of integration. Since the Picard-Fuchs operator $\mathcal{L}_\Gamma$ acts as

$$\mathcal{L}_\Gamma\pi_\Gamma(\underline{s},\underline{\xi}^2,D) = \int_{\gamma_n}\mathcal{L}_\Gamma\Omega_F = \int_{\gamma_n}d(\beta_\Gamma) = 0 \tag{3.3.3}$$

this integral vanishes because the cycle $\gamma_n = \{|x_1| = \cdots = |x_n| = 1\}$ has no boundaries $\partial\gamma_n = \emptyset$. In the case of the Feynman integral I_Γ this is not longer true as

$$\mathcal{L}_\Gamma I_\Gamma(\underline{s},\underline{\xi}^2,D) = \int_{\Delta_n}d(\beta_\Gamma) = \int_{\partial\Delta_n}\beta_\Gamma = \mathcal{S}_\Gamma \neq 0\,. \tag{3.3.4}$$

The boundary contributions arises from the configuration with some of the Schwinger coordinate $x_i = 0$ vanishing which corresponds to the so-called reduced topologies that are known to arises when applying the integration-by-part algorithm (see [113–115] for instance).

We illustrate this logic on some elementary examples of differential equations for multi-valued integrals relevant for the one- and two-loop massive sunset integrals discussed in this text.

3.3.1. Toward a systematic approach to differential equations. In the previous section we have presented a way of deriving a differential equation for the maximal cut of a Feynman integral based on the knowledge of its representation as an hypergeometric function. One can as well use the series expansion for determining the differential operator acting on this series. The Griffith method presented in §3.3.1.2 using the reduction of polynomial in the cohomology unfortunately does not work when there are non-isolated singularities. This is unfortunately a rather generic case for Feynman integrals beyond the special case of the two-loop sunset. In principle one can resolve the singularity by deforming the graph polynomial by introduction a finite number of deformation parameters $\underline{\lambda} = \{\lambda_i\}$ until one has only isolated singularities. The Picard-Fuchs operator $L^{\underline{\lambda}}$. Taking the limit $\underline{\lambda} \to \underline{0}$ would give a differential operator $L^{\underline{0}}$ that would factorise on the minimal order Picard-Fuchs operator L acting on the Feynman integral. Unfortunately this approach is not very useful in practice as the deformation increases a lot the complexity of the computation.

At more systematic approach is the use of the telescoping method introduced by [116] and developed by [117,118] and [119]. One wants to derive an annihilating operator L for the maximal cut integral π_Γ in (2.1.16).

Let first illustrate the idea of the method on a multi-parameter one dimensional integral

$$I(\underline{\xi}) = \int_a^b f(x, \underline{\xi}) \,. \tag{3.3.5}$$

We want to construct an operator P of the form

$$P := T(\underline{\xi}, \partial_{\underline{\xi}}) + \frac{d}{dx} C(x, \underline{\xi}, \partial_{\underline{\xi}}) \tag{3.3.6}$$

This clearly implies that

$$T(\underline{\xi}, \partial_{\underline{\xi}}) I(\underline{\xi}) = -\int_a^b \frac{d}{dx} C(x, \underline{\xi}, \partial_{\underline{\xi}}) f(x, \underline{\xi}) = -C(x, \underline{\xi}, \partial_{\underline{\xi}}) f(x, \underline{\xi}) \Big|_{x=a}^{x=b} . \tag{3.3.7}$$

In the case that the Griffith method work this gives the same answer but the advantage of the method is that the algorithm works as well when there are non-isolated singularities. For the case of multidimensional variables integrals one just apply the same method iteratively.

Because we know that Feynman integrals are annihilated by a finite order differential operator because the dimension of master integrals (or the system of Gauss-Manin connection) is finite [38, 96] the creative telescoping algorithm in [116, 117, 117, 119] will finish in a finite time although this may take a long time for complicated Feynman graphs.

This provides a direct approach for finding the differential equation for Feynman integrals.

3.3.1.1. *The logarithmic integral.* We consider the integral

$$I_1(t) = \int_a^b \frac{dx}{x(x-t)} \,, \tag{3.3.8}$$

and its cut integral

$$\pi(t) = \int_\gamma \frac{dx}{x(x-t)} \,, \tag{3.3.9}$$

where γ is a cycle around the point $x = t$. Clearly we have

$$\frac{d}{dx}\left(\frac{1}{t-x}\right) = \frac{1}{x(x-t)} + t\frac{d}{dt}\left(\frac{1}{x(x-t)}\right), \tag{3.3.10}$$

therefore the integral $\pi(t)$ satisfies the differential equation

$$t\frac{d}{dt}\pi(t) + \pi(t) = \int_\gamma \frac{d}{dx}\left(\frac{1}{t-x}\right) = 0, \tag{3.3.11}$$

and the integral $I_1(t)$ satisfies

$$t\frac{d}{dt}I_1(t) + I_1(t) = \int_a^b \frac{d}{dx}\left(\frac{1}{t-x}\right) = \frac{1}{b(b-t)} - \frac{1}{a(a-t)}. \tag{3.3.12}$$

Changing variables from t to p^2 or an internal mass will give the familiar differential equation for the one-loop bubble that will be commented further in §3.4.4.

3.3.1.2. *Elliptic curve.* The second example is the differential equation for the period of an elliptic curve $\mathcal{E} : y^2z = x(x-z)(x-tz)$ which is the geometry of the two-loop sunset integral. Consider the differential of the first kind on the elliptic curve

$$\omega = \frac{dx}{\sqrt{x(x-1)(x-t)}}, \tag{3.3.13}$$

this form can be seen as a residue evaluated on the elliptic curve $\omega = \mathrm{Res}_{\mathcal{E}}\Omega$ of the form on the projective space $\mathbb{P}^2$

$$\Omega = \frac{\Omega_0}{y^2z - x(x-z)(x-tz)}. \tag{3.3.14}$$

where $\Omega_0 = zdx \wedge dy + ydz \wedge dx + xdy \wedge dz$ is the natural top form on the projective space $[x : y : z]$. Systematic ways of deriving Picard-Fuchs operators for elliptic curve is given by Griffith's algorithm [120]. Consider the second derivative with respect to the parameter t

$$\frac{d^2}{dt^2}\Omega = 2\frac{x^2(x-z)^2z^2}{(y^2z - x(x-z)(x-tz))^2}\Omega_0 \tag{3.3.15}$$

the numerator belongs to the Jacobian ideal[1] of the polynomial $p(x, y, z) := y^2z - x(x-z)(x-tz)$, $J_1 = \langle \partial_x p(x, y, z) = -3x^2 + 2(t+1)xz - tz^2 2, \partial_y p(x, y, z) = 2yz$, $\partial_z p(x, y, z) = (t+1)x^2 + y^2 - 2txz\rangle$, since

$$x^2(x-z)^2z^2 = m_x^1\partial_x p(x, y, z) + m_y^1\partial_y p(x, y, z) + m_z^1\partial_z p(x, y, z). \tag{3.3.16}$$

[1] An ideal I of a ring R, is the subset $I \subset R$, such that 1) $0 \in I$, 2) for all $a, b \in I$ then $a + b \in I$, 3) for $a \in I$ and $b \in R$, $a \cdot b \in R$. For $P(x_1, \dots, x_n)$ an homogeneous polynomial in $R = \mathbb{C}[x_1, \dots, x_n]$ the Jacobian ideal of P is the ideal generated by the first partial derivative $\{\partial_{x_i} P(x_1, \dots, x_n)\}$ [121]. Given a multivariate polynomial $P(x_1, \dots, x_n)$ its Jacobian ideal is easily evaluated using `Singular` command `jacob(P)`. The hypersuface $P(x_1, \cdots, x_n) = 0$ for an homogeneous polynomial, like the Symanzik polynomials, is of codimension 1 in the projective space $\mathbb{P}^{n-1}$. The singularities of the hypersurface are determined by the irreducible factors of the polynomial. This determines the cohomology of the complement of the graph hypersurface and the number of independent master integrals as shown in [96].

This implies that

$$\frac{d^2}{dt^2}\Omega = \frac{\partial_x m_x^1 + \partial_y m_y^1 + \partial_z m_z^1}{(y^2z - x(x-z)(x-tz))^2}\Omega_0 + d\left(\frac{(ym_z^1 - zm_y^1)dx + (zm_x^1 - xm_z^1)dy + (xm_y^1 - ym_x^1)dz}{(y^2z - x(x-z)(x-tz))^2}\right) \tag{3.3.17}$$

therefore

$$\frac{d^2}{dt^2}\Omega + p_1(t)\frac{d}{dt}\Omega = \frac{-p_1(t)x(x-z)z + \partial_x m_x^1 + \partial_y m_y^1 + \partial_z m_z^1}{(y^2z - x(x-z)(x-tz))^2}\Omega_0 + d\left(\frac{(ym_z^1 - zm_y^1)dx + (zm_x^1 - xm_z^1)dy + (xm_y^1 - ym_x^1)dz}{(y^2z - x(x-z)(x-tz))^2}\right). \tag{3.3.18}$$

One easily derives that $\partial_x m_x^1 + \partial_y m_y^1 + \partial_z m_z^1$ is in the Jacobian ideal generated by J_1 and $x(x-z)z$ with the result that

$$\partial_x m_x^1 + \partial_y m_y^1 + \partial_z m_z^1 = m_x^2\partial_x p(x,y,z) + m_y^2\partial_y p(x,y,z) + m_z^2\partial_z p(x,y,z) + \frac{2t-1}{t(t-1)}x(x-z)z, \tag{3.3.19}$$

therefore $p_1(t) = \frac{2t-1}{t(t-1)}$ and the Picard-Fuchs operator reads

$$\frac{d^2}{dt^2}\Omega + \frac{2t-1}{t(t-1)}\frac{d}{dt}\Omega - \frac{\partial_x m_x^2 + \partial_y m_y^2 + \partial_z m_z^2}{(y^2z - x(x-z)(x-tz))^2}\Omega_0 = d\left(\frac{(ym_z^1 - zm_y^1)dx + (zm_x^1 - xm_z^1)dy + (xm_y^1 - ym_x^1)dz}{(y^2z - x(x-z)(x-tz))^2}\right) + d\left(\frac{(ym_z^2 - zm_y^2)dx + (zm_x^2 - xm_z^2)dy + (xm_y^2 - ym_x^2)dz}{y^2z - x(x-z)(x-tz)}\right). \tag{3.3.20}$$

since $\partial_x m_x^2 + \partial_y m_y^2 + \partial_z m_z^2 = -\frac{1}{4t(t-1)}$ we have that

$$\left(4t(t-1)\frac{d^2}{dt^2} - 4(2t-1)\frac{d}{dt} + 1\right)\omega = -2\partial_x\left(\frac{y}{(x-t)^2}\right). \tag{3.3.21}$$

For α and β a (sympletic) basis of $H_1(\mathcal{E},\mathbb{Z})$ the period integrals $\varpi_1(t) := \int_\alpha \omega$ and $\varpi_2(t) := \int_\beta \omega$ both satisfy the differential equation

$$\left(4t(t-1)\frac{d^2}{dt^2} - 4(1-2t)\frac{d}{dt} + 1\right)\varpi_i(t) = 0. \tag{3.3.22}$$

Again this differential operator acting on an integral with a different domain of integration can lead to an homogeneous terms as this is case for the two-loop sunset Feynman integral.

3.4. Toric geometry and Feynman graphs

We will show how the toric approach provides a nice way to obtain this maximal cut integral. The maximal cut integral $\pi_\Gamma(\underline{s}, \underline{\xi}^2, D)$ is the particular case of generalised Euler integrals

$$\int_\sigma \prod_{i=1}^r P_i(x_1, \ldots, x_n)^{\alpha_i} \prod_{i=1}^n x_i^{\beta_i} dx_i \tag{3.4.1}$$

studied by Gel'fand, Kapranov and Zelevinski (GKZ) in [63,64]. There $P_i(x_1, \ldots, x_n)$ are Laurent polynomials, α_i and β_i are complex numbers and σ is a cycle. The cycle entering the maximal cut integral in (2.1.21) is the product of circles $\sigma = \{|x_1| = |x_2| = \cdots = |x_n| = 1\}$. But other cycles arise when considering different cuts of Feynman graphs. The GKZ approach provides a totally combinatorial approach to differential equation satisfied by these integrals.

As well in the case when $P(\underline{x}, \underline{z}) = \sum_i z_{i_1,\ldots,i_r} \prod_{i=1}^n x_i^{\alpha_i}$ is the Laurent polynomial defining a Calabi-Yau hypersurface $\{P(\underline{x}, \underline{z}) = 0\}$, Batyrev showed that there is one canonical period integral [122, 123]

$$\Pi(\underline{z}) := \frac{1}{(2i\pi)^n} \int_{|x_1| = \cdots = |x_n| = 1} \frac{1}{P(\underline{x}, \underline{z})} \prod_{i=1}^n \frac{dx_i}{x_i} \,. \tag{3.4.2}$$

This corresponds to the maximal cut integral (2.1.21) In the case where $\omega = D/2 = 1$ which is satisfied by the $(n-1)$-loop sunset integral $D = 2$ dimensions. The graph hypersurface of the $(n-1)$-loop sunset (see (3.4.28)) is always a Calabi-Yau $(n-1)$-fold. See for more comments about this at the end of §**??**. We refer to the reviews [124, 125] for some introduction to toric geometry for physics.

3.4.1. The GKZ approach : a review. In the section we briefly review the GKZ construction based on [63, 64] see as well [126]. We consider the Laurent polynomial of $n-1$ variables $P(z_1, \ldots, z_r) = \mathcal{F}^{tor}\underline{z}, x_1, \ldots, x_n/(x_1 \cdots x_n)$ from the toric polynomial of §3.4.3. The coefficients of monomials are z_i (by homogeneity we set $x_n = 1$)

$$P(z_1, \ldots, z_r) = \sum_{\mathbf{a}=(a_1,\ldots,a_{n-1})\in\mathbf{A}} z_{\boldsymbol{a}} \prod_{i=1}^{n-1} x_i^{a_i}, \tag{3.4.3}$$

with $\mathbf{a} = (a_1, \ldots, a_{n-1})$ is an element of $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_r)$ a finite subset of $\mathbb{Z}^{n-1}$. The number of elements in A is r the number of monomials in $P(z_1, \ldots, z_r)$.

We consider the natural fundamental period integral [127]

$$\Pi(\underline{z}) := \frac{1}{(2i\pi)^{n-1}} \int_{|x_1| = \cdots = |x_{n-1}| = 1} P(z_1, \ldots, z_r)^m \prod_{i=1}^{n-1} \frac{dx_i}{x_i}, \tag{3.4.4}$$

which is the same as maximal cut π_Γ in (2.1.21) for $D = 2\omega = -m$. The derivative with respect to $z_{\mathbf{a}}$ reads

$$\frac{\partial}{\partial z_{\boldsymbol{a}}} \Pi(\underline{z}) = \frac{1}{(2i\pi)^{n-1}} \int_{|x_1| = \cdots = |x_{n-1}| = 1} m P(z_1, \ldots, z_r)^{m-1} \prod_{i=1}^{n-1} x_i^{a_i} \frac{dx_i}{x_i}, \tag{3.4.5}$$

therefore for every vector $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r) \in \mathbb{Z}^{n-1}$ such that

$$\ell_1 + \cdots + \ell_r = 0, \qquad \ell_1 \mathbf{a}_1 + \cdots + \ell_r \mathbf{a}_r = \boldsymbol{\ell} \cdot \mathbf{A} = 0, \tag{3.4.6}$$

then holds the differential equation

$$\left(\prod_{l_i>0}\partial_{z_i}^{l_i}-\prod_{l_i<0}\partial_{z_i}^{-l_i}\right)\Pi(\underline{z})=0. \tag{3.4.7}$$

Introducing the so-called $\mathcal{A}$-hypergeometric functions[2] $\Phi_{\mathbb{L},\gamma}(z_1,\dots,z_r)$ of r complex variables $(z_1,\dots,z_r)\in\mathbb{C}^r$

$$\Phi_{\mathbb{L},\boldsymbol{\gamma}}(z_1,\dots,z_r)=\sum_{(\ell_1,\dots,\ell_r)\in\mathbb{L}}\prod_{j=1}^{r}\frac{z_j^{\gamma_j+\ell_j}}{\Gamma(\gamma_j+\ell_j+1)}, \tag{3.4.8}$$

depending on the complex parameters $\boldsymbol{\gamma}:=(\gamma_1,\dots,\gamma_r)\in\mathbb{C}^r$ and the lattice

$$\mathbb{L}:=\{(\ell_1,\dots,\ell_r)\in\mathbb{Z}|\sum_{i=1}^{r}\ell_i\mathbf{a}_i=0,\ell_1+\cdots+\ell_r=0\}, \tag{3.4.9}$$

with r elements $\{\mathbf{a}_1,\dots,\mathbf{a}_r\}\in\mathbb{Z}^n$. These functions are solutions of the so-called $\mathcal{A}$-hypergeometric system of differential equations given by a vector $\mathbf{c}\in\mathbb{C}^n$ and :

- For every $\boldsymbol{\ell}=(\ell_1,\dots,\ell_r)\in\mathbb{L}$ there is one differential operator

$$\Box_{\boldsymbol{\ell}}:=\prod_{\ell_i>0}\partial_{z_i}^{\ell_i}-\prod_{\ell_i<0}\partial_{z_i}^{-\ell_i}, \tag{3.4.10}$$

  such that $\Box_{\boldsymbol{\ell}}\Phi_{\mathbb{L},\gamma}(z_1,\dots,z_r)=0$
- n differential operators $\mathbf{E}:=(E_1,\dots,E_{n-1})$

$$\mathbf{E}:=\mathbf{a}_1z_1\frac{\partial}{\partial z_1}+\cdots+\mathbf{a}_rz_r\frac{\partial}{\partial z_r}, \tag{3.4.11}$$

  such that for $\mathbf{c}=(c_1,\dots,c_{n-1})$ we have

$$(\mathbf{E}-\mathbf{c})\Phi_{\mathbb{L},\gamma}(z_1,\dots,z_r)=0. \tag{3.4.12}$$

  Notice that $E_1=\sum_{i=1}^n z_i\frac{\partial}{\partial z_i}$ is the Euler operator and c_1 is the degree of homogeneity of the hypergeometric function.

  These operators satisfy the commutation relations

$$\begin{aligned}\mathbf{z^uE}-\mathbf{Ez^u}&=-(\mathbf{A}\cdot\mathbf{u})\,\mathbf{z^u},\\ \partial_z^{\mathbf{u}}\mathbf{E}-\mathbf{E}\partial_z^{\mathbf{u}}&=(\mathbf{A}\cdot\mathbf{u})\,\partial_z^{\mathbf{u}},\end{aligned} \tag{3.4.13}$$

  with $\mathbf{z^u}:=\prod_{i=1}^r z_r^{u_r}$ and $\partial_z^{\mathbf{u}}:=\prod_{i=1}^r\partial_{z_r}^{u_r}$.

Using the GKZ construction one can easily derive a system of differential operator annihilating the maximal of any Feynman integral after identification of the toric variables with the physical parameters. The system of differential operators obtained from the GKZ system can be massaged into a set of Picard-Fuchs differential operators in a spirit similar to the one used in mirror symmetry [121,129,130].

Since it is rather complicated to restrict differential operators but it is easier to restrict functions, it is therefore preferable to determine the $\mathcal{A}$-hypergeometric representation of the maximal cut integral and derive the minimal differential operator annihilating this integral. For well chosen vector $\boldsymbol{\ell}\in\mathbb{L}$ the differential operator factorises with a factor being given by the minimal (Picard-Fuchs) differential operator acting on the Feynman integral.

[2] The convergence of these series is discussed in [128, §3-2] and [126, §5.2].

An important remark is that the maximal cut integral

$$\pi_\Gamma = \int_{|x_1|=\cdots=|x_{n-1}|=1} \frac{1}{\mathcal{F}_\Gamma} \prod_{i=1}^{n-1} dx_i, \tag{3.4.14}$$

is a particular case of fundamental period $\Pi(\underline{z})$ in (3.4.4) with $m=-1$ and therefore is given by a $\mathcal{A}$-hypergeometric function once we have identified the toric variables z_i with the physical parameters.

In the next section we illustrate this approach on some simple but fundamental examples.

3.4.2. Hypergeometric functions and GKZ system. The relation between hypergeometric functions and the GKZ differential system can be simply understood as follows (see [126, 131, 132]).

3.4.2.1. *The Gauß hypergeometric series.* Consider the case of $\mathbb{L}=(1,1,-1,1)\mathbb{Z}\subset\mathbb{Z}^4$ and the vector $\gamma=(0,c-1,-a,-b)\in\mathbb{C}^4$ and c a positive integer. The GKZ hypergeometric function is

$$\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)=\sum_{n\in\mathbb{Z}}\frac{u_1^n u_2^{1-c+n}u_3^{-a-n}u_4^{-b-n}}{\Gamma(1+n)\Gamma(c+n)\Gamma(1-n-a)\Gamma(1-n-b)}\,, \tag{3.4.15}$$

which can be rewritten as

$$\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)=\frac{u_2^{c-1}u_3^{-a}u_4^{-b}}{\Gamma(c)\Gamma(1-a)\Gamma(1-b)}\,{}_2F_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle|\frac{u_1u_2}{u_3u_4}\right). \tag{3.4.16}$$

The GKZ system is

$$\begin{aligned}
\left(\frac{\partial^2}{\partial u_1\partial u_2}-\frac{\partial^2}{\partial u_3\partial u_4}\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)&=0,\\
\left(u_1\frac{\partial}{\partial u_1}-u_2\frac{\partial}{\partial u_2}+1-c\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)&=0,\\
\left(u_1\frac{\partial}{\partial u_1}+u_3\frac{\partial}{\partial u_3}+a\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)&=0,\\
\left(u_1\frac{\partial}{\partial u_1}+u_4\frac{\partial}{\partial u_4}+b\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)&=0\,.
\end{aligned} \tag{3.4.17}$$

By differentiating we find

$$\begin{aligned}
\left(u_2\frac{\partial^2}{\partial u_1\partial u_2}-u_1\frac{\partial^2}{\partial u_1^2}+c\frac{\partial}{\partial u_1}\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)&=0,\\
\left(u_3u_4\frac{\partial^2}{\partial u_3\partial u_4}-\left(u_1\frac{\partial}{\partial u_1}+a\right)\left(u_1\frac{\partial}{\partial u_1}+b\right)\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)&=0\,.
\end{aligned} \tag{3.4.18}$$

combining these equations one finds

$$\begin{aligned}
\left(u_1^2\frac{\partial}{\partial u_1}+(1+a+b)u_1\frac{\partial}{\partial u_1}+ab\right)&\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4)\\
&=\frac{u_3u_4}{u_2}\left(u_1\frac{\partial^2}{\partial u_1^2}+c\frac{\partial}{\partial u_1}\right)\Phi_{\mathbb{L},\gamma}(u_1,\dots,u_4).
\end{aligned} \tag{3.4.19}$$

Setting $F(z) = \Gamma(c)\Gamma(1-a)\Gamma(1-b)\Phi_{\mathbb{L},\gamma}(z,1,1,1)$ gives that $F(z) = {}_2F_1\left({a\,b \atop c}|z\right)$ satisfies the Gauß hypergeometric differential equation

$$z(z-1)\frac{d^2F(z)}{dz^2} + ((a+b+1)z+c)\frac{dF(z)}{dz} + abF(z) = 0\,. \tag{3.4.20}$$

3.4.3. Toric polynomials and Feynman graphs. The second Symanzik polynomial $\mathcal{F}(\underline{s}, \underline{\xi}^2, x_1, \dots, x_n)$ defined in (**??**) is a specialisation of the homogeneous (toric) polynomial[3] of degree $l+1$ at most quadratic in each variables in the projective variables $(x_1, \dots, x_n) \in \mathbb{P}^{n-1}$

$$\mathcal{F}_l^{toric}(\underline{z}, x_1, \dots, x_n) = \mathcal{U}_l^{tor}(x_1, \dots, x_n)\left(\sum_{i=1}^n \xi_i^2 x_i\right) - \mathcal{V}_l^{tor}(x_1, \dots, x_n), \tag{3.4.21}$$

where for $l \leq n$

$$\mathcal{U}_l^{tor}(x_1, \dots, x_n) := \sum_{\substack{0\leq r_i\leq 1\\ r_1+\cdots+r_n=l}} u_{i_1,\dots,i_n}\prod_{i=1}^n x_i^{r_i}, \tag{3.4.22}$$

where the coefficients $u_{i_1,\dots,i_n} \in \{0,1\}$. The expression in (3.4.21) is the most generic form compatible with the properties of the Symanzik polynomials listed in §2.1.1.

There are $\frac{n!}{(n-l)!l!}$ independent coefficient in the polynomial $\mathcal{U}_l^{tor}(x_1, \dots, x_n)$. Of course this is a huge over counting, as this does not take into account the symmetries of the graphs and the constraints on the non-vanishing of some coefficients. This will be enough for the toric description we are using here. In order to keep most of the combinatorial power of the toric approach we will only do the specialisation of the toric coefficients with the physical slice corresponding of Feynman graph polynomial at the end on solutions. This will avoid having to think at constrained system of differential equations which is a difficult problem discussed recently in [**96**].

The kinematics part has the toric polynomial

$$\mathcal{V}_l^{tor}(x_1, \dots, x_n) := \sum_{\substack{0\leq r_i\leq 1\\ r_1+\cdots+r_n=l+1}} z_{i_1,\dots,i_n}\prod_{i=1}^n x_i^{r_i}, \tag{3.4.23}$$

where the coefficients $z_{i_1,\cdots,i_n} \in \mathbb{C}$. The number of independent toric variables $z_{\underline{i}}$ in $\mathcal{V}^{tor}(x_1, \dots, x_n)$ is $\frac{n!}{(n-l-1)!(l+1)!}$.

---

[3] Consider an homogeneous polynomial of degree d

$$P(\underline{z}, \underline{x}) = \sum_{\substack{0\leq r_i\leq n\\ r_1+\cdots+r_n=d}} z_{i_1,\dots,i_n}\prod_{i=1}^n x_i^{r_i}$$

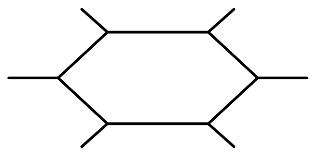
this is called a *toric polynomial* if it is invariant under the following actions

$$z_i \to \prod_{j=1}^n t_i^{\alpha_{ij}} z_i; \qquad x_i \to \prod_{j=1}^n t_i^{\beta_{ij}} x_i$$

for $(t_1, \dots, t_n) \in \mathbb{C}^n$ and α_{ij} and β_{ij} integers. The second Symanzik polynomial have a natural torus action acting on the mass parameters and the kinematic variables as we will see on some examples below. We refer to the book [**121**] for more details.

3.4.3.1. *Some important special cases.* There are few important special cases.

- At one-loop order $l = 1$ and the number of independent toric variables in $\mathcal{V}^{tor}(x_1, \ldots, x_n)$ is exactly the number of independent kinematics for an n-gon one-loop amplitude



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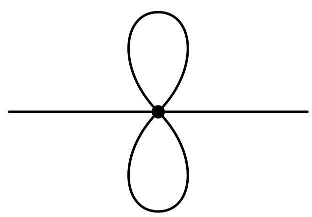
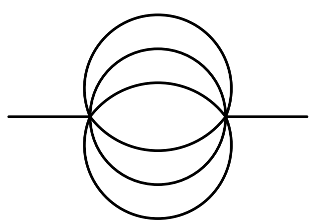
In this case the most general toric one-loop polynomial is

$$\mathcal{F}_1^{tor}(x_1, \cdots, x_n) = \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \xi_i^2 x_i\right) - \mathcal{V}_1^{tor}(x_1, \cdots, x_n). \tag{3.4.24}$$

- For $l = n$ there is only one vertex the graph is n-bouquet which is a product of n one-loop graphs. These graphs contribute to the reduced topologies entering the determination of the inhomogeneous term $\mathcal{S}_\Gamma$ of the Picard-Fuchs equation (3.3.1). They don't contribute to the maximal cut π_Γ for $l > 1$.



- The case $l = n-1$ corresponds to the $(n-1)$-loop two-point sunset graphs



.

In that case the kinematic polynomial is just

$$\mathcal{V}_{n-1}^{tor}(x_1, \ldots, x_n) = z_{1,\ldots,1} x_1 \cdots x_n, \tag{3.4.25}$$

and the toric polynomial

$$\mathcal{F}_{n-1}^{tor}(x_1, \ldots, x_n) = x_1 \cdots x_n \left(\sum_{i=1}^{n} \frac{u_{1,\ldots,0,\ldots,1}}{x_i}\right)\left(\sum_{i=1}^{n} \xi_i^2 x_i\right) - z_{1,\ldots,1} x_1 \cdots x_n, \tag{3.4.26}$$

where the index 0 in $u_{1,\ldots,0,\ldots,1}$ is at position i. Actually by redefining the parameter $z_{1,\ldots,1}$ the generic toric polynomial associated to the sunset graph are

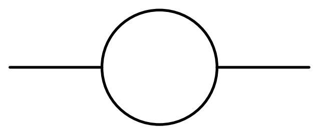
$$\mathcal{F}^{tor}_{\ominus}(x_1,\dots,x_n)=x_1\cdots x_n\left(\sum_{\substack{1\leq i,j\leq n\\ i\neq j}} z_{ij}\frac{x_i}{x_j}-z_0\right), \tag{3.4.27}$$

where $z_{ij}\in\mathbb{C}$ and $z_0\in\mathbb{C}\backslash\{0\}$. This polynomial has $1-n+n^2$ parameters where a sunset graph as $n+1$ physical parameters given by n masses and one kinematics invariant

$$\mathcal{F}^{l}_{\ominus}(p^2,\underline{\xi}^2,\underline{x})=x_1\cdots x_{l+1}\left(\sum_{i=1}^{l+1}\frac{1}{x_i}\right)\left(\sum_{i=1}^{l+1}\xi_i^2x_i\right)-p^2x_1\cdots x_{l+1}\,. \tag{3.4.28}$$

So there are too many parameters from $n\geq 3$ but this generalisation will be useful for the GKZ description used in the next sections.

3.4.4. The massive one-loop graph. In this section we show how to apply the GKZ formalism on the one-loop bubble integral



3.4.4.1. *Maximal cut.* The one-loop sunset (or bubble) graph as the graph polynomial

$$\mathcal{F}_{\circ}(x_1,x_2,t,\xi_1^2,\xi_2^2)=p^2x_1x_2-(\xi_1^2x_1+\xi_2^2x_2)(x_1+x_2)\,. \tag{3.4.29}$$

The most general toric degree two polynomial in $\mathbb{P}^2$ with at most degree two monomial is given by

$$\mathcal{F}^{tor}_{\circ}(x_1,x_2,z_1,z_2,z_3)=z_1x_1^2+z_2x_2^2+z_3x_1x_2\,. \tag{3.4.30}$$

This toric polynomial has three parameters which is exactly the number of independent physical parameters. The identification of the variables is given by

$$z_1=-\xi_1^2,\qquad z_2=-\xi_2^2,\qquad z_3=p^2-(\xi_1^2+\xi_2^2)\,, \tag{3.4.31}$$

We consider the equivalent toric Laurent polynomial

$$P(x_1,x_2)=\frac{\mathcal{F}^{tor}_{\circ}}{x_1x_2}=\sum_{i=1}^{3} z_ix_1^{a_i^1}x_2^{a_i^2}\,, \tag{3.4.32}$$

so that p^2 in (3.4.29) corresponds to the constant term (or the origin the Newton polytope) and setting $\mathbf{a}_i=(1,a_i^1,a_i^2)$ we have

$$\mathbf{A}_{\circ}=\begin{pmatrix}\mathbf{a}_1\\ \mathbf{a}_2\\ \mathbf{a}_3\end{pmatrix}=\begin{pmatrix}1&-1&1\\1&1&-1\\1&0&0\end{pmatrix}. \tag{3.4.33}$$

The lattice is defined by

$$\mathbb{L}_{\circ}:=\{\boldsymbol{\ell}:=(\ell_1,\ell_2,\ell_3)\in\mathbb{Z}^3|\ell_1\mathbf{a}_1+\ell_2\mathbf{a}_2+\ell_3\mathbf{a}_3=\boldsymbol{\ell}\cdot\mathbf{A}_{\circ}=0\}\,. \tag{3.4.34}$$

This means that the elements of $\mathbb{L}_{\circ}$ are in the kernel of $\mathbf{A}_{\circ}$. This lattice in $\mathbb{Z}^3$ has rank one

$$\mathbb{L}_{\circ}=(1,1,-2)\mathbb{Z}\,. \tag{3.4.35}$$

Notice that all the elements automatically satisfy the condition $\ell_1 + \ell_2 + \ell_3 = 0$.

Because the rank is one the GKZ system of differential equations is given by

$$\begin{aligned} e_1 &:= \frac{\partial^2}{\partial z_1 \partial z_2} - \frac{\partial^2}{(\partial z_3)^2}, \\ d_1 &:= \sum_{r=1}^{3} z_r \frac{\partial}{\partial z_r}, \\ d_2 &:= z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2}, \end{aligned} \tag{3.4.36}$$

By construction for $\alpha \in \mathbb{C}$

$$\begin{aligned} e_1 (\mathcal{F}_\circ^{tor})^\alpha &= 0, \\ d_1 (\mathcal{F}_\circ^{tor})^\alpha &= \alpha\, (\mathcal{F}_\circ^{tor})^\alpha, \end{aligned} \tag{3.4.37}$$

and

$$d_2 (\mathcal{F}_\circ^{tor})^\alpha = \frac{1}{2} \left(\partial_{x_1} (x_1 (\mathcal{F}_\circ^{tor})^\alpha) - \partial_{x_2} (x_2 (\mathcal{F}_\circ^{tor})^\alpha) \right) , \tag{3.4.38}$$

therefore the action of the derivative d_2 vanishes on the integral but not the integrand

$$d_2 \int_\gamma (\mathcal{F}_\circ^{tor})^\alpha = 0 \qquad \text{for} \qquad \partial\gamma = \emptyset\,. \tag{3.4.39}$$

The GKZ hypergeometric series is defined as for $\gamma_i \notin \mathbb{Z}$

$$\Phi^\circ_{\mathbb{L},\boldsymbol{\gamma}} = \sum_{\boldsymbol{\ell} \in \mathbb{L}_\circ} \prod_{i=1}^{3} \frac{z_i^{l_i + \gamma_i}}{\Gamma(l_i + \gamma_i + 1)}, \tag{3.4.40}$$

in this sum we have $\boldsymbol{\ell} = n(1,1,-2)$ with $n \in \mathbb{Z}$, and the condition $\sum_{i=1}^{3} \gamma_i \mathbf{a}_i = (0,0,-1)$ which can be solved using $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) = \gamma(1,1,-2) + (0,0,-1)$, leading to

$$\Phi^\circ_{\mathbb{L},\boldsymbol{\gamma}} = \frac{1}{z_3} \sum_{n \in \mathbb{Z}} \frac{u_1^n}{\Gamma(n + \gamma + 1)^2 \Gamma(-2n + \gamma)}, \tag{3.4.41}$$

where we have introduced the new toric coordinate

$$u_1 := \frac{z_1 z_2}{z_3^2} = \frac{\xi_1^2 \xi_2^2}{\left(p^2 - (\xi_1^2 + \xi_2^2)\right)^2}\,. \tag{3.4.42}$$

This is the natural coordinate dictated by the invariance of the period integral under the transformation $(x_1, x_2) \to (\lambda x_1, \lambda x_2)$ and $(z_1, z_2, z_3) \to (z_1/\lambda, z_2/\lambda, z_3/\lambda)$.

This GKZ hypergeometric function is a combination of ${}_3F_2$ hypergeometric functions

$$\begin{aligned} \Phi^\circ_{\mathbb{L},\boldsymbol{\gamma}} = \frac{1}{z_3^{1-2\gamma}} \Bigg(& \frac{u_1^{\gamma-1}}{\Gamma(\gamma)\Gamma(\gamma+2)}\, {}_3F_2 \left({1,\, 1-\gamma,\, 1-\gamma \atop 1 + \frac{\gamma}{2},\, \frac{3}{2} + \frac{\gamma}{2}} \Big| \frac{1}{4u_1} \right) \\ & + \frac{u_1^\gamma}{\Gamma(\gamma+1)^2}\, {}_3F_2 \left({1,\, \frac{1}{2} - \frac{\gamma}{2},\, \frac{1}{2} - \frac{\gamma}{2} \atop 1+\gamma,\, 1+\gamma} \Big| 4u_1 \right) \Bigg)\,. \end{aligned} \tag{3.4.43}$$

For $\gamma = 0$ the series is trivially zero as the system is resonant and needs to be regularised [128, 133] . The regularisation is to use the functional equation for the Γ-function $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ to replace the pole term by

$$\lim_{\epsilon\to 0} \frac{\Gamma(\epsilon)}{\Gamma(-2n+\epsilon)} = \Gamma(1+2n), \qquad n \in \mathbb{Z}\backslash\{0\}, \tag{3.4.44}$$

and write the associated regulated period as

$$\pi_\circ = \lim_{\epsilon\to 0} \frac{1}{z_3} \sum_{n\in\mathbb{N}} \frac{u_1^n \Gamma(\epsilon)}{\Gamma(n+1)^2\Gamma(-2n+\epsilon)}\,, \tag{3.4.45}$$

which is easily shown to be

$$\begin{aligned} \pi_\circ(z_1,z_2,z_3) &= \frac{1}{z_3}\, {}_2F_1\left(\begin{matrix}\frac12 & 1\\ & 1\end{matrix}\middle| 4u_1\right) = \frac{1}{\sqrt{z_3^2-4z_1z_2}}\,,\\ &= \frac{1}{\sqrt{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}}\,. \end{aligned} \tag{3.4.46}$$

This expression of course matches the expression for the maximal cut (2.1.21) integral $\pi_\circ(p^2,\xi_1^2,\xi_2^2,2)$ in two dimensions

$$\pi_\circ(p^2,\xi_1^2,\xi_2^2,2) = \frac{1}{(2i\pi)^2}\int_{|x_1|=|x_2|=1} \frac{dx_1dx_2}{\mathcal{F}_\circ(x_1,x_2)}\,. \tag{3.4.47}$$

3.4.4.2. *The differential operator.* From the expression of the maximal cut $\pi_\circ$ in (3.4.46) as an hypergeometric series, which satisfies a second order differential equation (3.4.20), we can extract a differential operator with respect to p^2 or the masses ξ_i^2 annihilating the maximal cut. This differential equation is not the minimal one as it can be factorised leaving minimal order differential operators are annihilating the maximal cut are such that $L^\circ_{PF,(1)}\pi_\circ(p^2,\xi_1^2,\xi_2^2) = 0$ and $L^\circ_{PF,(2)}\pi_\circ(p^2,\xi_1^2,\xi_2^2) = 0$ with

$$L^\circ_{PF,(1)} = p^2\frac{d}{dp^2} + \frac{p^2(p^2-\xi_1^2-\xi_2^2)}{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}\,, \tag{3.4.48}$$

and

$$L^\circ_{PF,(2)} = \xi_1^2\frac{d}{d\xi_1^2} - \frac{\xi_1^2(p^2-\xi_1^2+\xi_2^2)}{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}\,, \tag{3.4.49}$$

with of course a similar operator with the exchange of ξ_1 and ξ_2. These operators do not annihilate the integrand but lead to total derivatives

$$L^\circ_{PF,(1)}\frac{1}{\mathcal{F}_\circ(\underline{x},p^2,\underline{\xi}^2)} = \partial_{x_1}\left(\frac{p^2(2\xi_2^2-(p^2-(\xi_1^2+\xi_2^2))x_1)}{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)\mathcal{F}_2(x_1,1,p^2,\underline{\xi}^2)}\right)\,, \tag{3.4.50}$$

and

$$L^\circ_{PF,(2)}\frac{1}{\mathcal{F}_\circ(\underline{x},p^2,\underline{\xi}^2)} = \partial_{x_1}\left(\frac{((p^2-\xi_2^2)^2-\xi_1^2(p^2+\xi_2^2))x_1-\xi_2^2(p^2+\xi_1^2-\xi_2^2)}{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)\,\mathcal{F}_2(x_1,1,p^2,\underline{\xi}^2)}\right)\,. \tag{3.4.51}$$

These operators can be obtained from the operator $td/dt+1$ derived in §3.3.1.1 and the change of variables $t = \frac{\sqrt{(p^2-\xi_1^2-\xi_2^2)^2-4\xi_1^2\xi_2^2}}{\xi_1^2}$. For the boundary term one needs to pay attention that the shift induces a dependence on the physical parameters in the domain of integration.

3.4.4.3. *The massive one-loop sunset Feynman integral.* Having determined the differential operators acting on the maximal cut it is now easy to obtain the action of these operators on the one-loop integral. The action of the Picard-Fuchs operators on the Feynman integral $I_\circ(p^2,\xi_1^2,\xi_2^2,2)$ are given by

$$L^\circ_{PF,(1)}\, I_\circ(p^2,\xi_1^2,\xi_2^2,2) = -\frac{2}{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}\,, \tag{3.4.52}$$

and

$$L^\circ_{PF,(2)}\, I_\circ(p^2,\xi_1^2,\xi_2^2,2) = \frac{\xi_1^2-\xi_2^2-p^2}{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}\,. \tag{3.4.53}$$

It is then easy to obtain that in $D=2$ dimensions the one-loop massive bubble evaluates to

$$\begin{aligned} I_\circ(p^2,\xi_1^2,\xi_2^2) = {} & \frac{1}{\sqrt{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}} \\ & \times \log\left(\frac{p^2-(\xi_1^2+\xi_2^2)-\sqrt{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}}{p^2-(\xi_1^2+\xi_2^2)+\sqrt{(p^2-(\xi_1+\xi_2)^2)(p^2-(\xi_1-\xi_2)^2)}}\right)\,. \end{aligned} \tag{3.4.54}$$

3.4.5. The two-loop sunset. The sunset graph polynomial is the most general cubic in $\mathbb{P}^2$ with maximal order two degree for each variables
(3.4.55)
$$\mathcal{F}_\ominus(x_1,x_2,x_3,t,\underline{\xi}^2) = x_1x_2x_3\left(p^2-(\xi_1^2x_1+\xi_2^2x_2+\xi_3^2x_3)\left(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)\right)\,,$$

which corresponds to the toric polynomial

$$\mathcal{F}^{tor}_\ominus = x_1x_2x_3\left(\frac{x_3z_1}{x_1}+\frac{x_2z_2}{x_1}+\frac{x_3z_3}{x_2}+\frac{x_1z_4}{x_3}+\frac{x_2z_5}{x_3}+\frac{x_1z_6}{x_2}+z_7\right)\,. \tag{3.4.56}$$

To the contrary to the one-loop case there are more toric parameters z_i than physical variables. The identification of the physical variables is
(3.4.57)
$$-\xi_1^2 = z_4 = z_6,\quad -\xi_2^2 = z_2 = z_5,\quad -\xi_3^2 = z_1 = z_3,\quad p^2-(\xi_1^2+\xi_2^2+\xi_3^2) = z_7,$$

xAs before writing the toric polynomial as

$$P_\ominus = \sum_{i=1}^{7} z_i x_1^{a_i^1} x_2^{a_i^2} x_3^{a_i^3}\,, \tag{3.4.58}$$

and setting $\mathbf{a}_i = (1,a_i^1,a_i^2,a_i^3)$ we have

$$\mathbf{A}_\ominus = \begin{pmatrix}\mathbf{a}_1\\ \vdots \\ \mathbf{a}_7\end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 1\\ 1 & -1 & 1 & 0\\ 1 & 0 & -1 & 1\\ 1 & 1 & 0 & -1\\ 1 & 0 & 1 & -1\\ 1 & 1 & -1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\,, \tag{3.4.59}$$

The lattice is now defined by

$$\mathbb{L}_\ominus := \{\boldsymbol{\ell} := (\ell_1,\dots,\ell_7) \in \mathbb{Z}^7 | \ell_1 \mathbf{a}_1 + \cdots + \ell_7 \mathbf{a}_7 = \boldsymbol{\ell}\cdot \mathbf{A}_\ominus = 0\}\,. \tag{3.4.60}$$

This lattice in $\mathbb{Z}^7$ has rank four $\mathbb{L}_\ominus = \oplus_{i=1}^4 L_i\mathbb{Z}$ with the basis

$$\begin{pmatrix} L_1\\ \vdots\\ L_4\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & -3\\ 0 & 1 & 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 1 & 0 & 1 & 0 & -2\\ 0 & 0 & 0 & 1 & -1 & -1 & 1\end{pmatrix}, \tag{3.4.61}$$

From this we derive the sunset GKZ system

$$\begin{aligned} e_1 &:= \frac{\partial^3}{\partial z_1\partial z_5\partial z_6} - \frac{\partial^3}{(\partial z_7)^3},\\ e_2 &:= \frac{\partial^2}{\partial z_2\partial z_6} - \frac{\partial^2}{(\partial z_7)^2},\\ e_3 &:= \frac{\partial^2}{\partial z_3\partial z_5} - \frac{\partial^2}{(\partial z_7)^2},\\ e_4 &:= \frac{\partial^2}{\partial z_4\partial z_7} - \frac{\partial^2}{\partial z_5\partial z_6} \end{aligned} \tag{3.4.62}$$

by construction $e_i(\mathcal{F}^{tor}_\ominus)^\alpha = 0$ with $\alpha \in \mathbb{C}$ for $1 \leq i \leq 4$. We have as well this second set of operators from the operators

$$\begin{aligned} d_1 &:= \sum_{r=1}^7 z_r \frac{\partial}{\partial z_r},\\ d_2 &:= z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} - z_4\frac{\partial}{\partial z_4} - z_6\frac{\partial}{\partial z_6},\\ d_3 &:= z_2\frac{\partial}{\partial z_2} - z_3\frac{\partial}{\partial z_3} + z_5\frac{\partial}{\partial z_5} - z_6\frac{\partial}{\partial z_6},\\ d_4 &:= z_1\frac{\partial}{\partial z_1} + z_3\frac{\partial}{\partial z_3} - z_4\frac{\partial}{\partial z_4} - z_5\frac{\partial}{\partial z_5} \end{aligned} \tag{3.4.63}$$

The interpretation of these operators is the following

- The Euler operator $d_1\mathcal{F}^\alpha_{tor} = \alpha\,\mathcal{F}^\alpha_{tor}$ for $\alpha\in\mathbb{C}$.
- To derive the action of these operators on the maximal cut period integral

$$\pi^{tor}_\ominus(z_1,\dots,z_7) = \frac{1}{(2i\pi)^3}\int_\gamma \frac{1}{\mathcal{F}^{tor}_\ominus}\prod_{i=1}^3 dx_i\,, \tag{3.4.64}$$

  we remark that if $\mathcal{F}^{tor}_\ominus = x_1x_2x_3\,P_\ominus$ we have

$$\begin{aligned} d\left(\frac{1}{P_\ominus}\frac{dx_1}{x_1}\right) &= \frac{-z_1x_1/x_2 + z_3x_2 + z_4x_2/x_1 - z_6/x_2}{P_\ominus^2}\,\frac{dx_1}{x_1}\wedge\frac{dx_2}{x_2},\\ d\left(\frac{1}{P_\ominus}\frac{dx_1}{x_1}\right) &= -\frac{z_1x_1/x_2 + z_2x_1 - z_4x_2/x_1 - z_5/x_1}{P_\ominus^2}\,\frac{dx_1}{x_1}\wedge\frac{dx_2}{x_2}, \end{aligned} \tag{3.4.65}$$

therefore since the cycle γ has no boundary

$$\begin{aligned}
d_2\pi^{tor}_{\ominus} &= \int_\gamma d\left(\frac{1}{P_\ominus}\frac{dx_1}{x_1}\right) = 0,\\
d_3\pi^{tor}_{\ominus} &= -\int_\gamma d\left(\frac{1}{P_\ominus}\frac{dx_2}{x_2}\right) = 0,\\
d_4\pi^{tor}_{\ominus} &= \int_\gamma d\left(\frac{1}{P_\ominus}\left(\frac{dx_1}{x_1}+\frac{dx_2}{x_2}\right)\right) = 0\,.
\end{aligned} \tag{3.4.66}$$

- The natural toric coordinates are

$$u_1 := \frac{z_1z_5z_6}{z_7^3}, \qquad u_2 := \frac{z_2z_6}{z_7^2}, \qquad u_3 := \frac{z_3z_5}{z_7^2}, \qquad u_4 := \frac{z_4z_7}{z_5z_6}\,, \tag{3.4.67}$$

which reads in terms of the physical parameters

$$\begin{aligned}
u_2 &= \frac{\xi_1^2\xi_2^2}{\left(p^2-(\xi_1^2+\xi_2^2+\xi_3^2)\right)^2}, \quad u_3 = \frac{\xi_2^2\xi_3^2}{\left(p^2-(\xi_1^2+\xi_2^2+\xi_3^2)\right)^2},\\
u_4 &= \frac{p^2-(\xi_1^2+\xi_2^2+\xi_3^2)}{\xi_2^2}, \quad u_1 = u_2u_3u_4.
\end{aligned} \tag{3.4.68}$$

They are the natural variables associated with the toric symmetries of the period integral

$$\begin{aligned}
(x_1,x_2)\to(\lambda x_1,x_2), &\quad (z_1,z_2,z_3,z_4,z_5,z_6,z_7)\to(z_1/\lambda,z_2/\lambda,z_3,z_4\lambda,z_5\lambda,z_6,z_7),\\
(x_1,x_2)\to(x_1,\lambda x_2), &\quad (z_1,z_2,z_3,z_4,z_5,z_6,z_7)\to(z_1\lambda,z_2,z_3/\lambda,z_4/\lambda,z_5,z_6\lambda,z_7),\\
(x_1,x_2)\to(\lambda x_1,\lambda x_2), &\quad (z_1,z_2,z_3,z_4,z_5,z_6,z_7)\to(z_1,z_2/\lambda,z_3/\lambda,z_4,z_5\lambda,z_6\lambda,z_7).
\end{aligned} \tag{3.4.69}$$

The sunset GKZ hypergeometric series is defined as for $\gamma_i \notin \mathbb{Z}$ with $1\leq i\leq 7$

$$\Phi^{\ominus}_{\mathbb{L},\boldsymbol{\gamma}}(z_1,\dots,z_7) = \sum_{\boldsymbol{\ell}\in\mathbb{L}}\prod_{i=1}^{7}\frac{z_i^{l_i+\gamma_i}}{\Gamma(l_i+\gamma_i+1)}, \tag{3.4.70}$$

in this sum we have $\boldsymbol{\ell} = \sum_{i=1}^4 n_iL_i$ with $n_i\in\mathbb{Z}$, and the condition $\sum_{i=1}^7\gamma_i\mathbf{a}_i = (-1,0,0,0)$ which can be solved using $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_7) = \sum_{i=1}^4\gamma_i\mathcal{L}_i + (0,\dots,0,-1)$. Using the leading to toric variables the solution reads

$$\begin{aligned}
\Phi^{\ominus}_{\mathbb{L},\boldsymbol{\gamma}}(z_1,\dots,z_7) &= \frac{1}{z_7}\sum_{(n_1,\dots,n_4)\in\mathbb{Z}}\frac{u_1^{n_1+\gamma_1}u_2^{n_2+\gamma_2}u_3^{n_3+\gamma_3}u_4^{n_4+\gamma_4}}{\prod_{i=1}^4\Gamma(n_i+\gamma_i+1)}\times\\
&\times\frac{1}{\Gamma(n_1+n_2-n_4+\gamma_1+\gamma_2-\gamma_4+1)\Gamma(n_1+n_3-n_4+\gamma_1+\gamma_3-\gamma_4+1)}\\
&\times\frac{1}{\Gamma(-3n_1-2n_2-2n_3+n_4-3\gamma_1-2\gamma_2-2\gamma_3+\gamma_4)}\,.
\end{aligned} \tag{3.4.71}$$

With $\boldsymbol{\gamma} = (0,0,0,0,0,0,0)$ the series is trivially zero as being resonant. The resolution is to the regularise the term has a zero by using for $\ell_7<0$

$$\lim_{\epsilon\to0}\frac{\Gamma(\epsilon)}{\Gamma(\ell_7+\epsilon)} = (-1)^{\ell_7}\Gamma(1-\ell_7)\,, \tag{3.4.72}$$

and write the associated regulated period as

$$\pi_\ominus^{(2)}(p^2,\underline{\xi}^2)=\lim_{\epsilon\to0}\sum_{(n_1,n_2,n_3,n_4)\in\mathbb{N}}\frac{(\xi_1^2)^{n_1+n_2}(\xi_2^2)^{n_1+n_2+n_3-n_4}(\xi_3^2)^{n_1+n_3}}{\prod_{i=1}^4\Gamma(1+n_i)}$$
$$\times\frac{(p^2-(\xi_1^2+\xi_2^2+\xi_3^2))^{-3n_1-2n_2-2n_3+n_4-1}(-1)^{-3n_1-2n_2-2n_3+n_4}\Gamma(\epsilon)}{\Gamma(1+n_1+n_2-n_4)\Gamma(1+n_1+n_3-n_4)\Gamma(-3n_1-2n_2-2n_3+n_4+\epsilon)}\,. \tag{3.4.73}$$

One can expand this expression as a series near $t=\infty$ to get that

$$\pi_\ominus^{(2)}(p^2,\xi_1^2,\xi_2^2,\xi_3^2)=\sum_{n\geq0}(p^2)^{-n-1}\sum_{n_1+n_2+n_3=n}\left(\frac{n!}{n_1!n_2!n_3!}\right)^2\xi_1^{2n_1}\xi_2^{2n_2}\xi_3^{2n_3}\,, \tag{3.4.74}$$

which is the series expansion of the maximal cut integral

$$\pi_\ominus^{(2)}(p^2,\underline{\xi}^2)=\frac{1}{(2i\pi)^3}\int_\gamma\frac{1}{\mathcal{F}_\ominus}\prod_{i=1}^3dx_i\,, \tag{3.4.75}$$

where $\gamma=\{|x_1|=|x_2|=|x_3|=1\}$.

3.4.5.1. *The differential operators.* Now that we have the expression for the maximal cut it is easy to derive the minimal order differential operator annihilating this period. There are various methods to derive the Picard-Fuchs operator from the maximal cut. One method is to use the series expansion of the period around $s=1/t=0$. Another method is to reduce the GKZ system of differential operator in similar fashion as shown for the hypergeometric function in §3.4.2.1. This method leads to a fourth order differential operator which factorises a minimal second order operator. We notice that this approach is similar to the integration-by-part based approach

The minimal order differential operator is of second order

$$\mathcal{L}_{PF}^\ominus=\left(p^2\frac{d}{dp^2}\right)^2+q_1(p^2,\underline{\xi}^2)\left(p^2\frac{d}{dp^2}\right)+q_0(p^2,\underline{\xi}^2)\,, \tag{3.4.76}$$

with the coefficients given in [3, 134]. The action of this differential operator on the maximal cut is given by

$$\mathcal{L}_{PF}^\ominus\pi_\ominus^{(2)}=\frac{1}{(2i\pi)^3}\int_\gamma\mathcal{L}_{PF}^\ominus\frac{1}{\mathcal{F}_\ominus}\prod_{i=1}^3dx_i=\frac{1}{(2i\pi)^3}\int_\gamma\left(\sum_{i=1}^3\partial_i\beta_i\right)\prod_{i=1}^3dx_i=0\,. \tag{3.4.77}$$

The action of this operator on the Feynman integral is given by then we find that that full differential operator acting on the two-loop sunset integral is given by

$$\mathcal{L}_{PF}^\ominus\,I_\ominus(p^2,\underline{\xi}^2)=\int_{\substack{x_1\geq0\\x_2\geq0}}\left(\sum_{i=1}^3\partial_i\beta_i\right)\,\delta(x_3=1)\prod_{i=1}^3dx_i=\mathcal{S}_\ominus\,, \tag{3.4.78}$$

where the inhomogeneous term reads

$$\mathcal{S}_\ominus=\mathcal{Y}_\ominus(p^2,\underline{\xi}^2)+c_1(p^2,\underline{\xi}^2)\log\left(\frac{m_1^2}{m_3^2}\right)+c_2(p^2,\underline{\xi}^2)\log\left(\frac{m_2^2}{m_3^2}\right)\,, \tag{3.4.79}$$

with the Yukawa coupling[4]

$$\mathcal{Y}_\ominus(p^2,\underline{\xi}^2)=\frac{6(p^2)^2-4p^2(\xi_1^2+\xi_2^2+\xi_3^2)-2\prod_{i=1}^4\mu_i}{(p^2)^2\prod_{=1}^4(p^2-\mu_i^2)}\,, \tag{3.4.80}$$

where $(\mu_1,\dots,\mu_4)=((-\xi_1+\xi_2+\xi_3)^2),(\xi_1-\xi_2+\xi_3)^2),(\xi_1+\xi_2-\xi_3)^2),(\xi_1+\xi_2+\xi_3)^2))$. A geometric interpretation is the integral [3]

$$\mathcal{Y}_\ominus(p^2,\underline{\xi}^2)=\int_{\mathcal{E}_\ominus}\Omega_\ominus\wedge p^2\frac{d}{p^2}\Omega_\ominus\,, \tag{3.4.81}$$

where $\Omega_\ominus$ is the sunset residue differential form

$$\Omega_\ominus=\mathrm{Res}_{\mathcal{E}_\ominus=0}\frac{x_1dx_2\wedge dx_3+x_3dx_1\wedge dx_2+x_2dx_3\wedge dx_1}{\mathcal{F}_\ominus}\,, \tag{3.4.82}$$

on the sunset elliptic curve

(3.4.83)

$$\mathcal{E}_\ominus:=\{p^2x_1x_2x_3-(\xi_1^2x_1+\xi_2^2x_2+\xi_3^2x_3)(x_1x_2+x_1x_3+x_2x_3)|(x_1,x_2,x_3)\in\mathbb{P}^2\}\,.$$

The Yukawa coupling satisfies the differential equation

$$p^2\frac{d}{p^2}\mathcal{Y}_\ominus(t)=(2-q_1(p^2,\underline{\xi}^2))\mathcal{Y}_\ominus(p^2,\underline{\xi}^2)\,. \tag{3.4.84}$$

The coefficients c_1 and c_2 in (3.4.79) are the integral of the residue one form between the marked points on $Q_1=[0,-\xi_3^2,\xi_2^2]$, $Q_2=[-\xi_3^2,0,\xi_1^2]$ and $Q_3=[-\xi_2^2,\xi_1^2,0]$ on the elliptic curve [3]

$$c_1(p^2,\underline{\xi}^2):=p^2\frac{d}{p^2}\int_{Q_1}^{Q_3}\Omega_\ominus,\qquad c_2(p^2,\underline{\xi}^2):=p^2\frac{d}{p^2}\int_{Q_2}^{Q_3}\Omega_\ominus\,. \tag{3.4.85}$$

3.4.6. The generic case. In this section we show how to determine the differential equation for the l-loop sunset integral from the knowledge of the maximal cut. The maximal cut of the l-loop sunset integral is given by

$$\pi_\ominus^{(l)}(p^2,\underline{\xi}^2)=\sum_{n\geq 0}t^{-n-1}\,A_\ominus(l,n,\xi_1^2,\dots,\xi_{l+1}^2)\,, \tag{3.4.86}$$

with

$$A_\ominus(l,n,\xi_1^2,\cdots,\xi_{l+1}^2):=\sum_{r_1+\cdots+r_{l+1}=n}\left(\frac{n!}{r_1!\cdots r_{l+1}!}\right)^2\prod_{i=1}^{l+1}\xi_i^{2r_i}\,. \tag{3.4.87}$$

3.4.6.1. *The all equal mass case.* For the all equal mass case one can easily determine the differential equation to all order [7] using the Bessel integral representation of [27]. We present here a different derivation.

For the all equal masses the coefficient of the maximal cut satisfies a nice recursion [137]

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[4]This quantity is the usual Yukawa coupling of particle physics and string theory compactification. The Yukawa coupling is determined geometrically by the integral of the wedge product of differential forms over particular cycles [135]. The Yukawa couplings which depend non-trivially on the internal geometry appear naturally in the differential equations satisfied by the periods of the underlying geometry as explained for instance in these reviews [124, 136].

(3.4.88)
$$\sum_{k\geq 0}\left(n^{l+2}\sum_{1\leq i\leq k}\sum_{\substack{a_i+b_i=l+2\\ 1<a_{i+1}+1<a_i\leq l+1}}\prod_{i=1}^{k}(-a_ib_i)\left(\frac{n-i}{n-i+1}\right)^{a_i-1}\right)A_\ominus(l,n-k,\underline{1})=0,$$

where $a_i\in\mathbb{N}$. Standard method gives that the associated differential operator acting on $t\pi^l_\ominus(t,1,\dots,1)=\sum_{n\geq0}(p^2)^{-n}A(l+1,n,1,\dots,1)$ reads

(3.4.89)
$$\mathcal{L}^{(l),1mass}_{PF,\ominus}=\sum_{k\geq0}(p^2)^k\sum_{1\leq i\leq k}\sum_{\substack{a_i+b_i=l+2,a_{k+1}=0\\ 1<a_{i+1}+1<a_i\leq l+1}}\left(k-p^2\frac{d}{p^2}\right)^{l+2-a_1}\\ \times\prod_{i=1}^{k}(-a_ib_i)\left(k-i-p^2\frac{d}{dp^2}\right)^{a_i-a_{i+1}}.$$

This operator has been derived in [7, §9] using different method.

They are differential operators of order l, the loop order, in d/dp^2 and the coefficients are polynomials of degree $l+1$

(3.4.90)
$$\mathcal{L}^{(l),1mass}_{PF}=(-p^2)^{\lceil l/2\rceil-1}\prod_{i=1}^{\lfloor l/2\rfloor+1}(p^2-\mu_i^2)\left(\frac{d}{dp^2}\right)^l+\cdots$$

where $\mu_i^2:=(\pm1\pm1\cdots\pm1)^2$ is the set of the different thresholds. The operator $\mathcal{L}^{(2),1mass}_{PF}$ is the Picard-Fuchs operator of the family of elliptic curves for $\Gamma_1(6)$ for the all equal mass sunset [1], the operator $\mathcal{L}^{(3),1mass}_{PF}$ of the family of $K3$ surfaces [2]. Having determined the Picard-Fuchs operator it is not difficult to derive its action on the Feynman integral with the result that [7]

(3.4.91)
$$\mathcal{L}^{(l),1mass}_{PF}(I_\ominus(p^2,1,\dots,1))=-(l+1)!\,.$$

3.4.6.2. *The general mass case.* For unequal masses the recursion relation does not close only on the coefficients (3.4.87) and no simple closed formula is known for the differential operator on the maximal cut. The minimal differential operator annihilating the $\pi^{(l)}_\ominus(t,\underline{\xi}^2)$ can be obtained using the GKZ hypergeometric function discussed in the previous section.

For the l-loop sunset integral the GKZ lattice has rank l^2, $\mathbb{L}=\sum_{i=1}^{l^2}n_i\,L_i$. For instance for the three-loop sunset the regulated hypergeometric series representation of the maximal cut reads

(3.4.92)
$$\pi^{(3)}_\ominus(p^2,\underline{\xi}^2)=-\lim_{\epsilon\to0}\sum_{(n_1,\dots,n_9)\in\mathbb{N}^9}\frac{(\xi_1^2)^{n_1+n_2+n_3}(\xi_2^2)^{n_1+n_3+n_4+n_6-n_7-n_8+n_9}}{\prod_{i=1}^9\Gamma(1+n_i)}\\ \times\frac{(\xi_3^2)^{n_2+n_5+n_8}(\xi_4^2)^{n_1+n_4+n_6}}{\Gamma(n_1+n_4+n_6-n_7-n_8+1)\Gamma(n_2+n_5-n_6+n_8-n_9+1)}\\ \times\frac{}{\Gamma(n_1+n_3-n_5+n_6-n_7-n_8+n_9+1)}\\ \times\frac{(-p^2+\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)^{-3n_1-2n_2-2n_3-2n_4-n_5-2n_6+n_7-n_9-1}\Gamma(\epsilon)}{\Gamma(-3n_1-2n_2-2n_3-2n_4-n_5-2n_6+n_7-n_9+\epsilon)}.$$

The minimal order differential operator annihilating the maximal cut $p^2\pi^{(3)}_{\ominus}(p^2,\underline{\xi}^2)$ with generic mass configurations, $\xi_1 \neq \xi_2 \neq \xi_3 \neq \xi_4$ and all the masses non vanishing, is an operator of order 6, with polynomial coefficients $c_k(t)$ of degree up to 29

$$L^3_{PF,\ominus} = \sum_{k=0}^{6} c_k(t) \left(t\frac{d}{t}\right)^k . \tag{3.4.93}$$

For instance the differential operator for the mass configuration $\xi_i = i$ with $1 \leq i \leq 4$ is given by

$$\begin{aligned} c_6 &= (t-100)(t-36)(t-64)(t-4)^2(t-16)^2\\ &\quad\times\big(345t^{12} - 10275t^{11} + 243243t^{10} + 700860t^9 - 289019444t^8 + 9517886160t^7\\ &\quad - 169244843904t^6 + 2163112875520t^5 - 24375264125952t^4\\ &\quad + 198627459010560t^3 - 896517312217088t^2\\ &\quad + 1570362910310400t - 1192050032640000\big), \end{aligned} \tag{3.4.94}$$

and

$$\begin{aligned} c_5 &= (t-4)(t-16)\big(7245t^{17} - 1461150t^{16} + 108842709t^{15} - 4073021820t^{14}\\ &\quad + 79037467036t^{13} + 706049613520t^{12} - 122977114948800t^{11}\\ &\quad + 4897976525794560t^{10} - 118057966435402752t^9\\ &\quad + 2042520337021317120t^8 - 28129034886941589504t^7\\ &\quad + 321784682881513881600t^6 - 2877522528057659228160t^5\\ &\quad + 17978948962533528043520t^4 - 69950845277551433089024t^3\\ &\quad + 151117855778012806512640 0t^2 - 182250696371318292480000t\\ &\quad + 96676211287130112000000\big), \end{aligned} \tag{3.4.95}$$

and

$$\begin{aligned} c_4 &= 2\big(23460t^{19} - 4086975t^{18} + 273974766t^{17} - 9833465295t^{16}\\ &\quad + 173874227860t^{15} + 3780156754180t^{14}\\ &\quad - 419091386081744t^{13} + 16647873781420800t^{12}\\ &\quad - 425729411677916160t^{11} + 8098824799795968000t^{10}\\ &\quad - 125136842089603031040t^9 + 1631034274362173030400t^8\\ &\quad + 17364390414642101354496t^7 + 140612615518097533829120t^6\\ &\quad - 807868060015143792148480t^5 + 3100095209313936311582720t^4\\ &\quad - 7563751451192001262780416t^3 + 11448586013594218187980800t^2\\ &\quad - 9812428506034109153280000t + 3374878648568905728000000\big), \end{aligned} \tag{3.4.96}$$

and

$$
\begin{aligned}
c_3 = 12\big(&8970t^{19} - 1147050t^{18} + 56442264t^{17} - 1477273050t^{16} - 447578647t^{15} \\
&+ 2416587481200t^{14} - 130189239609348t^{13} + 4001396495500560t^{12} \\
&- 86975712270293184t^{11} + 1511724058206439680t^{10} \\
&- 22690173944998831104t^{9} + 289974679497600921600t^{8} \\
&- 2900762618196498137088t^{7} + 20882244400635484241920t^{6} \\
&- 101090327023260610854912t^{5} + 308760428925736546467840t^{4} \\
&- 559057237244267332632576t^{3} + 533177283118109609164800t^{2} \\
&- 133034777312420167680000t - 14061994369037107200000\big),
\end{aligned}
\tag{3.4.97}
$$

and

$$
\begin{aligned}
c_2 = 24\big(&3105t^{19} - 260100t^{18} + 8740695t^{17} - 121279200t^{16} - 8982728081t^{15} \\
&+ 771645247175t^{14} - 29786960482306t^{13} + 741851366254700t^{12} \\
&- 14140682364004072t^{11} + 237224880534337760t^{10} \\
&- 3605462277123620992t^{9} + 44725169880349560320t^{8} \\
&- 405767142088142927872t^{7} + 2549108215435181793280t^{6} \\
&- 11307241496864563101696t^{5} + 40972781273200446013440t^{4} \\
&- 141797614014479525216256t^{3} + 363118631232748702924800t^{2} \\
&- 415180490608717332480000t + 210929915535556608000000\big),
\end{aligned}
\tag{3.4.98}
$$

and

$$
\begin{aligned}
c_1 = 24\big(&345t^{19} - 15000t^{18} + 345675t^{17} + 7323600t^{16} - 3165461083t^{15} \\
&+ 184943420750t^{14} - 5084383561348t^{13} + 91042473303800t^{12} \\
&- 1344824163401536t^{11} + 17444484465759680t^{10} \\
&- 146155444722244096t^{9} - 426434786380119040t^{8} \\
&+ 31798683088486989824t^{7} - 488483076656283893760t^{6} \\
&+ 5136134162164414021632t^{5} - 40834519838668015534080t^{4} \\
&+ 222597043391679285952512t^{3} - 685074395310881085849600t^{2} \\
&+ 830360981217434664960000t - 421859831071113216000000\big),
\end{aligned}
\tag{3.4.99}
$$

and

$$\begin{aligned} c_0 = 1728 \big(&21908444t^{15} - 1482071825t^{14} + 40507170144t^{13} - 668436089250t^{12} \\ &+ 8209054542408t^{11} - 65000176183240t^{10} - 503218239747392t^9 \\ &+ 3196270830386752 0t^8 - 619576476284137472t^7 + 7554395788685281280t^6 \\ &- 73455221906789646336t^5 + 571135922816871792640t^4 \\ &- 3095113137012548304896t^3 + 9514922157095570636800t^2 \\ &- 11532791405797703680000t + 5859164320432128000000\big)\,. \end{aligned} \tag{3.4.100}$$

A systematic study of the differential operators for the l loop sunset integral will appear in [4–6].

3.5. The extended Griffith-Dwork reduction

We work with the regularised parametric representation of a Feynman integral attached to a graph Γ, with the notation that makes explicit the regularisation used in physics

$$I_\Gamma^{\epsilon,\kappa} = \int_{x_i \geq 0} \Omega_\Gamma^{\epsilon,\kappa}; \qquad \Omega_\Gamma^{\epsilon,\kappa} = \omega_\Gamma^{\epsilon,\kappa} dx_1 \cdots dx_n \tag{3.5.1}$$

with

$$\omega_\Gamma^{\epsilon,\kappa} = \frac{\mathcal{U}_\Gamma^{\nu_1+\cdots+\nu_n - \frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma^{\nu_1+\cdots+\nu_n - \frac{LD}{2}}} \prod_{i=1}^n x_i^{\nu_i - 1}, \tag{3.5.2}$$

with $D = 2\delta - 2\epsilon$ with δ a positive integer for dimensional regularisation and $\nu_i = \mathsf{v}_i + \mu_i \kappa$ for analytic regularisation. In the case when $\epsilon = \kappa = 0$ and δ a positive integer, the exponents in (3.5.2) are integers and we have a rational differential form. One may then use the Griffiths-Dwork pole reduction [138–142] applied to the case of Feynman integrals [3, 5, 8, 17, 143, 144] for determining the minimal order differential operators associated with a given Feynman integral. In integer space-time dimensions and without analytic regulator, the integrand of the Feynman integral is a rational differential form to which one can apply the generalised Griffith-Dwork algorithm [5]. When working in dimensional regularisation, i.e. $\epsilon \neq 0$, or analytic regularisation, i.e. $\kappa \neq 0$, the integrand is a twisted differential form.

In the case of dimensional regularisation the integrand is given by

$$\omega_\Gamma^{\epsilon,0} = \frac{\mathcal{U}_\Gamma^{\nu_1+\cdots+\nu_n-(L+1)d}}{\mathcal{F}_\Gamma^{\nu_1+\cdots+\nu_n-Ld}} \left(\frac{\mathcal{U}_\Gamma^{L+1}}{\mathcal{F}_\Gamma^L}\right)^\epsilon \prod_{i=1}^n x_i^{\nu_i-1}, \tag{3.5.3}$$

and for analytic regularisation

$$\omega_\Gamma^{0,\kappa} = \frac{\mathcal{U}_\Gamma^{\nu_1+\cdots+\nu_n-(L+1)d}}{\mathcal{F}_\Gamma^{\nu_1+\cdots+\nu_n-Ld}} \prod_{i=1}^n \left(\frac{x_i \mathcal{U}_\Gamma}{\mathcal{F}_\Gamma}\right)^{b_i\kappa} \prod_{i=1}^n x_i^{\mathsf{v}_i-1}. \tag{3.5.4}$$

In both cases the twist is the power of an homogeneous degree 0 rational function in $\mathbb{P}^{n-1}$.

One possible approach is a direct application of the Griffiths-Dwork pole reduction [143, 144] or the creative telescoping algorithm [117, 118, 145, 146] but this approach leads to large linear systems limiting its use for Feynman graphs with many legs or many loops. Therefore, in this work, we give an extension of the Griffiths-Dwork reduction algorithm which make an essential use of the fact that the twist is build from the Symanzik graph polynomials $\mathcal{U}_\Gamma$ and $\mathcal{F}_\Gamma$. This reduces the size of the linear system to be solved for determining the coefficients of the differential operator. Because this linear system is generically dense and of large size, we use the finite field package `FiniteFlow` [147] to derive analytic solutions.

This way we can analyse how the space-time dimension or the analytic regulator affects the minimal order of the differential operators.

Let us consider r parameters from the set of internal masses and independent kinematics, $\underline{z} := \{z_1, \dots, z_r\} \in \vec{m} \cup \vec{s}$. We seek differential operators annihilating the differential form $\Omega_\Gamma^{\epsilon,\kappa}$

$$\left(\sum_{a_1=0}^{o_1} \sum_{a_r=0}^{o_r} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}\right) \Omega_\Gamma^{\epsilon,\kappa} = d\beta_\Gamma^{\epsilon,\kappa}, \tag{3.5.5}$$

where $c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa)$ are rational functions of the physical parameters, but they are independent of the edge variables $x_1, \dots, x_n$. The inhomogeneous term is a total derivative in x_i's where the only allowed poles are those already present in $\Omega_\Gamma^{\epsilon,\kappa}$ [5]. Because the domain of integration of the Feynman integral does not depend on the physical parameters, we then deduce

$$\left(\sum_{a_1=0}^{o_1} \sum_{a_r=0}^{o_r} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}\right) I_\Gamma^{\epsilon,\kappa} = \mathscr{S}_\Gamma^{\epsilon,\kappa}, \tag{3.5.6}$$

where $\mathscr{S}_\Gamma^{\epsilon,\kappa}$ is an inhomogeneous term obtained by integrating $d\beta_\Gamma^{\epsilon,\kappa}$ over the boundary of orthant (2.1.15). This is a non-trivial task because one needs to blow-up the intersections between the graph hypersurface and the domain of integration, so the integral is well-defined [3, 47, 60, 144]. For instance, Section 3.2 of [3] gives a detailed derivation of the inhomogeneous term for the two-loop sunset integral along these lines. If the integration is done over a cycle $\mathcal{C}$, like the one defined by the maximal cut $\mathcal{C}_{\max} := \{|x_1| = \cdots = |x_n| = 1\}$, the resulting integral is annihilated by the action of the differential operator [8]

$$\left(\sum_{a_1=0}^{o_1} \sum_{a_r=0}^{o_r} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}\right) \int_{\mathcal{C}} \Omega_\Gamma^{\epsilon,\kappa} = 0\,. \tag{3.5.7}$$

The ideal generated by these differential operators is a differential module (or D-module). Thus, the differential equations we are seeking can be obtained by deriving annihilators of $\Omega_\Gamma^{\epsilon,\kappa}$, i.e., partial differential operators that annihilate the integrand by acting on the physical parameter and the edge variables.

3.5.1. Griffiths-Dwork reduction for twisted differential forms. The differentiation of $\Omega_\Gamma^{\epsilon,\kappa}$ leads to expressions of the type

$$\sum_{\substack{\mathsf{a}=a_1+\cdots+a_r\\ a_i\geq 0}} c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r} \Omega_\Gamma^{\epsilon,\kappa} = \sum_{\substack{\mathsf{a}=a_1+\cdots+a_r\\ a_i\geq 0}} \frac{c_{a_1,\dots,a_r}(\vec{m}, \vec{s}, \epsilon, \kappa) P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{\mathsf{a}}} \Omega_\Gamma^{\epsilon,\kappa}, \tag{3.5.8}$$

where $P^{(a_1,\dots,a_r)}(\underline{x})$ is a homogeneous polynomial of degree $(L+1)(a_1+\cdots+a_r)$ in the edge variables $\underline{x}$. The sum is over the differential operators of order $a_1 \geq 0,\dots,a_r \geq 0$ and fixed total order $\mathsf{a} := a_1+\cdots+a_r$. The pole order in the second Symanzik polynomial $\mathcal{F}_\Gamma$ has increased by a. To derive Eq. (3.5.5) one needs to find the coefficient $c_{a_1,\dots,a_r}(\vec{m},\vec{s},\epsilon,\kappa)$. From now on we consider the case where $\nu_1=\cdots=\nu_r=1$ so that $\nu=n$. The case with $\nu_i\neq 1$ is an immediate generalisation.

3.5.1.1. *The pole reduction for dimensional regularisation.* We adapt the Griffiths-Dwork pole reduction to the case of the twisted differential form (**??**) in dimensional regularisation (i.e. $\kappa=0$ and $\epsilon\neq 0$). The starting point of the algorithm is the reduction of polynomial $P^{(a_1,\dots,a_r)}(\underline{x})$ in the numerator of (3.5.8)

$$P^{(a_1,\dots,a_r)}(\underline{x}) = \vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\mathcal{F}_\Gamma\,, \tag{3.5.9}$$

where we have introduced the gradient $\vec{\nabla}\mathcal{F}_\Gamma := (\partial_{x_1}\mathcal{F}_\Gamma(\underline{x}),\dots,\partial_{x_n}\mathcal{F}_\Gamma(\underline{x}))$. The components of the size n vector $\vec{C}_{\mathsf{a}}(\underline{x})$ are homogeneous polynomials of degree $\mathsf{a}(L+1)-L$ in $\underline{x}$. We generalise the construction by Griffiths [139,140] to include the twist factor for $\mathsf{a}>1$

$$\beta^{(a_1,\dots,a_r)} = \sum_{1\leq i<j\leq n}\frac{x_iG^j_{\mathsf{a}}(\underline{x})-x_jG^i_{\mathsf{a}}(\underline{x})}{\mathcal{F}_\Gamma^{\mathsf{a}-1}}\,dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\,. \tag{3.5.10}$$

To take into account the general dimensional case, we have introduced the vectors of twisted forms

$$\vec{G}_{\mathsf{a}}(\underline{x}) := \vec{C}_{\mathsf{a}}\,\omega_\Gamma^\epsilon\,, \tag{3.5.11}$$

whose components are of homogeneous degree $(\mathsf{a}-1)(L+1)+1-n$. Following the same steps as in [138], we have

$$\begin{aligned} d\beta_\Gamma^{(a_1,\dots,a_r)} &= -(\mathsf{a}-1)\sum_{1\leq i<j\leq n}\frac{x_iG^j_{\mathsf{a}}(\underline{x})-x_jG^i_{\mathsf{a}}(\underline{x})}{\mathcal{F}_\Gamma^{\mathsf{a}}}\,d\mathcal{F}_\Gamma\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n \\ &+\sum_{1\leq i<j\leq n}\frac{d(x_iG^j_{\mathsf{a}}(\underline{x})-x_jG^i_{\mathsf{a}}(\underline{x}))}{\mathcal{F}_\Gamma^{\mathsf{a}-1}}\wedge dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n\,. \end{aligned} \tag{3.5.12}$$

From the degree of homogeneity of $\mathcal{F}_\Gamma$ and the components of $\vec{G}_{\mathsf{a}}(\underline{x})$

$$\begin{aligned} &\sum_{i=1}^n x_i\frac{\partial\mathcal{F}_\Gamma(\underline{x})}{\partial x_i} = (L+1)\mathcal{F}_\Gamma(\underline{x}),\\ &\sum_{i=1}^n x_i\frac{\partial\vec{G}_{\mathsf{a}}(\underline{x})}{\partial x_i} = ((\mathsf{a}-1)(L+1)+1-n)\vec{G}_{\mathsf{a}}(\underline{x})\,, \end{aligned} \tag{3.5.13}$$

we find that

$$d\beta_\Gamma^{(a_1,\dots,a_r)} = (\mathsf{a}-1)\frac{\vec{G}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\mathcal{F}_\Gamma}{\mathcal{F}_\Gamma^{\mathsf{a}}}\Omega_0^{(n)} - \frac{\vec{\nabla}\cdot\vec{G}_{\mathsf{a}}(\underline{x})}{\mathcal{F}_\Gamma^{\mathsf{a}-1}}\Omega_0^{(n)}\,. \tag{3.5.14}$$

Using the definition of $\vec{G}_{\mathsf{a}}$ in (3.5.11) we have reduced the pole order of $\mathcal{F}_\Gamma$ in (3.5.8)

$$(\mathsf{a}-1)\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\Omega_\Gamma^\epsilon = \frac{\vec{\nabla}\cdot\vec{G}_{\mathsf{a}}(\underline{x})}{\mathcal{F}_\Gamma^{\mathsf{a}-1}}\Omega_0^{(n)} + d\beta_\Gamma^{(a_1,\dots,a_r)}. \tag{3.5.15}$$

We now expand the first term in the right-hand-side

$$\vec{\nabla}\cdot\vec{G}_{\mathsf{a}}(\underline{x})=\vec{\nabla}\cdot\vec{C}_{\mathsf{a}}(\underline{x})\frac{\mathcal{U}_\Gamma^{\lambda_U}}{\mathcal{F}_\Gamma^{\lambda_F}}+\vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\left(\frac{\mathcal{U}_\Gamma^{\lambda_U}}{\mathcal{F}_\Gamma^{\lambda_F}}\right),\tag{3.5.16}$$

where we have defined

$$\lambda_U=n-(L+1)(\delta-\epsilon),\qquad \lambda_F=n-L(\delta-\epsilon).\tag{3.5.17}$$

The second term in this equation can be evaluated using

$$\begin{aligned}\vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\left(\frac{\mathcal{U}_\Gamma^{\lambda_U}}{\mathcal{F}_\Gamma^{\lambda_F}}\right)&=\Big(\lambda_U\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathcal{U}_\Gamma-\lambda_F\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathcal{F}_\Gamma\Big)\frac{\mathcal{U}_\Gamma^{\lambda_U}}{\mathcal{F}_\Gamma^{\lambda_F}}\\&=\left[-\lambda_F\frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma}+\lambda_U\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathcal{U}_\Gamma\right]\frac{\mathcal{U}_\Gamma^{\lambda_U}}{\mathcal{F}_\Gamma^{\lambda_F}}\,,\end{aligned}\tag{3.5.18}$$

where we have used Eq. (3.5.9) in the second equality. Therefore,

$$\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\Omega_\Gamma^\epsilon=\frac{\vec{\nabla}\cdot\vec{C}_{\mathsf{a}}(\underline{x})+\lambda_U\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathcal{U}_\Gamma}{(\mathsf{a}-1+\lambda_F)\;\mathcal{F}_\Gamma^{\mathsf{a}-1}}\Omega_\Gamma^\epsilon+\frac{1}{\mathsf{a}-1+\lambda_F}d\beta_\Gamma^{(a_1,\dots,a_r)}.\tag{3.5.19}$$

This expression involves the term $\vec{C}_{\mathsf{a}}\cdot\vec{\nabla}\log\mathcal{U}_\Gamma$ which has a pole in $\mathcal{U}_\Gamma$. We then perform a second reduction by demanding that

$$\vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\mathcal{U}_\Gamma=c_{\mathsf{a}}(\underline{x})\,\mathcal{U}_\Gamma\,,\tag{3.5.20}$$

where $c_{\mathsf{a}}(\underline{x})$ is a homogeneous polynomial of degree $(\mathsf{a}-1)(L+1)$. This is equivalent to the computation of syzygies of $\mathrm{Jac}(\mathbf{U}_\Gamma):=\langle\vec{\nabla}\mathcal{U}_\Gamma(\underline{x})\rangle$. Indeed, using the homogeneity of $\mathcal{U}_\Gamma$ we can rewrite the previous equation as

$$\Big(L\vec{C}_{\mathsf{a}}(\underline{x})-c_{\mathsf{a}}(\underline{x})\vec{x}\Big)\cdot\vec{\nabla}\mathcal{U}_\Gamma=0\,,\tag{3.5.21}$$

which are examples of syzygies of the Jacobian of $\mathbf{U}_\Gamma$. Using this reduction in Eq. (3.5.19) leads to

$$\left(\frac{\partial}{\partial z_1}\right)^{a_1}\cdots\left(\frac{\partial}{\partial z_r}\right)^{a_r}\Omega_\Gamma^\epsilon=\frac{M^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{\mathsf{a}-1}}\Omega_\Gamma^\epsilon+\frac{\mathsf{a}-1}{\mathsf{a}+n-L(\delta-\epsilon)}d\beta_\Gamma^{(a_1,\dots,a_r)}\tag{3.5.22}$$

with the numerator given by the polynomial of homogeneous degree $(\mathsf{a}-1)(L+1)$

$$M^{(a_1,\dots,a_r)}(\underline{x}):=\frac{\vec{\nabla}\cdot\vec{C}_{\mathsf{a}}(\underline{x})+\lambda_U\,c_{\mathsf{a}}(\underline{x})}{\mathsf{a}-1+\lambda_F},\tag{3.5.23}$$

with λ_U and λ_F the powers of the $\mathcal{U}_\Gamma$ and the $\mathcal{F}_\Gamma$ polynomials respectively given in (3.5.17).

To perform the pole reduction, we have to solve the linear system

$$\begin{cases}\vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\mathcal{F}_\Gamma=P^{(a_1,\dots,a_r)}(\underline{x})\\\vec{C}_{\mathsf{a}}(\underline{x})\cdot\vec{\nabla}\mathcal{U}_\Gamma=c_{\mathsf{a}}(\underline{x})\mathcal{U}_\Gamma\end{cases},\tag{3.5.24}$$

for determining the coefficients of $\vec{C}_{\mathsf{a}}(\underline{x})$ and $c_{\mathsf{a}}(\underline{x})$. The system (3.5.24) has a solution when its rank is positive. We have a linear system of the n components of $\vec{C}_{\mathsf{a}}(\underline{x})$ which are homogeneous polynomial of degree $\deg(C)=\deg(P^{(a_1,\dots,a_r)})-L$ in $\underline{x}$ and $c_{\mathsf{a}}(\underline{x})$ which is a polynomial of homogeneous degree $\deg(C)-1$. Since the number of coefficients of a homogeneous polynomial of degree d in n variables is $\binom{d+n-1}{d}$, the system has

$$n\binom{\deg(P^{(a_1,\dots,a_r)})-L+n-1}{\deg(P^{(a_1,\dots,a_r)})-L}+\binom{\deg(P^{(a_1,\dots,a_r)})-L+n-2}{\deg(P^{(a_1,\dots,a_r)})-L-1} \tag{3.5.25}$$

unknown variables for

$$\binom{\deg(P^{(a_1,\dots,a_r)})+n-1}{\deg(P^{(a_1,\dots,a_r)})}+\binom{\deg(P^{(a_1,\dots,a_r)})+n-2}{\deg(P^{(a_1,\dots,a_r)})-1} \tag{3.5.26}$$

equations. Since the $\deg(P^{(a_1,\dots,a_r)})=\mathsf{a}(L+1)$, the rank of the system (3.5.24) is

$$\begin{aligned}\text{rank} &= (3.5.25)-(3.5.26)\\ &= n\binom{(L+1)(\mathsf{a}-1)+n}{(L+1)(\mathsf{a}-1)+1}+\binom{(L+1)(\mathsf{a}-1)+n-1}{(L+1)(\mathsf{a}-1)}\\ &\quad-\binom{(L+1)\mathsf{a}+n-1}{(L+1)\mathsf{a}}-\binom{(L+1)\mathsf{a}+n-2}{(L+1)\mathsf{a}-1}\end{aligned} \tag{3.5.27}$$

For fixed values of loops L and number of edges n there is always a value of the number of derivatives a such that the system has positive rank.

A few comments are in order. In practice for Feynman integrals, the polynomial $P^{(a_1,\dots,a_r)}(\underline{x})$ is not a generic homogeneous polynomial, so the number of equations is smaller or equal than (3.5.26). We remark that this way of solving the linear system includes implicitly the freedom given by the syzygies of $\mathrm{Jac}(\mathcal{F}_\Gamma):=\langle\vec{\nabla}\mathcal{F}_\Gamma(\underline{x})\rangle$ and $\mathrm{Jac}(\mathcal{U}_\Gamma)$ since they belong to the kernel of equation (3.5.9) and (3.5.20) respectively.[5] One important property of that reduction is that the differential form $\beta_\Gamma^{(a_1,\dots,a_n)}$ is that it does not have poles that are not poles of $\mathcal{F}_\Gamma$ which is guaranteed by construction. We refer to Section 3 of [5] for a discussion of the pole constraints.

The system of linear equation (3.5.24) is dense since, in general, all coefficients in $\vec{C}_{\mathsf{a}}(\underline{x})$ and $c_{\mathsf{a}}(\underline{x})$ are non-vanishing. Moreover we are interested in analytic solutions of these systems. We thus benefit from the dense solver implemented in the Mathematica package `FiniteFlow`, described in detail in Sec.4 of [147]. Specifically, in Mathematica have used the command `FFDenseSolve`.

Part 4. Function space

A large class of amplitudes evaluate to (multiple) polylogarithms. In this case a study of the discontinuities of the amplitude can give access to interesting algebraic structures [148]. As well elliptic integrals arise from multiloop amplitudes [1, 134, 149–152] . One example is the sunset Feynman integral studied in section 4.3.2. The value of the integral is obtained from a variation of mixed Hodge structure when the external momentum is varying [1]. Pierre V.: update that paragraph

4.1. Polylogarithms

A very clear motivic approach to polylogarithms is detailed in the article by Beilinson and Deligne [153]. We only refer to the main points needed for the

[5] It was noticed in [5], that in the rational case, only the first order syzygies are needed to take into account the non-isolated singularities of Feynman integrals.

present discussion, for details we refer to the articles [153, 154]. The iterated integral definition of the polylogarithms

$$\begin{aligned} \mathrm{Li}_1(z) &:= -\log(1-z) = \int_0^z \frac{dt}{1-t} \\ \mathrm{Li}_{k+1}(z) &:= \int_0^z \mathrm{Li}_k(z)\,\frac{dt}{t}, \qquad k\geq 1 \end{aligned} \tag{4.1.1}$$

imply that they provide multivalued function on $\mathbb{P}^1\backslash\{0,1,\infty\}$. These multivalued function have monodromy properties. To this end defined the lower triangular matrix of size $n\times n$ as

$$A(z) := \begin{pmatrix} 1 & 0 & \cdots & & & \cdots & 0 \\ -\mathrm{Li}_1(z) & 1 & 0 & \cdots & & \cdots & 0 \\ -\mathrm{Li}_2(z) & \log z & 1 & 0 & \cdots & \cdots & 0 \\ -\mathrm{Li}_3(z) & \frac{(\log z)^2}{2!} & \log z & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots \end{pmatrix} \mathrm{diag}(1, 2i\pi, \dots, (2i\pi)^n)\,. \tag{4.1.2}$$

so that $A_{1k}(z) = -\mathrm{Li}_k(z)$ for $1\leq k\leq n$, and $A_{pq}(z) = (2i\pi)^{p-1}(\log z)^{q-p}/(q-p)!$ for $2\leq p<q\leq n$.

For a fixed value of z this matrix is the period matrix associated with the mixed Hodge structure for the polylogarithms, the columns are the weight and the lines are the Hodge degree.

A determination of this matrix $A(z)$ depends on the path γ in $\mathbb{P}^1\backslash\{0,1,\infty\}$ and a point $z\in]0,1[$. For a counterclockwise path γ_0 around 0 or γ_1 around 1 the determination of $A(z)$ is changed as

$$A_{\gamma\gamma_i}(z) = A_\gamma(z)\,\exp(e_i); \qquad i = 0,1 \tag{4.1.3}$$

where e_i are the nilpotent matrices

$$e_0 := \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & & \\ 0 & \cdots & 0 & \ddots & 0 \end{pmatrix}; \quad e_1 := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ & & & \\ 0 & \cdots & \ddots & \cdots \end{pmatrix}. \tag{4.1.4}$$

The matrix $A(z)$ satisfies the differential equation

$$dA(z) = (e_0\,d\log(z) + e_1 d\log(z-1))\;A(z)\,. \tag{4.1.5}$$

This differential equation defined over $\mathbb{C}\backslash\{0,1\}$ defined the *nth polylogarithm local system*. This local system underlies a good variation of mixed Hodge structure whose weight graded quotients are canonically isomorphic to $\mathbb{Q},\mathbb{Q}(1),\dots,\mathbb{Q}(n)$ [154, theorem 7.1].

One can define single-valued real analytic function on $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$, and continuous on $\mathbb{P}^1(\mathbb{C})$. The first important example is the *Bloch-Wigner dilogarithm* defined as [155, 156]

$$D(z) := \Im\mathrm{m}\left(\mathrm{Li}_2(z) + \log|z|\log(1-z)\right). \tag{4.1.6}$$

The Bloch-Wigner dilogarithm function satisfies the following functional equations

$$\begin{aligned} D(z) &= -D(\bar z) = D(1-z^{-1}) = D((1-z)^{-1}) \\ &= -D(z^{-1}) = -D(1-z) = -D(-z(1-z)^{-1})\,. \end{aligned} \tag{4.1.7}$$

The differential of the Bloch-Wigner dilogarithm $D(z)$ is given by

$$dD(z) = \log|z| d\arg(1-z) - \log|1-z| d\arg(z)\,. \tag{4.1.8}$$

At higher-order there is no unique form for the real analytic version of the polylogarithm. A particularly nice version with respect to Hodge structure provided by Beilinson and Deligne in [153] is given by

$$\mathcal{L}_m(z) := \sum_{k=0}^{m-1} \frac{B_k}{k!} (\log(z\bar{z}))^k \times \begin{cases} \Re\mathrm{e}(\mathrm{Li}_{m-k}(z)) & \text{for } m = 1 \mod 2 \\ \Im\mathrm{m}(\mathrm{Li}_{m-k}(z)) & \text{for } m = 0 \mod 2\,. \end{cases} \tag{4.1.9}$$

where B_k are Bernoulli numbers $x/(e^x - 1) = \sum_{k\geq 0} B_k\, x^k/k!$.

A *dilogarithm Hodge structure*, relevant to one-loop amplitudes in four dimensions, has been defined in [84] as a mixed Tate Hodge structure such that for some integer n, $gr^W_{2p} H = (0)$ for $p \neq n, n+1, n+2$.

4.1.1. Polylogarithms and Feynman integrals. The construction of the mixed Hodge structure of the massive triangle in section 2.2.2 was shown in [60,84] to correspond to the dilogarithm Hodge structure describe above.

Are all the Feynman integral expressible as polylogarithms or multiple polylogarithms in several variables? It is conjectured in [157] that using integration by parts one could always express Feynman integrals as a combination of a finite set of master integrals satisfying the differential equation (4.1.5), therefore leading to multiple polylogarithm functions. Various high-loop graphs have been shown to evaluate to multiple polylogarithms [78,148,158–160].

Counter-examples leading to elliptic functions are known in the massless case [151, 161,162] or the massive case by the sunset graph [31,32,134,150] and three-loop sunset graph with all equal internal masses [2]. The sunset graphs with all equal internal masses lead to elliptic polylogarithms discussed below.

Different functions are expected from other classes of Feynman graphs. Determining and evaluating Feynman integrals are open and difficult questions, where one can hope that mixed Hodge structures or motivic methods be useful.

4.2. Elliptic polylogarithms

In this section we recall the main properties of the elliptic polylogarithms following [155,163–166].

Let $\mathcal{E}(\mathbb{C})$ be an elliptic curve over $\mathbb{C}$. The elliptic curve can be viewed either as the complex plane modded by a two-dimensional lattice $\mathcal{E}(\mathbb{C}) \cong \mathbb{C}/(\mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2)$. A point $z \in \mathbb{C}/(\mathbb{Z}\varpi_1 + \mathbb{Z}\varpi_2)$ is associated to a point $P := (\wp(z), \wp'(z))$ on $\mathcal{E}(\mathbb{C})$ where $\wp(z) = z^{-2} + \sum_{(m,n)\neq(0,0)} \left((z + m\varpi_1 + n\varpi_2)^{-2} - (m\varpi_1 + n\varpi_2)^{-2}\right)$ is the Weierstraß function and $q := \exp(2i\pi\tau)$ with $\tau = \varpi_2/\varpi_1$ the period ratio in the upper-half plane $\mathbb{H} = \{\tau | \Re\mathrm{e}(\tau) \in \mathbb{R}, \Im\mathrm{m}(\tau) > 0\}$. Or we can see the elliptic curve as $\mathcal{E}(\mathbb{C}) \cong \mathbb{C}^\times/q^{\mathbb{Z}}$. A point P on the elliptic curve is then mapped to $x := e^{2i\pi z}$.

One defines an *elliptic polylogarithm* $\mathcal{L}^{\mathcal{E}}_n : \mathcal{E}(\mathbb{C}) \to \mathbb{R}$ as the average of the real unvalued version of the polylogarithms

$$\mathcal{L}^{\mathcal{E}}_m(P) := \sum_{n\in\mathbb{Z}} \mathcal{L}_m(x\, q^n) \tag{4.2.1}$$

where $q := \exp(2i\pi\tau)$ with $\tau \in \mathfrak{h} := \{\tau|\Re\mathrm{e}(\tau) \in \mathbb{R}, \Im\mathrm{m}(\tau) > 0\}$. This series converges absolutely with exponential decay and is invariant under the transformation $x \mapsto qx$ and $x \mapsto q^{-1}x$.

If we have a collection of points P_r on the elliptic curve one can consider a linear combination of the elliptic polylogarithms. Such objects play an important role when computing regulators for elliptic curves, and in the so-call Beilinson conjecture relating the value of the regulator map to the value of L-function of the elliptic curve [155, 167–170].

Interestingly, as explained in [1], elliptic polylogarithms from Feynman graphs differ from (4.2.1). A simple physical reason is that the Feynman integral is a multivalued function therefore cannot be build from a real analytic version of the polylogarithms. We will need the following sums of the elliptic dilogarithms

$$\sum_{r=1}^{n_r} c_r \sum_{n\geq 0} \mathrm{Li}_2\left(q^n z_r\right) \tag{4.2.2}$$

where z_r is a finite set of points on the elliptic curve and c_r are rational numbers. This expression is invariant under $z \mapsto qz$ and $z \mapsto q^{-1}z$ only for a *very special* choice of set of points depending on the (algebraic) geometry of the graph. A more precise definition of the quantity appearing from the two-loop sunset Feynman graph is given in equation (4.3.9).

4.2.1. Mahler measure. A *logarithmic Mahler measure* is defined by

$$\mu(F) := \oint_{|x_1|=\cdots=|x_n|=1} \log|F(x_1,\ldots,x_n)| \prod_{i=1}^{n} \frac{dx_i}{2i\pi x_i}\,, \tag{4.2.3}$$

and the *Mahler measure* is defined by $M(F) := \exp(\mu(F))$. In the definition $F(x_1,\ldots,x_n)$ is a Laurent polynomial in x_i.

Numerical experimentations by Boyd [171] pointed out to a relation between the logarithmic Mahler measure for certain Laurent polynomials F and values of L-functions of the projective plane curve $C_F : F(x_1,\ldots,x_n) = 0$

$$\mu(F) = \mathbb{Q}^\times\, L'(Z_F, 0)\,. \tag{4.2.4}$$

In [172] (see as well [173, 174]) Rodrigez-Villegas showed that the logarithmic Mahler measure is given by evaluating the Bloch regulator leading to expressions given by the Bloch-Wigner dilogarithm. The relation in (4.2.4) is then a consequence of the conjectures by Bloch [155] and Beilinson [167] relating regulators for elliptic curves to the values of L-functions (see [169, 170] for some review on these conjectures).

Let consider the logarithmic Mahler measure defined using the second Symanzik polynomial $\mathcal{F}_2(x,y;t) = (1+x+y)(x+y+xy) - txy$ for the two-loop sunset

$$\mu_\ominus(t) = \frac{1}{(2i\pi)^2}\int_{|x|=|y|=1} \log(|\mathcal{F}_2(x,y;t)|)\frac{dxdy}{xy}\,. \tag{4.2.5}$$

The Mahler measure associated with this polynomial has been studied by Stienstra [175, 176] and Lalin-Rogers in [177].

Consider the field $F = \mathbb{Q}(\mathcal{E}_\ominus)$ where $\mathcal{E}_\ominus = \{(x,y) \in \mathbb{P}^2 | \mathcal{F}_2(x,y;t) = 0\}$ is the sunset elliptic curve, and consider a Néron $\widehat{\mathcal{E}_\ominus}$ model of $\mathcal{E}_\ominus$ over $\mathbb{Z}$. The regulator map is an application from the higher regulator $K_2(\widehat{\mathcal{E}_\ominus})$ to $H^1(\mathcal{E}_\ominus,\mathbb{R})$ [155]. The regulator map is defined by

$$\begin{aligned} r: K_2(\mathcal{E}_\ominus) &\to H^1(\mathcal{E}_\ominus,\mathbb{R}) \\ \{x,y\} &\mapsto \{\gamma\to\int_\gamma \eta(x,y)\}\,, \end{aligned} \tag{4.2.6}$$

where

$$\eta(x,y)=\log|x|d\arg(y)-\log|y|d\arg x\,. \tag{4.2.7}$$

Notice that $\eta(x,1-x)=dD(x)$ the differential of the Bloch-Wigner dilogarithm.

If x and y are non-constant function on $\mathcal{E}_\ominus$ with divisors $(x)=\sum_i x_i(a_i)$ and $(y)=\sum_i n_i(b_i)$ one associates the quantity $(x)\diamond(y)=\sum_{i,j} m_in_i(a_i-b_j)$

A theorem by Beilinson states that if $\omega\in\Omega^1(\mathcal{E}_\ominus)$ then

$$\int_{\mathcal{E}_\ominus(\mathbb{C})}\omega\wedge\eta(x,y)=\varpi_r\,R_\tau((x)\diamond(y)) \tag{4.2.8}$$

where $R_\tau(z)$ is the Kronecker-Eisenstein series [155, 178] defined as

$$R_\tau(e^{2i\pi(a+b\tau)}):=\frac{\Im\mathrm{m}(\tau)^2}{\pi^2}\sum_{(p,q)\neq(0,0)}\frac{e^{2i\pi(aq-pb)}}{(p+q\tau)^2(p+q\bar\tau)}\,. \tag{4.2.9}$$

The logarithmic Mahler measure for the sunset graph is expressed as a sum of elliptic-dilogarithm evaluated at torsion points on the elliptic curve

$$\mu_\ominus(t)=-3\,\Im\mathrm{m}\left(R_\tau(\zeta_6)+R_\tau(\zeta_6^2)\right) \tag{4.2.10}$$

with $\zeta_6=\exp(i\pi/3)$ is a sixth root of unity and $\tau=\varpi_2/\varpi_1$ is the period ratio of the elliptic curve. This relation is true for t large enough so that the elliptic curve $\mathcal{E}_\ominus$ does not intersect the torus $\mathbb{T}^2=\{|x|=|y|=1\}$.

The Beilinson conjecture [167, 169, 170] implies that the Mahler measure is rationally related to the value of the Hasse-Weil L-function for the sunset elliptic curve evaluated at $s=2$

$$\mu_\ominus(t)=\mathbb{Q}^\times\,L(\mathcal{E}_\ominus(t),2)\,. \tag{4.2.11}$$

which can be easily numerically checked using [106].

Differentiating the Mahler measure with respect to t gives

$$g_1(t):=-t\frac{d\mu_\ominus(t)}{dt}=\frac{1}{(2i\pi)^2}\oint_{|x|=1}\oint_{|y|=1}\frac{t\,dxdy}{\mathcal{F}_2(x,y;t)}\,. \tag{4.2.12}$$

This quantity is actually a period of the elliptic curve [175] since we are integrating the two-form $\omega=\frac{t\,dxdy}{\mathcal{F}_2(x,y;t)}$ over a two-cycle given by the torus $\mathbb{T}^2=\{|x|=1,|y|=1\}$ (for t large enough so that the elliptic curve does not intersect the torus).

4.3. Some explicit solutions for the all equal masses sunset graphs

In [179] Broadhurst provided a mixture of proofs and numerical evidences that up to and including four loops the special values $t=K^2/m^2=1$ for the all equal mass sunset graphs are given by values of L-functions.

For generic values of $t=K^2/m^2\in[0,(n+1)^2]$, the solution is expressible as an elliptic dilogarithm at $n=2$ loops order [1] and elliptic trilogarithm at $n=3$ loops order [2]. The situation at higher-order is not completely clear.

In the following we present the one- and two-loop order solutions.

4.3.1. The massive one-loop bubble. In $D = 2$ dimensions the one-loop sunset graph, is the massive bubble, which evaluates to

$$I_2^2(m_1, m_2, K^2) = \frac{\log(z^+) - \log(z^-)}{\sqrt{\Delta}} \tag{4.3.1}$$

where

$$\begin{aligned} z^{\pm} &= (K^2 - m_1^2 - m_2^2 \pm \sqrt{\Delta})/(2m_1^2) \\ \Delta_\circ &= (K^2)^2 + m_1^4 + m_2^4 - 2(K^2 m_1^2 + K^2 m_2^2 + m_1^2 m_2^2) \end{aligned} \tag{4.3.2}$$

where $\Delta_\circ$ is the discriminant of the equation

$$(m_1^2 x + m_2^2)(1 + x) - K^2 x = m_1^2 (x - z^+)(x - z^-) = 0 \tag{4.3.3}$$

In the single mass case $m_1 = m_2 = m_3 = 1$ the integral reads

$$I_2^2(t) = -4\, \frac{\log\left(\sqrt{t} + \sqrt{t-4}\right) + \log 2}{\sqrt{t(t-4)}}\,. \tag{4.3.4}$$

This expression satisfies the differential equation for $n = 2$ in table **??**.

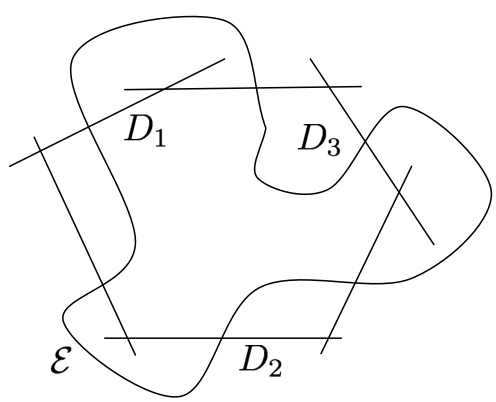

FIGURE 2. After blowup, the coordinate triangle becomes a hexagon in P with three new divisors D_i. The elliptic curve $X_\ominus = \{\mathcal{F}_2(x, y; t) = 0\}$ now meets each of the six divisors in one point.

4.3.2. The sunset integral. The domain of integration for the sunset is the triangle $\Delta = \{[x, y, z] \in \mathbb{P}^2 | x, y, z \geq 0\}$ and the second Symanzik polynomial $\mathcal{F}_2(x, y, z; t) = (x + y + z)(xy + xz + yz) - txyz$. The integral is given by

$$I_3^2(t) = \int_\Delta \frac{zdx \wedge dy + xdy \wedge dz - ydx \wedge dz}{\mathcal{F}_2(x, y, z; t)}\,. \tag{4.3.5}$$

This integral is very similar to the period integral in equation (4.2.12) for the elliptic curve $\mathcal{E}_\ominus := \{\mathcal{F}_2(x, y; t) = 0\}$. The only difference between these two integrals is the domain of integration. In the case of the period integral in (4.2.12) on integrates over a two-cycle and, for well chosen values of t, the elliptic curve has no intersection with the domain of integration, and therefore is a period of *a pure Hodge structure*.

In the case of the Feynman integral the domain of integration has a boundary, so it is not a cycle, and for *all* values of t the elliptic curve intersects the domain of integration. This is precisely because the domain of integration of Feynman graph integral is given as in (2.1.15) that *Feynman integrals lead to period of mixed Hodge structures.*

As explained in section 2.2.1 one needs to blow-up the points where the elliptic curve $\mathcal{E}_\ominus := \{\mathcal{F}_2(x,y,z;t) = 0\}$ (the graph polar part) intersects the boundary of the domain of integration $\partial\Delta \cap \mathcal{E}_\ominus = \{[1,0,0],[0,1,0],[0,0,1]\}$. The blown-up domain is the hexagon $\mathfrak{h}$ in figure 2. The associated mixed Hodge structure is given by [1] for the relative cohomology $H^2(\mathcal{P}-\mathcal{E}_\ominus, \mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h})$

$$0 \to H^1(\mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h}) \to H^2(\mathcal{P}-E, \mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h}) \to H^2(\mathcal{P}-\mathcal{E}_\ominus, \mathbb{Q}) \to 0 \tag{4.3.6}$$

and for the domain of integration we have the dual sequence

$$0 \to H_2(P-E) \to H_2(P-\mathcal{E}_\ominus, \mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h}) \to H_1(\mathfrak{h}-\mathcal{E}_\ominus\cap\mathfrak{h}) \to 0 \tag{4.3.7}$$

The Feynman integral for the sunset graph coincides with $I_3^2(t) = \langle\omega, s(1)\rangle$ where ω in $F^1H^1(\mathcal{E}_\ominus,\mathbb{C})$ is an element in the smallest Hodge filtration piece $F^2H^1(\mathcal{E}_\ominus,\mathbb{C})(-1)$, and $s(1)$ is a section in $H^1(\mathcal{E}_\ominus,\mathbb{Q}(2))$ [1].

The integral is expressed as the following combination of elliptic dilogarithms

$$-\frac{I_3^2(t)}{6} = -i\frac{\pi}{6}\,\varpi_r(t))(1-2\tau) + \frac{\varpi_r(t)}{\pi}\,E_\ominus(\tau)\,, \tag{4.3.8}$$

where the Hauptmodul $t = \frac{\pi}{\sqrt{3}}\,\eta(\tau)^6\eta(2\tau)^{-3}\eta(3\tau)^{-2}\eta(6\tau)$, the real period $\varpi_r(t) = \frac{\pi}{\sqrt{3}}\,\eta(\tau)^6\eta(2\tau)^{-3}\eta(3\tau)^{-2}\eta(6\tau)$ and τ is the period ratio for the elliptic curve $\mathcal{E}_\ominus$. Using $q := \exp(2i\pi\tau)$ the elliptic dilogarithm is given by

$$\begin{aligned} E_\ominus(\tau) &= -\frac{1}{2i}\sum_{n\geq 0}\left(\mathrm{Li}_2\left(q^n\zeta_6^5\right) + \mathrm{Li}_2\left(q^n\zeta_6^4\right) - \mathrm{Li}_2\left(q^n\zeta_6^2\right) - \mathrm{Li}_2\left(q^n\zeta_6\right)\right) \\ &+ \frac{1}{4i}\left(\mathrm{Li}_2\left(\zeta_6^5\right) + \mathrm{Li}_2\left(\zeta_6^4\right) - \mathrm{Li}_2\left(\zeta_6^2\right) - \mathrm{Li}_2\left(\zeta_6\right)\right)\,. \end{aligned} \tag{4.3.9}$$

which we can write as well as q-expansion

$$E_\ominus(\tau) = \frac{1}{2}\sum_{k\in\mathbb{Z}\backslash\{0\}}\frac{(-1)^{k-1}}{k^2}\,\frac{\sin(\frac{n\pi}{3}) + \sin(\frac{2n\pi}{3})}{1-q^k}\,. \tag{4.3.10}$$

As we mentioned earlier this integral is not given by an elliptic dilogarithm obtained by evaluating the real analytic function $D(z)$ to the contrary to the Mahler measure described in section 4.2.1.

The amplitude is closely related to the *regulator* in arithmetic algebraic geometry [**155**, **167**, **169**, **170**]. Let $conj : M_\mathbb{C} \to M_\mathbb{C}$ be the real involution which is the identity on $M_\mathbb{R}$ and satisfies $conj(c\,m) = \bar{c}\,m$ for $c\in\mathbb{C}$ and $m\in M_\mathbb{R}$. With notation as above, the extension class $s(1) - s_F \in H^1(\mathcal{E}_\ominus,\mathbb{C})$ is well-defined up to an element in $H^1(\mathcal{E}_\ominus,\mathbb{Q}(2))$ (i.e. the choice of $s(1)$). Since $conj$ is the identity on $H^1(\mathcal{E}_\ominus,\mathbb{Q}(2))$, the projection onto the minus eigenspace $(s(1)-s_F)^{conj=-1}$ is canonically defined. The regulator is then

$$\langle\omega, (s(1)-s_F)^{conj=-1}\rangle \in \mathbb{C}\,. \tag{4.3.11}$$

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Institut de Physique Théorique,, CEA, IPhT, F-91191 Gif-sur-Yvette, France, CNRS, URA 2306, F-91191 Gif-sur-Yvette, France

Email address: pierre.vanhove@ipht.fr