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An S-matrix approach to gravitational-wave physics

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The detection of gravitational waves emitted by binary systems has opened a new astronomical window into the Universe. We describe recent advances in the field of scattering amplitudes applied to the post-Minkowskian expansion, and the extraction of the effective two-body gravitational potential. The techniques presented here apply to any effective field theory of gravity and are not restricted to four-dimensional Einstein gravity.

1. Motivation

Our conception of the force of gravity conditions our vision of the shape and dynamic of our Universe. The current cosmological paradigm relies on General Relativity, Einstein's theory of gravity formulated as a relativistic theory of curved space-time, which posits that the force of gravity is universal as it couples to all types of matter and energy at all scales. But General Relativity fails to explain the observed dynamics and stability of the galaxies or the accelerated expansion of our Universe. Dark matter, and dark energy of unknown nature have been introduced to quantify our lack of understanding. The recent tension in the value for the acceleration rate of our Universe [1], between the local astrophysical measurements and from the cosmic microwave background, is a strong invitation to consider modifications of the law of gravity over large astrophysical scales. Because gravity is the weakest of all forces, direct detection of gravitational effects is very challenging. For a long time, there was a huge observational gap in the scales where gravity could be measured with precision between the solar system range and the cosmological range. This has now changed thanks to the detection of gravitational waves.

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The landmark detection of gravitational waves by the LIGO and Virgo collaborations give information about gravity at an intermediate scale in our Universe. We then have new tools to learn about physics in the extreme gravity environment of coalescing binary black holes or neutron stars, and confront the predictions of Einstein's theory of gravity, and to search for new physics.

The detection of the first gravitational-wave signal has opened an area of *precision* gravity. The gravitational-wave signals can tell about gravity at various scales, involving the dynamics of black holes. Ultimately, this would tell us how good we understand gravity both in the weak and strong coupling regimes. The binary mergers detected until now by the LIGO-Virgo [2,3] collaboration are clean sources of gravitational waves and the gravitational-wave signal is currently modelled by general relativity in vacuum, at accuracy close to second order in the mass ratio parameter. The next-generation gravitational-wave detectors, such as the Laser Interferometer Space Antenna (LISA) will be influenced by the environment of the sources and the signal will be more "dirty" [4]. The lack of clear predictions for non-linearities (from the accretion disk for instance) in the post-merger phase means that these could be confused with modifications of the signal predicted by theories beyond general relativity.

The profile of the wave depends on the physics of the mergers and how the waves have propagated in space. There are three typical regimes for the merger: (a) the inspiral phase when the two binaries are far apart with relatively slow motion and weak gravitational interaction, (b) the merger phase in the strong gravity regime and (c) the final ring-down phase where the final state of the system is relaxing. After the binary system has merged into the final black hole, the ring-down phase tells about the structure of the horizon of the black hole. We estimate that there are more than 100 million black holes in our galaxy and 100 billion supermassive black holes in our Universe. These numbers indicate the potential of gravitational-wave detections from black-hole binary mergers at intermediate distances in our Universe. For neutron star binary mergers, one can compare the propagation of the gravitational waves to the radio signals and visible emissions. The analysis of gravitational waves will allow for precision tests of General Relativity, and of the black-hole paradigm itself. The combined gravitational waves and electromagnetic signals will allow us to pin down the properties of the binary and its environment. The recently enhanced observatories (LIGO, Virgo, and KAGRA) and the vastly improved sensitivity of the third generation of gravitational-wave observatories, the Einstein Telescope and the Cosmic Explorer, and the future space-based LISA, [4,5] will permit detailed measurements of the sources' physical parameters and will complement, in a multi-messenger approach, the observation of signals emitted by cosmological sources obtained through other kinds of telescopes.

"We wouldn't call it a tension or a problem, but rather a crisis," commented the Nobel prize winner David Gross, about the different measurements of the expansion rate of the Universe from local and cosmological observations [6]. The signal contains information about the sources and how the gravitational waves have propagated in space and time till their detections on Earth. The knowledge of the mass distribution of sources of gravitational waves can be used to infer cosmological parameters in the absence of redshift measurements obtained from electromagnetic observations [7–9].

Black holes and compact stars and gravitational waves are amongst the most spectacular predictions of general relativity. It is therefore natural to use them as probes of the most fundamental principles of Einstein's theory [10,11]. The gravitational-wave event constrain a plethora of mechanisms associated with the generation and propagation of gravitational waves, including the activation of scalar fields [12], gravitational leakage into large extra dimensions [13], the variability of Newton's constant [14,15], the speed of gravitational waves, their propagation [16], gravitational Lorentz violation and the strong equivalence principle [17] and higher-derivative corrections [18]. The LIGO Scientific Collaboration and the Virgo Collaboration have verified that the observations are consistent with Einstein's theory of gravity, constraining the presence of certain parametric anomalies in the signal [9]. However, the true potential for gravitational-wave detections to both rule out exotic objects and constrain physics beyond General Relativity is severely limited by the lack of understanding of the coalescence regime in almost all relevant modified gravity theories.

Current predictions for gravitational-wave signals are based on a variety of complementary theoretical approaches: the weak field and small velocity expansion in the inspiral regime, numerical relativity for the merger, and the quasi-normal modes for the relaxation of the final black hole. They have been used for analysing successful detections, but they have their limitations. Astrophysical evidence suggests that black holes can have a variety of intrinsic angular momenta, including close to maximally allowed values. The presence of spin can lead to qualitative changes in the dynamics of a binary system, such as the orbital-plane precession when the spins are not aligned with the orbital angular momentum. Such an effect would lead, in particular, to a modulation of the amplitude, frequency and phase of the observed gravitational-wave signal.

For this, we need to produce accurate theoretical gravitational waveform templates. We have to clarify how much can be understood from exact theoretical computations. We have to answer the questions about how much we understand gravity in the weak and strong coupling regimes. And whether we can learn about gravitational physics beyond Einstein gravity, like modified gravity scenarios (extra dimensions, massive gravity, . . .), or quantum effects.

The detected gravitational waves are classical, but in parallel to the other forces of Nature one assumes the existence of a massless particle, the graviton, which carries the force of gravity. The quantum gravity effects have been elusive so far although they play a fundamental role in our understanding of cosmology and the nature of black holes. There are indications that gravitational-wave spectroscopy could provide a way to test some paradigms for the microscopic description of a black hole, like the "fuzzball" scenario [19], and lead to a better understanding of the early time cosmology and the inflationary scenario [9].

Although the status of the high-energy behaviour of quantum gravity is still open, considering effective field theory of gravity at low energy does not pose a problem. One can safely extract low-energy physics from the quantization of the gravitational interactions observables that are independent of the high-energy behaviour. In this context, it is important to work with gravity effective field theories [20–24]. There are the classical contributions (post-Minkowskian expansion) but as well infra-red effects which depend only on the low-energy degrees of freedom. As argued by J. D. Bjorken [25] *[...] as an open theory, quantum gravity is arguably our best quantum field theory, not the worst.*

In the present text, I review the approach for evaluating classical gravity observables from a quantum S-matrix approach developed in the works [26–33].

2. Classical gravitational scattering from quantum scattering

Einstein's theory of gravity is the first term of an effective field theory coupling gravity to matter

$$\mathcal{S}=\int d^4x\sqrt{-g}\left[\frac{R}{16\pi G_N}+g^{\mu\nu}T^{\rm matter}_{\mu\nu}+\mathcal{L}_{\rm corrections}\right]. \tag{2.1}$$

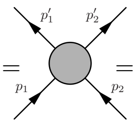
We assume that the effective field theory satisfies the standard requirements of locality, unitarity and Lorentz invariance, and of course that the theory is diffeomorphism invariant (i.e. we have the symmetries of General relativity). The low-energy degrees of freedom are the massless graviton and the usual massive matter fields.

We are interested in extracting physical observables from the gravitational interactions[1] between two massive body of masses m_i and spin S_i with $i=1,2$ interacting via the exchange of

[1] One could as well include electro-magnetic interactions as considered in [34] or standard model contributions as in e.g. [35] but here we will only consider the exchange of the graviton and focus on the gravitational sector.

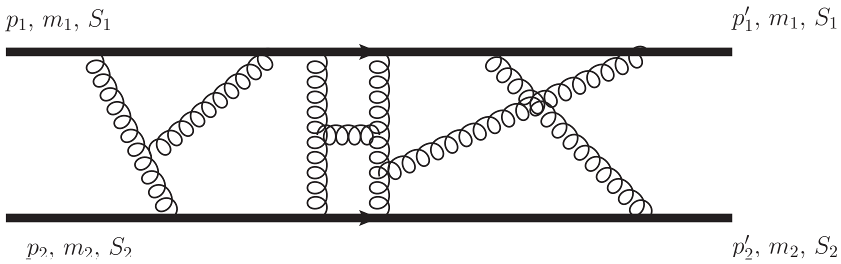
massless spin-2 graviton [36–40]. The two-body scattering matrix can be expanded in perturbation

$$\mathcal{M}(p_1,p_2,p_1',p_2') = $$



$$= \sum_{L=0}^{\infty} G_N^{L+1} \mathcal{M}_L(\gamma, q^2). \tag{2.2}$$

The quantum scattering matrix $\mathcal{M}(p_1,p_2,p_1',p_2')$ depends on the incoming energy $\gamma := p_1 \cdot p_2/(m_1 m_2)$, the momentum transfer $(p_1 - p_1')^2 =: q^2$ and $\hbar$. At a given order in perturbation one gets the exchange of gravitons (curly lines) between massive external matters (solid lines)



A traditional argument (see for instance [41]) gives that the L-loop contribution is of order $\mathcal{M}_L(\gamma, q^2) = \mathcal{O}(\hbar^{L-1})$. A different behaviour emerges when keeping fixed the wave-number $\underline{q} = q/\hbar$ and taking both the $\hbar \to 0$ and the small momentum transfer $q \to 0$ limit [27,42,43]. The L-loop two-body scattering amplitude has the Laurent expansion around four dimensions

$$\mathcal{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{1}{\hbar^{L-1} |\underline{q}|^{\frac{L(4-D)}{2}}} \sum_{r \geq -2} (\hbar |\underline{q}|)^r \mathcal{M}_L^{(-L+1+r)}(\gamma, \underline{q}^2)\,. \tag{2.3}$$

The classical piece of the amplitude that will contribute to the classical interactions depicted in the above figure is the contribution of order $1/\hbar$

$$\mathcal{M}_L(\gamma, \underline{q}, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, \underline{q}^2)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, \underline{q}^2)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0). \tag{2.4}$$

The full quantum amplitude contains three types of contributions: (1) the term of order $1/\hbar^r$ with $L+2 \leq r \leq 2$ that are more singular than the classical piece, (2) the classical piece of order $1/\hbar$ and (3) the quantum corrections of order $\hbar^r$ with $r \geq 0$. All these contributions are constrained by the unitarity of the S-matrix as we will explain in section (b).

The scattering amplitude approach completes the post-Newtonian computations by providing information beyond its regime of validity and leads to surprising results connecting the conservative part and gravitational radiation effects [28,44–50]. It gives a new perspective on the traditional methods [21,22,51–53] used for computing the gravitational-wave templates. This approach allows connecting the re-summed post-Newtonian results [23,54,55] and the high-energy behaviour [49,56]. We can then explore the behaviour of the post-Minkowskian expansion for higher-dimensional gravity. One can as well consider higher derivative corrections induced from string theory for instance, and study their effects on the gravitational-wave signals [57–59] or various quantum effects [26] induced from the higher-order terms in the $\hbar$ expansion in (2.4).

For gravitational-wave physics, we need to extract the classical potential from the knowledge of the quantum scattering amplitudes. One wants to extract the classical Hamiltonian $\mathcal{H}_{\rm PM}(p,r)$ for the gravitational interactions between two classical massive body

$$\mathcal{H}_{\rm PM}(r,p) = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + \sum_{L \geq 0} G_N^{L+1} \mathcal{V}_{\rm L+1}(r,p). \tag{2.5}$$

The relativistic potential $V_{\rm L+1-PM}(p,r) = \sum_{n \geq 0} c_{L,n} v^n$, with $p = p_1 - p_1' = (E, v)$, sums an infinite number of post-Newtonian contributions in the small velocity expansion $v/c \ll 1$.

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(a) The eikonal formalism

One route for relating the potential $\mathcal{V}_{L+1}$ to the scattering amplitude $\mathcal{M}_L$ is to use the eikonal formalism for extracting the *classical* scattering angle χ from the *quantum* scattering amplitude calculation. For this, one converts the amplitude to the b-space[2] by performing a Fourier transform with respect to the momentum transfer

$$\mathcal{M}_L(\gamma,b)=\frac{1}{4m_1m_2\sqrt{\gamma^2-1}}\int_{\mathbb{R}^{D-2}}\frac{d^{D-2}\vec{\underline{q}}}{(2\pi)^{D-2}}\mathcal{M}_L(p_1,p_2,p_1',p_2')e^{i\vec{\underline{q}}\cdot\vec{b}}. \tag{2.6}$$

The classical eikonal phase $\delta(\gamma,b)$ is defined by the exponentiation of the S-matrix

$$1+i\mathcal{T}=(1+i2\Delta)e^{\frac{2i\delta(\gamma,b)}{\hbar}}. \tag{2.7}$$

The eikonal phase has the perturbation expansion

$$\delta(\gamma,b)=\sum_{L\geq 0}\delta_L(\gamma,b)G_N^{L+1}, \tag{2.8}$$

which is then connected to the $\hbar$ Laurent expansion of the scattering amplitude in (2.4), through the expansion of the full scattering matrix in b-space

$$1+i\mathcal{T}=1+i\sum_{L\geq 0}G_N^{L+1}\mathcal{M}_L(\gamma,b). \tag{2.9}$$

Having determined the classical eikonal contribution at a given loop order one can then evaluate the scattering angle at this order in perturbation [28,29,47,49]

$$\sin\left(\frac{\chi}{2}\right)\Big|_L=-\frac{\sqrt{(p_1+p_2)^2}}{m_1m_2\sqrt{\gamma^2-1}}\frac{\partial\delta_L(\gamma,b)}{\partial b}. \tag{2.10}$$

And from the scattering angle, one reconstructs the classical potential $\mathcal{V}_{L+1}(r,p)$ in (2.5) by matching the expression for the angle from the Hamilton-Jacobi formalism [60,61]. The scattering angle is the main link for connecting the scattering amplitudes to the dynamics of the two-body system. It has been derived in many independent methods in different regimes [23,47–50,62–65].

Unfortunately, this approach leads to complicated computations. In the first place, the eikonal exponentiation in (2.7) is obtained after a careful separation, order by order, of the various terms that go into the exponent and those terms that remain as prefactor at the linear level. A second complication is that after exponentiation in impact-parameter space one must apply the inverse transformation and seek from it two crucial ingredients: (1) the correct identification of the transverse momentum transfer $\vec{\underline{q}}$ in the centre-of-mass frame and (2) the correct identification of the scattering angle from the saddle point. At low orders in the eikonal expansion, this procedure works well but it hinges on the impact-parameter transformation being able to undo the convolution product of the momentum-space representation. When $\underline{q}^2$-corrections are taken into account it is well-known that this procedure requires amendments. This motivates why alternative pathways are rooted in the WKB approximation [30,55,66].

(b) An exponential representation of the S-matrix

The Hamilton-Jacobi equations applied to the principal function

$$\mathcal{S}(r,\varphi;E,J)=J\varphi+\int p_r(r;E,J)dr \tag{2.11}$$

leads to the radial action in the post-Minkowskian expansion [60]

$$I_{\text{radial}}(E,J)=\oint p_r(r;E,J)dr. \tag{2.12}$$

[2] This is not the impact parameter b_J orthogonal to the asymptotic momentum in the centre-of-mass frame. The relation between the two quantities is $b_J=b\cos(\chi/2)$ [47,49].

It has been proposed in [55] a relation between the scattering amplitude and the radial action (see [66] for the probe limit case)

$$i\mathcal{M} \propto \int \left(e^{iI_{\text{radial}}(E,J)} - 1\right) dJ. \tag{2.13}$$

This relation looks similar to the eikonal approach described in the previous section, but the systematic of the effective field theory approach, applied to the potential region [55], makes the evaluation of the classical contributions more efficient.

Another approach, introduced in [30], uses an exponential representation of the S-matrix at the operator level

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right), \tag{2.14}$$

with the completeness relation

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{2} \frac{d^{D-1}k_i}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{k_1}} \prod_{j=1}^{n} \frac{d^{D-1}\ell_j}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_j}} \left|k_1, k_2; \ell_1, \dots \ell_n\right\rangle \left\langle k_1, k_2; \ell_1, \dots \ell_n\right|, \tag{2.15}$$

which includes all the exchange of gravitons for $n \geq 1$ entering the radiation-reaction contributions $\hat{N}^{\text{rad}}$. With this exponential representation of the S-matrix, we systematically relate matrix elements of the operator in the exponential $\hat{N}$ to ordinary Born amplitudes minus pieces provided by unitarity cuts [30]. This is seen by the perturbation expansion

$$\begin{aligned}
&\hat{N}_0 = \hat{T}_0, \qquad \hat{N}_0^{\text{rad}} = \hat{T}_0^{\text{rad}},\\
&\hat{N}_1 = \hat{T}_1 - \frac{i}{2\hbar}\hat{T}_0^2, \qquad \hat{N}_1^{\text{rad}} = \hat{T}_1^{\text{rad}} - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_0^{\text{rad}} + \hat{T}_0^{\text{rad}}\hat{T}_0),\\
&\hat{N}_2 = \hat{T}_2 - \frac{i}{2\hbar}(\hat{T}_0^{\text{rad}})^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_1 + \hat{T}_1\hat{T}_0) - \frac{1}{3\hbar^2}\hat{T}_0^3,
\end{aligned} \tag{2.16}$$

and similarly for higher orders. The simplicity of this method seems very appealing and suggests that it may be used to streamline post-Minkowskian amplitudes in gravity by means of a diagrammatic technique that systematically avoids the evaluation of the cut diagrams that must be subtracted, but simply discards them at the integrand level. This decomposition is in correspondence with the $1/\hbar|\underline{q}|$ expansion of the scattering amplitude in (2.4). The scattering matrix operator $\hat{T}$ is related to the scattering amplitude $\mathcal{M}_L \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_L|p_1', p_2'\rangle$. The tree-level matrix element for the two-body scattering $\mathcal{M}_0 \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_0|p_1', p_2'\rangle$ is of order $\mathcal{O}(1/\hbar)$. At one-loop order amplitude decomposes into two pieces

$$\mathcal{M}_1 \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_1|p_1', p_2'\rangle \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{N}_1|p_1', p_2'\rangle + \frac{i}{2\hbar^2}\langle p_1, p_2|\hat{T}_0^2|p_1', p_2'\rangle. \tag{2.17}$$

By unitarity the coefficient of the $\mathcal{O}(1/\hbar^2)$ contribution in the scattering amplitude is $\langle p_1, p_2|\hat{T}_0^2|p_1', p_2'\rangle$, and the matrix element $\langle p_1, p_2|\hat{N}_1|p_1', p_2'\rangle$ is given by the classical piece is of order $\mathcal{O}(1/\hbar)$. Therefore, for the classical two-body scattering only the matrix elements of $\hat{N}$ are needed. They are extracted from the scattering amplitude by the *velocity cuts* introduced recently [28,29] which are a practical way of realising the decomposition (2.17) at the amplitude level. These velocity cuts provide a natural way to organise amplitude calculations [67].

By construction, the scattering angle is reproduced in perturbation theory. The completeness relation implies that the two-body scattering contains the multi-graviton exchanges. Therefore, the result is not limited to what is known as the potential region of the multi-loop amplitudes [23, 54,55] but include also radiation reaction pieces [28,29,48,49,56].

(c) The classical effective potential from a Lippmann-Schwinger approach

It is a classical problem in perturbation scattering theory to relate the scattering amplitude $\mathcal{M}$ to an interaction potential $\mathcal{V}$. This is typically phrased in terms of non-relativistic quantum mechanics, but it is readily generalized to the relativistic case. Crucial in this respect is the fact that we shall consider particle solutions to the relativistic equations only. There will thus be, in the language of old-fashioned (time-ordered) perturbation theory, no back-tracking diagrams corresponding to multi-particle intermediate states. This is trivially so since we neither wish to treat the macroscopic classical objects such as heavy neutron stars as indistinguishable particles with their corresponding antiparticles nor do we wish to probe the scattering process in any potential annihilation channel. The classical objects that scatter will always be restricted to classical distance scales.

In the Lippman-Schwinger [68] approach the classical piece in the quantum scattering amplitude $\mathcal{M}$ and the $\mathcal{V}$ are recursively related by

$$\mathcal{M}(p,p') = \mathcal{V}(p,p') + \int \frac{d^3k}{(2\pi)^3}\frac{\mathcal{V}(p,k)\mathcal{M}(k,p')}{E_p - E_k + i\epsilon}\,, \tag{2.18}$$

in the centre-of-mass frame with $p_1 = (E_1, \vec{p})$, $p_1' = (E_1, \vec{p}\,')$, $p_2 = (E_2, -\vec{p})$ and $p_2' = (E_2, -\vec{p}\,')$. The position space potential $\mathcal{V}(r,p)$ is obtained after Fourier transform

$$\mathcal{V}(r,p) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r}\mathcal{V}(p,q). \tag{2.19}$$

This framework is a relativistic extension, through a one-particle Hamiltonian and associated Salpeter equation, of the conventional approach to determining the interaction potential in perturbation gravity. The Lippmann-Schwinger equations are used to derive straightforwardly the needed Born subtractions at arbitrary loop order. We remark that the EFT approach of [23,24] is a Lippman-Schwinger approach applied to the potential region of the two-body scattering.

The resulting post-Minkowskian Salpeter equation is not an effective low-energy theory (momentum is not limited), but rather a small $|\underline{q}|/m$ approximation where small momentum is exchanged and only particle states are summed over. Although systematic, this approach leads to rather cumbersome computations. A more direct way of deriving the effective potential is to connect the potential to the scattering angle.

(d) The Effective-One-Body formalism

One route to connect the scattering regime to the bound-state regime is based on the Effective One-Body (EOB) formalism [69,70], suitably adapted from post-Newtonian to post-Minkowskian formulations [33,44,45,60].

One important lesson from the scattering amplitude approach to gravitational scattering in general relativity is that at least up to and including third post-Minkowskian order, there exists, in isotropic coordinates, a very simple relationship between centre-of-mass momentum p and the effective classical potential $V_{\text{eff}}(r,p)$ of the form [61]

$$p^2 = p_\infty^2 - \mathcal{V}_{\text{eff}}(r,E); \qquad \mathcal{V}_{\text{eff}}(r,E) = -\sum_{n\geq 1} f_n \left(\frac{G_N(m_1+m_2)}{r}\right)^n \tag{2.20}$$

where the coefficients f_n are directly extracted from the scattering angle

$$\chi = \sum_{k\geq 1} \frac{2b}{k!}\int_0^\infty du \left(\frac{d}{du^2}\right)^k \left[\frac{1}{u^2+b^2}\left(\frac{\mathcal{V}_{\text{eff}}\left(\sqrt{u^2+b^2}\right)(u^2+b^2)}{\gamma^2-1}\right)^k\right]. \tag{2.21}$$

By computing the two-body scattering in perturbation one derives a Lorentz invariant expression valid in all regime of relative velocity between the two interacting massive bodies. Importantly, the relation between the scattering amplitude and the Effective-One-Body Effective potential in (2.21) is valid in any space-time dimension and applies to gravity in higher dimensions [71,72].

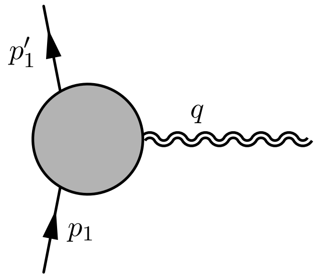
3. The Schwarzschild-Tangherlini black holes

The Nobel prize citation for Roger Penrose states that "black-hole formation is a robust prediction of the general theory of relativity". Subrahmanyan Chandrasekhar explained that they are the most perfect macroscopic objects there are in the universe since the only elements in their construction are our concepts of space and time. Black-hole solutions are a perfect playground to validate the formalism of deriving classical gravity from quantum scattering amplitudes. This also opens new avenues for studying black holes in generalized theories of gravity.

In 1973, Duff analysed [73] the question of the classical limit of quantum gravity by extracting the Schwarzschild black-hole metric from quantum tree graphs to G_N^3 order. This was a consistency check on the way classical Einstein's gravity is embedded into the standard massless spin-2 quantization of the gravitational interactions.

Considering the importance of the Schwarzschild black-hole metric solution both for observation and theoretical considerations, we present a derivation of this metric from a scattering amplitude approach.

By evaluating the vertex function of the emission of a graviton from a particle of mass m, spin S and charge Q, in d dimensions



$$= -\frac{i\sqrt{32\pi G_N}}{2} \sum_{l\geq 0} \langle T^{(l)\,\mu\nu}(q^2)\rangle \epsilon_{\mu\nu} \tag{3.1}$$

one can extract the metric of physical black holes [74]

- Schwarzschild black hole: Scalar field $S=0$, mass m [31,75–77]
- Reissner-Nordström black hole: Scalar field $S=0$, charge Q, mass m [34]
- Kerr-Newman black hole: Fermionic field $S=\frac{1}{2}$, charge Q, mass m [34,77]

At each loop order, we extract the l-loop contribution to the transition density of the stress-energy tensor $\langle T_{\mu\nu}(q^2)\rangle = \sum_{l\geq 0}\langle T^{(l)}_{\mu\nu}(q^2)\rangle$

$$i\mathcal{M}_3^{(l)}(p_1, q) = -\frac{i\sqrt{32\pi G_N}}{2}\langle T^{(l)\,\mu\nu}(q^2)\rangle \epsilon_{\mu\nu}, \tag{3.2}$$

where $\epsilon^{\mu\nu}$ is the polarization of the graviton with momentum $q = p_1 - p_2$ is the momentum transfer. The relation between the metric perturbation and the stress-energy tensor reads

$$h^{(l+1)}_{\mu\nu}(\vec{x}) = -16\pi G_N \int \frac{d^{D-1}\vec{q}}{(2\pi)^d} e^{i\vec{q}\cdot\vec{x}} \frac{1}{\vec{q}^2}\left(\langle T^{(l)}_{\mu\nu}\rangle^{\text{class.}}(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle^{\text{class.}}(q^2)\right). \tag{3.3}$$

In this relation enters the classical contribution at l loop order $\langle T^{(l)}_{\mu\nu}\rangle^{\text{class.}}(q^2)$ defined by the classical limit of the quantum scattering amplitude [27,42,43].

In [31], the Schwarzschild-Tangherlini metric up to G_N^4 was obtained in four, five and six dimensions. The amplitude computation contains quantum corrections that are induced from the higher orders in the $\hbar$ expansion. These corrections can be included in the metric [34,77].

4. Discussion

General relativity can be considered in space-times of various dimensions. Gravity is richer in higher dimensions as black-hole solutions develop non-trivial properties in general dimensions [78,79]. It is therefore important to validate our current understanding of the connection between the quantum scattering amplitudes and classical general relativity in general dimensions [71,72]. By reproducing the classical Schwarzschild-Tangherlini metric from scattering amplitudes in four, five and six dimensions, we validate the procedure for extracting the classical piece from the quantum scattering amplitudes. The method can be applied to derive other black-hole metrics, like the Kerr-Newman and Reissner-Nordström metrics by considering the vertex function of the emission of the graviton from a massive particle with spin and charge [34,77,80–85].

The scattering amplitude approach presented in this work can be applied to any effective field theory of gravity coupled to matter fields. One can include quantum corrections in (3.3) and examine the impact of quantum effects on the black-hole solutions [77], the effects of modified gravity models [86] or study the impact of higher derivative contributions [57–59] to the gravitational-wave templates.

The amplitudes computations, being performed in general dimensions, lead to results that have an analytic dependence on the space-time dimensions. It is remarkable that in this approach classical gravity physics contributions are determined by unitarity of the quantum amplitudes [27].

We have improved our understanding of the relation between general relativity and the quantum theory of gravity. This leads to many new exciting developments leading to a better understanding of the gravitational interactions in a binary system. This provides new techniques that can be applied to any gravitational effective field theories which have amplitude description: opening the possibility to search for deviation from Einstein gravity.

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