Modular pre-processing for automated reasoning in dependent type theory

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- Abstract

The power of modern automated theorem provers can be put at the service of interactive theorem proving. But this requires in particular bridging the expressivity gap between the logics these provers are respectively based on. This paper presents the implementation of a modular suite of pre-processing transformations, which incrementally bring certain formulas expressed in the Calculus of Inductive Constructions closer to the first-order logic of Satisfiability Modulo Theory solvers. These transformations address issues related to the axiomatization of inductive types, to polymorphic definitions or to the different implementations of a same theory signature. This suite is implemented as a plugin for the Coq proof assistant, and integrated to the SMTCoq toolchain.

2012 ACM Subject Classification Theory of computation \rightarrow Automated reasoning; Theory of computation \rightarrow Higher order logic; Theory of computation \rightarrow Type theory

Keywords and phrases interactive theorem proving, Calculus of Inductive Constructions, Coq, automated theorem provers, Satisfiability Modulo Theory solvers, inductive types, polymorphism, arithmetic, pre-processing

Digital Object Identifier 10.4230/LIPIcs...

1 Introduction

Automated provers, such as first-order provers, satisfiability provers, or satisfiability modulo theories (SMT) provers, are fast and versatile: these highly optimized pieces of software are used for a variety of applications, ranging from program verification [12] to combinatorics [13]. When they export a trace of their output, the latter can be used to construct a formal verification thereof [7], thus significantly increasing the trust in the proofs they produce. This feature in turn can be exploited to make the power of automated provers available to the users of *interactive* theorem provers, and help with the gradual construction of broader formal proofs. Automating formal proofs is an absolute need in program (formal) verification, and thus in endeavors like the CompCert compiler [16] or the Verified Software Toolchain [1]. In fact, they can significantly boost the productivity of users in a broader range of applications, by discarding mundane proof obligations and by increasing the robustness of proof scripts.

Interactive theorem provers typically implement a flavor of higher-order logic, a strict superset of the fragments of first-order logic handled by automated theorem provers. Therefore, a statement expressed as a goal in the interactive theorem prover should first undergo a translation process, before being handed to an automated prover. This paper discusses the design and the implementation of this pre-processing phase, in the context of interactive theorem provers based on the Calculus of Inductive Construction (CIC), a dependent



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LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

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type theory with inductive types. The central idea is to rely on a modular suite of atomic transformations, rather than on a general encoding of the Calculus of Inductive Constructions into a target dialect of first order logic. We have implemented a new Coq plugin, called Sniper, which embraces this methodology and provides a tactic, called snipe, for first-order automation.

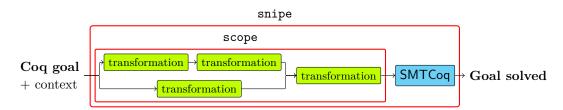


Figure 1 Sniper methodology

As illustrated on Figure 1, snipe in turn extends the SMTCoq plugin [11], for interacting with SMT solvers, with a modular and extensible pre-processing strategy, called scope. In particular, scope currently makes use of Trakt, a new plugin which may also be used as a standalone tool, for translating goals into a user-defined representation of integers and of logical connectives and relations. On the implementation side, our main contributions are thus the trakt and snipe tactics.

The article is organized as follows. First, we showcase on some examples the possibilities of our tool (§ 2), and recall some background on the tools involved in our implementation (§ 3). Then, after a presentation of our general methodology (§ 4), we provide a more in-depth description of the main novel pre-processing transformations (§ 5). Finally, we show some benchmarks about our implementation before concluding.

The accompanying source code, together with installation instructions and an overview of the code, is available here: https://github.com/smtcoq/sniper/tree/itp22.

2 Motivating examples

In this section, we first showcase our new tactic **snipe** on general examples, before entering into the details of its methodology.

2.1 Showcases

2.1.1 CompCert

As Xavier Leroy and Sandrine Blazy pointed out when reasoning about memory models for CompCert [17], many Coq developements are actually first-order reasoning and would benefit from automation, since the specifications of a compiler are mostly based on decidable structures and first-order predicates. For instance, consider the following examples taken from the CompCert repository:

```
Inductive memory_chunk : Type :=
   | Mint8signed
   | Many64. (* 8 other cases are skipped *)

Definition size_chunk (chunk : memory_chunk) : Z :=
   match chunk with
   | Mint8signed => 1
   | Many64 => 8
```

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```
end.

Definition perm_order : permission -> permission -> bool :=
  match (p, p') with
  | (Writable, Writable) => true
  | (Readable, Readable) => true
  | (Freeable, _) => true
  | (Writable, Readable) => true
  | (_, Nonempty) => true
  | _ => false
  end.
```

The tactic snipe automatically proves:

```
Lemma size_chunk_pos : forall chunk, size_chunk chunk > 0.

Proof. snipe. Qed.

Lemma perm_order_trans : forall p1 p2 p3,
    perm_order p1 p2 -> perm_order p2 p3 -> perm_order p1 p3.

Proof. snipe. Qed.
```

Lemma perm_order_trans deals directly with a Boolean relation, but we shall see that the tactic trakt comes along with a "translation database", which may be enriched manually by the user, allowing them to automatically cast some propositional predicates into Boolean ones.

2.1.2 Day-to-day proofs

More generally, automation helps Coq users by discharging tedious parts of proofs.

During the proof, the user still has to build the difficult parts manually (finding the correct mathematical reasoning needed for the proof), but they can benefit from our automation tools to get rid of the easier parts. Here is an example of a lemma that states that the length of the concatenation of two lists is the sum of their lengths:

```
Lemma app_length (A : Type) (HA : CompDec A) (1 1' : list A) :
  length (1 ++ 1') = length 1 + length 1'.
Proof. induction 1; snipe. Qed.
```

In this example, the user needs to perform the induction manually, but then the lemma can be proved automatically by the snipe tactic.

Similar examples can be found in the supplementary materials, and we encourage users to experiment with the tactic.

2.2 Under the hood

2.2.1 Overview

Let us come back to the example of app_length, to detail how the snipe tactic proceeds on both branches of the induction on the first list.

1. It generates first-order (often polymorphic) statements about inductive datatypes and functions in the local context. In our example, we will get (among other statements):

```
forall (B : Type), @length B [] = 0. (* implicit argument for clarity purposes *)
forall (B : Type) (1: list B) (x : B), length x :: xs = length xs + 1.
forall (B : Type) (1 : list B), [] ++ 1 = 1.
forall (B : Type) (1 1' : list B) (x : B), (x :: 1) ++ 1' = x :: (1 ++ 1').
forall (B : Type) (1 : list B) (x : B), x::1 <> [].
```

2. It eliminates prenex polymorphism by instantiating the generated lemmas; here, the mentioned lemmas are instantiated by the type variable A:

```
@length A [] = 0.
forall (1: list A) (x : A), length x :: xs = length xs + 1.
forall (1 : list A), [] ++ 1 = 1.
forall (1 l' : list A) (x : A), (x :: l) ++ l' = x :: (1 ++ l').
forall (1 : list A) (x : A), x::l <> [].
```

3. It transforms the goal and the hypotheses in order to use datatypes understandable by SMT solvers: the logic must be in bool and the integers must be of type z; the goal from our example becomes the following:

4. It finally calls the SMT solver veriT on the Coq goal and context and turns its proof certificate into a Coq proof.

Our tactic is thus a combination of pre-processing transformations, followed by a call to an SMT solver. We now detail the role of this pre-processing.

2.2.2 Pre-processing needs

In the standard input format of SMT solvers, the SMT-LIB language, the various SMT theories are represented as specific sets of types and symbols. For instance, integers are identified with a type Int, equipped with standard operations, such as addition or multiplication. Delegating Coq proofs to SMT solvers involves a translation of the proof statements from Coq to SMT-LIB. This translation selects standard Coq types to be mapped to the standard SMT-LIB types. For example, the type Z modeling integers in binary representation has been chosen as the Coq counterpart of the type Int in SMT-LIB. Arithmetic goals expressed in Z using the recognized standard symbols can therefore be proved with SMTCoq, such as the following:

```
Goal forall (x : Z), x + 0 = x.
Proof. smt. Qed.
```

However, Coq users have a lot of freedom in the expression of the theoretical concepts they need to leverage in their proofs, as the type theory backing the proof assistant lives at a way higher level of abstraction than SMT-LIB.

This freedom introduces the need to pre-process the goal before applying the translation to SMT-LIB, because the default translation, when run as is, leaves a lot of terms uninterpreted, thus losing information that can be crucial for the SMT solvers to find a proof. While state-of-the-art pre-processing tools are able to handle part of this added complexity, they were not designed to target SMT solvers, so there is still much space for improvement. The work presented in this article is a first step towards solving this problem.

For instance, the pre-processing plugin called Trakt, corresponding to step 3 in the previous subsection, enables SMTCoq to prove goals such as the following (the intermediate goal sent to SMTCoq is given as a comment):

```
Goal forall (f : int -> int) (x : int), (f (x + 0) == f x)%R.

Proof. trakt Z bool. smt. Qed.
(* forall (f' : Z -> Z) (x' : Z), Z.eqb (f' (x' + 0)%Z) (f' x') = true *)
```

3 Background

In this section, we recall a few relevant facts about the Coq proof assistant, the vehicle of our implementation, including a brief overview of the various approaches to meta-programming available in Coq. We illustrate in particular a few salient features which make the interaction with external first-order oracles more complex. At several places in this paper, we mention examples from the Mathematical Components library [19]: this development makes use of several advanced formalization techniques which challenge this interaction. We also include a brief presentation of the SMTCoq plugin [11], the current backend of our automated tactic.

3.1 Coq

The Coq proof assistant [26] implements a dependent type theory with inductive types, the Calculus of Inductive Constructions (CIC) [6]. In this functional programming language, programs are also used as a data structure for representing proofs, and provability coincides with type inhabitation. The default logical foundations of Coq are constructive, which means that the prover is compatible both with intuitionistic logic and with the assumption of non-constructive axioms, such as the excluded middle principle.

The official distribution of Coq comes with a standard library, which provides in particular various definitions of data structures, most often inductive types, of a general interest, together with a corpus of proved properties. For instance, the library features a type nat for unary natural numbers, a type z for binary integers, etc. Users shall also define their own alternative data structures: for instance, the Mathematical Components library defines a type int for unary integers. Because Coq can also be used to write (and evaluate) programs, a usual situation is to work with (at least) two different data structures for a same concept, one appropriate for computations (e.g., binary integers), and one better-behaved for reasoning (e.g., unary integers).

Modern Coq libraries most often make use of typeclass mechanisms, based either on enhanced inference of implicit parameters [23] or on unification hints [18]. These mechanisms allow in particular to share notations and theories among the instances of a hierarchy of structures. A particular effect of using typeclass based notations is that the latter are typically elaborated as the beta-expanded forms of a given operation. For instance, a term displayed as (x + y) * x for two integers x, y: int is in fact (@GRing.mul int_Ring (@GRing.add int_ZmodType x y) x). The latter term is in turn convertible to zmul (zadd x y) x, but the expanded version exposes the instances witnessing that the type int is endowed with a structure of commutative group and of ring, as required to interpret the infix notations (_ + _) and (_ * _) respectively. Because of this overloading technique, one should take into account possibly syntactically different versions of a same operation or constant when matching the signature of a given theory (e.g., linear integer arithmetic).

3.2 Meta-programming

Coq offers various approaches to *meta-programming*, that is, to implement programs operating on the syntax of arbitrary Coq terms. In particular, *quoting* and *unquoting* operations, respectively turn a Coq term into a term of the target meta-language, and reciprocally. The present contribution combines three such meta-languages, which offer in particular different levels of control on the syntax for terms:

■ Ltac [26] is a tactic language available at top-level, which provides pattern matching on Coq terms, goals and context, backtracking and recursion. We use Ltac mainly to prove

the statements we construct and to glue together the various transformations.

- Coq-Elpi [25] is a plugin that makes the Elpi language available in Coq to define new commands and tactics. Elpi is an OCaml implementation of λProlog, a language well suited to manipulate abstract syntax trees containing binders and unification variables. In Elpi, Coq terms are represented with an inductive type using the Higher-Order Abstract Syntax [20] encoding. The Coq-Elpi plugin gives access to primitives of Coq, so as to be able to look up previously defined terms, to declare new constants, to call various unification strategies, etc. Its quotation and anti-quotation syntax turns Elpi into a full-blown meta-language for Coq.
- MetaCoq [22] is a plugin which provides a quoted syntax of Coq terms, defined as a Coq inductive type. This plugin also provides a tactic quote_term, which turns a Gallina term into its quoted counterpart. MetaCoq is useful for performing a very fine-grained analysis on the syntax of Coq terms, and we use it to create new statements from the initial goal (e.g., the eliminators for inductive types). The technicalities of de Bruijn indices and the lack of pretty-printing or notation tools sometimes limit its usability.

3.3 SMTCoq

We target state-of-the-art SMT solvers through the SMTCoq plugin. This plugin provides communication with the SAT solver ZChaff and the SMT solvers veriT and CVC4, through a translation of their *proof witnesses* as *certificates* in a common format, which are checked in Coq using computational reflection. Since, among them, only the veriT solver has support for quantifiers in its proof witnesses, we currently use this solver.

Targeting SMT solvers offers built-in theories. Through veriT, SMTCoq offers support for the theories of equality and linear integer arithmetic (for the type z). We believe that many Coq formalizations require arithmetic, often mixed with propositional reasoning, which makes this support useful in many cases.

SMTCoq solves goals directly expressed in first-order logic without existential quantification. To use it, one thus needs to translate goals into this fragment of the logic, which is the heart of this article. Since SMTCoq does solve goals in this fragment, it is sufficient for this translation to relate its resulting goal with the initial goal, which we do through modular, proof-producing transformations.

Finally, even if part of the reasoning of SMT solvers is inherently classical (e.g., CNF computation), SMTCoq does not require classical logic in general: it only needs the predicates and equalities that appear in the goal to be decidable. It is then the choice of the user to either prove these properties, or just admit them. However, the decidability of equality is automatically proved for many types through the typeclass mechanism: types must be members of a class called CompDec, mainly specifying that they are inhabited, totally ordered, and that the equality on them is indeed decidable. In SMTCoq, preliminary tactics used to translate propositional goals into Boolean goals, but they are now completely subsumed by our new transformation Trakt presented below.

4 Methodological manifesto

4.1 General methodology

Our methodology is presented in Figure 1. To automatically solve a Coq goal, we first translate it into a first-order formula, then try to discharge it using first-order provers. The originality of our approach lies in the translation: the key idea is to design small *logical*

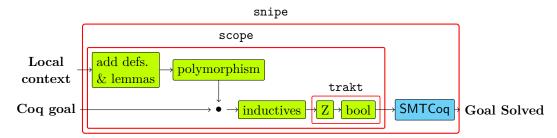
transformations, each one dealing with one aspect of the logic (e.g., datatypes), and compose them, possibly in various ways.

In Coq, tactics transform Coq goals into a (possibly empty) set of new goals, together with a proof that the validity of the latter implies the validity of the original goal. Our logical transformations are a particular case of such tactics whose role is to encode one aspect of Coq logic that is present in the goal, so that their combination end up in first-order logic. Moreover, these transformations are heuristics, they do not need human expertise, although they allow some optional arguments as we shall see. The motivations are detailed below.

- Extensibility: Since each transformation deals with one aspect of Coq logic, it is by design easy to start with transformations for parts of CIC close to first-order logic, experiment with them, and progressively extend with more and more complex features of CIC.
- Modularity: Transformations can then be composed in different ways, that we call strategies, depending on the expected behavior of the tactic. In particular, in the future, it should allow targeting different kinds of automatic provers, beyond SMT solvers, by plugging in or not transformations that may help them.
- **Predictability**: These strategies make the tactic predictable, meaning that users know when it is likely to solve the goal. We think that it is very important for users of proof assistants.

4.2 The scope tactic

Our Sniper plugin currently targets SMT solvers, by providing a strategy called scope that implements small transformations dealing currently with polymorphism, algebraic datatypes, and arithmetic, as summed up below:



We have highlighted in § 2.2.2 that targeting SMT solvers requires pre-processing. First, in order to benefit from propositional and theory-specific reasoning offered by these solvers, a crucial block, called trakt (the last two transformations in the figure), allows us to switch between representations of propositions and datatypes. On the contrary, state-of-the-art proof-producing SMT solvers do not support polymorphism or algebraic datatypes, for instance. This is why we implement transformations to explicit the meaning of Coq constructions, such as datatypes and pattern matching, or to remove polymorphism. Let us now enter into the details.

The first block of transformations examines the context of the current goal and turns it into statements that will be useful to the automatic solver. This happens by applying the following 4 consecutive transformations:

- = for each defined function symbol f, this definition is turned into an equation of the form f = fun x1 ... xn \Rightarrow ... that is added to the local context;
- the generated higher-order equalities are replaced with universally quantified equations of the form forall x1 ... xn, f x1 ... xn = ...;

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- each fixpoint construct appearing in the resulting equation is replaced with the name of the function that is defined by this fixpoint;
- for every pattern-matching operation, one equation per branch is generated, by specializing the matched term to each of its possible values.

The second block deals with the polymorphic assumptions present in the context. Each of these facts is instantiated with well-chosen types appearing in the current goal and the instances are added to the local context.

The third block generates facts about the inductive types that appear in the goal. These facts correspond to the three fundamental properties of inductive types:

- the direct images of the constructors are pairwise disjoint (no-confusion property),
- **the constructors are** *injective*,
- **each** term is *generated* by one of the constructors (inversion principle).

The third property deserves more attention, since the natural way of expressing it would be through existential statements (for each element of the inductive type, there exists arguments such that the element is some contructor applied to these arguments). Since existential statements are not handled by SMTCoq, we define a notion of eliminators on inductive types that eliminates the need for existential quantifications. More details are provided in § 5.2.

Finally, the last block is the Trakt transformation. As explained is § 3.3, SMTCoq makes a link between Coq and SMT solvers through the type z for arithmetic and the type bool for propositions. The role of the Trakt transformation is to target this fragment, which is the object of § 5.1. More details about the other transformations can be found in previous work [4].

5 Focus on two transformations

5.1 Trakt

Let us start with Trakt, a general pre-processing tool that makes SMTCoq available to Coq users, regardless of the representation of arithmetic and logic they use in their proofs. The transformation harnesses information given in the form of translation tuples, in which the user gives source and target values, along with a justification allowing the substitution of the latter for the former in the translated goal. In the following section, we shall mainly illustrate Trakt used in the context of SMTCoq, but the tool remains available in a standalone mode for other uses.

5.1.1 A goal-pre-processing tool for SMTCoq

The main purpose of Trakt is to make the goal more palatable for SMT solvers by rewriting it in a flavour that involves the types that they can recognise. The trakt tactic provided by the tool performs two kinds of modifications on the goal, which are crucial to extend the scope of SMTCoq: casting integers into type z, and translating propositional logic to Boolean logic when possible Trakt is certifying: from the initial goal G, it generates both a substitute goal G' and a proof of $G' \to G$ to allow the substitution. The generation is done by traversing the source goal, and depends on knowledge given by the user beforehand.

Arithmetic. To express integers in \mathbb{Z} , we need to be able to detect them in the source goal. Therefore, the user needs to declare the types to be considered as *integer types*. The required declaration for an integer type is a translation tuple $(T, e, \bar{e}, \mathrm{id}_1, \mathrm{id}_2)$, where T is a type,

 $e: T \to \mathbb{Z}$ and $\bar{e}: \mathbb{Z} \to T$ are embedding functions (*i.e.*, explicit casts) in both ways, and id₁ and id₂ are proofs that both of their compositions are identities.

Not only can we embed into \mathbb{Z} every value living in a declared integer type, but we can also process values in functional types containing integers (*i.e.*, uninterpreted functions), as well as universal quantifiers. For example, if a type I is an integer type, then the translation turns a quantifier such as $\forall f: I \to I$ into $\forall f': \mathbb{Z} \to \mathbb{Z}$.

To complete arithmetic proofs, we also need to recognise standard arithmetic operations and values from the SMT-LIB format when they are expressed in an integer type, so that they do not remain uninterpreted. This is the case for standard operators and values (e.g., addition, zero) in these types, which can be mapped to the ones in \mathbb{Z} , but this is also the case for any operation that composes them (e.g., a fused multiply-add operator). These terms can be declared as symbols with a (s, s', p) tuple, where s and s' are two symbols, and p is an embedding property:

$$\frac{s: T_1 \to \cdots \to T_n \to T_o \qquad s': T_1' \to \cdots \to T_n' \to T_o'}{p: \forall (t_1: T_1) \cdots (t_n: T_n), \mathbf{e}_o^? \left(s \ t_1 \cdots t_n\right) = s' \left(\mathbf{e}_1^? \ t_1\right) \cdots \left(\mathbf{e}_n^? \ t_n\right)}$$

Here, e_i^2 is a meta-notation for an optional embedding function from T_i to \mathbb{Z} . If $T_i' = T_i$, it is the identity. Otherwise $T_i' = \mathbb{Z}$ and it is the embedding function from T_i to \mathbb{Z} , provided that the user declared it before declaring the symbol. This property is the one that allows replacing the symbols in various integer types with their counterparts in \mathbb{Z} . For example, addition in an integer type I has type $I \to I \to I$, so the expected type for the embedding property is the following (with e the embedding function for type I):

$$\forall x \ y : I, e \ (x +_I \ y) = e \ x +_{\mathbb{Z}} e \ y$$

▶ Example 1.
$$\forall (f:I \rightarrow I \rightarrow I)(x \ y:I), x = y \rightarrow f \ x \ (y +_I 0_I) = f \ y \ x.$$

In order to preprocess this goal, we need to know that I is an integer type and the symbols $+_I$ and 0_I can be mapped to $+_{\mathbb{Z}}$ and $0_{\mathbb{Z}}$. If all the declarations are present, the goal is translated into the following:

$$\forall (f': \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z})(x' \ y': \mathbb{Z}), x' = y' \to f \ x' \ (y' +_{\mathbb{Z}} 0_{\mathbb{Z}}) = f \ y' \ x'$$

Logic. Trakt is also able to translate logical connectors, equalities, and various other n-ary predicates into their Boolean equivalents. To detect these user-defined n-ary predicates in the source goal, we need to declare them as relations. This is done by providing a translation tuple (R, R', p), where R and R' are the relation and its Boolean equivalent, and p is a proof of equivalence:

$$\frac{R:T_1\to\cdots\to T_n\to L \qquad R':T_1'\to\cdots\to T_n'\to L'}{p:\forall (t_1:T_1)\ \cdots\ (t_n:T_n), R\ t_1\ \cdots\ t_n\sim_{L,L'} R'\ (e_1^?\ t_1)\ \cdots\ (e_n^?\ t_n)}$$

L	L'	$\sim_{L,L'}$		
bool	bool	$\lambda b. \ \lambda b'. \ b = b'$		
Prop	bool	$\lambda P. \ \lambda b. \ P \leftrightarrow b = \mathbf{true}$		
Prop	Prop	$\lambda P. \ \lambda Q. \ P \leftrightarrow Q$		

Here, L and L' are logical types (*i.e.*, either Prop or bool) and $\sim_{L,L'}$ is a way to express equivalence depending on these logical types. e_i^2 is the same concept of optional embedding function as used above to define the embedding property for symbols. For instance, to associate the equality on an integer type I with the Boolean equality on \mathbb{Z} , the proof of

equivalence must have type $\forall x \ y : I, x =_I y \leftrightarrow e \ x =_{\mathbb{Z},\mathbb{B}} e \ y = \mathbf{true}$, where $=_{\mathbb{Z},\mathbb{B}}$ denotes Boolean equality on \mathbb{Z} .

So far, pre-processing a term does not modify its structure, because we only considered equivalence. The notion of integer type described above can be extended, so that Trakt is also able to handle non-surjective embeddings $(e.g., \mathbb{N} \text{ into } \mathbb{Z})$. In this case, the embedding is partial, because $e \circ \bar{e}$ is not always an identity. The declaration for integer types is enriched with an alternative tuple $(T, e, \bar{e}, C, p_C, \mathrm{id}_1, \mathrm{id}_{2C})$, where C is a restricting predicate on the embedded values (which we call an embedding condition), p_C is a proof that it is true on every embedded value, and id_{2C} is id_2 restricted to the condition C:

$$id_{2C}: \forall z: \mathbb{Z}, C \ z \to e \ (\bar{e} \ x)$$

When an embedding is partial, every value embedded from this type into \mathbb{Z} adds a condition to the output formula.

▶ Example 2.
$$\forall (f : \mathbb{N} \to I)(n : \mathbb{N}), f \ n +_I 0_I =_I f \ n.$$

In order to preprocess this goal, we need to have the same declarations as in example 1, but also an embedding declaration for \mathbb{N} and a relation declaration for the equality on type I. If everything is present, the goal is translated into the following:

$$\forall (f': \mathbb{Z} \to \mathbb{Z})(n': \mathbb{Z}), n' \geq_{\mathbb{Z}, \mathbb{B}} 0_{\mathbb{Z}} = \mathbf{true} \to f' \ n' +_{\mathbb{Z}} 0_{\mathbb{Z}} =_{\mathbb{Z}, \mathbb{B}} f' \ n' = \mathbf{true}$$

5.1.2 Technical aspects

To build a tool that meets these requirements, three technical challenges arise. The first one is term traversal: inspecting a Coq term and building a new term at the same time is not a trivial task, and the optimal meta-language might be different according to the needs of the algorithm. The second issue is data collection, storage and retrieval. Indeed, we need a technical tool to build a database from user knowledge, Coq commands to fill it with translation tuples, then a mechanism to perform lookups while our pre-processing algorithm runs. The third challenge is term matching. We need to define the level of flexibility which our pattern matching operates with, ranging from a purely syntactic matching strategy to the full-fledged Coq conversion, keeping in mind that this impacts performance. In view of these requirements, our choice was to go with Coq-Elpi. Let us now illustrate these needs and give some insight as to why it was a good choice of meta-language to satisfy them.

Term traversal and reconstruction. Our algorithm takes the shape of a Coq-Elpi predicate with various cases according to the shape of the inspected term. Binders are one of the main obstacles in the traversal of a Coq term, but Coq-Elpi offers an easy way to manage them. The Elpi language offers β -reduction and β -abstraction at the meta level, along with a pi keyword that introduces a locally bound variable that can be used in subsequent calls, while letting us express the outputs of these calls as a function of this variable.

Let us give a simplified snippet taken from the code of the main pre-processing algorithm of Trakt. Consider a predicate preprocess which takes the input term and outputs the pre-processed term along with the certification of the goal substitution.

```
preprocess (prod N T F) Out Proof :- !,
  pi x\ decl x _ T => preprocess (F x) (F' x) (ProofF x),
  % ...
```

This case of the predicate describes how we pre-process universal quantifiers: prod N T F is the constructor of forall N: T, F N in the Coq-Elpi inductive type representing the AST

of Coq terms. Coq-Elpi uses the HOAS encoding for Coq terms, so F has type term \rightarrow term. Here, we use pi to create a new Elpi variable x to represent the variable that is bound in the universal quantifier. Then, we use Prolog-like implication to execute the recursive call with the meta-level information that x has type T.

Notice that the recursive call is done on the term under the universal quantifier through β -reducing F with the fresh variable, and the outputs are β -abstracted with respect to this variable too. Several constructors are available in the Coq-Elpi term inductive type to forge terms from these meta-functions (to build the output values Out and Proof). In this way, no term can escape the scope of x, as the recursive call only gives back variables F' and ProofF, which have type term -> term. Therefore, the meta-language allows handling Coq terms without ever having to manage any context or de Bruijn indices.

Detecting known terms. A case of the Coq-Elpi pre-processing predicate is executed only if its head clause can be unified with the given input, through a purely syntactic unification algorithm \grave{a} la Prolog. A quote and unquote mechanism is available in Coq-Elpi to ease the expression of the expected input for each case, as a pattern with Prolog variables standing in for Coq unification variables in Ltac. For instance, here is a simplified version of the head clause of the pre-processing predicate for the arrow case:

```
preprocess {{ lp:A -> lp:B }} TVar _ {{ lp:A' -> lp:B' }} Proof :-
```

We can see that a pair of double brackets is used to quote a Coq term in Coq-Elpi, and an lp: prefix can be used to introduce Coq-Elpi code into the quoted Coq term. In this way, this case of the algorithm only matches terms for which the arrow notation is valid. It also identifies both sides of the arrow with A and B respectively, so that we can use them in the body of the predicate to build variables A' and B', connecting them back with the same arrow to craft the output term.

Nevertheless, purely syntactic matching does not cover all our needs. Indeed, thanks to the powerful notation system and type inference in Coq, it is possible to define notations that apply to several types. For example, in the Mathematical Components library, notations are used to simulate ad hoc polymorphism for arithmetic operations and neutral values. This allows unifying notations across different types, by writing the addition as + no matter what the underlying datatype is, and the actual Coq term behind it is a projection on an automatically inferred instance of a general ring structure. Thus, behind the term x + 0 actually lies the more complex $OGRing.add_x (OGRing.zero_)$, the holes being filled with an instance of the ring structure according to the type of x. This means that even though the user declares one of the concrete ring types as an integer type and standard operations on it as known symbols, if Trakt operates with a purely syntactic matching strategy, it does not recognize terms like the one above, because the head term, OGRing.add, is uninterpreted. This is why Trakt uses Coq conversion only in this restricted case.

Knowledge database and user API. Before making proofs, the user can communicate knowledge to Trakt through four Coq commands, one for each kind of information declared: integer types, relations, symbols, and terms that can trigger conversion. Each of these commands is associated to a Coq-Elpi database with a predicate, and every call to the command adds an instance of the associated predicate to the database. This allows the user

¹ In this case, the inference mechanism is canonical structures.

to statically fill the database and then freely call the trakt tactic that will harness this added knowledge by performing lookups at runtime.

Let us show a simplified syntax of the four available commands. Integer types are declared through one of the following commands:

```
Trakt Add Embedding (T) (Z) (e) (e') (id1) (id2).

Trakt Add Embedding (T) (Z) (e) (e') (id1) (id2C) (pC).
```

The variable names copy the ones in § 5.1.1, except for e, to replace \bar{e} . Notice that the embedding condition is missing. It is due to the fact that this type can actually be inferred from p_C or id_{2C} . This kind of simplification is performed everytime it is possible, to relieve the user from useless repetitions.

Relations and symbols follow a similar model. The terms triggering conversion are the ones described in the previous paragraph (i.e., structure projections). The command just requires the term as an argument.

```
Trakt Add Symbol (s) (s') (p).
Trakt Add Relation (R) (R') (p).
Trakt Add Conversion (t).
```

Trakt beyond SMTCoq In this section, we have presented the use of Trakt in the context of SMTCoq, showing that the tool is able to map integer types to \mathbb{Z} and output Boolean logic when possible. In fact, the approach followed in Trakt is more general. What we presented as integer types is an instance of the more general concept of *embeddable types*, where the output type is not necessarily \mathbb{Z} . It basically means that Trakt can also manage embeddings on other theories than arithmetic. As a result, in the command trakt Z bool, type Z may very well be replaced with other target embedding types. Similarly, relations accept predicates on any type (not necessarily an integer type). In particular, this means that we can register decidable equalities on unknown types, which is a useful feature in SMTCoq.

We have presented the processing of logic as a transformation from Prop to bool, as SMTCoq requires Booleans for its translation to the SMT-LIB format. But this might not be the case for all proof automation tools (e.g., lia works best with propositional logic). This is why Trakt is also able to do the pre-processing the other way around, turning Boolean logic into propositional logic when possible. The real trakt tactic can be controlled through parameters specifying the target type for embeddings and logic. Through this broader picture, Trakt can be seen as a general pre-processor, acting like a type-level funnel on any theory that is relevant to a proof automation tactic to be called afterwards.

5.2 Eliminators for inductive types

We explain in this paragraph how we define eliminators for inductive types, avoiding altogether the need for existential statements in the inversion principles.

In order to illustrate our transformation, we consider the case of the list inductive type. In the Coq standard library, list is an inductive with two constructors, [] (the empty list) and _ :: _ (a list with one element added to another list), which enjoy the following inversion property:

$$\forall A (l: \mathtt{list} A), \ l = [] \lor \exists x \ l', \ l = x :: l'$$

This statement, written in the most natural way, uses existentials. But existentials are an obstacle to automation as they are not handled very well by automated provers. Indeed, automated provers are usually based on classical logic, in which existentials are just negation

of universals, which is not valid in constructive logic and in particular, in Coq. Thus, most SMT certificates dealing with existentials cannot be used to build a Coq proof term. For this reason, scope generates in fact an alternative formulation of G_T , which is usable in both classical and constructive settings.

To achieve this, the idea is to introduce new definitions in the local context for *eliminators*. In general, scope introduces one such eliminator per argument of each constructor of a given inductive type. Then, eliminators are used to describe how the terms of the inductive type can be generated from its constructors. Let us continue our example with list.

When given a type A and a term a of type A, our transformation get_eliminators generates and proves this statement:

```
\forall (l: \mathtt{list}\, A), l = [\,] \lor l = \mathtt{elim}_{2.1}\, A\, a\, l :: \mathtt{elim}_{2.2}\, A\, [\,]\, l.
```

Suppose that our inductive features n constructors C_1,\ldots,C_n and each constructor has respectively k_1,\ldots,k_n (non dependent) arguments. The function for the i-th constructor, and the j-th argument (with $j \leq k_i$), performs a pattern matching on the inductive term. If this term corresponds to the i-th constructor, the function $\mathtt{elim}_{i,j}$ returns its j-th projection, i.e., $\mathtt{elim}_{i,j}(C_i(\ldots,x_j,\ldots)) = x_j$. Otherwise, it returns a default term of the expected type, i.e., $\mathtt{elim}_{i,j}(C_{i'}(\ldots)) = \mathtt{default}$. This default term is found automatically by an auxiliary tactic².

Thus, they are as many functions generated as the sum of the arguments of each constructor (parameters excluded). In the case of list, we have:

These eliminators are uninterpreted symbols for the external solver: they are just a trick to eliminate existential quantifiers in the statement G_T . Even without these information, the external solver can perform case disjunction over an inductive term once the statement has been generated and proved.

To sum up, this transformation works in three steps:

- 1. It generates the eliminators and poses them in the local context;
- 2. It generates and proves a general statement (still abstracting over the default terms);
- 3. Whenever it is possible, it finds an inhabitant for each required type and generates a statement with no quantification over default terms.

6 Benchmarks

6.1 Quantitative benchmarks

We tested snipe on the actual List module from the Coq standard library and compared it to hammer (on a 4 core Intel i5-10310U with 16GB RAM). As most of the statements are

² As SMTCoq requires to work on inhabited types, the tactic looks for this canonical inhabitant or finds one already present in the context.

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polymorphic, we needed to add a CompDec hypothesis on the type variable. Furthermore, we translated the In and the lel predicate (lel 1 m stands when length 1 \leftarrow length m) into a Boolean version.

For snipe, four ways of completing the proofs were considered: using the standalone tactic, using it after performing an induction on a suitable variable, using it with previous lemmas given as parameters, or using it with a combination of both.

For testing hammer, we used either hammer alone, after an induction, or twice in the same goal when it is an equivalence, with split. As this tactic is undeterministic, the results depend on the prover which found the proof first. Here, we used Vampire [15] 4.6.1 and CVC4 [2] 1.6.

	tactic	Induction	Lemmas	Induction	Split +	Total proved	Average
	alone	+tactic	+ tactic	+lemmas	induction	lemmas	time (s)
				+tactic	+tactic		
snipe	27	28	11	6	not	72/331	3.60
					concerned		
hammer	93	135	not	not	2	230/331	1.89
			concerned	concerned			

For now, hammer performs better in average than snipe because existentials, higher-order and inductive predicates are not handled by our tactic. We believe that new transformations will increase the fragment of CIC that the tactic can translate into first-order logic. It is also faster because each transformation of snipe is certifying and because the proof reconstruction in hammer works differently (see § 7) and does not work everytime. Consider:

```
Lemma app_eq_nil (1 1':list' A) : 1 ++ 1' = [] -> 1 = [] /\ 1' = [].

Proof. snipe. Qed.
```

The hammer tactic calls an external solver but is then not able to reconstruct the proof, but snipe works here because our transformations generate a proof and does not change the overall structure of the goal. In addition, the proof given by the external solver is reconstructed in Coq step by step by the SMTCoq plugin, which causes less failures.

6.2 Arithmetic example

The snipe tactic uses the built-in theories of SMT solvers to produce proofs. Here, this lemma can be proved in less than 2s:

► Example 3. Arithmetic in **z**:

```
Lemma arith_and_uninterpreted_symbol
  (T : Type) (HT : CompDec T) (x y : nat) (b : bool) (f : nat -> T) :
    True /\ b = true \/ f (x + y) = f (y + x).
Proof. snipe. Qed.
```

The trakt transformation translates the statement in nat into an equivalent one in Z as veriT needs statements with integers in Z. Then, we can benefit from the built-in arithmetic of this solver. We can compare this to hammer, which is slower here (and sometimes cannot reconstruct the proof depending on the libraries loaded) because it has to search for arithmetic lemmas in Coq database and give them to the solver instead of using its built-in arithmetic when it has one. As hammer treats all Coq terms indifferently, as uninterpreted ones, some arithmetic proofs may be harder to find or to reconstruct.

7 Related works and perspectives

Conclusion We presented an automation tactic for Coq based on modular pre-processing transformations from aspects of CIC to FOL. The main transformation we introduced is Trakt, which tackles the problem of different Coq representations of the same type. It allows transforming all representations of integers into z, which is part of the interpreted theories of the SMT solvers, and transforming propositional statements into Boolean ones. Other aspects of the snipe plugin deal with types and functions uninterpreted for the solver, by generating proved first-order statements about them. The combination of this different transformations increases Coq automation and will be enriched with new ones in the future.

Related works Putting automated theorem proving at the service of the users of interactive theorem provers has been a long-lasting research program. This line of research took a leap with *hammers*, initially developed for provers of the HOL family: sledgehammer for Isabelle/HOL [21], HolyHammer for HOL-Light and HOL4 [14]. Hammers are designed to be used without requiring *any* configuration from the users, and they roughly follow the same general pattern:

- 1. select a large set of possibly useful results from lemmas libraries, possibly using machine learning;
- 2. transform the goal into a formula which can be handled by automatic provers, possibly using an unsound translation;
- 3. send the transformed goal to a bunch of first-order automatic provers;
- **4.** if one of the solvers finds a certificate, use it as a hint for a formal-proof-producing first-order prover, implemented by the proof assistant.

The hammer approach has been transposed to Coq in a project called CoqHammer [9]. In this case, step 2 involves a shallow (incomplete) embedding of CIC into first-order logic [8] and the automated provers targeted at step 3 do not involve theory-specific decision procedures. By contrast, the latter is precisely the focus of the present work, motivated in particular by applications in program verification (cf. § 2.1.1). Our approach to step 2 and step 4 is also different. Instead of a monolithic translation, step 2 is performed via a modular composition of atomic, formally validated transformations, which reduces step 4 to the sheer reconstruction of the exact first-order problem handed to the prover. This modular approach is specially useful to avoid committing automation to a restricted class of data structures. As argued in § 4, we believe that it also enhances predictability and helps diagnostic in case the tactic fails. It is also extensible: additional atomic transformations can be easily added to the schema in order to make the whole tactic more expressive. Conversely, some transformations may also be unplugged to target other automatic provers, such as the extension of SMT solvers to higher-order logic [27], that we plan to explore.

Step 1 of the hammer strategy is rather orthogonal to the work presented here, which does not discuss relevance filters. Such filters would certainly expand the automation power, and we will explore this in future work. However, we expect to face the need for a finer-tuning, as on one hand, SMT solvers can only handle a smaller portion of the available theory lemmas, but on the other, they do not need to know the theory specific ones. In fact, we see this approach as complementary to hammers: CoqHammer shall be run in the background (as sledgehammer) and be helpful when a valid proof is found, while our approach targets knowledgeable users for them to automatically discharge well identified parts of their proofs.

The present paper significantly extends a preliminary description by some of the authors [4] with the following novel contributions:

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- we effectively target SMT solvers, for their theory reasoning capabilities, which stresses the need for adequate pre-processing of arithmetic and Boolean connectives (under uninterpreted function symbols). This motivated the design Trakt, one of our main contributions here;
- we implement other new transformations, such as eliminators for algebraic data-types, and we improve the robustness of the whole strategy;
- we evaluate the resulting snipe tactic both on performance and expressivity benchmarks. Coq tactics dedicated to integer arithmetic, notably lia, benefit from a pre-processing procedure called zify [3], implemented using typeclasses. This extensible pre-processing is meant to "canonize" a formula by normalizing the data structures used for the theories under interest, e.g., integers. Properties on arbitrary user-defined versions of arithmetic can be proved automatically, provided that suitable instances of the zify typeclasses have been declared. For instance, the mczify library declares the relevant instances to data structures defined in the Mathematical Components library. The zify pre-processing was however not designed to support uninterpreted symbols, which is a major limitation in the context of automation based on SMT, as it prevents combining arithemtics with congruence.

Finally, the verified pre-processing performed by Trakt also relates to automated transfer of properties from one representation of a mathematical concept to another has been studied under the perspective of parametricity, and implemented, e.g., in the CoqEAL library [5]. Homotopy equivalences, at the core of homotopy type theory, make precise the types for which any such property can be transferred, and univalent parametricity makes possible to perform this transfer in practice, as partially implemented in the companion prototype to Tanter et al.'s paper [24]. Part of the proof reconstruction features of Trakt could in principle be addressed by univalent parametricity, under the assumption of the univalence axiom, but an appropriate implementation for this purpose is not yet available.

Future improvements It is left for future work an implementation of the snipe tactic which can integrate user-defined (and possibly parametric) tactics, at any suitable place in the chain of transformations.

Another possible improvement would be the generation of fewer instances while eliminating prenex polymorphism. This could go with different version of <code>snipe</code>, with different strategies of instantiation.

New transformations In Coq developments, users tend to write logical properties in the form of an inductive relation in Prop, even when they are decidable, rather than with a Boolean function. For now, if they want to benefit from Sniper automation, they have to write manually the equivalent Boolean function and prove its completeness and correctness, and add these to the Trakt database. We plan to automatize this work for some cases, by taking inspiration from the work of David Delahaye and al.[10], based on a mode consistency analysis of an inductive relation.

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