

Weightwise Almost Perfectly Balanced Functions, Construction From A Permutation Group Action View.

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Abstract. The construction of Boolean functions with good cryptographic properties over a subset of vectors with fixed Hamming weight is significant in lightweight stream ciphers like FLIP. We present a general idea to construct a class of Weightwise Almost Perfectly Balanced (WAPB) Boolean functions by using the action of a cyclic permutation group on \mathbb{F}_2^n . Further, considering a particular permutation group $\langle \psi_n \rangle$ (where ψ_n is a special type of permutation on n elements), we present a class of WAPB Boolean functions on n variables. This class generalizes the Weightwise Perfectly Balanced (WPB) Boolean function construction by Liu and Mesnager. Further, we studied the nonlinearity and weightwise nonlinearities of this class of functions.

Keywords: Boolean function, Weightwise perfectly balanced (WPB), Weightwise almost perfectly balanced (WAPB), Nonlinearity

1 Introduction

An n -variable Boolean function f is a mapping from the n -dimensional vector space \mathbb{F}_2^n to \mathbb{F}_2 , where \mathbb{F}_2 denotes a finite field with two elements $\{0, 1\}$. Depending upon the underlying algebraic structure, the symbol ‘+’ is to indicate addition operations in both \mathbb{F}_2 and \mathbb{R} . Typically, the cryptographic criteria of the Boolean functions are defined over the entire domain of vector space \mathbb{F}_2^n . However, the study of Boolean functions over restricted domains gained interest following the introduction of the stream cipher FLIP in 2016 [MJSC16]. In FLIP, the inputs to the filter function have a fixed Hamming weight of $\frac{n}{2}$. An initial cryptographic study of Boolean function in a restricted domain was conducted by Carlet et al. in [CMR17]. Boolean functions that are balanced over the subsets of \mathbb{F}_2^n containing vectors with constant Hamming weight are said to be Weightwise Perfectly Balanced (WPB). Such functions exist only when n is a power of two; otherwise, only functions that are *almost balanced* can exist. Here, *almost balanced* means that the counts of preimages of 0 and 1 differ by at most one. Functions that are almost balanced on each subset of constant Hamming weight are termed Weightwise Almost Perfectly Balanced (WAPB).

The first weightwise perfectly balanced (WPB) Boolean function construction was introduced in [CMR17] in 2017. Multiple studies and constructions [TL19, LM19, LS20, MS21, Su21, ZS22, GM22a, MPJ⁺22, MKL22, GM22b, MSLZ22, GS22, GM23c, GM23b, ZLC⁺23, DM23, ZS23, ZJZQ23, YCL⁺23, GM23a, DM24, Méa24] of WPB and WAPB functions are available in the literature. Liu and Mesnager [LM19] presented a class of WPB Boolean functions that are 2-rotation symmetric. These functions have the best weightwise nonlinearities compared to the currently available constructions. In this article, we have generalize this construction to create a class of WAPB Boolean functions for any number of variables.

[PM: add more detailed contribution part]

2 Preliminaries

Let \mathbb{F}_2^n be the vector space of dimension n over the binary field \mathbb{F}_2 . For any two vectors $a = (a_1, a_2, \dots, a_n)$ and $b = (b_1, b_2, \dots, b_n)$ in \mathbb{F}_2^n , the dot product is defined as $a \cdot b = a_1b_1 + a_2b_2 + \dots + a_nb_n \bmod 2$.

A Boolean function of n variables is a map from \mathbb{F}_2^n to \mathbb{F}_2 and \mathcal{B}_n is the set of all n -variable Boolean functions. The 2^n -length binary sequence $(f(v_0), f(v_1), \dots, f(v_{2^n-1}))$ is called as the truth table of the Boolean function f , it corresponds to the ordered vectors of \mathbb{F}_2^n as $v_0 = (0, 0, \dots, 0), v_1 =$

$(0, 0, \dots, 1), \dots, v_{2^n-1} = (1, 1, \dots, 1)$. The Hamming weight of a vector $x \in \mathbb{F}_2^n$, denoted by $w_H(x)$, is the number nonzero coordinates (i.e., 1s) in the vector x . The support of $f \in \mathcal{B}_n$, denoted by $\text{supp}(f) = \{x \in \mathbb{F}_2^n : f(x) = 1\}$. Therefore, the Hamming weight of the Boolean function f , is cardinality of $\text{supp}(f)$ and is denoted by $w_H(f)$. The function f is called balanced if $w_H(f) = 2^{n-1}$. Let $f(x), g(x) \in \mathcal{B}_n$, then The Hamming distance between two Boolean functions $f, g \in \mathcal{B}_n$ is defined by $d_H(f, g) = |\{x \in \mathbb{F}_2^n : f(x) \neq g(x)\}|$ i.e., $d_H(f, g) = w_H(f + g)$.

An n -variable Boolean function f can be expressed as a polynomial in the ring $\mathbb{F}_2[x_1, x_2, \dots, x_n] / \langle x_1^2 + x_1, x_2^2 + x_2, \dots, x_n^2 + x_n \rangle$, i.e. $f(x) = \sum_{u \in \mathbb{F}_2^n} c_u x^{u_1} x^{u_2} \dots x^{u_n}$, where c_u are the coefficients with a value in \mathbb{F}_2 . It is called as the algebraic normal form or ANF and the number of variables in the highest order monomial with nonzero coefficient is called the *algebraic degree* of the function f , and denoted as $\deg(f)$. A function f is called as affine function if $f(x) = a \cdot x + b$ for $a \in \mathbb{F}_2^n$ and $b \in \mathbb{F}_2$. If $b = 0$, then f is also called a linear Boolean function. \mathcal{A}_n denotes the set of all n -variable affine functions.

Definition 1 (Walsh-Hadamard Transform). *The Walsh-Hadamard transform of a function on \mathbb{F}_2^n is the map $W_f : \mathbb{F}_2^n \rightarrow \mathbb{R}$, defined by*

$$W_f(w) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + w \cdot x}.$$

Definition 2 (Nonlinearity). *The nonlinearity of $f \in \mathcal{B}_n$ denoted as $NL(f)$, is the minimum Hamming distance of f to any affine function. That is, $NL(f) = \min_{g \in \mathcal{A}_n} d_H(f, g)$. It can be verified that $NL(f) = 2^{n-1} - \frac{1}{2} \max_{w \in \mathbb{F}_2^n} |W_f(w)|$.*

We denote $E_{k,n} = \{x \in \mathbb{F}_2^n : w_H(x) = k\}$ and $w_{k,n}(f) = |\{x \in E_{k,n} : f(x) = 1\}| = |\text{supp}(f) \cap E_{k,n}|$. Accordingly, the Hamming distance of two functions $f, g \in \mathcal{B}_n$ on $E_{k,n}$ denoted as $d_{k,n}(f, g) = |\{x \in E_{k,n} : f(x) \neq g(x)\}|$. The cryptographic criteria like balancedness, nonlinearity and algebraic immunity of a function f defined over \mathbb{F}_2^n can also be defined, if we restrict f to the set $E_{k,n}$. For two integers m, n with $m \leq n$, we define $[m, n] = \{m, m+1, \dots, n\}$.

Definition 3 (Weightwise Almost Perfectly Balanced (WAPB)). *A Boolean function $f \in \mathcal{B}_n$ is said to be weightwise almost perfectly balanced (WAPB) if for all $k \in [0, n]$,*

$$w_{k,n}(f) = \begin{cases} \frac{\binom{n}{k}}{2} & \text{if } \binom{n}{k} \text{ is even,} \\ \frac{\binom{n}{k} \pm 1}{2} & \text{if } \binom{n}{k} \text{ is odd.} \end{cases}$$

Definition 4 (Weightwise Perfectly Balanced (WPB)). *A Boolean function $f \in \mathcal{B}_n$ is said to be weightwise perfectly balanced (WPB) if for all $k \in [1, n-1]$,*

$$w_{k,n}(f) = \frac{\binom{n}{k}}{2},$$

and $f(0, 0, \dots, 0) = 0 = 1 + f(1, 1, \dots, 1)$.

Using Lucas' Theorem [Fin47], we have that a WPB function exists only if, n is a power of 2. Hence, there are $\prod_{k=1}^{n-1} \left(\frac{\binom{n}{k}}{\binom{n}{k}/2} \right)$ WPB Boolean functions.

Definition 5 (Restricted Walsh Transform). *Let f be an n -variable Boolean function, then its Walsh transform $\mathcal{W}_{f,k}(a)$ is defined as:*

$$\mathcal{W}_{f,k}(a) = \sum_{x \in E_{k,n}} (-1)^{f(x) + a \cdot x}.$$

Definition 6 (Weightwise Nonlinearity). The nonlinearity of $f \in \mathcal{B}_n$ over $\mathbb{E}_{k,n}$, denoted as $\text{NL}_k(f)$, is the Hamming distance of f to the set of all affine functions \mathcal{A}_n when evaluated over $\mathbb{E}_{k,n}$. That is, $\text{NL}_k(f) = \min_{g \in \mathcal{A}_n} d_{k,n}(f, g) = \min_{g \in \mathcal{A}_n} w_{k,n}(f + g)$.

The following identity and upper bound on the nonlinearity of a Boolean function over $\mathbb{E}_{k,n}$ can be derived. The upper bound is further improved by Mesnager et al. in [MZD19].

Lemma 1 ([CMR17], Propositions 4 and 5). If $f \in \mathcal{B}_n$ then for $k \in [0, n]$,

$$\text{NL}_k(f) = \frac{|\mathbb{E}_{k,n}|}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |\mathcal{W}_{f,k}(a)|, \text{ and}$$

$$\text{NL}_k(f) \leq \frac{1}{2} [|\mathbb{E}_{k,n}| - \sqrt{|\mathbb{E}_{k,n}|}] = \frac{1}{2} \left[\binom{n}{k} - \sqrt{\binom{n}{k}} \right].$$

Definition 7 (Algebraic immunity and weightwise algebraic immunity). The algebraic immunity of a Boolean function $f \in \mathcal{B}_n$, denoted as $\text{Al}(f)$, is defined as:

$$\text{Al}(f) = \min_{g \neq 0} \{\deg(g) \mid fg = 0 \text{ or } (f+1)g = 0\},$$

where $\deg(g)$ is the algebraic degree of g . The function g is called an annihilator of f (or $f+1$).

The weightwise algebraic immunity of $f \in \mathcal{B}_n$ for $k \in [0, n]$, denoted as $\text{Al}_k(f)$, is defined as:

$$\text{Al}_k(f) = \min_{g \neq 0 \text{ over } \mathbb{E}_{k,n}} \{\deg(g) \mid fg = 0 \text{ or } (f+1)g = 0\}.$$

2.1 Permutation symmetric Boolean functions

We denote \mathbb{S}_n be the symmetric group on n elements. For a permutation $\pi \in \mathbb{S}_n$, the action of permutation group $P = \langle \pi \rangle$ on \mathbb{F}_2^n makes a partition of \mathbb{F}_2^n , called orbits. The set of orbits is denoted as \mathcal{O} . For $x \in \mathbb{F}_2^n$, we denote the orbit containing x as $O(x) = \{y \in \mathbb{F}_2^n \mid \pi^k(y) = x \text{ for some } k \in \mathbb{Z}\}$. Now we define two special permutations as follows.

1. **Rotation (or, cyclic) permutation (σ):** The permutation $\sigma \in \mathbb{S}_n$ is called rotation permutation if $\sigma((x_1, x_2, \dots, x_n)) = (x_n, x_1, \dots, x_{n-1})$ for every $(x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n$. We denote the rotation permutation on n elements by σ (or, σ_n to specify the number n). The cyclic group $P = \langle \sigma \rangle$ is called rotation symmetric group.
2. **Distinct binary-cycle permutation(ψ):** Let n be a positive integer with binary representation as

$$n = n_1 + n_2 + \dots + n_w \text{ where } n_1 = 2^{a_1}, n_2 = 2^{a_2}, \dots, n_w = 2^{a_w} \text{ and } 0 \leq a_1 < a_2 < \dots < a_w. \quad (1)$$

A permutation $\psi \in \mathbb{S}_n$ is called a distinct binary-cycle permutation if its disjoint cycle form contains cycles of length n_1, n_2, \dots, n_w . We denote the permutation on n elements by ψ (or, ψ_n to specify the number n). Without loss of generality, we consider

$$\psi = (x_1, x_2, \dots, x_{n_1})(x_{n_1+1}, x_{n_1+2}, \dots, x_{n_1+n_2}) \cdots (x_{n-n_w+1}, x_{n-n_w+2}, \dots, x_n) = \sigma_{n_1} \sigma_{n_2} \cdots \sigma_{n_w}. \quad (2)$$

Note that, $\sigma = \psi$ when $n = 2^m$ is a power of 2. For $\pi \in \mathbb{S}_n$, we define π symmetric Boolean functions and $2\text{-}\pi$ symmetric Boolean functions as follows.

Definition 8 (π Symmetric Boolean function (πS)). Let $\pi \in \mathbb{S}_n$ be a permutation on n elements with order $o(\pi)$. A Boolean function f is π symmetric (in short, πS) if and only if for any $(x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n$,

$$f(\pi^k(x_1, x_2, \dots, x_n)) = f(x_1, x_2, \dots, x_n) \text{ for every } 1 \leq k \leq o(\pi)$$

where $\pi^k = \pi \circ \pi^{k-1}$ for $k > 1$ and $\pi^1 = \pi$. Therefore, πS Boolean functions have the same truth value for all vectors in every orbit obtained by the action of permutation group $P = \langle \pi \rangle$ on \mathbb{F}_2^n .

For $\pi = \sigma$, the σ S Boolean functions are known as Rotation Symmetric Boolean functions (in short, RotS). The σ S Boolean functions are very well studied functions in the literature [PQ99, CS02, SM08]. However, no study on ψ S Boolean functions is available in the literature. When n is power of 2, the class of σ S Boolean functions and the class of ψ S Boolean functions are same.

Definition 9 (2- π Symmetric Boolean function(2- π S)). Let $\pi \in \mathbb{S}_n$ be a permutation on n elements and \mathcal{O} be the set of all orbits due to the group action of $P = \langle \pi \rangle$ on \mathbb{F}_2^n . A Boolean function f is 2- π symmetric (in short, 2- π S) if and only if for every orbit $\mathcal{O} \in \mathcal{O}$ with a fixed representative element ν_o ,

$$f(\pi^{2i}(\nu_o)) = f(\nu_o); \quad f(\pi^{2i+1}(\nu_o)) = f(\nu_o) + 1 \text{ for every } 0 \leq i < \lfloor \frac{|\mathcal{O}|}{2} \rfloor.$$

Note that, if $|\mathcal{O}|$ is odd then $f(\pi^{|\mathcal{O}|-1}(\nu_o))$ can be any value from \mathbb{F}_2 .

Therefore, 2- π S Boolean functions have the alternative truth value for the lexicographically ordered vectors in every orbit obtained by the action of permutation group $P = \langle \pi \rangle$ on \mathbb{F}_2^n . As example, a 2-RotS Boolean function (i.e., 2- σ S) on $n = 5$ satisfies $f(00001) = f(00100) = f(10000)$ and $f(00010) = f(01000) = 1 + f(00001)$ for the orbit $\{00001, 00010, \dots, 10000\}$ with representative 00001.

A construction of a class of 2- σ S WPB Boolean functions is presented by Liu and Mesnager [LM19].

Proposition 1. [LM19] For a Boolean function $f \in \mathcal{B}_n$ with n is power of 2, if $f(x^2) = f(x) + 1$ holds for all $x \in \mathbb{F}_{2^n} \setminus \{0, 1\}$, then f is WPB.

The Boolean function proposed in Proposition 1 is 2- σ S. Since, n is the power of 2 in the construction, the cardinality of all orbits in $\mathbb{F}_{2^n} \setminus \{0, 1\}$ are even. Therefore, $f(x^2) = f(x) + 1, x \in \mathbb{F}_{2^n} \setminus \{0, 1\}$ is well defined and hence, the truth value 1 and 0 can be assigned alternatively to the half of the vectors in the each orbit. This can not be assigned when n is not a power of 2 as there are some orbits with cardinality odd and hence the $f(x^2) = f(x) + 1$ can not be defined. However, we have proposed an generalization of this concept to construct WAPB Boolean function on any n where n is a natural number in Section 4. When n is power of 2, the class of 2- σ S Boolean functions and the class of 2- ψ S Boolean functions are same.

2.2 Some Results on Krawtchouk Polynomials

We present some results on Krawtchouk polynomials and sequences which are useful for later results.

Definition 10 (Krawtchouk polynomial). For a positive integer n , the Krawtchouk polynomial [MS78, Page 151] of degree k is given by

$$K_k(x, n) = \sum_{j=0}^k (-1)^j \binom{x}{j} \binom{n-x}{k-j} \text{ for } k = 0, 1, \dots, n.$$

Some nice properties of the Krawtchouk polynomials from [DMS06, Proposition 4, Corollary 1] are presented in the following Proposition.

- Proposition 2.**
1. $K_0(l, n) = 1, K_1(l, n) = n - 2l$.
 2. $K_k(l, n) = (-1)^l K_{n-k}(l, n)$ (that implies, $K_{\frac{n}{2}}(l, n) = 0$ for n even and l odd).
 3. $K_k(l, n) = (-1)^k K_k(n-l, n)$, (that implies, $K_k(\frac{n}{2}, n) = 0$ for n even and k odd).
 4. For n odd, $|K_k(1, n)| \geq |K_k(l, n)|$ where $0 \leq k \leq n$ and $1 \leq l \leq n-1$,
 5. For n even, $|K_k(1, n)| \geq |K_k(l, n)|$ where $0 \leq k \leq n$ and $1 \leq l \leq n-1$ except $k = \frac{n}{2}$ or $l = \frac{n}{2}$.
 6. $(n-l)K_k(l+1, n) = (n-2k)K_k(l, n) - lK_k(l-1, n)$.

The following relations are derived from some results in [DMS06, GM22a],

Theorem 1 (Krawtchouk polynomials relations). For integers $n > 0, 0 \leq k \leq n$ and fixed $a \in \mathbb{F}_2^n$ such that $w_H(a) = \ell$, the following relations hold.

1. $\sum_{x \in \mathbb{E}_{k,n}} (-1)^{a \cdot x} = K_k(\ell, n).$
2. If $l_{a,b}(x) = a \cdot x + b$, where $a \in \mathbb{F}_2^n, b \in \mathbb{F}_2$, be an affine Boolean function then

$$w_{k,n}(l_{a,b}) = \frac{1}{2}(|\mathbb{E}_{k,n}| - (-1)^b K_k(\ell, n)).$$

Now we present some results on the sequences from Krawtchouk polynomial.

Lemma 2. Let n be a positive integer and $k \in [0, n]$. Then

1. $K_k(1, n) \geq 0$ if $k \leq \lfloor \frac{n}{2} \rfloor$ and $K_k(1, n) < 0$ if $k > \lfloor \frac{n}{2} \rfloor$. In particular, $K_k(1, n) = 0$ if n is even and $k = \frac{n}{2}$.
2. $K_k(2, n) \geq 0$ iff $k \leq \frac{n}{2} - \frac{\sqrt{n}}{2}$ or, $k \geq \frac{n}{2} + \frac{\sqrt{n}}{2}$.
In particular, $K_k(2, n) = 0$ if n is an even square integer and $k = \frac{n}{2} \pm \frac{\sqrt{n}}{2}$.
3. $K_k(l+1, n) + K_k(l, n) = 2K_k(l, n-1)$ and $K_k(l+1, n) - K_k(l, n) = -2K_{k-1}(l, n-1)$.
4. $K_k(l, n) + K_{k-1}(l, n) = K_k(l, n+1)$ and $K_k(l, n) - K_{k-1}(l, n) = K_k(l+1, n+1)$.

Proof. 1. The proof can be derived from the expression $K_k(1, n) = \binom{n-1}{k} - \binom{n-1}{k-1}$.

$$2. K_k(2, n) = \binom{n-2}{k} - 2\binom{n-2}{k-1} + \binom{n-2}{k-2} = \frac{(n-2)!((2k-n)^2-n)}{k!(n-k)!}.$$

Hence, $K_k(2, n) \geq 0 \iff (2k-n)^2 - n \geq 0 \iff (2k-n)^2 \geq n \iff 2k-n \geq \sqrt{n}$ or, $2k-n \leq -\sqrt{n} \iff k \geq \frac{n}{2} + \frac{\sqrt{n}}{2}$ or, $k \leq \frac{n}{2} - \frac{\sqrt{n}}{2}$. The particular case follows similarly.

3. We have,

$$\begin{aligned} K_k(l+1, n) + K_k(l, n) &= \sum_{j=0}^k (-1)^j \binom{l+1}{j} \binom{n-l-1}{k-j} + \sum_{j=0}^k (-1)^j \binom{l}{j} \binom{n-l}{k-j} \\ &= \sum_{j=0}^k (-1)^j \left[\binom{l+1}{j} \binom{n-l-1}{k-j} + \binom{l}{j} \binom{n-l}{k-j} \right] \\ &= \sum_{j=0}^k (-1)^j \left[\binom{l+1}{j} \binom{n-l-1}{k-j} + \binom{l}{j} \left(\binom{n-l-1}{k-j} + \binom{n-l-1}{k-j-1} \right) \right] \\ &= \sum_{j=0}^k (-1)^j \left[\left(\binom{l+1}{j} + \binom{l}{j} \right) \binom{n-l-1}{k-j} + \binom{l}{j} \binom{n-l-1}{k-j-1} \right] \\ &= \sum_{j=0}^k (-1)^j \left[\left(\binom{l+1}{j} + \binom{l}{j} \right) \binom{n-l-1}{k-j} + \sum_{j=1}^{k+1} (-1)^{j-1} \binom{l}{j-1} \binom{n-l-1}{k-j} \right] \\ &= \sum_{j=0}^k (-1)^j \left[\left(\binom{l+1}{j} + \binom{l}{j} \right) \binom{n-l-1}{k-j} + \sum_{j=0}^k (-1)^{j-1} \binom{l}{j-1} \binom{n-l-1}{k-j} \right] \\ &= \sum_{j=0}^k (-1)^j \left(\binom{l+1}{j} + \binom{l}{j} - \binom{l}{j-1} \right) \binom{n-l-1}{k-j} \\ &= 2 \sum_{j=0}^k (-1)^j \binom{l}{j} \binom{n-l-1}{k-j} = 2K_k(l, n-1). \end{aligned}$$

The second part of this Item be proved using the similar technique as above.

4. These can be proved by adding and subtracting the equalities in Item 3. □

We present the minimum of the sequence $K_k(l, n), 0 \leq l \leq n$ for a fixed k and n in the following theorem.

Theorem 2. 1. Let $n = 2m + 1$ be an odd integer for some $m \in \mathbb{Z}^+$. Then for $k \in [0, n]$

$$\min_{0 \leq l \leq n-1} K_k(l, n) = \begin{cases} K_k(n-1, n) = -K_k(1, n) & \text{for } k \in [0, \frac{n-1}{2}] \text{ and odd,} \\ K_k(1, n) & \text{for } k \in [\frac{n+1}{2}, n]. \end{cases}$$

In particular, $\min_{0 \leq l \leq n-1} K_m(l, n) = K_m(2, n) = -K_m(1, n)$.

2. Let $n = 2m$ be an even integer for some $m \in \mathbb{Z}^+$.

Then for $k \in [m+1, n]$ and even, $\min_{0 \leq l \leq n} K_k(l, n) = K_k(1, n)$. In particular,

(a) $\min_{0 \leq l \leq n} K_m(l, n) = K_m(2, n)$ if m is even;

(b) for $n \geq 10$, $\min_{0 \leq l \leq n} K_{m-1}(l, n) = K_{m-1}(2, n)$ if m is odd.

3. $\min_{0 \leq l \leq n} K_2(l, n) = -\lfloor \frac{n}{2} \rfloor$.

4. $\max_{0 \leq l \leq n} K_k(l, n) = K_k(0, n) = \binom{n}{k}$.

5. $K_k(n, n) = \begin{cases} \max_{0 \leq l \leq n} K_k(l, n) & \text{if } k \text{ is even;} \\ \min_{0 \leq l \leq n} K_k(l, n) & \text{if } k \text{ is odd.} \end{cases}$

Proof. 1. From Proposition 2[Item 4], for $k \in [0, n]$, we have $|K_k(1, n)| \geq |K_k(l, n)|$ for all $l \in [1, n-1]$.

For $k \in [\frac{n+1}{2}, n]$, $K_k(1, n) < 0$ (from Lemma 2[Item 1]), we have $\min_{1 \leq l \leq n-1} K_k(l, n) = K_k(1, n)$.

Further, k being odd in $[0, \frac{n-1}{2}]$, using Proposition 2[Item 3], we have $K_k(1, n) = -K_k(n-1, n)$. Hence, for $k \in [0, \frac{n-1}{2}]$, as $K_k(1, n) \geq 0$ (from Lemma 2[Item 1]), $\max_{1 \leq l \leq n-1} K_k(l, n) = K_k(1, n)$ i.e., $\min_{1 \leq l \leq n-1} K_k(l, n) = K_k(n-1, n)$.

As $K_k(0, n) = \binom{n}{k} > 0$, we have $\min_{0 \leq l \leq n-1} K_k(l, n) = \min_{1 \leq l \leq n-1} K_k(l, n)$ and the result of the first part of the Theorem.

As $n = 2m+1$, we have from Lemma 2[Item 3 and Item 1] that $K_m(2, n) + K_m(1, n) = 2K_m(1, n-1) = 0$. That implies, $K_m(2, n) = -K_m(1, n) \leq 0$.

As $K_m(0, n) = \binom{n}{m} > 0$, $\min_{0 \leq l \leq n} K_m(l, n) = \min_{1 \leq l \leq n-1} K_m(l, n) = K_m(2, n)$ (from Proposition 2[Item 4]).

Hence the second part is done.

2. As $k \geq m+1$, from Proposition 2[Item 5], we have $|K_k(1, n)| \geq |K_k(l, n)|$ for $l \in [1, n-1]$. As $K_k(1, n) \leq 0$ (from Lemma 2[Item 1]) and $K_k(0, n) = K_k(n, n) = \binom{n}{k} > 0$, $\min_{0 \leq l \leq n} K_k(l, n) = \min_{1 \leq l \leq n-1} K_k(l, n) = K_k(1, n)$.

Hence, the first part is done.

Further from Lemma 2[Item 2]), we have $K_m(2, n) \leq 0$ as $m = \frac{n}{2}$.

(a) Here, $m = \frac{n}{2}$ is even. We will show $|K_m(2, n)| \geq |K_m(l, n)|$ for $1 \leq l \leq n-1$. At first, we will use induction on l to show it for l even and $2 \leq l \leq \frac{n}{2}$.

For $l = 2$, it is already $|K_m(2, n)| = |K_m(l, n)|$.

Assume that $|K_m(2, n)| \geq |K_m(l, n)|$ for some even l and $2 \leq l \leq \frac{n}{2} - 2$. Then, we need to prove that $|K_m(2, n)| \geq |K_m(l+2, n)|$. From Proposition 2[Item 6], for $2 \leq l \leq \frac{n-4}{2} = \frac{n}{2} - 2$, we have

$$\begin{aligned} (n - (l+1))K_m(l+2, n) &= (n - 2m)K_m(l+1, n) - (l+1)K_m(l, n) \\ &= -(l+1)K_m(l, n), \text{ as } n = 2m \\ \implies |(n - l - 1)K_m(l+2, n)| &= |(l+1)K_m(l, n)| \\ \implies |K_m(l+2, n)| &= \frac{l+1}{n-l-1} |K_m(l, n)| \leq |K_m(l, n)| \leq |K_m(2, n)| \text{ as } \frac{l+1}{n-l-1} \leq 1. \end{aligned}$$

Therefore, $|K_m(2, n)| \geq |K_m(l, n)|$ for all $2 \leq l \leq \frac{n}{2}$ and even. From Proposition 2[Item 2], $K_m(l, n) = 0$ for l odd. Hence, $|K_m(2, n)| \geq |K_m(l, n)|$ for all $1 \leq l \leq \frac{n}{2}$. Further, from Proposition 2[Item 3], $K_m(l, n) = K_m(n-l, n)$ as m even. Hence, $|K_m(2, n)| \geq |K_m(l, n)|$ for all $1 \leq l \leq n-1$. Therefore, as $K_m(2, n) < 0$ and $K_m(0, n) = K_m(n, n) = \binom{n}{m} > 0$, $\min_{0 \leq l \leq n} K_m(l, n) = \min_{1 \leq l \leq n-1} K_m(l, n) = K_m(2, n)$ for m even.

- (b) Now $m = \frac{n}{2}$ is odd. At first, we will show that $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(l, n)|$ for $2 \leq l \leq \frac{n}{2}$ using induction on l .

For $l = 2$, it is already $|\mathbf{K}_{m-1}(2, n)| = |\mathbf{K}_{m-1}(l, n)|$.

We will prove it for $l = 3$ i.e. $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(3, n)|$. This is needed as we will see later that the induction has a recursion with depth two. It can be checked from Lemma 2[Item 2] that $\mathbf{K}_{m-1}(2, n) \leq 0$ for $n \geq 4$. To prove $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(3, n)|$, we need to show that $\mathbf{K}_{m-1}(2, n) \leq \mathbf{K}_{m-1}(3, n) \leq -\mathbf{K}_{m-1}(2, n)$ i.e., $\mathbf{K}_{m-1}(3, n) - \mathbf{K}_{m-1}(2, n) \geq 0$ and $\mathbf{K}_{m-1}(3, n) + \mathbf{K}_{m-1}(2, n) \leq 0$.

From Lemma 2[Item 3 and Item 1], we have $\mathbf{K}_{m-1}(3, n) + \mathbf{K}_{m-1}(2, n) = 2\mathbf{K}_{m-1}(2, n-1) = 2(2\mathbf{K}_{m-1}(1, n-2) - \mathbf{K}_{m-1}(1, n-1)) = -2\mathbf{K}_{m-1}(1, n-1) \leq 0$.

Similarly, $\mathbf{K}_{m-1}(3, n) - \mathbf{K}_{m-1}(2, n) = -2\mathbf{K}_{m-2}(2, n-1) \geq 0$ if $m-2 \geq \frac{n-1-\sqrt{n-1}}{2}$ i.e., if $n \geq 10$ (Lemma 2[Item 2]). Hence, $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(3, n)|$ if $n \geq 10$.

Assume that $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(l-1, n)|$ and $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(l, n)|$ for some $3 \leq l \leq \frac{n}{2} - 1$. Then, we need to prove that $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(l+1, n)|$.

Using Proposition 2[Item 6], we have

$$\begin{aligned} (n-l)\mathbf{K}_{m-1}(l+1, n) &= (n-2(m-1))\mathbf{K}_{m-1}(l, n) - l\mathbf{K}_{m-1}(l-1, n) \\ &= 2\mathbf{K}_{m-1}(l, n) - l\mathbf{K}_{m-1}(l-1, n) \\ \implies (n-l)|\mathbf{K}_{m-1}(l+1, n)| &\leq 2|\mathbf{K}_{m-1}(l, n)| + l|\mathbf{K}_{m-1}(l-1, n)| \leq (2+l)|\mathbf{K}_{m-1}(2, n)| \\ \implies |\mathbf{K}_{m-1}(l+1, n)| &\leq \frac{l+2}{n-l}|\mathbf{K}_{m-1}(2, n)| \\ \implies |\mathbf{K}_{m-1}(l+1, n)| &\leq |\mathbf{K}_{m-1}(2, n)| \quad \text{as } \frac{l+2}{n-l} \leq 1 \text{ for } 2 \leq l \leq \frac{n}{2} - 1. \end{aligned}$$

Hence, $|\mathbf{K}_{m-1}(2, n)| \geq |\mathbf{K}_{m-1}(l, n)|$ for $2 \leq l \leq \frac{n}{2}$.

Then $|\mathbf{K}_{m+1}(2, n)| \geq |\mathbf{K}_{m+1}(l, n)|$ for $2 \leq l \leq \frac{n}{2}$ as $|\mathbf{K}_{m+1}(l, n)| = |\mathbf{K}_{m-1}(l, n)|$ (from Proposition 2[Item 2]).

Now, we will show that $\min_{0 \leq l \leq n} \mathbf{K}_{m+1}(l, n) = \mathbf{K}_{m+1}(1, n)$. From Lemma 2[Item 3 and Item 1], we have

$\mathbf{K}_{m+1}(2, n) - \mathbf{K}_{m+1}(1, n) = -2\mathbf{K}_m(1, n-1) > 0$. Similarly, $\mathbf{K}_{m+1}(2, n) + \mathbf{K}_{m+1}(1, n) = 2\mathbf{K}_{m+1}(1, n-1) \leq 0$. That implies $-\mathbf{K}_{m+1}(1, n) \leq \mathbf{K}_{m+1}(2, n) < \mathbf{K}_{m+1}(1, n)$ i.e., $|\mathbf{K}_{m+1}(2, n)| \leq |\mathbf{K}_{m+1}(1, n)|$.

From Lemma 2[Item 1], we have $\mathbf{K}_{m+1}(1, n) \leq 0$. Hence $\min_{0 \leq l \leq n} \mathbf{K}_{m+1}(l, n) = \mathbf{K}_{m+1}(1, n)$.

3. We have

$$\mathbf{K}_k(l, n) = \sum_{j=0}^k (-1)^j \binom{l}{j} \binom{n-l}{k-j} = \sum_{j=0}^k \binom{l}{j} \binom{n-l}{k-j} - 2 \sum_{\substack{j=0 \\ j:\text{odd}}}^k \binom{l}{j} \binom{n-l}{k-j} = \binom{n}{k} - 2 \sum_{\substack{j=0 \\ j:\text{odd}}}^k \binom{l}{j} \binom{n-l}{k-j}. \quad (3)$$

Hence, $\mathbf{K}_2(l, n) = \binom{n}{2} - 2\binom{l}{1}\binom{n-l}{1} = \binom{n}{2} - 2l(n-l)$. For real value of l , the function $\mathbf{K}_2(l, n)$ has minima at $l = \frac{n}{2}$ as $\frac{d(\mathbf{K}_2(l, n))}{dl} = 4l - 2n = 0$ at $l = \frac{n}{2}$ and $\frac{d^2(\mathbf{K}_2(l, n))}{dl^2} = 4 > 0$. In our case, as $l = \mathbf{w}_H(a + \psi^{-1}(a))$ for $a \in \mathbb{F}_2^n$ is an integer, we have $\min_{0 \leq l \leq n} \mathbf{K}_2(l, n)$ is $\binom{n}{2} - 2(\frac{n}{2})^2 = -\frac{n}{2}$ when n is even. For n is odd, it can be checked that $\mathbf{K}_2(l, n)$ has minimum at $l = \frac{n-1}{2}$ and $l = \frac{n+1}{2}$ with value $\min_{0 \leq l \leq n} \mathbf{K}_2(l, n) = \binom{n}{2} - 2\frac{n-1}{2}\frac{n+1}{2} = -\frac{n-1}{2}$. Hence, combining both the cases, we have $\min_{0 \leq l \leq n} \mathbf{K}_2(l, n) = -\lfloor \frac{n}{2} \rfloor$.

4. From Equation 3, we have $\mathbf{K}_k(l, n) = \binom{n}{k} - 2 \sum_{\substack{j=0 \\ j:\text{odd}}}^k \binom{l}{j} \binom{n-l}{k-j} \leq \binom{n}{k} = \mathbf{K}_k(0, n)$. Hence, we are done.
5. From Proposition 2[Item 3], we have $\mathbf{K}_k(n, n) = (-1)^k \mathbf{K}_k(0, n)$. Hence, for k is even, $\mathbf{K}_k(n, n) = \mathbf{K}_k(0, n)$ is maximum in $\mathbf{K}_k(l, n), l \in [0, n]$. Like Equation 3, we have $\mathbf{K}_k(l, n) = 2 \sum_{\substack{j=0 \\ j:\text{even}}}^k \binom{l}{j} \binom{n-l}{k-j} - \binom{n}{k} \geq -\binom{n}{k} = -\mathbf{K}_k(0, n)$. Hence, for odd k , we have $\mathbf{K}_k(n, n) = -\mathbf{K}_k(0, n) \leq \mathbf{K}_k(l, n)$ for $l \in [0, n]$. Hence we are done. \square

3 Construction of WAPB Boolean functions using Group action

In this section we will present a construction of 2- π S WAPB Boolean functions using the group action of a cyclic permutation group $P = \langle \pi \rangle$. Let $P = \langle \pi \rangle$ be a cyclic subgroup of the symmetric group \mathbb{S}_n on n elements. Let the group action of P on \mathbb{F}_2^n partitions the set into g_n number of orbits. The orbit generated by $x \in \mathbb{F}_2^n$ is denoted as $O_\pi(x) = \{g(x) : g \in P\} = \{x, \pi(x), \pi^2(x), \dots, \pi^{l-1}(x)\}$ where l is the order of the permutation π . As $w_H(\pi^i(x)) = w_H(x)$ for $1 \leq i \leq l-1$, the group action P splits each $E_{k,n}$ into orbits and let $g_{k,n}$ be the number of orbits in $E_{k,n}$. Denote $\nu_{k,n,i}$ be the orbit representative of i -th orbit $E_{k,n}$ with some ordering. The construction of 2- π S WAPB Boolean functions is presented in Construction 1.

Construction 1 Construction of 2- π S WAPB Boolean function

Input: $\pi \in \mathbb{S}_n$

Output: A 2- π S WAPB Boolean function $f_\pi \in \mathcal{B}_n$

```

Initiate  $\text{supp}(f_\pi) = \phi$ 
 $t = 0$ 
for  $k \leftarrow 0$  to  $n$  do
  for  $i \leftarrow 1$  to  $g_{k,n}$  do
     $u = \nu_{k,n,i}$ ;  $l = |O_\pi(u)|$ 
    if  $l$  is even then
      for  $j \leftarrow 1$  to  $\frac{l}{2}$  do
         $\text{supp}(f_\pi).\text{append}(u)$ 
         $u \leftarrow \pi \circ \pi(u)$ 
      end for
    else
       $u = \pi^t(u)$ 
      for  $j \leftarrow 1$  to  $\lceil \frac{l-t}{2} \rceil$  do
         $\text{supp}(f_\pi).\text{append}(u)$ 
         $u \leftarrow \pi \circ \pi(u)$ 
      end for
      Update  $t \leftarrow 1 - t$ 
    end if
  end for
end for
return  $f_\pi$ 

```

Construction 1 ensures a balanced WAPB Boolean function. The binary variable t indicates whether the partially constructed is balanced (when $t = 0$) or having an extra 1 (when $t = 1$) during each iteration of orbits.

Example 1. Consider $n = 5$ and the permutation $\pi = \sigma$ is the rotation permutation. Then considering the orbits with representatives 00000, 00001, 00011, 00101, 00111, 01011, 01111, 11111, we have the resultant function $f_\sigma \in \mathcal{B}_5$ of Construction 1 as $\text{supp}(f_\sigma) = \{00000, 00010, 01000, 00011, 01100, 10001, 01010, 01001, 00111, 11100, 10011, 10110, 11010, 01111, 11101, 10111\}$. Hence, f_σ is a 2-RotS (i.e., 2- σ S) WAPB Boolean function.

Given a permutation $\pi \in \mathbb{S}_n$, let denote \mathcal{O}_e and \mathcal{O}_o be the set of orbits of even cardinality and odd cardinality of the group action $P = \langle \pi \rangle$ on \mathbb{F}_2^n respectively. Further, let denote \mathcal{O}_k be the set of orbits of cardinality k for $k \in [0, n]$. Here, \mathcal{O}_1 is the set of orbits of cardinality 1. Since every orbit in \mathcal{O}_1 contains exactly one element, abusing the notation, we also denote $\mathcal{O}_1 = \{x \in \mathbb{F}_2^n : x = \pi(x)\}$.

Proposition 3. For $\pi \in \mathbb{S}_n$ where $n \in \mathbb{Z}^+$, let $\mathcal{O}_1 = \{x \in \mathbb{F}_2^n : \pi(x) = x\}$ (i.e., the set of vectors in \mathbb{F}_2^n of orbit length 1). Then, for $c \in \mathbb{F}_2^n$ and $k \in [0, n]$,

1. $|\{x \in \mathbb{F}_2^n \setminus \mathcal{O}_1 : c \cdot (x + \pi(x)) = 1\}| = \begin{cases} 2^{n-1} & \text{if } c \in \mathbb{F}_2^n \setminus \mathcal{O}_1 \\ 0 & \text{if } c \in \mathcal{O}_1. \end{cases}$
2. $|\{x \in \mathbb{E}_{k,n} \setminus \mathcal{O}_1 : c \cdot (x + \pi(x)) = 1\}| = \frac{1}{2}(|\mathbb{E}_{k,n}| - \mathbf{K}_k(l, n))$, where $l = \mathbf{w}_H(c + \pi^{-1}(c))$.

Proof. Now for any $x = (x_1, x_2, \dots, x_n), c = (c_1, c_2, \dots, c_n) \in \mathbb{F}_2^n$,

$$c \cdot (x + \pi(x)) = c \cdot x + c \cdot \pi(x) = c \cdot x + \pi^{-1}(c) \cdot x = (c + \pi^{-1}(c)) \cdot x \quad (4)$$

Therefore, $c \cdot (x + \pi(x))$ is a linear Boolean function on n variables. Then $c \cdot (x + \pi(x))$ is the zero Boolean function if and only if $c = \pi^{-1}(c)$ i.e., $c \in \mathcal{O}_1$. Hence,

$$|\{x \in \mathbb{F}_2^n : c \cdot (x + \pi(x)) = 1\}| = \mathbf{w}_H(c \cdot (x + \pi(x))) = \begin{cases} 2^{n-1} & \text{if } c \in \mathbb{F}_2^n \setminus \mathcal{O}_1 \\ 0 & \text{if } c \in \mathcal{O}_1 \end{cases} \quad (5)$$

Further, if $x \in \mathcal{O}_1$, then $\pi(x) = x$ and that implies $c \cdot (x + \pi(x)) = 0$. Hence,

$$\begin{aligned} & |\{x \in \mathcal{O}_1 : c \cdot (x + \pi(x)) = 0\}| = |\mathcal{O}_1| \\ \implies & |\{x \in \mathcal{O}_1 : c \cdot (x + \pi(x)) = 1\}| = 0. \end{aligned} \quad (6)$$

Now combining Equation 5 and Equation 6 we have the desired result of Item 1

$$|\{x \in \mathbb{F}_2^n \setminus \mathcal{O}_1 : c \cdot (x + \pi(x)) = 1\}| = \begin{cases} 2^{n-1} & \text{if } c \in \mathbb{F}_2^n \setminus \mathcal{O}_1 \\ 0 & \text{if } c \in \mathcal{O}_1. \end{cases}$$

Moreover, as $c \cdot (x + \pi(x)) = (c + \pi^{-1}(c)) \cdot x$ is linear in $\mathbb{E}_{k,n}$, using Theorem 1, we have

$$|\{x \in \mathbb{E}_{k,n} : c \cdot (x + \pi(x)) = 1\}| = \mathbf{w}_{n,k}((c + \pi^{-1}(c)) \cdot x) = \frac{1}{2}(|\mathbb{E}_{k,n}| - \mathbf{K}_k(l, n))$$

where $l = \mathbf{w}_H(c + \pi^{-1}(c))$. If $x \in \mathcal{O}_1$ then $c \cdot (x + \pi(x)) = 0$. Hence,

$$|\{x \in \mathbb{E}_{k,n} \setminus \mathcal{O}_1 : c \cdot (x + \pi(x)) = 1\}| = \frac{1}{2}(|\mathbb{E}_{k,n}| - \mathbf{K}_k(l, n)).$$

□

Theorem 3. Let n be a positive integer and $\pi \in \mathbb{S}_n$. Then $\mathbf{NL}(f_\pi) \geq 2^{n-2} - \frac{|\mathcal{O}_o|}{2}$.

Proof. Here $\mathcal{O}_e, \mathcal{O}_o$ and \mathcal{O}_1 are the set of orbits of even cardinalty, odd cardinality and single elements of the group action $G = \langle \pi \rangle$ on \mathbb{F}_2^n respectively. Then the Walsh spectrum of f_π at $a \in \mathbb{F}_2^n$ is as follows.

$$W_{f_\pi}(a) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f_\pi(x) + a \cdot x} = \sum_{\mathcal{O} \in \mathcal{O}} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x) + a \cdot x} = \sum_{\mathcal{O} \in \mathcal{O}_e} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x) + a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_o} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x) + a \cdot x} \quad (7)$$

$$\begin{aligned} \sum_{\mathcal{O} \in \mathcal{O}_e} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x) + a \cdot x} &= \frac{1}{2} \left[\sum_{\mathcal{O} \in \mathcal{O}_e} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x) + a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_e} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(\pi(x)) + a \cdot \pi(x)} \right] \\ &= \frac{1}{2} \left[\sum_{\mathcal{O} \in \mathcal{O}_e} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x) + a \cdot x} + (-1)^{f_\pi(\pi(x)) + a \cdot \pi(x)} \right] \\ &= \frac{1}{2} \left[\sum_{\mathcal{O} \in \mathcal{O}_e} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x)} \left((-1)^{a \cdot x} - (-1)^{a \cdot \pi(x)} \right) \right] \quad (\text{as } f_\pi(\pi(x)) = 1 + f_\pi(x)). \end{aligned}$$

There are some vectors x in even orbits such that $((-1)^{a \cdot x} - (-1)^{a \cdot \pi(x)}) = 0$ i.e., $a \cdot (x + \pi(x)) = 0$. As these vectors contributes 0 to the sum, we now remove them in the equation. Hence, we have

$$\begin{aligned} \sum_{\mathbf{O} \in \mathcal{O}_e} \sum_{x \in \mathbf{O}} (-1)^{f_\pi(x) + a \cdot x} &= \frac{1}{2} \left[\sum_{\mathbf{O} \in \mathcal{O}_e} \sum_{x \in \mathbf{O}: a \cdot (x + \pi(x)) = 1} (-1)^{f_\pi(x)} ((-1)^{a \cdot x} - (-1)^{a \cdot \pi(x)}) \right] \\ &= \frac{1}{2} \left[\sum_{\mathbf{O} \in \mathcal{O}_e} \sum_{x \in \mathbf{O}: a \cdot (x + \pi(x)) = 1} 2 \times (-1)^{f_\pi(x) + a \cdot x} \right] = \sum_{\mathbf{O} \in \mathcal{O}_e} \sum_{x \in \mathbf{O}: a \cdot (x + \pi(x)) = 1} (-1)^{f_\pi(x) + a \cdot x} \end{aligned} \quad (8)$$

Similarly, $\sum_{\mathbf{O} \in \mathcal{O}_o} \sum_{x \in \mathbf{O}} (-1)^{f_\pi(x) + a \cdot x} = \sum_{\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1} \sum_{x \in \mathbf{O}} (-1)^{f_\pi(x) + a \cdot x} + \sum_{\mathbf{O} \in \mathcal{O}_1} \sum_{x \in \mathbf{O}} (-1)^{f_\pi(x) + a \cdot x}$

$$= \sum_{\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1} \sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} + \sum_{\mathbf{O} \in \mathcal{O}_1} (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} \quad (9)$$

where $\nu_{\mathbf{O}}$ is the orbit representative of the orbit \mathbf{O} . Then, for an orbit $\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1$, if $f(\pi^{|\mathbf{O}|-1}(\nu_{\mathbf{O}})) = 1 + f(\pi^{|\mathbf{O}|-2}(\nu_{\mathbf{O}})) = f(\nu_{\mathbf{O}})$, we have

$$\begin{aligned} \sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} &= \frac{1}{2} \left(\sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} + \sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} \right) \\ &= \frac{1}{2} \left(\sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} + \sum_{i=1}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} \right) + \frac{(-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}}}{2} \\ &= \frac{1}{2} \left(\sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} + \sum_{i=1}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}})) + a \cdot \pi^i(\nu_{\mathbf{O}})} + (-1)^{1 + f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} \right) \\ &\quad + \frac{(-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} - (-1)^{1 + f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}}}{2} \\ &= \frac{1}{2} \left(\sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}}))} \left((-1)^{a \cdot \pi^i(\nu_{\mathbf{O}})} - (-1)^{a \cdot \pi^{i+1}(\nu_{\mathbf{O}})} \right) \right) + (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} \\ &= \frac{1}{2} \left(\sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}}))} \left((-1)^{a \cdot \pi^i(\nu_{\mathbf{O}})} - (-1)^{a \cdot \pi^{i+1}(\nu_{\mathbf{O}})} \right) \right) + (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}}. \end{aligned} \quad (10)$$

If $f(\pi^{|\mathbf{O}|-1}(\nu_{\mathbf{O}})) = f(\pi^{|\mathbf{O}|-2}(\nu_{\mathbf{O}})) = 1 + f(\nu_{\mathbf{O}})$, consider $\hat{\nu}_{\mathbf{O}} = \pi^{-1}(\nu_{\mathbf{O}})$ as the representative element. We have now the earlier situation with the representative element $\hat{\nu}_{\mathbf{O}}$ i.e., $f(\pi^{|\mathbf{O}|-1}(\hat{\nu}_{\mathbf{O}})) = 1 + f(\pi^{|\mathbf{O}|-2}(\hat{\nu}_{\mathbf{O}})) = f(\hat{\nu}_{\mathbf{O}})$. Hence we will have Equation 10 for the representative element $\hat{\nu}_{\mathbf{O}}$. Hence, Equation 9 becomes

$$\begin{aligned} \sum_{\mathbf{O} \in \mathcal{O}_o} \sum_{x \in \mathbf{O}} (-1)^{f_\pi(x) + a \cdot x} &= \frac{1}{2} \sum_{\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1} \left(\sum_{i=0}^{|\mathbf{O}|-1} (-1)^{f_\pi(\pi^i(\nu_{\mathbf{O}}))} \left((-1)^{a \cdot \pi^i(\nu_{\mathbf{O}})} - (-1)^{a \cdot \pi^{i+1}(\nu_{\mathbf{O}})} \right) \right) + \sum_{\mathbf{O} \in \mathcal{O}_o} (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} \\ &= \frac{1}{2} \sum_{\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1} \sum_{x \in \mathbf{O}} (-1)^{f_\pi(x)} \left((-1)^{a \cdot x} - (-1)^{a \cdot \pi(x)} \right) + \sum_{\mathbf{O} \in \mathcal{O}_o} (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} \\ &= \frac{1}{2} \sum_{\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1} \sum_{x \in \mathbf{O}: a \cdot (x + \pi(x)) = 1} 2 \times (-1)^{f_\pi(x) + a \cdot x} + \sum_{\mathbf{O} \in \mathcal{O}_o} (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}} \\ &= \sum_{\mathbf{O} \in \mathcal{O}_o \setminus \mathcal{O}_1} \sum_{x \in \mathbf{O}: a \cdot (x + \pi(x)) = 1} (-1)^{f_\pi(x) + a \cdot x} + \sum_{\mathbf{O} \in \mathcal{O}_o} (-1)^{f_\pi(\nu_{\mathbf{O}}) + a \cdot \nu_{\mathbf{O}}}. \end{aligned} \quad (11)$$

Now substituting the values in Equation 8 and Equation 11 in Equation 7, we have

$$\begin{aligned}
W_{f_\pi}(a) &= \sum_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{O}_1} \sum_{x \in \mathcal{O}: a(x+\pi(x))=1} (-1)^{f_\pi(x)+a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_o} (-1)^{f_\pi(\nu_o)+a \cdot \nu_o} \\
\Rightarrow |W_{f_\pi}(a)| &\leq \left| \sum_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{O}_1} \sum_{x \in \mathcal{O}: a(x+\pi(x))=1} (-1)^{f_\pi(x)+a \cdot x} \right| + \left| \sum_{\mathcal{O} \in \mathcal{O}_o} (-1)^{f_\pi(\nu_o)+a \cdot \nu_o} \right| \\
&\leq \left| \sum_{\mathcal{O} \in \mathcal{O} \setminus \mathcal{O}_1} \sum_{x \in \mathcal{O}: a(x+\pi(x))=1} (-1)^{f_\pi(x)+a \cdot x} \right| + |\mathcal{O}_o| \\
\Rightarrow |W_{f_\pi}(a)| &\leq \begin{cases} 2^{n-1} + |\mathcal{O}_o| & \text{if } a \in \mathbb{F}_2^n \setminus \mathcal{O}_1 \\ |\mathcal{O}_o| & \text{if } a \in \mathcal{O}_1. \end{cases} \quad (\text{from Proposition 3}).
\end{aligned}$$

Hence, the nonlinearity of f_π satisfies

$$\begin{aligned}
\text{NL}(f_\pi) &= 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |W_{f_\pi}(a)| \geq 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} \{2^{n-1} + |\mathcal{O}_o|, |\mathcal{O}_o|\} = 2^{n-1} - 2^{n-2} - \frac{|\mathcal{O}_o|}{2} \\
\Rightarrow \text{NL}(f_\psi) &\geq 2^{n-2} - \frac{|\mathcal{O}_o|}{2}.
\end{aligned}$$

□

Theorem 4. Let $\pi \in \mathbb{S}_n$ for some $n \in \mathbb{Z}^+$. Then

$$\text{NL}_k(f_\pi) \geq \frac{1}{4} \left(\binom{n}{k} - 2|\mathcal{O}_{o,k}| + \min_{\substack{2 \leq l \leq n \\ l \text{ even}}} K_k(l, n) \right).$$

Proof. Let \mathcal{O}_k be the set of all orbits of the group action $G = \langle \pi \rangle$ on $\mathbb{E}_{k,n}$. Let $\mathcal{O}_{e,k}$, $\mathcal{O}_{o,k}$ and $\mathcal{O}_{1,k}$ be the set of all orbits in \mathcal{O}_k of cardinality even, odd and one respectively. The restricted Walsh spectrum of f_π at $a \in \mathbb{F}_2^n$ is

$$W_{f_\pi,k}(a) = \sum_{x \in \mathbb{E}_{k,n}} (-1)^{f_\pi(x)+a \cdot x} = \sum_{\mathcal{O} \in \mathcal{O}_{e,k}} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x)+a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_{o,k}} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x)+a \cdot x}. \quad (12)$$

Following the similar process in the proof of Theorem 3, we have

$$\sum_{\mathcal{O} \in \mathcal{O}_{e,k}} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x)+a \cdot x} = \sum_{\mathcal{O} \in \mathcal{O}_{e,k}} \sum_{\substack{x \in \mathcal{O} \\ a \cdot (x+\pi(x))=1}} (-1)^{f_\pi(x)+a \cdot x} \quad (13)$$

and

$$\sum_{\mathcal{O} \in \mathcal{O}_{o,k}} \sum_{x \in \mathcal{O}} (-1)^{f_\pi(x)+a \cdot x} = \sum_{\mathcal{O} \in \mathcal{O}_{o,k} \setminus \mathcal{O}_{1,k}} \sum_{\substack{x \in \mathcal{O} \\ a \cdot (x+\pi(x))=1}} (-1)^{f_\pi(x)+a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_{o,k}} (-1)^{f_\pi(\nu_o)+a \cdot \nu_o}, \quad (14)$$

where ν_o is the orbit representative of the orbit \mathcal{O} . Hence, substituting the values in Equation 13 and Equation 14 in Equation 12, we have

$$\begin{aligned}
W_{f_\pi,k}(a) &= \sum_{\mathcal{O} \in \mathcal{O}_{e,k}} \sum_{\substack{x \in \mathcal{O} \\ a \cdot (x+\pi(x))=1}} (-1)^{f_\pi(x)+a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_{o,k} \setminus \mathcal{O}_{1,k}} \sum_{\substack{x \in \mathcal{O} \\ a \cdot (x+\pi(x))=1}} (-1)^{f_\pi(x)+a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_{o,k}} (-1)^{f_\pi(\nu_o)+a \cdot \nu_o} \\
&= \sum_{\mathcal{O} \in \mathcal{O}_k \setminus \mathcal{O}_{1,k}} \sum_{\substack{x \in \mathcal{O} \\ a \cdot (x+\pi(x))=1}} (-1)^{f_\pi(x)+a \cdot x} + \sum_{\mathcal{O} \in \mathcal{O}_{o,k}} (-1)^{f_\pi(\nu_o)+a \cdot \nu_o} \\
\Rightarrow |W_{f_\pi,k}(a)| &\leq \left| \sum_{\mathcal{O} \in \mathcal{O}_k \setminus \mathcal{O}_{1,k}} \sum_{\substack{x \in \mathcal{O} \\ a \cdot (x+\pi(x))=1}} (-1)^{f_\pi(x)+a \cdot x} \right| + \left| \sum_{\mathcal{O} \in \mathcal{O}_{o,k}} (-1)^{f_\pi(\nu_o)+a \cdot \nu_o} \right| \\
&\leq \frac{1}{2} (|\mathbb{E}_{k,n}| - K_k(l, n)) + |\mathcal{O}_{o,k}|
\end{aligned}$$

Hence, the nonlinearity of f_π satisfies

$$\begin{aligned} \text{NL}_k(f_\pi) &= \frac{|\mathbf{E}_{k,n}|}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n} |\mathcal{W}_{f_\pi,k}(a)| \geq \frac{|\mathbf{E}_{k,n}|}{2} - \frac{1}{4} \max_{a \in \mathbb{F}_2^n} (|\mathbf{E}_{k,n}| - \mathbf{K}_k(l, n) + 2|\mathcal{O}_{o,k}|) \\ \implies \text{NL}_k(f_\pi) &\geq \frac{1}{4} \left(|\mathbf{E}_{k,n}| - 2|\mathcal{O}_{o,k}| + \min_{a \in \mathbb{F}_2^n} \mathbf{K}_k(l, n) \right) = \frac{1}{4} \left(\binom{n}{k} - 2|\mathcal{O}_{o,k}| + \min_{0 \leq l \leq n} \mathbf{K}_k(l, n) \right). \end{aligned}$$

Further, $l = \mathbf{w}_H(a + \pi^{-1}(a))$ is always even as $\mathbf{w}_H(a) = \mathbf{w}_H(\pi^{-1}(a))$. Hence, we have

$$\text{NL}_k(f_\pi) \geq \frac{1}{4} \left(\binom{n}{k} - 2|\mathcal{O}_{o,k}| + \min_{\substack{0 \leq l \leq n \\ l \text{ even}}} \mathbf{K}_k(l, n) \right).$$

Using Theorem 2[Item 4], we have further,

$$\text{NL}_k(f_\pi) \geq \frac{1}{4} \left(\binom{n}{k} - 2|\mathcal{O}_{o,k}| + \min_{\substack{2 \leq l \leq n \\ l \text{ even}}} \mathbf{K}_k(l, n) \right).$$

□

In the following sections we will study 2- π S WAPB Boolean functions for two different permutations $\psi, \sigma \in \mathbb{S}_n$ for any positive integer n . Here, σ is the rotation permutation and ψ is defined in Section 4. Both functions generalize the Liu-Mesnager construction [LM19] of WPB Boolean functions on $n = 2^m$ variables.

4 Construction and Study of 2- ψ S WAPB Boolean functions

In this section, we present a class of 2- ψ S WAPB Boolean function which is a special case of the construction presented in Section 3. This construction extends the idea of Liu-Mesnager construction [LM19] to generate WAPB Boolean functions. As Liu-Mesnager construction outputs a WPB Boolean function, the form of n (the number of variable) needs to be a power of 2. However, in our case, the number of variables n can be any positive integer for generating a WAPB Boolean functions.

Let n be a positive integer with binary representation as $n = n_1 + n_2 + \dots + n_w$ as defined in Equation 1. We denote $\mathbf{w}_H(n) = w$ i.e., the number of 1's in the binary representation of n . For $x = (x_1, x_2, \dots, x_n)$, we have

$$\psi(x) = (\sigma_{n_1}(x_1, \dots, x_{n_1}), \sigma_{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}), \dots, \sigma_{n_w}(x_{n-n_w+1}, \dots, x_n)) \quad (15)$$

where σ_{n_i} is the rotation permutation on n_i elements. Here, $\text{ord}(\psi) = 2^{a_w} = n_w$. Hence, the cardinality of orbits obtained due the action of $P = \langle \psi \rangle$ on \mathbb{F}_2^n are of power of 2 i.e., $|O_\psi(x)| = 2^l$ where $0 \leq l \leq a_w$ for $x \in \mathbb{F}_2^n$. Hence, there are some orbits of cardinality 1 and the rest are of even cardinality.

Lemma 3. *Let n be a positive integer and $\psi \in \mathbb{S}_n$ as in Equation 2. Then there are 2^w orbits of cardinality 1 where $w = \mathbf{w}_H(n)$.*

Proof. For a vector $x \in \mathbb{F}_2^n$ is having an orbit of cardinality 1 i.e., $|O_\psi(x)| = 1$ if and only if the coordinates of x present in the cycles are of same value i.e.,

$$x_1 = x_2 = \dots = x_{n_1}; \quad x_{n_1+1} = x_{n_1+2} = \dots = x_{n_1+n_2}; \quad \dots, \quad x_{n-n_w+1} = x_{n-n_w+2} = \dots = x_n. \quad (16)$$

As each partition of coordinates can be either 0 or 1, there are 2^w vectors x in \mathbb{F}_2^n satisfying Equation 16 and hence $|O_\psi(x)| = 1$. □

Since every orbit contains the vectors of same weight, we denote the weight of an orbit is the weight of vectors in the orbit i.e., $w_H(O_\psi(x)) = w_H(x)$ for $x \in \mathbb{F}_2^n$. Further, for $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n) \in \mathbb{F}_2^n$, we say y covers x (i.e., $x \preceq y$), if $x_i \leq y_i, \forall 1 \leq i \leq n$ i.e., $y_i = 1$ if $x_i = 1, \forall 1 \leq i \leq n$. Similarly, given two positive integers n and k with binary representation $n = 2^{a_1} + 2^{a_2} + \dots + 2^{a_w}$ and $k = 2^{b_1} + 2^{b_2} + \dots + 2^{b_t}$, we denote $k \preceq n$ if $\{b_1, b_2, \dots, b_t\} \subseteq \{a_1, a_2, \dots, a_w\}$.

Lemma 4. *Let n be a positive integer and $\psi \in \mathbb{S}_n$ as in Equation 2. For $k \in [0, n]$, the number of orbits of weight k and cardinality 1 is 1 if $k \preceq n$, otherwise it is 0.*

Proof. Let $k = 2^{b_1} + 2^{b_2} + \dots + 2^{b_t}$ where $0 \leq b_1 < b_2 < \dots < b_t$.

Case I: Let $k \preceq n$ i.e., $\{b_1, b_2, \dots, b_t\} \subseteq \{a_1, a_2, \dots, a_w\}$. Since the only way of writing k as sum of powers of 2 is $k = 2^{b_1} + 2^{b_2} + \dots + 2^{b_t}$ and satisfying the condition in Equation 16, there is only one vector x with $w_H(x) = k$ and $|O_\psi(x)| = 1$. In this case, the coordinates of x in the partitions of cardinality $2^{b_1}, 2^{b_2}, \dots, 2^{b_t}$ are having value 1 and other coordinates have value 0.

Case II: Let $k \not\preceq n$, then $\{b_1, b_2, \dots, b_t\} \not\subseteq \{a_1, a_2, \dots, a_w\}$. Therefore, if $w_H(x) = k$, the nonzero coordinates of x can not be partitioned of (distinct) sizes from the set $\{2^{a_1}, 2^{a_2}, \dots, 2^{a_w}\}$. As a result, the coordinates of x will not satisfy the Equation 16. Hence, $|O_\psi(x)| > 1$. Hence, in this case there is no orbit of weight k and cardinality 1. \square

Let denote \mathcal{O} be the set of all orbits due the action of ψ on \mathbb{F}_2^n . Further, we denote \mathcal{O}_o be the set of orbits of odd cardinality (i.e., here 1) and \mathcal{O}_e be the set of orbits of even cardinality. Since the cardinality of all orbits of cardinality odd is 1, abusing the notation, we also denote \mathcal{O}_o as the set of all vectors belonging in the orbits of odd cardinality. Hence from Equation 16, $\mathcal{O}_o = \{(x_1, x_2, \dots, x_n) \in \mathbb{F}_2^n : x_1 = x_2 = \dots = x_n; x_{n_1+1} = x_{n_1+2} = \dots = x_{n_1+n_2}; \dots; x_{(n-n_w)+1} = x_{(n-n_w)+2} = \dots = x_n\}$. For example, if $n = 6$, there are there are $2^{w_H(6)} = 2^2 = 4$ orbits of weight 1 and $\mathcal{O}_o = \{000000, 000011, 111100, 111111\}$.

By choosing such permutation ψ for Construction 1, we have every slice $E_{k,n}$, $0 \leq k \leq n$, contains at most one orbit of odd cardinality (and i.e., 1). Therefore, it becomes easy to construct 2- ψ S WAPB Boolean functions as other orbits are of even cardinality. Hence, we have the following result.

Proposition 4. *Let n be a positive integer and $\psi \in \mathbb{S}_n$ as in Equation 2. For a Boolean function $f_\psi \in \mathcal{B}_n$, if $f_\psi(\psi(x)) = 1 + f_\psi(x)$ holds for all $x \in \mathbb{F}_2^n \setminus \mathcal{O}_o$ where \mathcal{O}_o is the set of vectors whose orbit cardinality is 1, then f_ψ is WAPB.*

Hence, when $n = 2^m$, a power of 2, $\psi = \sigma$ and Construction 1 on input $\psi \in \mathbb{S}_n$ results the 2-RotS WPB Boolean function by Liu and Mesnager [LM19]. A simplified version of Construction 1 is presented in Construction 4 for input ψ .

Construction 2 Construction of 2- ψ S WAPB Boolean function using $\psi \in \mathbb{S}_n$

Input: $\psi \in \mathbb{S}_n$ as in Equation 2

Output: A 2- ψ S WAPB Boolean function $f_\psi \in \mathcal{B}_n$

For every orbit \mathcal{O} in \mathbb{F}_2^n due to the action of $P = \langle \psi \rangle$, do the following:

if $|\mathcal{O}|$ is even **then**

f satisfies $f_\psi(\psi(x)) = 1 + f(x)$ for $x \in \mathcal{O}$

end if

if $|\mathcal{O}| = 1$ **then**

 assign $f_\psi(x) = 0$ or 1 to make f balanced.

end if

return f_ψ

[PM: For construction 2, I propose the following modifications:

- in input we add a representative ν_i
- in input we add a binary vector v of length the number of orbits.
- then, for every orbit f takes the value v_i on ν_i , and we keep " f satisfies $f_\psi(\psi(x)) = 1 + f(x)$ for $x \in \mathcal{O}$ "
- we withdraw the last part, forcing f_ψ to be balanced.

The advantage of the extra inputs would be to define more easily each function later on (if we use an order to list the representatives, we can identify a function in n variables only from the the vector v). Regarding the balancedness, WAPB functions are not required to be balanced, so we would have a more general description (I do not think the balancedness is used in the proofs after). We would make a remark on the restriction that is sufficient to be balanced, or even than the weight of f is determined by the weight of v restricted to the orbits of size 1.]

Theorem 5. *The number of orbits generated due the action of ψ on \mathbb{F}_2^n is*

$$g_n = \frac{1}{n_w} \sum_{k=1}^{n_w} 2^{\gcd(n_1, k) + \gcd(n_2, k) + \dots + \gcd(n_w, k)}.$$

Proof. As $\text{ord}(\psi) = 2^{a_w} = n_w$, let denote $G = \langle \psi \rangle = \{\psi_n^1, \psi_n^2, \dots, \psi_n^{n_w}\}$ where $\psi_n^1 = \psi$ and $\psi_n^i = \psi \circ \psi_n^{i-1}$ for $i \geq 2$. From the disjoint cycle form of ψ as in Equation 15, we have

$$\psi(x) = (\sigma_{n_1}(x_1, \dots, x_{n_1}), \sigma_{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}), \dots, \sigma_{n_w}(x_{n-n_w+1}, \dots, x_n))$$

where σ_{n_i} is the rotation permutation on n_i elements. Hence, we denote, $\psi_n = (\sigma_{n_1}, \sigma_{n_2}, \dots, \sigma_{n_w})$ and for positive integers k , we have $\psi_n^k = (\sigma_{n_1}^k, \sigma_{n_2}^k, \dots, \sigma_{n_w}^k)$.

Now to apply Burnside's lemma, for every $k \in \{1, 2, \dots, n_w\}$, we need to compute the number of fixed vectors $z \in \mathbb{F}_2^n$ by ψ_n^k i.e., $\psi_n^k(z) = z$. That is, for every $k \in \{1, 2, \dots, n_w\}$, we need to compute the number of vectors $z \in \mathbb{F}_2^n$ such that $\rho_{n_1}^k(z_1) = z_1, \rho_{n_2}^k(z_2) = z_2, \dots, \rho_{n_w}^k(z_w) = z_w$ where $z = (z_1, z_2, \dots, z_w)$ and $z_1 \in \mathbb{F}_2^{n_1}, z_2 \in \mathbb{F}_2^{n_2}, \dots, z_w \in \mathbb{F}_2^{n_w}$.

Here, the number of permutation cycles in $\sigma_{n_i}^k = \gcd(n_i, k)$ for $1 \leq i \leq w$ and $1 \leq k \leq n_w$. So, the length of each permutation cycle in $\sigma_{n_i}^k$ is $\frac{n_i}{\gcd(n_i, k)}$. Therefore, the total number of permutation cycles in ψ_n^k is

$$\gcd(n_1, k) + \gcd(n_2, k) + \dots + \gcd(n_w, k).$$

As every permutation cycle fixes all 0's or all 1's, each permutation cycle has two choices. $\sigma_{n_i}^k$ fixes $2^{\gcd(n_i, k)}$ number of $z_i \in \mathbb{F}_2^{n_i}$. Therefore, ψ_n^k fixes $2^{\gcd(n_1, k) + \gcd(n_2, k) + \dots + \gcd(n_w, k)}$ number of $z \in \mathbb{F}_2^n$. Hence, by using the Burnside Lemma, the number of orbits is

$$g_n = \frac{1}{n_w} \sum_{\pi \in G} |\text{fix}_{\mathbb{F}_2^n}(\pi)| = \frac{1}{n_w} \sum_{k=1}^{n_w} 2^{\gcd(n_1, k) + \gcd(n_2, k) + \dots + \gcd(n_w, k)}.$$

□

The representative of each orbit can be assigned 0 or 1 and accordingly other vectors in the orbit are assigned. Hence, there are 2^{g_n} WAPB 2- ψ S Boolean functions on n variables. Further, we can count the number of balanced WAPB 2- ψ S Boolean functions on n variables. There are 2^w many orbits of cardinality 1 and remaining $g_n - 2^w$ orbits are having cardinality even. Further, 2^{w-1} orbits from the 2^w orbits of cardinality 1 are to be assigned 1 to make f_ψ balanced. Hence $\binom{2^w}{2^{w-1}} \times 2^{g_n - 2^w}$ balanced WAPB 2- ψ S Boolean functions can be generated using Construction 4. Now we will study some cryptographic properties of the function $f_\psi \in \mathcal{B}_n$.

Theorem 6. *Let $\psi \in \mathbb{S}_n$ as in Equation 2 with $w_H(n) = w$. Then $\text{NL}(f_\psi) \geq 2^{n-2} - 2^{w-1}$.*

Proof. As $w_H(n) = w$, from Lemma 3 there are 2^w orbits with cardinality 1 and remaining orbits are of even cardinality (i.e., $|\mathcal{O}_0| = |\mathcal{O}_1| = 2^w$). Then from Theorem 3, the nonlinearity bound of f_ψ is

$$\text{NL}(f_\psi) \geq 2^{n-2} - \frac{|\mathcal{O}_0|}{2} = 2^{n-2} - 2^{w-1}.$$

□

In the following table we have presented the maximum and minimum nonlinearity among all f_ψ for the number of variables $n = \{4, 5, \dots, 10\}$ along with the upperbound of balanced Boolean functions and lowerbound of f_ψ as per Theorem 6. We have searched all such Boolean functions for $n \leq 6$ and from 2^{20} randomly chosen such Boolean functions for $n > 6$.

n	4	5	6	7	8	9	10
Number of functions	$2^4 \times \binom{2}{1} = 2^5$	$2^8 \times \binom{4}{2} = 3 \times 2^9$	$2^{18} \times \binom{4}{2} = 3 \times 2^{19}$	$2^{36} \times \binom{8}{4} = 35 \times 2^{37}$	$2^{34} \times \binom{2}{1} = 2^{35}$	$2^{68} \times \binom{4}{2} = 3 \times 2^{69}$	$2^{138} \times \binom{4}{2} = 3 \times 2^{139}$
Max Nonlinearity	4	12	26	56	116	236	480
% functions at max nl	100	22.917	0.651042	0.304318	0.008297	0.072575	0.013638
Nonlinearity upper bound	4	12	26	56	116	240	492
Min Nonlinearity	4	6	14	28	64	192	328
% functions at min nl	100	4.17	0.260417	0.014687	0.006199	0.000191	2^{-20}
Nonlinearity lower bound	3	6	14	28	63	144	254

Now we will study the weightwise nonlinearity of f_ψ .

Theorem 7. Let $n \geq 2$ be an positive integer and $\psi \in \mathbb{S}_n$ as in Equation 2. Then

$$\text{NL}_k(f_\psi) \geq \begin{cases} \frac{1}{4} \left(\binom{n}{k} + \min_{\substack{2 \leq l \leq n \\ l \text{ even}}} K_k(l, n) \right) & \text{if } k \not\leq n \\ \frac{1}{4} \left(\binom{n}{k} + \min_{\substack{2 \leq l \leq n \\ l \text{ even}}} K_k(l, n) - 2 \right) & \text{if } k \leq n. \end{cases}$$

Proof. From Lemma 4, we have $|\mathcal{O}_{o,k}| = |\mathcal{O}_{1,k}| = 1$ if $k \leq n$ else it is 0. Hence from Theorem 4, we have

$$\text{NL}_k(f_\pi) \geq \begin{cases} \frac{1}{4} \left(\binom{n}{k} + \min_{\substack{2 \leq l \leq n \\ l \text{ even}}} K_k(l, n) \right) & \text{if } k \not\leq n \\ \frac{1}{4} \left(\binom{n}{k} + \min_{\substack{2 \leq l \leq n \\ l \text{ even}}} K_k(l, n) - 2 \right) & \text{if } k \leq n. \end{cases}$$

□

Corollary 1. Let $n \geq 2$ be an positive integer and $\psi \in \mathbb{S}_n$ as in Equation 2. If k is even or, n is odd then

$$\text{NL}_k(f_\psi) \geq \begin{cases} \frac{1}{4} \left(\binom{n}{k} + \min_{\substack{2 \leq l \leq n-1 \\ l \text{ even}}} K_k(l, n) \right) & \text{if } k \not\leq n \\ \frac{1}{4} \left(\binom{n}{k} + \min_{\substack{2 \leq l \leq n-1 \\ l \text{ even}}} K_k(l, n) - 2 \right) & \text{if } k \leq n. \end{cases}$$

Otherwise (i.e., k is odd and n is even), $\text{NL}_k(f_\psi) \geq 0$. [DKD: need to something for this case]

Proof. From Theorem 2[Item 5], if k is even then $\max_{0 \leq l \leq n} K_k(l, n) = K_k(n, n)$. Further, if n is odd then $K_k(n, n)$ is not included in the minimum finding. Hence we are done for the case k is even or, n is odd.

If k is odd and n is even, then $\min_{\substack{2 \leq l \leq n \\ l \text{ even}}} K_k(l, n) = K_k(n, n) = -\binom{n}{k}$. Further, in this case $k \not\leq n$. Hence, $NL_k(f_\psi) \geq 0$, which is infact always true. \square

$\begin{matrix} & k \\ n \end{matrix}$	2	3	4	5	6	7	8	9	10
	n-2	n-3	n-4	n-5	n-6	n-7	n-8	n-9	n-10
15	24 0	0 45	330 0	0 500	1215 0	0 1506	- -	- -	- -
16	28 8	0 0	443 228	0 0	1931 1502	0 0	3003 -	- -	- -
17	32 0	0 60	580 0	0 910	3003 0	0 4004	5720 0	- -	- -
18	36 8	0 0	750 340	0 0	4550 3094	0 0	10725 9724	0 0	- -
19	40 0	0 76	950 0	0 1530	6650 0	0 9282	18343 0	0 21879	- -
20	45 10	0 0	1190 484	0 0	9524 5814	0 0	30719 25194	0 0	43758 43758

Table 1. A lower bound of $NL_k(f_\psi)$ as per Theorem 7

Now substituting the results of Theorem 2[Item 1 and Item 2] in Corollary 7, we have nonlinearity bounds for some k .

Theorem 8. 1. If n is odd then

$$NL_k(f_\psi) \geq \begin{cases} \frac{1}{2} \binom{n-1}{k-1} & \text{if } k \in [0, \frac{n-1}{2}], \text{ odd and } k \not\leq n; \\ \frac{1}{2} \left(\binom{n-1}{k-1} - 1 \right) & \text{if } k \in [0, \frac{n-1}{2}], \text{ odd and } k \leq n; \\ \frac{1}{2} \binom{n-1}{k} & \text{if } k \in [\frac{n+1}{2}, n] \text{ and } k \not\leq n; \\ \frac{1}{2} \left(\binom{n-1}{k} - 1 \right) & \text{if } k \in [\frac{n+1}{2}, n] \text{ and } k \leq n. \end{cases}$$

2. If n is even and $k \in [\frac{n}{2} + 1, n]$ is even then

$$NL_k(f_\psi) \geq \begin{cases} \frac{1}{2} \binom{n-1}{k} & \text{if } k \not\leq n; \\ \frac{1}{2} \left(\binom{n-1}{k} - 1 \right) & \text{if } k \leq n. \end{cases}$$

As $E_{\lfloor \frac{n}{2} \rfloor, n}$ is a largest slice among all slices $E_{k, n}, k \in [0, n]$, studying the nonlinearity at this domain is useful for cryptographic purpose. The cipher FLIP also uses the domain $E_{\frac{n}{2}, n}$ for its design. We have the nonlinearity bounds for the largest slice (i.e., $k = \lfloor \frac{n}{2} \rfloor$) in the following Theorem.

Theorem 9. 1. If $n = 2m + 1$ is odd then

$$\text{NL}_m(f_\psi) \geq \begin{cases} \frac{1}{2} \binom{n-1}{m-1} & \text{if } m \not\leq n \text{ (i.e., } n \text{ is not of the form } 2^t - 1); \\ \frac{1}{2} \left(\binom{n-1}{m-1} - 1 \right) & \text{if } m \leq n \text{ (i.e., } n \text{ is of the form } 2^t - 1). \end{cases}$$

2. If $n = 2m$ is even then

(a) $\text{NL}_m(f_\psi) \geq \binom{n-2}{m}$ if m is even.

(b) $\text{NL}_{m-1}(f_\psi) \geq \frac{1}{2} \left(\binom{n-2}{m-1} + \binom{n-2}{m-3} \right)$ if m is odd and $n \geq 10$.

$\begin{smallmatrix} & k \\ n & \end{smallmatrix}$	2	3	4	5	6	7	8	9	10
	n-2	n-3	n-4	n-5	n-6	n-7	n-8	n-9	n-10
15	24 0	0 45	330 0	0 500	1215 0	0 1506	- -	- -	- -
16	28 8	0 0	443 228	0 0	1931 1502	0 0	3003 -	- -	- -
17	32 0	0 60	580 0	0 910	3003 0	0 4004	5720 0	- -	- -
18	36 8	0 0	750 340	0 0	4550 3094	0 0	10725 9724	0 0	- -
19	40 0	0 76	950 0	0 1530	6650 0	0 9282	18343 0	0 21879	- -
20	45 10	0 0	1190 484	0 0	9524 5814	0 0	30719 25194	0 0	43758 43758

Table 2. A lower bound of $\text{NL}_k(f_\psi)$ as per Theorem 7

Now we will study the restricted nonlinearity $\text{NL}_2(f_\psi)$.

Denote $\text{NL}_k^n = \min\{\text{NL}_k^n(f_\psi) | f_\psi \in \mathcal{B}_n \text{ constructed as in Proposition 4}\}$.

Theorem 10. Let $n \geq 2$ be an positive integer and $\psi \in \mathbb{S}_n$ as in Equation 2. Then

$$\text{NL}_2(f_\psi) \geq \begin{cases} \frac{n(n-2)}{8} & \text{if } n \text{ is even} \\ \lfloor \frac{(n-1)^2}{8} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Moreover, if $n = 2^m$ for $m \geq 1$, $\frac{n(n-2)}{8} \leq \text{NL}_2^n \leq \text{NL}_4^{2n} \leq \text{NL}_8^{2^2n} \leq \dots \leq \text{NL}_{2^{i+1}}^{2^i n} \leq \dots$,
and if $n = 2^{a_w} + 2^{a_{w-1}} + \dots + 2^{a_1}$ for $a_w > a_{w-1} > \dots > a_1 \geq 0$ as defined in 1,

$$\left\lfloor \frac{(n+1)(n-4) + n}{8} \right\rfloor \leq \text{NL}_2^n \leq \text{NL}_4^{2n} \leq \text{NL}_8^{2^2n} \leq \dots \leq \text{NL}_{2^{i+1}}^{2^i n} \leq \dots.$$

Proof. From Theorem 2, we have $\min_{0 \leq l \leq n} K_2(l, n) = -\lfloor \frac{n}{2} \rfloor$. Using it in Theorem 7 we have,

$$\text{NL}_2(f_\psi) \geq \begin{cases} \frac{1}{4} \left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor \right) & \text{if } 2 \not\leq n \\ \frac{1}{4} \left(\binom{n}{2} - \lfloor \frac{n}{2} \rfloor - 2 \right) & \text{if } 2 \leq n. \end{cases}$$

Therefore, for $2 \nmid n$, we have $\text{NL}_2(f_\psi) \geq \begin{cases} \frac{n(n-2)}{8} & \text{if } n \text{ is even (i.e., } n = 4t \text{ form)} \\ \frac{(n-1)^2}{8} & \text{if } n \text{ is odd (i.e., } n = 4t + 1 \text{ form)}, \end{cases}$

and for $2 \mid n$, we have $\text{NL}_2(f_\psi) \geq \begin{cases} \frac{n(n-2)}{8} - \frac{1}{2} & \text{if } n \text{ is even (i.e., } n = 4t + 2 \text{ form)} \\ \frac{(n-1)^2}{8} - \frac{1}{2} & \text{if } n \text{ is odd (i.e., } n = 4t + 3 \text{ form)}. \end{cases}$

We can check that for n even, $\frac{n(n-2)}{8}$ is always an integer and for n odd, $\frac{(n-1)^2}{8}$ is an integer iff $2 \nmid n$. Hence, combining the cases, we have

$$\text{NL}_2(f_\psi) \geq \begin{cases} \frac{n(n-2)}{8} & \text{if } n \text{ is even} \\ \lfloor \frac{(n-1)^2}{8} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

To prove the second part of the theorem, we use the technique followed in the proof of [LM19, Theorem-3.14]. It can be checked that $\text{NL}_R(f) \leq \text{NL}_E(f)$ if $R \subseteq E$. When $n = 2^m$, for a fixed integer $j \in [1, m]$, consider the set

$$R_j = \{(y, y) : y \in \mathbb{F}_2^{\frac{n}{2}}, w_H(y) = 2^{j-1}\} = \{(y, y) : y \in E_{2^{j-1}, n}\} \subseteq E_{2^j, n}.$$

It can be checked that for $x = (y, y) \in R_j$, the orbit containing x , $O_x = \{(z, z) : z \in O_y\}$. Then for a WPB $f \in \mathcal{B}_n$ satisfying Proposition 4, we have a WPB $g \in \mathcal{B}_{\frac{n}{2}}$ such that $g(y) = f(x)$ for all $y \in \mathbb{F}_2^{\frac{n}{2}}$. This implies,

$$\text{NL}_2^{\frac{n}{2}}(g) = \text{NL}_{E_{2, \frac{n}{2}}}(g) = \text{NL}_{R_2}(f) \leq \text{NL}_{E_{4, n}}(f) \leq \text{NL}_4^n(f).$$

The equality $\text{NL}_{E_{2, \frac{n}{2}}}(g) = \text{NL}_{R_2}(f)$ can be proved as follows.

$$\begin{aligned} \text{NL}_{R_2}(f) &= \frac{|R_2|}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^{\frac{n}{2}}} \sum_{x \in R_2} (-1)^{f(x) + a \cdot x} = \frac{|E_{2, \frac{n}{2}}|}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^{\frac{n}{2}}} \sum_{y \in E_{2, \frac{n}{2}}} (-1)^{g(y) + a \cdot (y, y)} \\ &= \frac{|E_{2, \frac{n}{2}}|}{2} - \frac{1}{2} \max_{a \in \mathbb{F}_2^{\frac{n}{2}}} \sum_{y \in E_{2, \frac{n}{2}}} (-1)^{g(y) + (a_l + a_r) \cdot y} \quad \text{where } a = (a_l, a_r) \text{ with } a_l, a_r \in \mathbb{F}_2^{\frac{n}{2}} \\ &= \frac{|E_{2, \frac{n}{2}}|}{2} - \frac{1}{2} \max_{b \in \mathbb{F}_2^{\frac{n}{2}}} \sum_{y \in E_{2, \frac{n}{2}}} (-1)^{g(y) + b \cdot y} = \text{NL}_{E_{2, \frac{n}{2}}}(g). \end{aligned}$$

Then we have the generalised result (by proving $\text{NL}_{E_{2^t, \frac{n}{2}}}(g) = \text{NL}_{R_{t+1}}(f)$) as

$$\frac{n(n-2)}{8} \leq \text{NL}_2^n \leq \text{NL}_4^{2n} \leq \text{NL}_8^{2^2 n} \leq \dots \leq \text{NL}_{2^{i+1}}^{2^i n} \leq \dots$$

Now consider $n = n_w + n_{w-1} + \dots + n_1 = 2^{a_w} + 2^{a_{w-1}} + \dots + 2^{a_1}$ such that $a_w > a_{w-1} > \dots > a_1 \geq 0$ as defined in 1. Let $y \in E_{2, n}$ for $y = y_{n_w} y_{n_{w-1}} \dots y_{n_1}$ where $y_{n_i} = (y_{n_{i+1}+1}, y_{n_{i+1}+2}, \dots, y_{n_{i+1}+n_i})$ and $w_H(y) = 2$. Let us define,

$$R = \{(y_{n_w}, y_{n_w})(y_{n_{w-1}}, y_{n_{w-1}}) \dots (y_{n_1}, y_{n_1}) : y = y_{n_w} y_{n_{w-1}} \dots y_{n_1} \in \mathbb{F}_2^n \text{ and } w_H(y) = 2\} \subseteq E_{4, 2n}.$$

Now for $x = (y_{n_w}, y_{n_w})(y_{n_{w-1}}, y_{n_{w-1}}) \dots (y_{n_1}, y_{n_1}) \in R$, we have

$$O_x = \{(z_{n_w}, z_{n_w})(z_{n_{w-1}}, z_{n_{w-1}}) \dots (z_{n_1}, z_{n_1}) : z_{n_w} z_{n_{w-1}} \dots z_{n_1} \in O_y\}$$

Then for a $f \in \mathcal{B}_{2n}$ satisfying Proposition 4, we have a WAPB $g \in \mathcal{B}_n$ such that $\forall y \in \mathbb{F}_2^n$, $g(y) = f(x)$ for $x \in R$. This implies, $\text{NL}_2^n(g) = \text{NL}_R(f) \leq \text{NL}_{E_{4, 2n}}(f) = \text{NL}_4^{2n}(f)$. If we generalise the result, we have

$$\left\lfloor \frac{(n+1)(n-4) + n}{8} \right\rfloor \leq \text{NL}_2^n \leq \text{NL}_4^{2n} \leq \text{NL}_8^{2^2 n} \leq \dots \leq \text{NL}_{2^{i+1}}^{2^i n}.$$

□

Thus, the above theorem provides a better lower bound for the weightwise nonlinearity $\text{NL}_k(f_\psi)$, as proved in the paper [LM19].

Proposition 5. [LM19] For any $n = 2^m \geq 8$ and f_ψ be a WPB Boolean function as defined in 1, then

$$\text{NL}_{2^i}^{(n)}(f_\psi) \geq \begin{cases} 5, & \text{if } 1 \leq i \leq m-3, \\ 6, & \text{if } i = m-2, \\ 19, & \text{if } i = m-1. \end{cases}$$

5 Construction and Study of 2-RotS WAPB Boolean functions

[DKD: We can add another section on WAPB Boolean function due to the action of cyclic group.

1. Need to compute number of orbits of size odd in \mathbb{F}_2^n
2. Need to compute number of orbits of size odd in $\mathbb{E}_{k,n}$
3. $\text{NL}(f_\sigma), \text{NL}_k(f_\sigma)$
4. ...

]

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