

A Proofs

In this section, we provide the proof of the propositions given in the article. For each proposition, we re-state the statement and we provide the corresponding proof.

A.1 Proof of Proposition 1.

Statement: A Boolean function f has an Al_S of 0 if and only if the restriction f_S on S is constant.

Proof. First, we show the reverse implication. We suppose that $f \in \mathcal{B}_n$ is constant on S , either $f(x) = 0$ for all $x \in S$, or $f(x) = 1$ for all $x \in S$. If f is not null everywhere but $f(x) = 0 \forall x \in S$, then the constant function $g(x) = 1$ is not zero on all S , and is it such that $g(x)f(x) = 0$ for all $x \in S$. Similarly, if for all $x \in S$ $f(x) = 1$, the same argument can be used for the function $f + 1$ on S .

Then, we show the direct implication. We suppose that a function $g \in \mathcal{B}_n$ is a non-zero-annihilator of f restricted to S of degree 0. Then, $g \neq 0$ and $\deg(g) = 0 \Rightarrow g = 1$, therefore since g annihilates f in S , $g(x)f(x) = 0$ for all $x \in S$, resulting in $f(x)g(x) = f(x) = 0$. Similarly, if g annihilates $f + 1$ on S instead of f , we have that $g(x)(f(x) + 1) = 0$ implying that $f + 1 = 0 \Rightarrow f(x) = 1$ for all $x \in S$. \square

A.2 Proof of Proposition 2.

Statement: Let $Z = \text{supp}(f^{(o)}) \cap S$. If $\text{rank}\left(G_{r,n}^{f_Z^{(o)},S}\right) < \text{rank}(G_{r,n}^S)$, then $f^{(o)}$ admits a non-zero-annihilator restricted to S of degree at most r .

Proof. Let $D_r^n = \sum_{i=0}^r \binom{n}{i}$. Since $D_r^n = \text{rank}(G_{r,n}^{f_S^{(o)},S}) + |\text{Ker}(G_{r,n}^{f_S^{(o)},S})| = \text{rank}(G_{r,n}^S) + |\text{Ker}(G_{r,n}^S)|$, we have that $\text{rank}(G_{r,n}^{f_S^{(o)},S}) < \text{rank}(G_{r,n}^S) \iff |\text{Ker}(G_{r,n}^{f_S^{(o)},S})| > |\text{Ker}(G_{r,n}^S)|$. Therefore $\exists v \in \text{Ker}(G_{r,n}^{f_S^{(o)},S})$ such that $v \notin \text{Ker}(G_{r,n}^S)$. Hence, it exists a Boolean function g of degree r , whose truth table is given by $v \cdot G_{r,n}$, which is such that $v \cdot G_{r,n} \neq 0$ and $v \cdot G_{r,n}^S \neq 0$, but $(g \cdot f_S^{(o)})_S = v \cdot G_{r,n}^{f_S^{(o)},S} = 0$. Since $\text{rank}\left(G_{r,n}^{f_S^{(o)},S}\right) = \text{rank}\left(G_{r,n}^{f^{(o)},S}\right)$, $\text{rank}(G_{r,n}^{f^{(o)},S}) < \text{rank}(G_{r,n}^S) \Rightarrow \text{rank}(G_{r,n}^{f_Z^{(o)},S}) < \text{rank}(G_{r,n}^S)$ and hence the condition $\text{rank}(G_{r,n}^{f_Z^{(o)},S}) < \text{rank}(G_{r,n}^S)$ is enough to justify the existence of such v giving the non-zero-annihilator g of $f^{(o)}$ restricted to S . \square

A.3 Proof of Proposition 3

Statement: Let $k < \min\{|Z|, |E_{D_n^n}|\}$ and let V_k constructed following Definition 13. Suppose $\ker(V_k) = \langle \hat{g}_k \rangle = \langle (\epsilon_1, \epsilon_2, \dots, \epsilon_k) \rangle$ and let $g_k \in \mathcal{B}_n$ be the Boolean

function having \hat{g}_k as ANF. Furthermore, assume there exists a z_{k+j} , for $j \geq 1$ such that $g_k(z_{k+j}) = 1$. Then, the matrix V'_{k+1} constructed as:

$$V'_{k+1} = \begin{pmatrix} & & & & z_{k+j}^{\alpha_1} \\ & & & & z_{k+j}^{\alpha_2} \\ & & V_k & & \vdots \\ & & & & z_{k+j}^{\alpha_k} \\ z_1^{\alpha_{k+1}} & z_2^{\alpha_{k+1}} & \dots & z_k^{\alpha_{k+1}} & z_{k+j}^{\alpha_{k+1}} \end{pmatrix}$$

is such that $(\epsilon_1, \epsilon_2, \dots, \epsilon_k, 0) \notin \ker(V'_{k+1})$.

Proof. Suppose $\ker(V_k) = \langle \hat{g}_k \rangle = \langle (\epsilon_1, \epsilon_2, \dots, \epsilon_k) \rangle$, and that there exists $j \geq 1$ such that $g_k(z_{k+j}) = 1$, but it is also such that $(\epsilon_1, \epsilon_2, \dots, \epsilon_k, 0) \in \ker(V'_{k+1})$.

If $(\epsilon_1, \epsilon_2, \dots, \epsilon_k, 0) \in \ker(V'_{k+1})$, then $(\epsilon_1, \epsilon_2, \dots, \epsilon_k, 0) \cdot V'_{k+1} = \hat{0}$. Specifically, the last equation in the system of equations described by $(\epsilon_1, \epsilon_2, \dots, \epsilon_k, 0) \cdot V'_{k+1} = \hat{0}$ is:

$$\epsilon_1 z_{k+j}^{\alpha_1} + \dots + \epsilon_k z_{k+j}^{\alpha_k} + 0 \cdot z_{k+j}^{\alpha_{k+1}} = \epsilon_1 z_{k+j}^{\alpha_1} + \dots + \epsilon_k z_{k+j}^{\alpha_k} = 0.$$

However, from our hypothesis, we have:

$$g_k(z_{k+j}) = 1 \iff g_k(z_{k+j}) = (\epsilon_1, \epsilon_2, \dots, \epsilon_k) \begin{pmatrix} z_{k+j}^{\alpha_1} \\ z_{k+j}^{\alpha_2} \\ \vdots \\ z_{k+j}^{\alpha_k} \end{pmatrix} = \epsilon_1 z_{k+j}^{\alpha_1} + \dots + \epsilon_k z_{k+j}^{\alpha_k} = 1.$$

This contradiction shows that it cannot be the case that $(\epsilon_1, \dots, \epsilon_k, 0)$ is in the kernel of V'_{k+1} . □

A.4 Proof of Proposition 4.

Statement: The sequence of matrices $(V_k)_k$, constructed as described in Proposition 3, ensures that:

$$\dim(\ker(V_k)) \leq 1, \quad \forall k \in [1, |Z|].$$

Proof. We prove the result by induction:

- Base case. The initial matrix $V_1 = (1)$ has full rank, so $\dim(\ker(V_1)) = 0 \leq 1$.
- Induction step. We suppose that $\dim(\ker(V_k)) \leq 1$. We show that $\dim(\ker(V_{k+1})) \leq 1$.
 - If $\dim(\ker(V_k)) = 0$, adding a new row and column to form V_{k+1} can increase the kernel by at most 1. Hence, $\dim(\ker(V_{k+1})) \leq 1$.
 - If $\dim(\ker(V_k)) = 1$, let \hat{z} be an element of $\ker(V_k)$. Then either:

$$* \hat{z} = (\hat{\epsilon}, 0):$$

$$\hat{z} \cdot V_{k+1} = 0 \implies \hat{\epsilon} \cdot V_k = 0 \implies \hat{\epsilon} \in \ker(V_k).$$

By Proposition 3, $\hat{\epsilon}$ can only be a trivial solution.

* $\hat{z} = (\hat{y}, 1)$, where $\hat{y} \in \mathbb{F}_2^k$: In this case $\hat{z} \cdot V_{k+1} = 0$ implies:

$$\hat{y} \cdot V_k + (z_1^{\alpha_{k+1}}, \dots, z_k^{\alpha_{k+1}}) = 0 \iff \hat{y} \cdot V_k = (z_1^{\alpha_{k+1}}, \dots, z_k^{\alpha_{k+1}}), \quad (1)$$

and

$$\hat{y} \cdot (z_{k+1}^{\alpha_1}, \dots, z_{k+1}^{\alpha_k})^T + z_{k+1}^{\alpha_{k+1}} = 0 \iff \hat{y} \cdot (z_{k+1}^{\alpha_1}, \dots, z_{k+1}^{\alpha_k})^T = z_{k+1}^{\alpha_{k+1}}. \quad (2)$$

Since $\dim(\ker(V_k)) = 1$, $\text{rank}(V_k) = k - 1$. Hence, there exists at most one particular solution \hat{y}_p for Equation 1. If \hat{y}_p also solves Equation 2, then \hat{y}_p is in $\ker(V_{k+1})$.

Since $(\hat{\epsilon}, 0)$ cannot be in $\ker(V_{k+1})$, the only possible element in $\ker(V_{k+1})$ is \hat{y}_p , if it exists. Therefore, $\dim(\ker(V_{k+1})) \leq 1$.

□

A.5 Proof of Proposition 5.

Statement: Let $\ker(V_{E_k, Z_k}) = \langle \hat{g}_k \rangle = \langle (\epsilon_1, \dots, \epsilon_k) \rangle$, where the corresponding Boolean function $g_k \in \mathcal{B}_n$ satisfies $g_k(x) = 0, \forall x \in Z$.

Define $E^* = (\alpha_1, \alpha_2, \dots, \alpha_{k-1}, \alpha_{k+1})$, a vector constructed from E_k by replacing the last element α_k with α_{k+1} . Let $V_{k+1} = V_{E_{k+1}^*, Z_k}$ be the matrix defined by replacing the monomial α_k in V_{E_k, Z_k} , with the monomial α_{k+1} :

$$V_{E_{k+1}^*, Z_k} = \begin{pmatrix} & & & z_k^{\alpha_1} \\ & & & z_k^{\alpha_2} \\ & & & \vdots \\ E_{k-1}, V_{Z_{k-1}} & & & z_k^{\alpha_{k-1}} \\ & & & z_k^{\alpha_{k+1}} \\ z_1^{\alpha_{k+1}} & z_2^{\alpha_{k+1}} & \dots & z_{k-1}^{\alpha_{k+1}} & z_k^{\alpha_{k+1}} \end{pmatrix}.$$

Then, either $\hat{g}_k \notin \ker(V_{E_{k+1}^*, Z_k})$, or \hat{g}_k is the only element (aside from $\hat{0}$) in the kernel of $V_{E_{k+1}^*, Z_k}$. Hence:

$$\dim(\ker(V_{E_{k+1}^*, Z_k})) \leq 1.$$

Proof. Proposition 3 ensures that the kernel of $V_{E_{k+1}^*, Z_k}$ does not contain elements derived from the kernel of $V_{E_{k-1}, Z_{k-1}}$. Therefore, $V_{E_{k-1}, Z_{k-1}}$ can be treated as a full-rank matrix. Modifying the last row of a matrix that contains a full-rank sub-matrix does not increase the size of the kernel. This is because the first k entries of the elements in the kernel of V_{E_k, Z_k} and $V_{E_{k+1}^*, Z_k}$ are the same, with the only difference arising from the last row. Since this row is the only one that can introduce linear dependence, the dimension of the kernel of $V_{E_{k+1}^*, Z_k}$ is bounded by 1. □

A.6 Proof of Proposition 6.

Statement: Let k be such that $1 < k < \min(|Z|, |E_{D_n^n}|)$ it holds that

$$EF(V_k) = {}_p EF \left(\begin{pmatrix} EF(V_{k-1}) & \begin{matrix} z_k^{\alpha_1} \\ z_k^{\alpha_2} \\ \vdots \\ z_k^{\alpha_{k-1}} \end{matrix} \\ (z_1^{\alpha_k} \ z_2^{\alpha_k} \ \dots \ z_{k-1}^{\alpha_k}) P_{k-1} & z_k^{\alpha_k} \end{pmatrix} \right).$$

where P_{k-1} is such that $U_{k-1}^{-1} = P_{k-1}$, where $V_{k-1} = L_{k-1} U_{k-1}$ with $EF(V_{k-1}) = {}_p L_{k-1}$

Proof. Let k be such that $1 < k < \min(|Z|, |E_{D_n^n}|)$. The LU -decomposition of V_{k-1} is $V_{k-1} = L_{k-1} U_{k-1}$, where U_{k-1} is an upper triangular matrix and $L_{k-1} = EF(V_{k-1})$ is the lower triangular matrix.

$$V_k = \begin{pmatrix} L_{k-1} U_{k-1} & \begin{matrix} z_k^{\alpha_1} \\ z_k^{\alpha_2} \\ \vdots \\ z_k^{\alpha_{k-1}} \end{matrix} \\ (z_1^{\alpha_k} \ z_2^{\alpha_k} \ \dots \ z_{k-1}^{\alpha_k}) & z_k^{\alpha_k} \end{pmatrix} = \begin{pmatrix} L_{k-1} & \begin{matrix} z_k^{\alpha_1} \\ z_k^{\alpha_2} \\ \vdots \\ z_k^{\alpha_{k-1}} \end{matrix} \\ (z_1^{\alpha_k} \ z_2^{\alpha_k} \ \dots \ z_{k-1}^{\alpha_k}) U_{k-1}^{-1} & z_k^{\alpha_k} \end{pmatrix} \begin{pmatrix} 0 \\ U_{k-1} \\ \vdots \\ 0 \\ 0 \ 0 \ \dots \ 0 \ 1 \end{pmatrix}.$$

Since U_{k-1} is an upper triangular matrix, $\begin{pmatrix} 0 \\ U_{k-1} \\ \vdots \\ 0 \\ 0 \ 0 \ \dots \ 0 \ 1 \end{pmatrix}$ is also an upper triangular

matrix, and therefore

$$EF(V_k) = {}_p EF \left(\begin{pmatrix} L_{k-1} & \begin{matrix} z_k^{\alpha_1} \\ z_k^{\alpha_2} \\ \vdots \\ z_k^{\alpha_{k-1}} \end{matrix} \\ (z_1^{\alpha_k} \ z_2^{\alpha_k} \ \dots \ z_{k-1}^{\alpha_k}) U_{k-1}^{-1} & z_k^{\alpha_k} \end{pmatrix} \right).$$

Taking $L_{k-1} = {}_p EF(V_{k-1})$ and $P_{k-1} = U_{k-1}^{-1}$:

$$EF(V_k) = {}_p EF \left(\begin{pmatrix} EF(V_{k-1}) & \begin{matrix} z_k^{\alpha_1} \\ z_k^{\alpha_2} \\ \vdots \\ z_k^{\alpha_{k-1}} \end{matrix} \\ (z_1^{\alpha_k} \ z_2^{\alpha_k} \ \dots \ z_{k-1}^{\alpha_k}) P_{k-1} & z_k^{\alpha_k} \end{pmatrix} \right).$$

□

B Algorithms

In this section, we provide the pseudo-codes of the algorithms described in Section 3 and Section 4.2.

Algorithm 2 Algebraic immunity of f restricted to the set S , Reed-Muller method.

Input: $x \in \mathbb{F}_2^{2^n}$ evaluation of the truth table of $f \in \mathcal{B}_n$ and the restricted set $S \subseteq \mathbb{F}_2^n$.

Output: $\text{Al}_S(f)$

```

1: function FIND_DEG_SMALLEST_S_ANNIHILATOR( $G_{n,n}^{f^{(o)},S}, G_{d,n}^S, D^n$ )
2:    $r \leftarrow 0$ ;
3:    $i \leftarrow 0$ ;
4:    $ef_f, ef_S \leftarrow \text{null}, \text{null}$ ;
5:    $previousIndex \leftarrow 0$ ;
6:   while  $r \leq |S|/2$  do
7:      $D_i^n \leftarrow D^n[i]$ ;
8:      $ef_f \leftarrow \text{ECHELON\_FORM\_LAST\_}D_i^n\text{-ROWS} \left( ef_f \parallel G_{n,n}^{f^{(o)},S}[previousIndex : D_i^n] \right)^1$ ;
9:      $ef_S \leftarrow \text{ECHELON\_FORM\_LAST\_}D_i^n\text{-ROWS} \left( ef_S \parallel G_{n,n}^S[previousIndex : D_i^n] \right)$ ;
10:     $r \leftarrow \text{rank}(ef_S)$ ;
11:    if  $\text{rank}(ef_f) < r$  then
12:      return  $i$ ;
13:    end if
14:     $previousIndex \leftarrow D_i^n$ ;
15:     $i \leftarrow i + 1$ ;
16:  end while
17: end function
18:
19:  $Z = \text{supp}(x) \cap S$ ;
20:  $Z_c = \text{supp}(x + 1) \cap S$ ;
21: if  $Z = \emptyset$  or  $Z_c = \emptyset$  then return 0;
22: end if
23:  $m_{max}, D^{\leq n_{max}} \leftarrow \text{READ\_VALUES\_FROM\_FILE}()$ ;
24:  $G_{n,n} \leftarrow m_{max}[n]$ ;
25:  $D^n \leftarrow D^{\leq n_{max}}[n]$ ;
26:  $G_{n,n}^{f_Z,S}, G_{n,n}^{(f+1)Z,S}, G_{n,n}^S \leftarrow \text{CONSTRUCT\_RESTRICTED\_GENERATOR\_MATRICES}(G_{n,n}, x, S)^2$ 
27:  $immunity_f, immunity_{f+1} \leftarrow ($ 
28:   FIND_DEG_SMALLEST_S_ANNIHILATOR( $G_{n,n}^{f_Z,S}, G_{n,n}^S, D^n$ )
29:   | FIND_DEG_SMALLEST_S_ANNIHILATOR( $G_{n,n}^{(f+1)Z,S}, G_{n,n}^S, D^n$ )
30: );
31: return  $\text{MIN}(immunity_f, immunity_{f+1})^3$ ;

```

Algorithm 3 Algorithm to compute the algebraic immunity of a function $f \in \mathcal{B}_n$ restricted to S

Input: $x \in \mathbb{F}_2^{2^n}$ evaluation of the truth table of $f \in \mathcal{B}_n$ and the restricted set $S \subseteq \mathbb{F}_2^n$.

Output: $\text{Al}_S(f)$

```

1:
2:  $Z \leftarrow \text{supp}(x) \cap S$ ;
3:  $Z_c \leftarrow \text{supp}(x+1) \cap S$ ;
4: if  $Z = \emptyset$  or  $Z_c = \emptyset$  then
5:   return 0;
6: end if
7:
8:  $E_{D_n^n} \leftarrow \text{CONSTRUCT\_MONOMIALS}(n = n, d = n)^4$ ;
9:  $r \leftarrow \frac{|Z|}{|Z_c|}$  if  $|Z| < |Z_c|$  else  $\frac{|Z_c|}{|Z|}$ ;
10: if  $r \geq \frac{1}{4}$  then
11:    $\text{immunity}_f, \text{immunity}_{f+1} \leftarrow ($ 
12:      $\text{FIND\_DEG\_SMALLEST\_S\_ANNIHILATOR}(Z = Z, E = E_{D_n^n}, S = S)$ 
13:      $| \text{FIND\_DEG\_SMALLEST\_S\_ANNIHILATOR}(Z = Z_c, E = E_{D_n^n}, S = S)$ 
14:    $);$ 
15:   return  $\text{MIN}(\text{immunity}_f, \text{immunity}_{f+1})$ ;
16: else
17:   return  $\text{FIND\_DEG\_SMALLEST\_S\_ANNIHILATOR\_SEQ}(Z = Z, Z_c, E = E_{D_n^n}, S =$ 
18:      $S))^5$ ;
19: end if
```

C Comparison between Algorithm 2 and Algorithm 3 on WAPB functions

In this part, we compare the execution times of Algorithm 2 and Algorithm 3 by running both algorithms on multiple WAPB functions restricted to the largest set $E_{\lceil n/2 \rceil, n}$, with different numbers of variables. Specifically, the experiment is done by running the

² The symbol $||$ is used as row-concatenation of matrices, meaning that $C = A||B$ is the matrix constructed by appending all the row of the matrix B to the matrix A .

³ $\text{CONSTRUCT_RESTRICTED_GENERATOR_MATRICES}$ is a function extracting $G_{n,n}^{f_{\text{supp}(f) \cap S}, S}$, $G_{n,n}^{f_{\text{supp}(f+1) \cap S}, S}$ and $G_{n,n}^S$ from the full Reed-Muller code generator matrix $G_{n,n}$ following Definition 11.

⁴ The function MIN takes care of *null* values: $\text{MIN}(\text{immunity}_f, \text{null}) = \text{immunity}_f$ and $\text{MIN}(\text{null}, \text{immunity}_{f+1}) = \text{immunity}_{f+1}$.

⁹ $\text{CONSTRUCT_MONOMIALS}$ generates all monomials up to the degree d with n variables. $E_{D_n^n}$ may also be pre-computed and the function $\text{CONSTRUCT_MONOMIALS}$ removed from the algorithm.

¹⁰ The implementation of the function $\text{FIND_DEG_SMALLEST_S_ANNIHILATOR_SEQ}$ is similar to the function $\text{FIND_DEG_SMALLEST_S_ANNIHILATOR}$ described in Algorithm 1 with the difference that the degree of the minimal-degree non-zero annihilator of f or of $f+1$ restricted on S , is found sequentially instead of in parallel.

algorithms with the number of variables n going from 1 to 12. For both algorithms, if $n \leq 4$, we calculate the average over the $\text{Al}_{\lceil n/2 \rceil}$ of all functions in \mathcal{WAPB}_n , whereas for $n > 4$, we calculate the average over the $\text{Al}_{\lceil n/2 \rceil}$ of a random sample of 10^4 \mathcal{WAPB}_n functions.

Figure 1 shows the mean execution time per value of n of the two algorithms. We denote by $T(G^{\leq n_{max}}) = T(G^{\leq 12})$ the necessary time to pre-compute all the necessary Reed-Muller codes's generator matrices following Equation (2): we recall that it is the necessary time to compute the sequence $(G_{n,n})_{n \in [1, 12=n_{max}]}$, not only the generator matrix $G_{n,n}$ for the current iteration n . In fact, $T(G^{\leq 12})$ is constant, that is because the computation occurs before beginning the experiment since it is expected that Algorithm 2 is able to run for any number of variable n up to an upper bound n_{max} , without having to compute the Reed-Muller code for it.

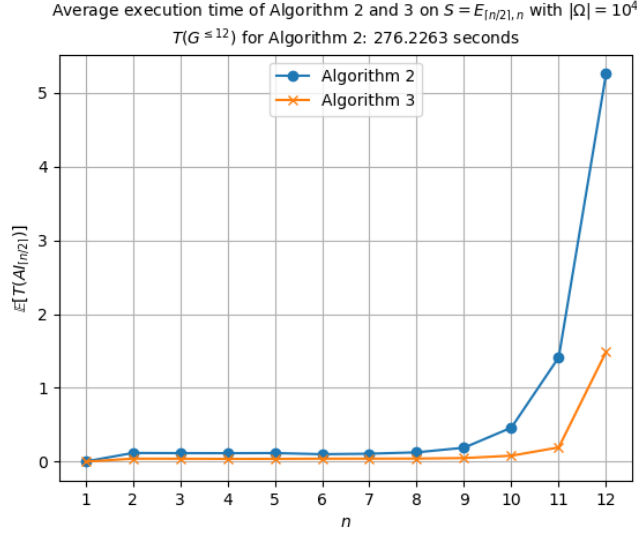


Fig. 1: Average time execution per number of variables of Algorithms 2 and 3 over 10^4 random \mathcal{WAPB}_n functions on the set $E_{\lceil n/2 \rceil, n}$. $T(G^{\leq 12})$ gives the time of the pre-computation of the sequence of all Reed-Muller matrices up to $G_{12,12}$.

The pre-computation times of $G^{\leq n_{max}}$ for Algorithm 2 are given in the Table 1. From Figure 1, we observe that Algorithm 3 outperforms Algorithm 2 for $n \leq 12$, even without considering the pre-computation time required for $G^{\leq n_{max}}$ and $D^{\leq n_{max}}$ in Algorithm 2. Furthermore, Algorithm 2 becomes impractical for $n > 12$ due to the excessive complexity of pre-computation. In contrast, Algorithm 3 remains feasible for larger values of n .

$G^{\leq 1}$	$G^{\leq 2}$	$G^{\leq 3}$	$G^{\leq 4}$	$G^{\leq 5}$	$G^{\leq 6}$	$G^{\leq 7}$	$G^{\leq 8}$	$G^{\leq 9}$	$G^{\leq 10}$	$G^{\leq 11}$	$G^{\leq 12}$
0.0126	0.0042	0.0017	0.004	0.0149	0.0470	0.1938	0.806	3.5456	14.9168	65.5014	276.2263

Table 1: Pre-computation time (in seconds) for $G^{\leq n_{max}}$ for n_{max} from 1 to 12.

Both implementations of the algorithms used to generate the plot in Figure 1 have been developed in *Python*, with the following enhancements:

- The linear algebra operations in Algorithm 2 are performed using the *SageMath* library, while those in Algorithm 3 are executed using an ad-hoc custom *Python* package implemented in *Rust*. This setup ensures comparability between the two implementations, as the *SageMath* library is highly optimized for linear algebra operations.
- The computation of $G^{\leq n_{max}}$ for Algorithm 2 is performed using the *generator_matrix* method of the *SageMath* object *BinaryReedMullerCodes*. While this computation could be more efficient if implemented in *Rust*, the current overall implementation still faces scalability issues. Specifically, for $n \geq 16$, the algorithm becomes impractical because $G_{16,16}$ cannot be stored in a native *Python* list or a *SageMath* matrix object. However, if the entire algorithm were implemented in *Rust*, it could potentially be feasible. Consequently, for larger values of n , Algorithm 2 is not usable, while Algorithm 3 remains viable.

D Plots of Al_k probability distributions of WPB functions and average execution times.

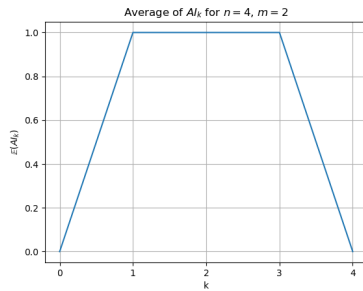


Fig. 2: Mean of Al_k over \mathcal{WPB}_2 .

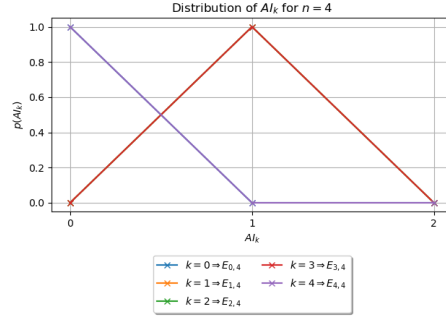


Fig. 3: Probability distribution of Al_k over \mathcal{WPB}_2 .

Fig. 4: Mean of Al_k and probability distribution of Al_k from Definition 16 of 4-variable WPB functions.

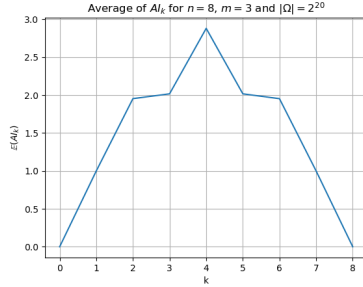


Fig. 5: Estimated average of Al_k over WPB_3 .

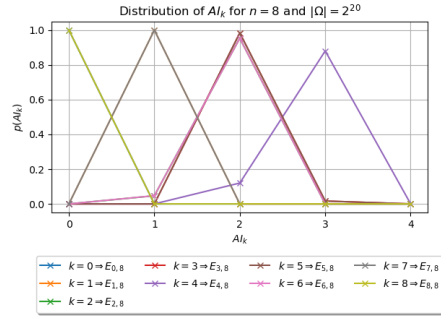


Fig. 6: Estimated Probability distribution of Al_k over WPB_3 .

Fig. 7: Estimated average of Al_k and probability distribution of Al_k over 8-variable WPB functions, with 2^{20} samples.

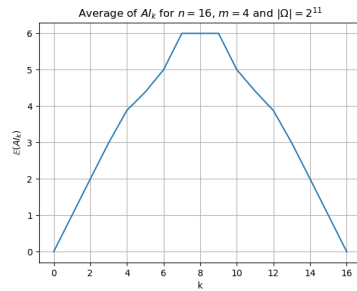


Fig. 8: Estimated average of Al_k over WPB_4 .

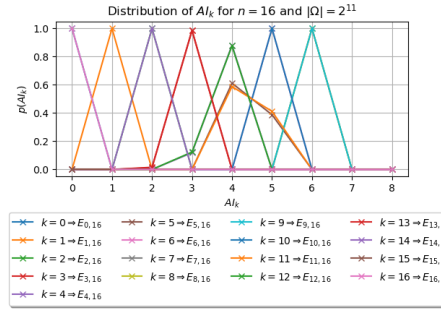


Fig. 9: Estimated Probability distribution of Al_k over WPB_4 .

Fig. 10: Estimated average of Al_k and probability distribution of Al_k over 8-variable WPB functions, with 2^{11} samples.

Table 2: Averages execution times and Al_k of \mathcal{WPB}_2 , \mathcal{WPB}_3 and \mathcal{WPB}_4 functions.(a) Average execution time and Al_k for $n = 4, m = 2$.

k	0	1	2	3	4
$\mathbb{E}[T(AI_k)]$	0.53×10^{-4}	0.0526	0.0487	0.0586	0.6×10^{-5}
$\mathbb{E}[AI_k]$	0.00	1.00	1.00	1.00	0.00

(b) Average execution time and Al_k for $n = 8, m = 3$ and $\text{sampleSize} = 2^{20}$.

k	1	2	3	4	5	6	7
$\mathbb{E}[T(AI_k)]$	0.5×10^{-3}	0.3×10^{-3}	0.7×10^{-3}	0.1×10^{-2}	0.7×10^{-3}	0.3×10^{-3}	0.4×10^{-3}
$\mathbb{E}[AI_k]$	1.00	1.95	2.02	2.88	2.02	1.95	1.00

(c) Average execution time and Al_k for $n = 16, m = 4$ and $\text{sampleSize} = 2^{11}$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\mathbb{E}[T(AI_k)]$	0.019	0.020	0.056	2.38	57.45	390.30	1807.83	4111.90	2262.56	432.50	74.77	3.20	0.068	0.019	0.018
$\mathbb{E}[AI_k]$	1.00	2.00	2.99	3.89	4.39	5.00	6.00	6.00	6.00	5.00	4.41	3.88	2.99	2.00	1.00