Systemic risk and contagion effects in Australian financial institutions and sectors

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1. Literature review

The starting point for the proposed research is the recent literature and the CIFR targeted areas and APRA aims and functions. This recent literature includes the following Adrian and Brunnermeier (2011), Acharya et al. (2012), Acharya et al. (2012) and Brownlees and Engle (2010). The proposed research aims to extend and apply these techniques particularly in relation to the entities regulated by APRA. Thus our broad aim is to develop, implement and bring to bear recent developments in stress testing on the aims of APRA and the CIFR targeted research areas detailed above.

2. Improved measures of contagion and systematic risk

CoVaR_q as proposed in Adrian and Brunnermeier (2011) is a basis for proposed measures contagion, exposure and systemic risk. It suffers from a number of drawbacks:

- Couched in terms of VaR_q containing the scale of the original measurements. It is worthwhile to have measures and techniques robust to scale.
- Conditioning on $\text{VaR}_{0.5}$ is undesirable and relatively intractable. In our proposal we reference stress with respect to the unconditional VaR_q. This permits a more transparent analysis and estimation.
- Our proposed approach separates out scale effects and interdependence effects and aims to relates these separately to external variables including shocks and drivers of systemic risk. Thus VaR_q movements due to scale are disentangled from movements due to codependence with separate driver responses.

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3. Significance of the project and policy implications

Understanding the impact of external shocks and their propagation through the financial system is vital for managing and remediating systemic risk. Effective regulation is dependent upon the development of a robust and reliable set of appropriate risk measures. We propose new measures of systemic risk that relate marginal and joint distributions separately to external drivers. This allows for more cogent and coherent stress testing as it includes the estimation of contagion effects, exposure effects and systemic risk across related entities and different financial sectors. Improved stress testing, estimation of risk effects and transmission of shocks through the financial system will make for more cogent prudential policy, prudential margin setting and better identify sources of risk to the financial system.

4. Percentile sensitivity and contagion

4.1. Theorem

Suppose x is a random vector with marginal distributions[1]

$$F(x) \equiv \{F(x_1), \ldots, F(x_p)\} \equiv (u_1, \ldots, u_p) \equiv u \ .$$

Further suppose $0 \leq q \leq 1$ is given and $\mathrm{Q}(x)$ is the vector of q–quantiles

$$F\{\mathrm{Q}(x)\} = q1 = \mathrm{Q}(u) \ ,$$

where 1 is a vector of p ones. Define the stress vector with respect to x_j as[2]

$$\frac{\mathrm{dQ}(x)}{\mathrm{d}x_j} \equiv \mathrm{Q}(x|u_j > q) - \mathrm{Q}(x) \ , \tag{1}$$

where $\mathrm{Q}(x|u_j > q)$ is the vector of q–quantiles of x given $u_j > q$. Then if $\mathrm{Q}(x)$ is linear in q then

$$\frac{\mathrm{dQ}(x_i)}{\mathrm{dQ}(x_j)} \equiv \frac{\mathrm{dQ}(x_i)/\mathrm{d}x_j}{\mathrm{dQ}(x_j)/\mathrm{d}x_j} = \frac{f_j}{f_i}\frac{q_{ij}}{q(1-q)} \ , \qquad q_{ij} = C_{ij}(q + q_{ij}, q) - q^2 \ , \tag{2}$$

where f_i and f_j are the densities of x_i and x_j evaluated at $\mathrm{Q}(x_i)$ and $\mathrm{Q}(x_j)$ and C_{ij} is the copula of (u_i, u_j). Further if

$$s_{ij} \equiv \frac{q_{ij}}{q(1-q)} \ ,$$

then $-1 \leq s_{ij} \leq 1$ with $s_{ij} = \pm 1$ if x_i and x_j are comonotonic and counter monotonic, respectively. If x_i and x_j are independent then $s_{ij} = 0$. If u_i and u_j are exchangeable then $s_{ji} = s_{ij}$.

[1] To economise on notation, write $F_j(x) \equiv F_j(x_j) \equiv F(x_j)$ and similar for other vector functions.

[2] In Adrian and Brunnermeier (2011) $\Delta CoVar_q \equiv q_{y|x=q_x} - q_y$. Variable y is generally the "financial system" and hence considered is the change in the VaR_q of the system when institution x stressed, with stress interpreted as $x = q_x$. On page 10 of their paper they incorrectly state "... $CoVaR$ conditions on the event that [an] institution is at its VaR level, which occurs with probability q."

4.2. Proof

By definition

$$\frac{\mathrm{dQ}(u_i)}{\mathrm{d}u_j} \equiv \mathrm{Q}(u_i|u_j > q) - q \equiv q_{ij} \ ,$$

where q_{ij} is such that

$$q = \frac{\mathrm{P}(u_i \leq q + q_{ij}, u_j > q)}{1-q} = \frac{q + q_{ij} - C_{ij}(q + q_{ij}, q)}{1-q} \ , \tag{3}$$

Rearranging yields the second equation in (2). Now

$$\frac{\mathrm{dQ}(x_i)}{\mathrm{d}x_j} \equiv \mathrm{Q}_{q+q_{ij}}(x_i) - \mathrm{Q}(x_i) \approx \mathrm{Q}'(x_i) q_{ij} = \frac{q_{ij}}{f_i} \ , \tag{4}$$

where $'$ denotes differentiation with respect to q and the subscript on Q indicates the revised q for the quantile calculation. The approximation follows from a first order Taylor expansion and is exact if the quantile is linear in q. Similarly

$$\frac{\mathrm{dQ}(x_j)}{\mathrm{d}x_j} \equiv \mathrm{Q}_{q+q(1-q)}(x_j) - \mathrm{Q}(x_j) \approx \frac{q(1-q)}{f_j} \ . \tag{5}$$

Dividing (4) by (5) yields the first equation in (2) and completes the proof.

4.3. Discussion

The critical result is that sensitivities factor into contributions from the ratios of the marginal densities and quantities calculated from the pairwise copulas. The via the implicit equation for q_{ij} in (2), solved using a root finding algorithm. The quantities q_{ij} are implicitly defined from the pairwise copulas. In summary the “sensitivity” matrix

$$S \equiv \frac{\mathrm{d}Q(x)}{\mathrm{d}Q'(x)} \ ,$$

where $'$ denotes transposition has ones on the diagonal quantities between ± 1 off the diagonal. The matrix S is called the sensitivity matrix and displays the sensitivity of the q–quantile of each component of x to stress in the same or other components.

Since f_i and f_j are the densities evaluated at $\mathrm{Q}(x_i)$ and $\mathrm{Q}(x_j)$ it follows that the ratio f_i/f_j can be replaced by the ratio of the hazards. This is useful in cases were it is practical to model the hazard rather than the density.

4.4. Implementation

Figure 1 displays the empirical copulas calculated from four weekly closing stock prices labelled anz, cba, mcq and wbc for $n = 761$ weeks from 2000 April 12 through to 2014 October 29. The empirical copulas are calculated by converting each observation to an empirical percentile and plotting the same against each of the other series percentiles.

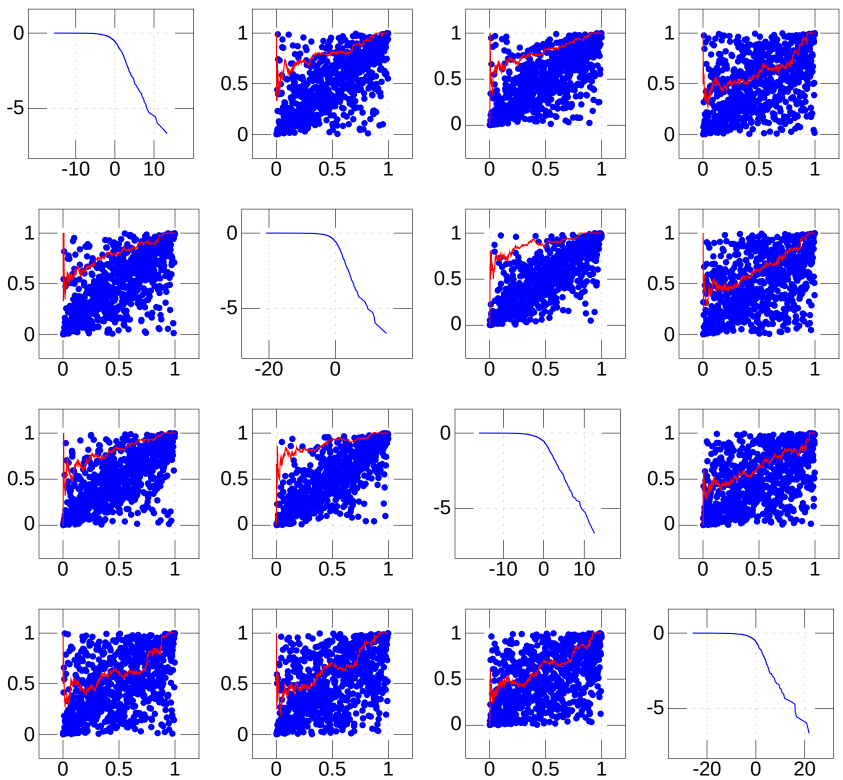

Figure 1: Pairwise copulas of bank stocks cba, anz, mqg, wbc, and the overall bank index. Red lines plot sensitivity s_{ij} as a function of q.

To estimate q_{ij} at a particular q, the second equation in (2) is iterated[3] starting from $q_{ij} = 0$ where copulas are estimated as

$$\hat{C}_{ij}(u_i, u_j) \equiv \hat{\mathrm{E}}\{(p_{ik} \leq u_i)(p_{jk} \leq u_j)\} \; .$$

Here $\hat{\mathrm{E}}$ computes the empirical mean over the cases $k = 1, \ldots, n$ and p_{ik} and p_{jk} are the empirical percentiles of case k of x_i and x_j, respectively.

4.5. Generalisations

Similar results apply when $\mathrm{Q}(x)$ is replaced by other risk measures such as $\mathrm{R}(x) \equiv \mathrm{E}\{xr(u)\}$ where r is a given function which acts componentwise and $xr(u)$ denotes componentwise multiplication. For example if $r(u) = mu^{m-1}$ then $\mathrm{R}(x) = \mathrm{E}\{\max(x^1, \ldots, x^m)\}$ where $x^1, \ldots, x^m$ are m independent copies of x and the risk measure is the expected worst outcome in m independent trials.

With $\mathrm{R}(x)$, the analogue of (1) is

$$\frac{\mathrm{dR}(x)}{\mathrm{d}x_j} \equiv \mathrm{R}(x|u_j > r_j) - \mathrm{R}(x_j) \; , \qquad r_j \equiv \mathrm{P}\{x_j \leq \mathrm{R}(x_j)\} \; . \tag{6}$$

This differs from (1) in that a different risk measure is used and r_j replaces q. If the components of $\mathrm{R}(x)$ are linear in the r_i then

$$\frac{\mathrm{dR}(x_i)}{\mathrm{dR}(x_j)} = \frac{f_j}{f_i} \frac{q_{ij}}{q(1-q)} \; , \qquad q_{ij} = C_{ij}(r_i + q_{ij}, r_j) - r_i r_j \; , \tag{7}$$

where f_i and f_j are the x_i and x_j densities at the r_i and r_i quantiles, respectively.

5. Conditionally Gaussian copula

Suppose $(u, v) = F_*(x, y)$ and $z = \Phi^-(v)$ where x and y are scalar random variables. Further write $t = 1, \ldots, n$ where the u_t are the ordered values of u and $q_t = \Phi^-(u_t)$. The correspondingly ordered values of v and z are denoted v_t and z_t, $t = 1, \ldots, n$.

Smoothing methods are here proposed to smooth and simulate z_t using a cubic spline. Of particular concern are simulated $\hat{z}_t$ for t near 1 or n corresponding to x extremes. Given simulated $\hat{z}_t$ for $t = 1, \ldots, n$, $\hat{v}_t = \Phi(\hat{z}_t)$ and $(x_t, \hat{y}_t) = F_*^-(u_t, \hat{v}_t)$, a draw from the joint. The u_t are uniform while the $\hat{v}_t$ are approximately uniform. The simulation singles out independent and dependent variables. Which should be which? If x and y are actual time series the simulation $(x_t, \hat{y}_t)$ is ordered back into the original time series order.

The conditionally Gaussian model for $z_t \equiv \Phi^-(v_t)$ given $q_t \equiv \Phi^-(u_t)$, $t = 1, \ldots, n$, is

$$z_t = \alpha_t + \beta q_t + s\epsilon_t \; , \qquad \alpha_{t+1} = \alpha_t + \delta_t + \frac{\eta_t}{3 - \sqrt{3}} \; , \qquad \delta_{t+1} = \delta_t + \eta_t \; , \tag{8}$$

[3]The second equation in (2) is a contraction mapping and hence has a fixed point.

where ϵ_t and η_t are zero mean disturbances with common variance σ^2. The q_t term is the Gaussian component, α_t a nonparametric deviation, and ϵ_t noise.

The second and third equations in (8) dictate α_t is cubic spline with knots at $t = 1, \ldots, n$ with increasing smoothness as $\sigma^2 \to 0$. In the limit α_t is a straight line. Thus $z_t - \beta q_t$ is, apart from noise $s\epsilon_t$, smooth with the degree of smoothness controlled by σ. The cubic spline model for the copula is the usual Gaussian copula if $\alpha_t \equiv 0$, is readily fit to percentile data, and provides a practical platform for copula simulation. Hence (8) provides a flexible yet easily implemented and useable extension of the Gaussian copula. Similar extensions can be imposed on say student–t quantiles as discussed in §**??**.

The parameters of (8) are s, β, σ and starting conditions α_0 and δ_0. The parameters δ_0 often plays a minor role and often appropriately set to zero. The term $\alpha_t + \beta q_t$ is the signal and ϵ_t, noise. The signal is relatively smooth and contains Gaussian component q_t if $\beta \neq 0$. Departures of z_t from the Gaussian component are the sum of a persistent component α_t and transient component, $s\epsilon_t$. Since ϵ_t and the driver of persistence η_t share σ, the smoothing parameter s measures relative importance of transience as opposed to persistence. The scale parameter σ relates to the degree of roughness since it is the variance of the changes in the local slope of α_t.

Model (8) provides an analysis of variance type decomposition for z_t:

$$\mathcal{V}(z_t) = \mathcal{V}(\alpha_t) + \beta^2 \mathcal{V}(q_t) + s^2 \mathcal{V}(\epsilon_t) \ , \qquad 1 = \mathcal{V}(\alpha_t) + \beta^2 + (s\sigma)^2 \ , \tag{9}$$

where $\mathcal{V}$, analogous to $\mathcal{E}$, computes the variance supposing t is uniform on $1, \ldots, n$. Decomposition (9) splits the total variance $\mathcal{V}(z_t) = 1$ into that due to the nonparametric component α_t, Gaussian component β^2 and residual noise, $(s\sigma)^2$.

The local correlation is the partial correlation between z_t and the Gaussian component q_t, after removing the effect of the nonparametric component α_t:

$$\rho_s \equiv \operatorname{cor}(z_t - \alpha_t, q_t) = \frac{\beta}{\sqrt{\operatorname{cov}(z_t - \alpha_t)}} = \pm \left\{ 1 + \left(\frac{s\sigma}{\beta} \right)^2 \right\}^{-1/2} \ , \tag{10}$$

where the sign is that of β. Using (9), ρ_s^2 equals the proportion of residual variance $\mathcal{V}(z_t) - \mathcal{V}(\alpha_t) = \beta^2 + (s\sigma)^2$ explained by the Gaussian component q_t.

The overall or global correlation is

$$\rho_v \equiv \operatorname{cor}(z_t, \alpha_t + \beta q_t) \ ,$$

reducing to (10) except that s is replaced by

$$v \equiv \mathcal{E}\left(\sum_{j=1}^{t-1} w_j^2 \right) = xxx \ , \qquad \alpha_t = \sum_{j=1}^{t-1} w_j \eta_{t-j}$$

The derivation of $v = xxx$ is shown appendix. Equivalently ρ_v^2 is the proportion of $\mathcal{V}(z_t)$ explained by the nonparametric and Gaussian component.

From (10)

$$\left(\frac{v}{s}\right)^2 = \frac{\rho_s^2/(1-\rho_s^2)}{\rho_v^2/(1-\rho_v^2)} \equiv \mathrm{OR} \ , \tag{11}$$

the odds ratio of local to global correlation. In general $v > s$ implying $\mathrm{OR} > xx$.

Table 1 sets out possible choices for s and β. These combinations are discussed in the further subsections.

Table 1: Special cases of the semi Gaussian copula

model	s	α_0	β	OR
empirical	0			∞
conormal	> 0		± 1	
conditionally Gaussian	> 0		$\neq 0$	> 1
linear probit	∞		0	
Gaussian	∞	0		
independence	∞	0	0	
co/counter monotonic	∞	0	± 1	

5.1. Empirical copula

The empirical copula results in (8) when $s = 0$ or the odds ratio (11) is infinite. In this case $z_t = \alpha_t + \beta q_t$. The deviation of $z_t - (\alpha_t + \beta q_t)$ can be made zero by selecting appropriate η_t. Hence $s = 0$ implies the fitted copula is the empirical copula. Note $\mathrm{E}(\alpha_t) = 0$ and $1 = \mathrm{cov}(z_t) = \mathrm{cov}(\alpha_t) + \beta^2$.

5.2. Large local variation

The other extreme $s \to \infty$ is displayed bottom rows of Table 1. Large s forces negligible σ and straight line behaviour for α_t yielding

$$z_t = \alpha_0 + \delta_0 t + \beta q_t + s\epsilon_t \ , \qquad t = 1, \ldots, n \ ,$$

where $s\epsilon_t$ has finite variance $(s\sigma)^2$. Since $\mathrm{E}(q_t) = \mathrm{E}(z_t) = 0$ any fitted relation is such that

$$\delta_0 = \frac{-2\alpha_0}{n+1} \ , \qquad z_t = \alpha_0(1 - 2u_t) + \beta q_t + s\epsilon_t \ .$$

Additionally $\mathrm{cov}(q_t) = \mathrm{cov}(z_t) = 1$ implying

$$1 = \frac{\alpha_0^2}{3} - 2\alpha_0\beta c + \beta^2 + (s\sigma)^2 \ , \qquad c \equiv \mathrm{cov}(u_t, q_t) \approx 2.82 \ , \tag{12}$$

and hence at most two of α_0, β and $s\sigma$ can be chosen freely.

5.2.1. Gaussian copula

If $s \to \infty$ and $\alpha_0 = 0$ then (q_t, z_t) is bivariate Gaussian with unit variances and correlation β and (8) reduces to a Gaussian copula with correlation β and the odds ration (11) is 1, that is local and ordinary correlation coincide and (12) reduces to $\beta = \pm\sqrt{1-(s\sigma)^2}$.

5.2.2. Probit copula

If $s \to \infty$ and $\beta = 0$ then (8) defines a probit copula. From (12), $\alpha_0 = \pm\sqrt{3\{1-(s\sigma)^2\}}$.

5.2.3. Independence

If $s \to \infty$ and $\alpha_0 = \beta = 0$ then the variables are independence. The situation is achieved either letting $\beta \to 0$ with a Gaussian copula or letting $\alpha_0 \to 0$ with a Probit copula. Under independence $s\sigma = 1$.

5.3. Conditionally Gaussian copula

This is the intermediate situation where $0 < s < \infty$. Testing $\alpha_0 = 1/s = 0$ is a test for a Gaussian copula. Goodness of fit is measured with

$$R^2 = 1 - (s\hat{\sigma})^2 = 1 - \frac{1}{n}\sum_{t=1}^{n}(z_t - \hat{z}_t)^2 \ , \tag{13}$$

where $\hat{z}_t$ is the estimate of z_t based on (8) using estimates of α_0, δ_0, β and $s\sigma$. Values of R^2 lie between 0 and 1 and have the usual interpretation: near 1 indicates the percentiles of u well explain z values of v. Note $R \to \beta$ if $\alpha_0 = 0$ and $s \to \infty$.

From (13)

$$\sum_{t=1}^{n}\left(\frac{z_t - \hat{z}_t}{s\hat{\sigma}}\right)^2 = n$$

suggesting the comparison of standardised residuals to ± 2 and perhaps plotting the percentile rank of the standardised residuals against u_t to identify lack of fit. Note R^2 differs from the correlation coefficient β since.

$$\frac{R}{\beta} = \pm\sqrt{1-(s\hat{\sigma})^2}\left\{1 + \left(\frac{s\sigma}{\beta}\right)^2\right\}$$

This emphasises the difference between R^2 and β. The former measures the goodness of fit of a semiparametric function of u to explain z while β measures the ability of a straight line in q to explain z. If $\beta \neq 0$ then q forms part of the semiparametric specification and hence if $\beta \neq 0$, $R^2 > \beta^2$.

The implied correlation between z_t and q_t is

$$\beta = \frac{\beta}{\sqrt{\beta^2 + (s\sigma)^2}} = \pm\left\{1 + \left(\frac{s\sigma}{\beta}\right)^2\right\}^{-1/2} \ .$$

The term $\beta/(s\sigma)$ is the signal to noise ratio. If $z_t, q_t \sim (0,1)$ then $\beta^2+(s\sigma)^2 = 1$ and

$$\beta = \pm\sqrt{1-(s\sigma)^2} = R = \beta \ , \qquad \frac{\beta}{s\sigma} = \frac{\pm\sqrt{1-(s\sigma)^2}}{s\sigma} = \frac{\beta}{\sqrt{1-\beta^2}}$$

.

Thus the empirical correlation between z_t and q_t is

$$\sum_{t=1}^{n} z_t q_t = \beta + \delta_0 \sum_{t=1}^{n} t q_t$$

By construction both z_t and q_t have mean zero and variance 1 implying the correlation between z_t and q_t is

If $\beta = 0$ then

$$z_t = \alpha_0 + n\delta_0 u_t + \beta\Phi^{-}(u_t) + s\epsilon_t \ ,$$

Testing $\delta_0 = 0$ amounts to testing for independence. If $\beta = \pm 1$ then $z_t - (\pm q_t) = \alpha_0 + t\delta_0$ implying $\alpha_0 = n\delta_0 = 0$ and $z_t = \pm q_t$, that is the variables are co or counter monotonic. Finally if $\beta \neq 0, \pm 1$ then $z_t = \alpha_0 + t\delta_0 + \beta q_t + s\epsilon_t$ since $\alpha_0 = \delta_0 = 0$, implied by the fact that Since both z_t and q_t are, by construction, standard normal, this amounts to prescribing a Gaussian copula with correlation indicated in Table 1. The ratio $\beta/(s\sigma)$ is the signal to noise ratio.

The middle column of the body of Table 1 displays alternatives for finite $s \neq 0$. If $\beta = 0$ the implied copula is nonparametric with α_t evolving smoothly as a function of t with the actual degree of smoothness increasing with s. A constraint is $\alpha_0 = \alpha_n = \sum_t \delta_t = 0$ and hence α_t forms a bridge. If $\beta = \pm 1$ then the signed difference $z_t - (\pm q_t)$ is modelled as a smooth. In a fit, the bridge structure is expected to be enforced by the data.

The scale σ quantifies roughness in the signal α_t but its size is determined by s with $s\sigma \leq 1$ implied by the standard normality of z_t. In fitting, s is either chosen a–priori, or estimated together with the other parameters. Given s, estimates of β, α_0, δ_0 and σ are closed form generalised least squares calculations. Thus s can be estimated using a one dimensional search. Given the values of the parameters, standard diagnostics are used to assess the fit and reveal areas of inadequate fit. These techniques are used and explained in the applications.

Use of q_t permits a partially Gaussian explanation of the copula with Gaussianity increasing with s. In the limit the enforced Gaussian copula has correlation R signed according to $\hat{\beta}$. Co or countermonotoniticity implies $R = \pm 1$ with the sign indicating the direction of the relationship. Setting $\beta = 0$ excludes the Gaussian copula component.

The fitted values $\hat{z}_t$ stacked in vector $\hat{z}$ provide a basis for simulating all percentiles including extreme percentiles. In particular suppose $z \sim N(\hat{z}, \Sigma)$ where Σ is the covariance matrix implied by the model. Then $v \sim \Phi_*(z)$ is a simulation of the the v_t. Bringing v into the original order yields (u_t, v_t), a simulation from the copula. Simulated percentiles are mapped back into the original scale as $F_*^{-}(u_t, v_t)$ to provide simulated (x_t, y_t) outcomes.

Table 2: Predicted z_t with conormal copula model

	$s = 0$	$0 < s < \infty$	$s \to \infty$
$\beta = 0$		α_t non parametric probit	$\alpha_0(1 - 2u_t)$ linear probit independence: $\alpha_0 = 0$
$\beta \neq 0, \pm 1$	z_t empirical	$\alpha_t + \beta q_t$ conormal	$\alpha_0(1 - 2u_t) + \beta q_t$ Gaussian if $\alpha_0 = 0$
$\beta = \pm 1$		$\alpha_t \pm q_t$ cointegrated departure	$\alpha_0(1 - 2u_t) \pm q_t$ co/counter monotonic if $\alpha_0 = 0$

6. Copula forcing

Even if data suggests a strict Gaussian copula it may be desirable to force stronger dependence, especially in the tails. This forcing is achieved with

7. Application to copula fitting

The following example illustrates the copula fitting technique.

8. Financial sensitivity and contagion

Suppose the singular value decomposition of the matrix S of sensitivities at a particular q is $S = UDV'$. Here D a diagonal matrix of singular values, arranged in decreasing order along the diagonal and U and V are orthonormal: $U'U = V'V = I$. If the series are independent then $U = D = V = I$. If the series are comonotonic then $U = V = (1, 0)$ while $D = \text{diag}(1, 0, \ldots, 0)$. Hence in the comonotonic case $S = 11'$.

So the best thing is to compare

$$Q = s_{i1}c_{j1} + \cdots + s_{ip}c_{jp}$$

If $Q = I$ then $S = C = I$. If $Q = 11'$, a matrix of ones, then $S = \sqrt{p}(1, 0)$ and $C = (c, 0)$ both have rank 1: here s and c are the leading columns of S and C, respectively and 0 a matrix of zeros.

The approximation $Q \approx sc'$ is exact if Q has rank 1 implying $Q = 11'$. The approximation $Q \approx sc'$ is inadequate if $Q = I$. Inadequacy can be judged from the largest singular value in D, as a proportion of p, the number of variables or rows in Q.

$$Q_* \equiv \frac{1}{p-1}(Q - I) = UDV' \ , \qquad s \equiv Q_* 1 \ , \qquad c \equiv Q_*' 1 \ .$$

$$Q = I + s1' + UDV' \ , \qquad s \equiv \frac{1}{p-1}(Q-I)1$$

Then s is the vector of average VaR_q sensitivity of each variable to all others. Further c is the average VaR_q impact of each variable on all others. The matrices U, D and V define the singular value (svd) decomposition of Q_*, arranged so that diagonal matrix D has the singular values on the diagonal in descending order. If $b = d_1 u_1$ and $c = v_1$ where u_1, d_1 and v_1 are the first column, top diagonal entry, and first column of U, D and V respectively, then for two variables $y \neq x$ in Q,

$$\Delta_x q_y = s_y + b_y c_x + \epsilon_{yx}$$

This states that percentage sensitivities are, apart from the "error" ϵ_{yx}, an average sensitivity plus a scaled response to the contagious effect of the x variable. The contagious effects contained in c are estimated by maximising the explanation of Q.

The vector $1'Q$ sums the changes in VaR_q when each of the column variables is stressed, and writes this as a proportion of the change in the variable being stressed. These proportional sums, subtracting 1 and divided by $p-1$ where p is the number of variables, measures the average contagion of each variable on all others: $c' = (p-1)^{-1}1'(Q-I)$ or $c = (p-1)^{-1}(Q-I)'1$

Alternatively the vector $Q1$ sums the changes in VaR_q of each row variable when all the column variables are stressed. Again it is appropriate to remove the effect of a variable on itself and consider the average over the remaining variables: $s \equiv (p-1)^{-1}(Q-I)1$. If $Q_* \equiv (p-1)^{-1}(Q-I)$ then $s = Q_*1$ and $c = \dot{q}_*1$ are the vectors of sensitivities and contagions, respectively. If $Q = I$ then $s = c = 0$. If $Q = 11'$ then $s = c = 1$.

9. Systemic risk and causal chains

A rank one approximation to the matrix Q is $Q \approx sc'$. Vector c is an index of the contagious impact of each of variables on the others while s measures the sensitivity of each variable to each of the others. The vectors s and c are derived from the singular value decomposition $Q = UDV'$ where $U'U = V'V = 1$ and where s and c are the first column of UD and V respectively, assuming the svd is organised so that the singular values in the diagonal matrix D are organised from largest to smallest. The appropriateness of the summarisation sc' is measured with $\text{tr}(Q - sc')$.

If the variables are independent then $Q = I$ and both s and c equal a column of the identity matrix with sc' a matrix of 0's except in a single diagonal position where it is 1. Then all but one variable has a contagion effect and only that variable is sensitive to the contagion provided by the variable.

If the variables are comonotonic then $Q = 11'$ and $s = 1$ and $c = 1$ where 1 denotes a vector of ones. Thus the rank 1 approximation is exact and each variable is equally contagious and equally sensitive.

Note that

$$Q^n \approx (s'c)^{n-1} sc' = \ ,$$

If the random variables are independent the $Q-I=0$ and $a=b=k=0$ and there is no error in the first order svd approximation. If the random variables are comonotonic then $Q=11'$ and $Q-I$ has ones everywhere except on the diagonal where it is zero. The vector of row means is then $p^{-1}(p-1)1$

Systemic risk in the system is measured with $b'k = d_1(u_1'v_1)$. In the case of comonotonic random variables $Q=11'$, $a=1$ and x, where p is the number of variables, $d_1=1$.

Furthermore we may define quantities such as $u^- \equiv \mathrm{VaR}_q(v|u \leq q)$ measuring the impact of a non distressed state in v. For brevity we do not dwell on these constructs in this writeup although the ramifications and potential uses of these constructs will be investigated in the research.

10. Generalising Sklar's Theorem

Skalr's Theorem states that if $F(x)$ is a a joint then there exists a C such that $F = C \circ F_*$ where $C_F = F$

11. Literature

We propose a measure for systemic risk: CoVaR, the value at risk (VaR) of financial institutions conditional on other institutions being in distress. We define an institution?s (marginal) contribution to systemic risk as the difference between CoVaR and the financial system?s VaR. From our estimates of CoVaR for characteristic-sorted portfolios of publicly traded financial institutions, we quantify the extent to which characteristics such as leverage, size, and maturity mismatch predict systemic risk contribution. We argue for macro-prudential regulation based on the degree to which such characteristics forecast systemic risk contribution.

12. Econometric implementation

The above development sets out our proposed broad framework for linking bivariate copulas and marginals to external variables and shocks study the impact of the same on stresses within the system and the contagious effects of crises. Proposed econometric analysis will implement and extend Brownlees and Engle (2010).

13. Data

We will employ publicly available data as published by APRA and other regulators.

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