

Systemic risk and contagion effects in Australian financial institutions and sectors

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Abstract

This article refines, builds on and extends SRISK methodology recently proposed in the literature. The refinement is to define SRISK in terms of a put on the Basel shortfall. The definition is built on by defining the systemic beta of a firm in terms of the departure of a stressed expectation from the normal expectation the put. By stressed expectation is meant an expectation that averages conditional expectations over outcomes defined by a stressor variable combined with a stressor function. The stressor variable and function are chosen by the practitioner and include functions of a market wide variable. Since stressed expectations are linear, a system systemic beta is naturally defined as linear in the firm specific systemic beta's. The methodology generalises automatically to situations where systemic stress arises from the interaction of multiple system wide variables. Application is made to Australian financial data.

1. Literature review

The starting point for the proposed research is the recent literature and the CIFR targeted areas and APRA aims and functions. This recent literature includes the following Adrian and Brunnermeier (2011), Acharya et al. (2012), Acharya et al. (2012) and Brownlees and Engle (2010). The proposed research aims to extend and apply these techniques particularly in relation to the entities regulated by APRA. Thus our broad aim is to develop, implement and bring to bear recent developments in stress testing on the aims of APRA and the CIFR targeted research areas detailed above.

2. Capital shortfall and leverage

If d_{it} and w_{it} are the debt and equity of a firm i at time t and k is the capital requirement, then the capital shortfall at time t is

$$k(d_{it} + w_{it}) - w_{it} = kd_{it} - (1 - k)w_{it} = kd_{it}(1 - e^{-\ell_{it}}) , \quad (1)$$

where

$$\ell_{it} \equiv \ln \frac{kd_{it}}{(1-k)w_{it}} = \ln \frac{d_{it}}{w_{it}} + \lg(k) , \quad \lg(k) \equiv \ln \frac{k}{1-k} .$$

The quantity ℓ_{it} is called k -log-leverage and $\ell_{it} > 0$ implies capital shortfall (1) is positive. Basel II assumes¹ $k = 0.08$ implying $\lg(k) = -2.44$ and ℓ_{it} is the actual log-leverage minus 2.44. As the actual log leverage of a firm increases to 2.44, the firm approaches a Basel breach.

Figure 2 displays Basel log-leverages for the four major and four minor Australian banks listed in Table 1 on the first trading day of each month from January 2003 through to December 2014. Note most banks are Basel compliant up to about 2008.

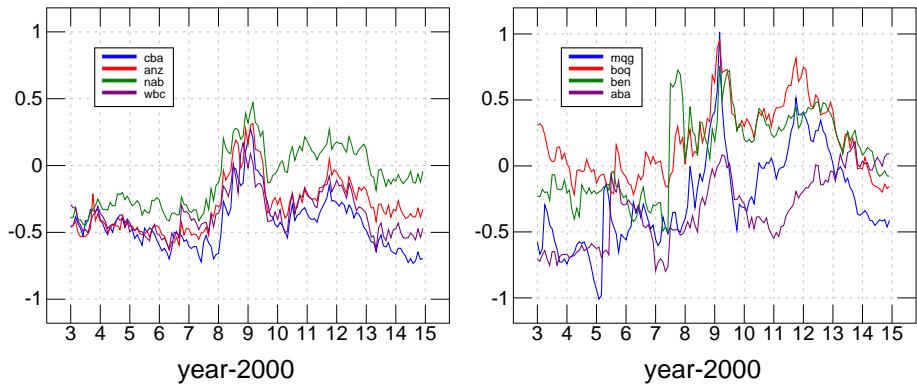


Figure 1: k -log-leverages for four major (left panel) and four minor (right panel) Australian banks from 2003 through to end of 2014. Note all banks contravene the Basel II threshold (corresponding to $\ell_{it} = 0$) around the start of 2009.

3. Future capital shortfall

Financial institutions and regulators are concerned with future Basel compliance and in assessing the likelihood of a future Basel breach. Future compliance depends on the future return on equity. Consider firm i at time $t+h$ where $h > 0$. Then²

$$\ell_{i,t+h} = \ln \frac{d_{i,t+h}}{w_{i,t+h}} + \lg(k) = \ell_{it} - \nu_{it} , \quad \nu_{it} \equiv \ln \frac{w_{i,t+h}}{w_{it}} ,$$

¹See for example Brownlees and Engle (2015). The value of k can and is varied to test for sensitivity etc.

²In the further development ν denotes a continuously compounded rate of return while e^ν is the continuously compounded return.

where it is assumed $d_{i,t+h} = d_{it}$. Hence ν_{it} is the rate of return on equity over $(t, t+h)$. The future return is unknown at time t but its distribution may be well understood.

The Basel II shortfall at time $t+h$ is,

$$d_{it}k(1 - e^{\nu_{it}-\ell_{it}}) , \quad (2)$$

and the shortfall decreases with ν_{it} . The Basel II limit is reached at $t+h$ if $\nu_{it} = \ell_{it}$ and breached if $\nu_{it} < \ell_{it}$. A breach is unlikely if ℓ_{it} is low i.e. either the log-leverage is low, or the constant k is low.

4. Systemic risk – SRISK

Brownlees and Engle (2015) define the systemic risk (SRISK) of a group of firms at time t as

$$k \sum_i d_{it} |1 - E_\phi(e^{\nu_{it}-\ell_{it}})|^+ , \quad |x|^+ \equiv \max(0, x) , \quad (3)$$

where k is the Basel constant, d_{it} and ν_{it} are, respectively, the debt and the rate of future return of firm i at time t , and E_ϕ denotes conditional expectation given a severe general market downturn. The contribution of firm i is thus a put on the conditional expected value of the Basel II shortfall of firm i at $t+h$.

The proportionate contribution of firm i to (3) is called the systemic risk of firm i at time t :

$$\text{SRISK}_{it} \equiv \frac{|1 - E_\phi(e^{\nu_{it}-\ell_{it}})|^+}{E\{|1 - E_\phi(e^{\nu_{it}-\ell_{it}})|^+\}} , \quad (4)$$

where E denotes debt weighted averaging. A large SRISK_i indicates a systemically important firm. The expression (4) depends on k only through its logit. The values of the “puts” $|1 - E_\phi(e^{\nu_{it}-\ell_{it}})|^+$ in (4) are known at time t : there is no uncertainty except for possible estimation uncertainty as to the conditional expected value.

If the distribution of $\nu_{it} - \ell_{it}$ is concentrated on the negative axis then $e^{\nu_{it}-\ell_{it}} < 1$ and

$$|1 - E_\phi(e^{\nu_{it}-\ell_{it}})|^+ = E_\phi(|1 - e^{\nu_{it}-\ell_{it}}|^+) .$$

If there is positive the probability of $\nu_{it} > \ell_{it}$ then the last relation only holds approximately. The SRISK contribution of a firm is zero whenever

$$\ln E_\phi(e^{\nu_{it}}) \leq \ell_{it} . \quad (5)$$

This inequality often holds even if the conditional expectation conditions on an extreme market downturn. If the conditional distribution of ν_{it} is normal with mean μ and variance σ^2 then

$$\ln E_\phi(e^{\nu_{it}}) = h\mu + h^2 \frac{\sigma^2}{2} ,$$

and (5) is equivalent to

$$\mu + h \frac{\sigma^2}{2} \leq \frac{\ell_{it}}{h} ,$$

indicating a Basel breach is unlikely if ℓ_{it} is very negative equivalent to the log-leverage well below 2.44, the Basel limit.

The system SRISK is zero if all firm the SRISK contributions are zero that is if (5) holds for every i . This would be usual in the Australian case.

5. Improved systemic risk measurement

For firm i define the Basel put and Basel risk as

$$p_{it} \equiv k |1 - e^{\nu_{it} - \ell_{it}}|^+ , \quad E(p_{it}) \geq 0 , \quad (6)$$

where E denotes normal or unstressed expectation. The Basel risk in firm i depends only k , the present leverage, and the lower tail of the ν_{it} distribution. The put value (6) is stated in terms of unit debt and increases with k , both directly from multiplication by k and indirectly through its logit, making a positive put outcome more likely.

Default put options similar to (6) have been discussed in the insurance literature as critical to an evaluation of a firm: see for example Merton (1977), Doherty and Garven (1986), Cummins (1988), Myers and Read (2001) and Sherris (2006).

The total Basel put value and hence Basel risk for firm i is $d_{it}E(p_{it})$. This is the cost of insuring firm i does not breach the Basel standard. In practice it may be appropriate to discount p_{it} in (6) by the interest rate over the period $(t, t+h)$ and value the put with risk neutral rather than normal expectation.

The put p_{it} is tradable since, given k , it relies only on the present leverage and the future return ν_{it} on equity w_{it} . Market participants can value the put in the same way as any other put and use the contract to diversify risk. Firms can buy puts in the market to hedge Basel default risk.

However regulators will value the puts p_{it} using a stressed, rather than risk neutral, expectation. Stressed expectations correspond closely to conditional expectation given a systemically important event or events. Regulators are concerned with the possibility of Basel breaches if such events occur and hence have an overbearing interest in monitoring stressed expectations. In the extreme regulators can charge firms all or a proportion of their stressed put values if there are implicit government bailout guarantees.

In terms of the put values $E(p_{it})$, the system Basel risk is defined as the debt weighted average of the individual firm's put values:

$$\mathcal{E}\{E(p_{it})\} = E\{\mathcal{E}(p_{it})\} , \quad (7)$$

since both E and \mathcal{E} are linear. The system Basel risk is stated per unit of system debt. The dollar value of Basel risk is the Basel risk multiplied by the

total system debt:

$$d_t \mathcal{E}\{\mathbb{E}(p_{it})\} = \sum_i d_{it} \mathbb{E}(p_{it}) , \quad d_t \equiv \sum_i d_{it} .$$

Thus the monetary Basel risk is a weighted average of firm specific monetary Basel risks.

6. Systemic stress and stressed expectation

In the above development \mathbb{E} denotes unstressed expectation: that is expectation that does not assume a stressful scenario. A stressed expectation, denoted \mathbb{E}_ϕ , is a linear combination of expectations, each assuming some level of stress

$$\mathbb{E}_\phi(p_{it}) \equiv \mathbb{E}\{\phi \mathbb{E}(p_{it}|\phi)\} = \mathbb{E}(\phi p_{it}) = \mathbb{E}(p_{it}) + \text{cov}\{\mathbb{E}(p_{it}|\phi), \phi\} . \quad (8)$$

Here $\phi \geq 0$ with $\mathbb{E}(\phi) = 1$ is thought of as a stress factor or “stressor,” defining events impacting p_{it} and serving as a basis for stress testing. Thus ϕ -stressing changes the expected value $\mathbb{E}(p_{it})$ by the covariance between the conditional expectation $\mathbb{E}(p_{it}|\phi)$ and the stressor ϕ .

The conditioning variable ϕ downplays or highlight different scenarios. The conditioning events are scenarios of interest and probabilistic weighting is according to the level of interest. The first equality in (8) from the definition of conditional probability (Whittle, 2000). The second equality follows from

$$\text{cov}(p_{it}, \phi) = \text{cov}\{\mathbb{E}(p_{it}|\phi), \phi\} .$$

The covariance is zero whenever p_{it} is independent of ϕ which occurs if $\phi = 1$.

The right hand side expression in (8) splits up the stressed expectation $\mathbb{E}_\phi(p_{it})$ into two components: “volatility stress” $\mathbb{E}(p_{it})$ and “systemic stress” $\text{cov}\{\mathbb{E}(p_{it}|\phi), \phi\}$. The volatility stress is the normal expectation of the put and reflects current, known, volatility. If the firm is currently Basel compliant than low current volatility implies minimal volatility stress. As current volatility increases, the put value rises reflecting the current increasing uncertainty, and in particular uncertainty as to tail events. This increasing uncertainty may be due to a variety of factors. However it is not stress arising from, for example, a given dire future financial scenario.

The second, systemic stress, component $\text{cov}\{\mathbb{E}(p_{it}|\phi), \phi\}$ in the right hand side of (8) does reflect the effect of a future dire financial scenario. It represents the increase in put values on account of the dire financial prognosis modelled with ϕ . Typically ϕ is assumed to impact a variety of firms i and hence its effect is systemic.

To a Bayesian the stressed expectation (8) is the mean of p_{it} in the posterior density

$$\int \phi f(p_{it}|\phi) f(\phi) d\phi = \int \phi f(p_{it}, \phi) d\phi = \mathbb{E}(\phi|p_{it}) f(p_{it}) \quad (9)$$

where f denotes density. Note (9) is a density since $\phi \geq 0$ and $E(\phi) = 1$. Any outcome p_{it} such that $E(\phi|p_{it})$ is large is amplified and vice versa. Further

$$E_s(p_{it}) = E \left\{ p_{it} \frac{\phi f(p_{it})}{f(p_{it})} \right\},$$

that is the density $f(p_{it})$ of p_{it} is replaced by the density $\phi f(p_{it})$. Hence more importance is given to those outcomes where ϕ is large.

7. Systemic beta's

Writing σ_ϕ as the standard deviation of ϕ , define the systemic beta of firm i with respect to stress ϕ as

$$\beta_{it} \equiv \frac{E_\phi(p_{it}) - E(p_{it})}{\sigma_\phi} = \frac{E\{(\phi - 1)p_{it}\}}{\sigma_\phi} = \frac{\text{cov}\{E(p_{it}|\phi), \phi\}}{\sigma_\phi}, \quad (10)$$

where σ_ϕ is the standard deviation of ϕ . Thus β_{it} is the systemic stress component in (8) standardised by σ_ϕ and

$$E_\phi(p_{it}) = E(p_{it}) + \beta_{it}\sigma_\phi, \quad E(p_{it}|\phi) \approx E(p_{it}) + \beta_{it}\frac{\phi - 1}{\sigma_\phi}. \quad (11)$$

The second, approximate, relationship is suggested from a regression interpretation of $\text{cov}(p_{it}, \phi)/\sigma_\phi^2$ as the regression coefficient of p_{it} on ϕ .

The expressions in (11) suggests two ways of thinking about the systemic beta's β_{it} . First, the left hand side equation shows systemic stress β_{it} serves to shift the mean of put prices by, on average, σ_ϕ . The shift $\beta_{it}\sigma_\phi$ is the market price of stress embedded in firm i at time t . Firms with large shifts embody stress risky firms.

The second interpretation is based on the right hand side approximation in (11) showing the change in the expected Basel put price if σ_ϕ units of stress ϕ are applied. The quantity $(\phi - 1)/\sigma_\phi$ is thought of as a "stress factor." The stress factor has mean 0 and standard deviation 1 and is scaled by β_{it} to yield the actual stress effect on p_{it} . Thus β_{it} is thought of a usual finance type "beta" with respect to the given ϕ . Stress is measured in standardised units. Units of stress take on different meaning depending on ϕ as discussed below. The distribution of ϕ determines the distribution of the stress factor. If the stress factor is normally distributed then a stress effect more than doubling of the put value occurs less than about 2.5% of the time.

Note the systemic beta's are stated in terms of additions to the normal put price $E(p_{it})$. Thus in an increasingly dire financial situation $E(p_{it})$ will increase. However stress β_{it} measures the change in current put values under further ϕ -stress. This varies from Brownlees and Engle (2015) where SRISK $_i$ does not distinguish between the current, possibly high, put prices $E(p_{it})$ and the further increment due to potential stress. Thus in an increasingly dire financial situation put prices are liable to increase on account of decreasing expected returns and

increasing volatility. However our definition of systemic stress using systemic beta's measures the further effect on account of potential extra stress imposed onto the system over and above the already existing stress. The systemic beta's capture "marginal" effects.

The monetary stress of a firm is $d_{it}\beta_{it}$ and represents the change in the monetary value of the Basel put if stress, as captured with ϕ , is applied.

8. Systemic beta for a group of firms

A measure of the total systemic stress is to consider the debt weighted put

$$p_t \equiv \mathcal{E}(p_{it}) \quad (12)$$

This "system" put (12) pays out zero if all firms meet the Basel requirement with payments increasing with the number of Basel breaches, the sizes of the breaches, and the relative debt sizes of the breaching firms.

In terms of (12), the systemic beta β_t for the system is defined as

$$\beta_t \equiv \frac{\mathbb{E}_\phi(p_t) - \mathbb{E}(p_t)}{\sigma_\phi} = \frac{\text{cov}[\mathbb{E}\{\mathcal{E}(p_{it})|\phi\}, \phi]}{\sigma_\phi} = \mathcal{E}(\beta_{it}). \quad (13)$$

Thus the system systemic beta is the debt weighted average of firm systemic beta's. The equalities in (13) follow from the fact that \mathbb{E} and \mathbb{E}_ϕ and \mathcal{E} are all linear. Similar to before, $\mathbb{E}(p_t) = \mathcal{E}\{\mathbb{E}(p_{it})\}$ is the volatility stress in the system while $\text{cov}\{\mathbb{E}(p_t|\phi), \phi\} = \mathcal{E}[\text{cov}\{\mathbb{E}(p_{it}|\phi), \phi\}]$ is the system systemic stress.

The monetary value system stress at time t is the sum of monetary firm stresses

$$d_t\beta_t = d_t\mathcal{E}(\beta_{it}) = \sum_i d_{it}\beta_{it} .$$

9. Determining systemic beta's via simulation

Firm specific stress β_{it} and system stress can be determined using simulation. For the former

$$\frac{1}{\sigma_\phi S} \sum_{s=1}^S (\phi^s - 1)p_{it}^s \rightarrow \beta_{it} , \quad S \rightarrow \infty , \quad (14)$$

where S is the simulation effort. The approximation is increasingly accurate as S becomes large. In this simulation, the (ϕ^s, p_{it}^s) are pairs of stress and put values generated from a model: for example ϕ^s may be zero unless there is a market drop below the α -percentile in which case it is $1/\alpha$. In this latter case the simulation (14) averages over all simulated put values p_{it}^s paired with an at least α -drop in the market.

10. Systemic beta's based on market return

The setup is illustrated with examples generalising and fine-tuning the approach of Engle.

10.1. Market return below a percentile threshold

Suppose the stress event is a market return in the bottom α -tail of the market return distribution. Then $\phi(u) = 1/\alpha$ for $u < \alpha$ and 0 otherwise and

$$E_\phi(p_{it}) \equiv E\{\phi E(p_{it}|\phi)\} = \frac{1}{\alpha} \int_0^\alpha E(p_{it}|u) du = E(p_{it}|\nu_{mt} < c_t) .$$

where c_t cuts out α probability in the lower tail of the ν_{mt} distribution. Then

$$\beta_{it} \approx \frac{e^{(\lg t \alpha)/2}}{S/\alpha} \sum_{u_{mt}^s < 1/\alpha} p_{it}^s ,$$

where the estimate of σ_ϕ implicit in the final expression is established using a direct calculation. In the right hand side $s = 1, \dots, S$ denotes simulations. The approximation decreases with the simulation effort S . The effective simulation effort is S/α and hence α small requires a large effort.

A fixed α cutoff implies the actual threshold depends on the distribution and hence on volatility and time. The actual threshold drop in the market $c_t = F_{mt}^-(\alpha)$, where F_{mt}^- is the inverse distribution function of ν_{mt} . The volatility associated with the latter distribution is approximately $\sqrt{h}\sigma_{mt}$ where σ_{mt} is current market volatility. Regulators and practitioners are well versed in working with VaR type calculations and hence a varying VaR type cutoff combining market volatility and stress is closely aligned to current practice.

10.2. Expected worst market return in n identical scenarios

If $\phi(u) = n(1-u)^{n-1}$ then $\phi(u)du = d\{1 - (1-u)^n\}$ and

$$E_\phi(p_{it}) \equiv E\{\phi E(p_{it}|\phi)\} = \int_0^1 E(p_{it}|u) d\{1 - (1-u)^n\} .$$

If u is the market return percentile then $1 - (1-u)^n$ is the distribution of the worst percentile outcome in n identical trials and hence the stressed expectation is that of the expected put price given the worst percentile market return in n identical trials. Further $\sigma_\phi = (n-1)/\sqrt{2n-1}$ and

$$\beta_{it} \approx \frac{n\sqrt{2n-1}}{(n-1)S} \sum_s (1 - u_{mt}^s)^{n-1} p_{it}^s .$$

Simulated returns ν_{mt}^s in the upper tail of the distribution have percentiles $u_{mt}^s \approx 1$ and hence for these returns the first factor in the sum is small if not negligible: the associated simulated put is heavily downweighted.

Note the contrast with the previous example where the bottom α proportion of simulated market returns are selected as the stressed sample. With the current specification for ϕ , every simulated value contributes to the stress computation, albeit with different weights.

10.3. Expected worst market return given a tail event

The above two situations can be combined. Suppose $\phi(u) = c(0.05 - u)^{19}$ for $u \leq 0.05$ and 0 otherwise and where c is such that $\phi(u)$ integrates to 1. Then

$$E(p_{it}) \approx \frac{1}{S/c} \sum_{u_{mt}^s < 1/20} \left(u_{mt}^s - \frac{1}{20} \right)^{20} p_{it}^s .$$

Similar to the first example, the stressed sample picks up the bottom 5% of market returns. The bottom 5% of market returns is further stressed by progressively downweighting returns as the percentile approaches 1/20. Large simulation effort S is required for a reasonable approximation since S/c is the effective simulation size and c is small.

10.4. Stress functions as weighted linear combinations of tail events

Brownlees and Engle (2015) define a systemic event as a market based downturn greater than a certain threshold and a systemic event depends on the chosen threshold. This arbitrariness is partially sidestepped by choosing a decreasing function $\phi = \phi(u_{mt})$ on the unit interval and noting

$$E\{\phi E(p_{it}|\phi)\} = -E\{u\phi'(u)E(p_{it}|u_{mt} \leq u)\}. \quad (15)$$

Thus a stressed expectation, stressed with a decreasing function of the market return, is equivalent to taking a weighted average of conditional lower tail expectations. This circumvents an explicit choice for the market threshold, enabling a mixture of thresholds. The equivalence of the left and right hand sides of (15) follows from

$$\int_0^1 \phi'(u) \int_0^u E(\nu_{it}|u_{mt} = v) dv du = \int_0^1 E(\nu_{it}|u_{mt} = v) \int_v^1 \phi'(u) du dv ,$$

provided $\phi(1) = 0$.

10.5. Copula stress functions

Stress can be based on a vector of variables x with distribution $F(x) = u$ and marginal densities $f(x_j)$. Suppose $c(u)$ is the copula density of x implying the density of x is $c(u) \prod_j f(x_j)$. If $\Phi(u)$ is another copula with density $\phi(u)$ consider

$$E_\phi(p_{it}) \equiv \int E(p_{it}|x) \phi(u) c(u) \prod_j \{f(x_j)\} dx . \quad (16)$$

If $\phi(u) = 1$ then $E_\phi(p_{it}) = E(p_{it})$, the ordinary expectation. The density $\phi(u)$ magnifies potentially stressful situation such as where all component of x are highly tail dependent. In the extreme case $\Phi(u) = \min_j(u_j)$ making the variables in x comonotonic.

The copula density stressor $\phi(u)$ can be combined with marginal stressors written as functions of x_j or percentiles u_j . If the latter then $\phi(u) \prod \phi(u_j)$

is the total stressor, corresponding to a density on the unit hypercube with non-uniform marginals $\phi(u_j) \neq 1$.

Copula stress effects are simulated as follows. Suppose x is a vector ν_{mt} of market factors over the period $(t, t+h)$. Simulations of a joint model are used to derive vectors (p_{it}^s, ν_{mt}^s) , $s = 1, \dots, S$. The ν_{mt}^s are converted to percentiles u^s and scalars

$$\phi^s \equiv \phi(u^s) \prod_j \phi(u_j^s), \quad s = 1, \dots, S.$$

The stress beta is then estimated as in (14).

Marginal distributions of r_m and r_n may be altered or unchanged. If unchanged then

$$f(r_n) = \int f(r_m, r_n) dr_m = \int f_\phi(r_m, r_n) dr_m$$

or

$$1 = \int f(r_m) c(u_m, u_n) dr_m = \int f(r_m) \phi(r_m, r_n) c(u_m, u_n) dr_m$$

since $f(r_m, r_n) = f(r_m) f(r_n) c(u_m, u_n)$.

10.6. Scenario based stress testing

Stress events captured with ϕ and used in (10) can be defined with respect any events, including a discrete number of scenarios labelled $k = 1, 2, \dots$. In this case

$$E_\phi(p_{it}) = \sum_k \pi_k \phi_k E(p_{it}|k), \quad \sum_k \pi_k \phi_k = E(\phi) = 1,$$

where ϕ_k is the weight assigned to scenario k . The $\pi_k \phi_k \geq 0$ are modified probabilities which weigh different scenarios according to level of interest. For example in standard stress testing the ϕ_k are chosen such that significant weight is given to scenarios k causing high firm distress. The “real world” probabilities π_k are usually ignored by simply choosing $\pi_k \phi_k$ to be of required magnitude.

In the discrete case β_{it} compares the expected value under an average of distress scenarios to the average without distress. The quantity σ_ϕ is now of limited relevance, signalling the volatility in the distress probabilities.

10.7. Market return

(Not sure if this will be of interest). A final simple stress function is $\phi = e^{-\lambda \nu_{mt}} \approx 1 - \lambda \nu_{mt}$ where λ is a parameter. In this case systemic stress models both positive or negative market impacts. (Not sure if this will be of interest).

If the forward return ν_{mt} is approximately normal with mean $h\mu_{mt}$ and variance $h\sigma_{mt}^2$ then

$$\beta_{it} \equiv \frac{\text{cov}(p_{it}, \phi)}{\sigma_\phi} \approx \frac{-\lambda \text{cov}(p_{it}, \nu_{mt})}{e^{\lambda h(\mu_{mt} + \sigma_{mt}^2/2)} \sqrt{e^{\lambda h \sigma_{mt}^2} - 1}},$$

(Need to put λ in here somewhere – Weihao can you check all this?).

11. Risk weighted asset valuation

Risk weighted asset calculations recognise that some assets are more susceptible to devaluation in times of stress. In a firm valuation risky assets are valued less and the resulting leverage is used to assess Basel compliance.

The systemic stress framework developed here can be used to formalise the risk weighting framework and quantify risk weights. Suppose total assets of firm i at time t are

$$a_{it} \equiv d_{it} + w_{it} = \sum_j a_{ijt} ,$$

where a_{ijt} is the value of the class j assets at time t . Similarly the total value assets at time $t+h$ is, denoting ν_{ijt} as the return on asset class j over the period $(t, t+h)$,

$$a_{i,t+h} = a_{it} \sum_j \frac{e^{\nu_{ijt}} a_{ijt}}{a_{it}} = a_{it} \mathcal{E}_a(e^{\nu_{ijt}}) ,$$

where \mathcal{E}_a denotes asset weighted averaging using asset values at time t . The log of the debt to equity ratio at time $t+h$ is thus, assuming debt is constant over the period,

$$\ell_{i,t+h} = -\ln \frac{a_{i,t+h} - d_{it}}{d_{it}} = -\ln \{(1 + e^{-\ell_{it}}) \mathcal{E}_a(e^{\nu_{ijt}}) - 1\} ,$$

and the Basel put equals, using the previous notation $\ell_{it}^* \equiv \ell_{it} + \text{lgt}(k)$,

$$p_{it} = k |1 - e^{\ell_{i,t+h}^*}|^+ = k \left| 1 - \frac{e^{\text{lgt}(k)}}{(1 + e^{-\ell_{it}}) \mathcal{E}_a(e^{\nu_{ijt}}) - 1} \right|^+ . \quad (17)$$

A return sample size S is assembled by simulating asset specific forward returns ν_{ijt}^s and stress factor ϕ^s and combining to arrive at stressed put prices. Initially assume the stress is a market return in the bottom α -tail of the market return distribution. Then realisations with market return in the lower tail are used to compute an average of put values (17). The average approaches $E_\phi(p_{it})$ as S becomes large. The average put value across all S is subtracted from the stressed sample average and divided by σ_ϕ to arrive at the estimate of β_{it} .

For more general ϕ it is convenient to estimate β_{it} using (14). Thus given ϕ , simulated asset returns ν_{ijt}^s are used to compute the corresponding simulated put value p_{it}^s as in (17) and weighted market percentile $\phi^s \equiv \phi(u_{mt}^s)$. These are then combined as in (14). If required, the stressed expectation is estimated using

$$E_\phi(p_{it}) = \mu_{it} + \sigma_\phi \beta_{it} , \quad \frac{1}{S} \sum_s p_{it}^s \rightarrow E(p_{it}) \equiv \mu_{it} .$$

The above framework formalises risk weighted asset methodology used in capital setting. Easily (dis)stressed assets have a return distribution sensitive to market downturn. This implies, in a downturn, $e^{\nu_{ijt}}$ is likely to be much less than 1 and the asset value a_{ijt} is downweighted in the computation of $\mathcal{E}(e^{\nu_{ijt}})$. In

turn the put price is increased. The responsiveness of the put price depends on the proportion a_{ijt}/a_{it} of assets in class j held by firm i at time t . Thus the $e^{\nu_{ijt}}$ can be viewed as risk weights where the averaging over put values corresponds to an implicit average risk weight. The implicit weights can be backed out to give risk weighted assets

$$a_{it}\mathcal{E}_a\left(\frac{1}{S}\sum_s \phi^s e^{\nu_{ijt}^s}\right) \rightarrow \mathbb{E}_\phi\{\mathcal{E}_a(e^{\nu_{ijt}})\}, \quad S \rightarrow \infty. \quad (18)$$

The expression in brackets in the left is the risk weight for asset j . Assets whose return ν_{ijt} tends to be very negative under stress ϕ have high risk and low weight. An asset with no volatility has $\nu_{ijt} \equiv 0$ and a risk weight of 1 since, on average, ϕ^s is 1.

The difference

$$k - \frac{1}{S} \sum_s \phi^s \mathcal{E}_a(e^{\nu_{ijt}^s}),$$

is the proportionate increase capital shortfall on account of risk weighting and as a proportion of total assets.

12. System resilience

System resilience aims to answer the question whether the system as a whole can absorb shocks. The capacity to absorb relies on implicit merging. When two firms merge the leverage of the resulting firm is less than that of the more highly leveraged firm. The merged firm is more able to withstand return on equity shocks unless negative shocks are more prone for the merged firm. Similarly when many firms merge Basel puts for the conglomerate become less valuable.

Firms do not necessarily merge, even if one or other is in dire financial circumstances. Hence the mergers spoken of here are hypothetical. From a regulators perspective, what would happen under a merger of all firms (or perhaps a group of firms) is nevertheless of interest. A system as as a whole near Basel non-compliance is more threatening than one where a few firms are near Basel non-compliant but the system itself is well away from non-compliance.

System compliance is partially monitored with the system put $p_t = \mathcal{E}(p_{it})$ and its stressed average $\mathbb{E}_\phi(p_t)$. The system put however simply monitors a debt weighted average of puts. It gives no guidance as to how much risk can be diversified away. Basing a put on system wide aggregated debt and equity and comparing the same to p_t gives at least some guidance.

To formalise the discussion suppose aggregate debt and equity for the system are d_t and w_t , respectively. The adjusted system log leverage is

$$\ell_t = \ln \frac{d_t}{w_t} + \text{lgt}(k) = \ln \sum_i \frac{w_{it}}{w_t} \left(\frac{d_{it}}{w_{it}} \times \frac{k}{1-k} \right) = \ln \mathcal{E}_w(e^{\ell_{it}}),$$

where \mathcal{E}_w denotes equity weighted averaging. A system wide breach occurs at $t + h$ if $\nu_t < \ell_t$ where ν_t is the forward rate of return on total equity w_t :

$$\nu_t = \ln \frac{w_{t+h}}{w_t} = \ln \mathcal{E}_w \left(\frac{w_{i,t+h}}{w_{it}} \right) = \ln \mathcal{E}_w(e^{\nu_{it}}), \quad (19)$$

and define a default put for the aggregate system

$$\pi_t \equiv k |1 - e^{\nu_t - \ell_t}|^+ = k \left| 1 - \frac{\mathcal{E}_w(e^{\nu_{it}})}{\mathcal{E}_w(e^{\ell_{it}})} \right|^+, \quad \pi_t \leq p_t \equiv \mathcal{E}(p_{it}).$$

(Weihao, can you prove the inequality?) The inequality is a result of diversification effects: low liquidity in one firm is offset by high liquidity in other firms.

A measure of the resilience of the system is

$$E_\phi(p_t) - E_\phi(\pi_t) = E(p_t - \pi_t) + \text{cov}\{E(p_t - \pi_t|\phi), \phi\}. \quad (20)$$

The first term on the right is the volatility resilience: that is the ability of the system to withstand – in aggregate – shocks from volatility. The covariance term on the right in (20) is the systemic stress resilience: that is whether systemic shocks are diversifiable.

13. Simulating future capital shortfall and market return

The estimation of systemic beta's requires projections of future capital shortfalls and future stressors. As in Brownlees and Engle (2015) in this article projections are constructed using time series models modelling the forward rates of return ν_{it} for firms i and ν_{mt} for the market. The market return is used as the stressor with different choices of the stress function ϕ modelling different stress scenarios.

The time series models used for projecting are stochastic volatility models based on the GARCH–DCC model of Engle (2002). and summarised in Appendix 18. It must be emphasised however that the GARCH–DCC framework is only one possible implementation. For example future return scenarios may be constructed in a more ad-hoc manner e.g. judiciously chosen scenarios decided on by regulators or policy makers. This emphasises that the systemic beta framework can be based on any method of generating future scenarios.

Figure 2 displays two snapshot outputs from GARCH–DCC projections: 6000 simulated one month ahead projected returns for CBA and the market, on two dates: the first trading day in January 2009 and the first trading day in December 2014. Both panels use the same horizontal and vertical scales. Thus based on the available data at those two points in time, policy makers and regulators faced entirely different projections based on the same time series model. The scatter of dots are the forward simulations. Note there is no lookahead bias as the model at each time point is based on the available data to that point in time. The red horizontal line in each panel indicates the projected Basel

limit: any firm return less than the limit would constitute a Basel breach. In the left panel there is little confidence that the Basel limit will be reached since it requires a return on equity approaching 20% – with confidence less than xxx the Basel standard will be breached. In the right panel there is no chance of a Basel breach under any of the projected return on equity view – the bank is “safe.”

The black dots in each panel indicate the actual outcome after one month. In the left panel the outcome is a decline in both the market and stock price. In the right panel the returns are xxx and xxx. Thus at the beginning of January 2009 the CBA bank was projected to be far from compliance in one month’s time while in December 2015 the one month projection is of complete compliance.

The two panels thus display very different volatility and slopes. These snapshots are used to compute systemic stress. Appendix xxx details the GARCH-DCC model used in this study.

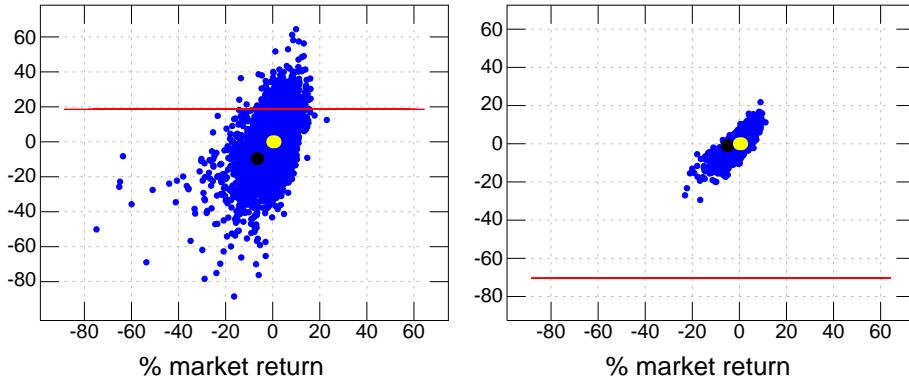


Figure 2: Six thousand simulated bivariate 1 month forward rate of returns (vertical axis) for CBA and the market rate of return (horizontal axis) at start of January 2009 (left panel) and December 2014 (right panel). Note scales in both panels are the same, the relatively large volatility in the left panel, and the left skew in the marginal distributions. Black and yellow dot in each panel indicates realized forward rate of return and origin, respectively. Red lines indicate Basel breach points – any firm return below the red line indicates a Basel breach given the bank leverage on the applicable date.

While the above discussion suggests, as obvious after the fact, that in December 2009 the CBA bank was under much stress. However the above calculations do not actually reflect real stress. In terms of the development of the previous sections the actual stress is beta given a stressor function ϕ and stressor variable. In this case the stressor variable is the market return, on the horizontal axis in each panel. Stress occurs if the market return is low. In the left panel low market outcomes are likely to lead to sharply lower CBA bank returns, as compared to the right panel. This is suggested by the slope of the scatter plots: the left slope appears steeper than the right and hence any stress in the market – in the sense of a market downturn is expected to lead to a larger drop in the

CBA bank return in the left scenario as compared to the right scenario. The magnitudes of the two responses depends on the stressor function. In essence the stressor function formalises the notion as to what constitutes a “big” drop. If a big drop is the expected worst in 20 similar market scenarios then the stress in the left and right are xxx and xxx, respectively.

14. Application to Australian financial institutions

In this study we consider financial data for the 8 Australian banks.

Table 1: Four major and four minor Australian banks

cba	Commonwealth Bank of Australia
anz	Australia & New Zealand Bank
nab	National Australia Bank
wbc	Westpac Banking Corporation
mqg	Macquarie Banking Group
boq	Bank of Queensland
ben	Bendigo and Adelaide Bank
aba	Auswide Bank

The analysis is based on daily return data is collected from 3 April 2000 through to 1 December 2014 for a total of 3848 trading days. Prices, adjusted for dividends are plotted in Figure 3.

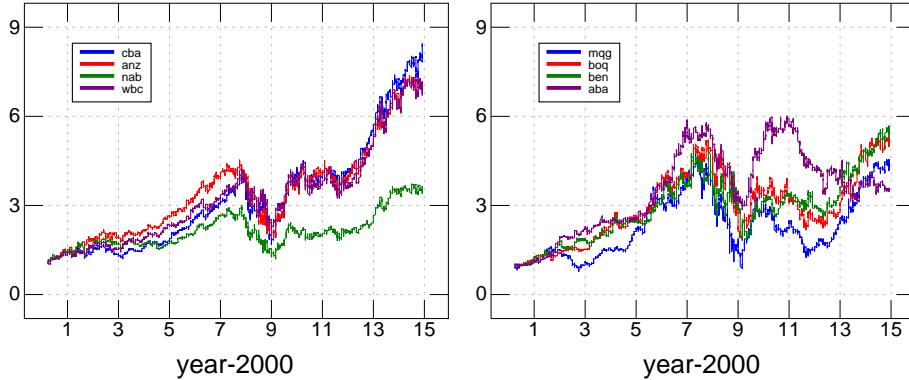


Figure 3: Basel log-leverages for four major (left panel) and four minor (right panel) Australian banks from April 2000 through to December 2014. Note all banks suffer falls starting the middle of 2008.

At each of the first trading day in the months from January 2003 through to December 2014, all prior returns are used estimate a TARCH-DCC model

described below. For each first trading day of the month, the fitted model is then used to simulate the forward return distribution over the next 22 trading days, corresponding to approximately one month. The details of the fitted models are described in the next subsection.

14.1. Forward return simulation

The forward simulations are implemented similar to Brownlees and Engle (2015). Given the latest available volatility and correlation estimates the filter recursions are moved forward in time using innovations randomly chosen from past standardised innovations. Thus the innovations are chosen to have the same marginal distributions as applicable in the past. The random choices are such that the same market return innovations are used in each bivariate analysis.

14.2. Systemic beta's

Figure 4 displays the system systemic beta (left panel) and the firm specific systemic beta's (right panel) for each month from January 2003 through to December 2014. The stressor is the ASX market return while the stressor function is the expected worst outcome in $n = 12$ identical scenarios: $\phi(u) = n(1-u)^{n-1}$.

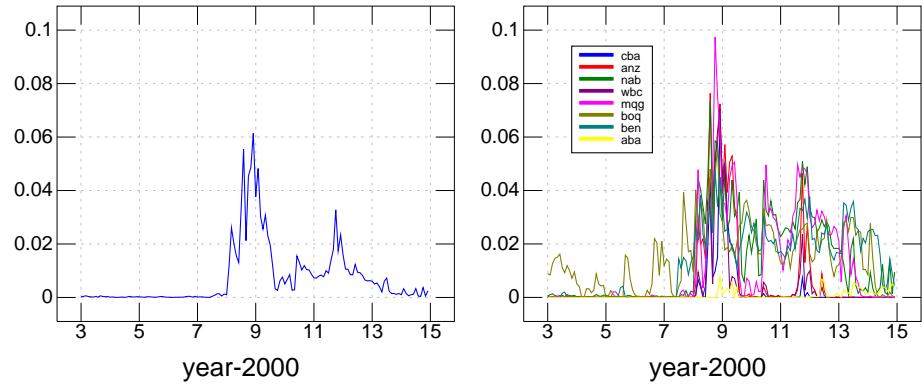


Figure 4: Systemic beta's. The left panel displays the system systemic beta. The right panel indicates systemic beta's for each of the eight banks. In each case the stressor is market return and the stress function is the expected worst outcome in 12 identical scenarios.

The left panel of Figure 4 indicates that up to about late 2008, there was no system systemic stress: The chance of a Basel default was virtually zero. However, indicated in the right panel, the BOQ bank did have substantial systemic risk – ie a substantial risk of sliding into Basel default under the systemic stress of a broad market downturn.

15. Systemic beta's compared to SRISK

The methodology developed and employed in this article departs in two respects from the SRISK methodology set out in Brownlees and Engle (2015). This section examines the force of these differences.

15.1. Put value versus put on expected value

The SRISK methodology, employs, apart from scaling by actual debt and the Basel constant k ,

$$\text{SRISK}_{it} = |1 - E_\phi(e^{\nu_{it} - \ell_{it}})|^+ \leq E_\phi(|1 - e^{\nu_{it} - \ell_{it}}|^+) \equiv \text{PRISK}_{it}, \quad (21)$$

where stressed expectation E_ϕ in the Brownlees and Engle (2015) article is conditional expectation given on a major market downturn. The “put” in the first expression is evaluated after the expectation E_ϕ and checks, in essence, whether on average, in a significant market downturn, the h period ahead return ν_{it} exceeds the log-leverage adjusted for k . Even if on average this holds true there may be substantial risk of a Basel breach, depending on the volatility of the return. Volatility in ν_{it} is ignored in the SRISK_{it} computation other than through the usual adjustment on account of continual compounding.

With PRISK_{it} volatility issues are taken into consideration. Other things equal, high volatility leads to higher put values since breach size is taken into consideration. Two identical situations save for volatility will lead to the same SRISK but different PRISK. Figure 2 emphasises volatility can vary significantly.

To evaluate the difference consider the percentage difference

$$100 \times \frac{\text{PRISK}_{it} - \text{SRISK}_{it}}{\text{PRISK}_{it}}$$

(to be continued – or maybe need some other measure)

15.2. Systemic stress versus volatility stress

This article measures stress using β_{it} , a scaled version of $E_\phi(p_{it}) - E(p_{it})$. Brownlees and Engle (2015) use E_ϕ as in (21). A stressed expectation may be high simply on account of put prices being high, again reflecting higher than normal volatility. Stress measure with E_ϕ alone picks up both current stress, due to high volatility, and the additional stress imparted by the stressing variable.

To gain insight into the relative sizes of these two components, consider the proportion of the stressed put price due to a possible future stress scenario

$$\frac{E_\phi(p_{it}) - E(p_{it})}{E_\phi(p_{it})} = \frac{\sigma_\phi \beta_{it}}{\mu_{it} + \sigma_\phi \beta_{it}}.$$

The denominator is total stress in firm i , made up of two components: the volatility stress $\mu_{it} = E(p_{it})$ and the systemic stress $\sigma_\phi \beta_{it} = E_\phi(p_{it}) - E(p_{it})$. Volatility stress arises on account of high volatility in the markets causing high

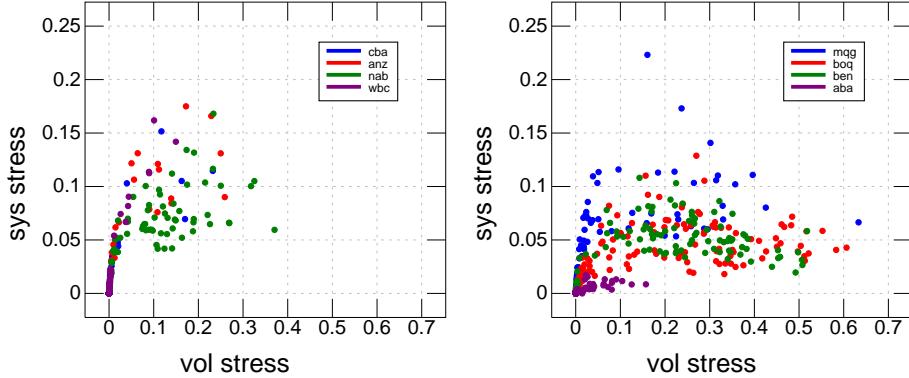


Figure 5: Systemic stress $\sigma_\phi \beta_{it} \equiv E_\phi(p_{it}) - E(p_{it})$ versus volatility stress $\mu_{it} \equiv E(p_{it})$ for four major banks (left panel) and four minor banks (right panel).

put prices. Systemic stress is the additional stress caused by a potential bad systemic event.

Figure 5 plots systemic stress versus volatility stress for the eight Australian banks of Table 1. Generally volatility stress is higher than systemic stress.

15.3. Flexible stress functions versus cutoffs

The further modification made to the SRISK methodology of Brownlees and Engle (2015) is to generalise the ‘‘cutoff’’ functional form for stressing. With the latter, stressed expectations are conditional expectations given a specified cutoff point. In Brownlees and Engle (2015) the cutoff is in absolute terms: a 10% fall in the market although this can varied. The cutoff methodology does not make explicit allowance for market volatility: in a volatile market a 10% reduction is more plausible than in a less volatile environment. This argues for specifying conditional behaviour in percentile terms relative to the then applicable market conditions. After all, current market volatility is known and there is a good reason to disentangle current market conditions from the evaluation of what is likely to happen if further stress arrives in the system.

Say something about the generalized functional forms ϕ as well as the fact that we have firmly embedded and/or aligned the methodology with more ad-hoc approaches to stress testing.

16. Standardised SRISK and layer dependence

Standardising SRISK by subtracting the unconditional expected capital shortfall and dividing by the same assuming maximum systemic risk yields a quantity which is independent of initial debt and equity. In addition standardised SRISK reflects the local dependence between an individual firm and the market at the selected threshold of the systemic event. Hence varying the threshold yields the

dependence structure of individual firm and market returns from benign events all the way to tail events.

The unconditional expected capital shortfall for firm i is $E(c_i) = kd_i - (1 - k)w_i\{1 + E(r_i)\}$, suggesting the standardised SRISK

$$\begin{aligned} \text{SRISK}_i^* &\equiv \frac{\text{SRISK}_i - E(c_i)}{\max(\text{SRISK}_i) - E(c_i)} = \frac{E(r_i) - E(r_i|u_m < a)}{E(r_i) - E(r_i|u_i < a)} \\ &= \frac{\text{cov}\{r_i, \bar{I}_a(u_m)\}}{\text{cov}\{r_i, \bar{I}_a(u_i)\}} = \frac{\text{cov}\{r_i, I_a(u_m)\}}{\text{cov}\{r_i, I_a(u_i)\}} \end{aligned}$$

where $\bar{I}_a(z) \equiv (z < a)$ and $I_a(z) \equiv (z > a)$ are indicators. Maximum SRISK occurs if firm return is comonotonic with the market return or $u_i = u_m$, u_i being the percentile rank of r_i . Standardised SRISK is hence layer dependence on the original scale, and mainly lies between 0 and 1 indicating independence and comonotonicity, respectively. Varying a reveals the local dependence between firm and market returns at different thresholds, with $a \rightarrow 0$ yielding the lower extreme tail dependence in a market crash of maximum severity.

Performing the same standardising on SRISK_i which is a weighted average of SRISK for firm i across various thresholds yields

$$\widetilde{\text{SRISK}}_i^* = \frac{\text{cov}\{r_i, \phi(u_m)\}}{\text{cov}\{r_i, \phi(u_i)\}}.$$

17. Comparing SRISK for different firms (not sure)

Suppose of interest is whether β_{it} varies similarly across firms as τ varies where τ is the threshold, k , or some other parameter used to compute β_{it} . High correlation implies high systemic risk since firms are simultaneously affected.

Define $\epsilon_{it} \equiv (\phi - 1)p_{it}$ and ϵ_{it} as the vector with components ϵ_{it} . Then $E(\epsilon_{it}|\tau) = \sigma_\phi(\tau)\beta_{it}(\tau)$, the stress in firm i at the given parameter setting τ and

$$\text{cov}\{E(\epsilon_{it}|\tau)\} = \text{cov}(\epsilon_t) - E\{\text{cov}(\epsilon_{it}|\tau)\},$$

is the covariance between stresses as τ , the stress parameter varies. Thus the covariances and correlations between firms as stress parameters vary can be computed from the covariability between the ϵ_{it} and the ...

18. Standardised SRISK and layer dependence

Standardising SRISK by subtracting the unconditional expected capital shortfall and dividing by the same assuming maximum systemic risk yields a quantity which is independent of initial debt and equity. In addition standardised SRISK reflects the local dependence between an individual firm and the market at the selected threshold of the systemic event. Hence varying the threshold yields the dependence structure of individual firm and market returns from benign events all the way to tail events.

The unconditional expected capital shortfall for firm i is $E(c_i) = kd_i - (1 - k)w_i\{1 + E(r_i)\}$, suggesting the standardised SRISK

$$\begin{aligned} \text{SRISK}_i^* &\equiv \frac{\text{SRISK}_i - E(c_i)}{\max(\text{SRISK}_i) - E(c_i)} = \frac{E(r_i) - E(r_i|u_m < a)}{E(r_i) - E(r_i|u_i < a)} \\ &= \frac{\text{cov}\{r_i, \bar{I}_a(u_m)\}}{\text{cov}\{r_i, \bar{I}_a(u_i)\}} = \frac{\text{cov}\{r_i, I_a(u_m)\}}{\text{cov}\{r_i, I_a(u_i)\}} \end{aligned}$$

where $\bar{I}_a(z) \equiv (z < a)$ and $I_a(z) \equiv (z > a)$ are indicators. Maximum SRISK occurs if firm return is comonotonic with the market return or $u_i = u_m$, u_i being the percentile rank of r_i . Standardised SRISK is hence layer dependence on the original scale, and mainly lies between 0 and 1 indicating independence and comonotonicity, respectively. Varying a reveals the local dependence between firm and market returns at different thresholds, with $a \rightarrow 0$ yielding the lower extreme tail dependence in a market crash of maximum severity.

Performing the same standardising on $\widetilde{\text{SRISK}}_i$ which is a weighted average of SRISK for firm i across various thresholds yields

$$\widetilde{\text{SRISK}}_i^* = \frac{\text{cov}\{r_i, \phi(u_m)\}}{\text{cov}\{r_i, \phi(u_i)\}}.$$

Appendix

Denote the daily (log) return for firm i at time t as

$$\delta_{it} = \mu_i + \sigma_{it}\epsilon_{it}, \quad \epsilon_{it} \sim (0, 1), \quad \epsilon_{it}^- \equiv I(\epsilon_{it} < 0) = I(\delta_{it} < \mu), \quad (22)$$

where I denotes the indicator function and hence ϵ_{it}^- indicates whether there is a negative return. The volatility σ_{it} is modelled as

$$\sigma_{i,t+1}^2 = \omega + \sigma_{it}^2\{\beta + (\alpha + \gamma\epsilon_{it}^-)\epsilon_{it}^2\}. \quad (23)$$

Hence the response of the variance to ϵ_{it}^2 is increased by γ if $\epsilon_{it} < 0$ compared to the response if $\epsilon_{it} > 0$. This is a simple threshold GARCH model: called TARCH. In the fits it is assumed the mean μ does not vary with time t and the terms σ_{it}^2 and ϵ_{it} in the right hand side of (23) are sufficient to capture volatility dynamics.

The model defined by (22) and (23) is estimated for each security i including the market, $i = m$. Correlations between securities and the market are modelled using positive definite recursions (Engle, 2002)

$$(Q_{i,t+1} - S) = \alpha(\eta_{it}\eta_{it}' - S) + \beta(Q_{it} - S), \quad \eta_{it} \equiv (\epsilon_{it}, \epsilon_{mt})',$$

The correlations ρ_{it} recovered from Q_{it} are used as input into the firm i and market return and volatility equations. All model fits in this paper have been performed with the R language (R Development Core Team, 2008) and in particular the package described by Ghalanos (2012). All other calculations performed using the J language (Iverson, 2003).

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