# Systemic risk and contagion effects in Australian financial institutions and sectors

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**Abstract**

This article refines, builds on and extends SRISK methodology recently proposed in the literature. The refinement is to define SRISK in terms of a put on the Basel shortfall. The definition is built on by defining the systemic beta of a firm in terms of the departure of a stressed expectation from the normal expectation the put. By stressed expectation is meant an expectation that averages conditional expectations over outcomes defined by a stressor variable combined with a stressor function. The stressor variable and function are chosen by the practitioner and include functions of a market wide variable. Since stressed expectations are linear, a system systemic beta is naturally defined as linear in the firm specific systemic beta's. The methodology generalises automatically to situations where systemic stress arises from the interaction of multiple system wide variables. Application is made to Australian financial data.

## 1. Literature review

The starting point for the proposed research is the recent literature and the CIFR targeted areas and APRA aims and functions. This recent literature includes the following Adrian and Brunnermeier (2011), Acharya et al. (2012), Acharya et al. (2012) and Brownlees and Engle (2010). The proposed research aims to extend and apply these techniques particularly in relation to the entities regulated by APRA. Thus our broad aim is to develop, implement and bring to bear recent developments in stress testing on the aims of APRA and the CIFR targeted research areas detailed above.

## 2. Capital shortfall and leverage

If $d_{it}$ and $w_{it}$ are the debt and equity of a firm $i$ at time $t$ and $k$ is the capital requirement, then the capital shortfall at time $t$ is

$$k(d_{it} + w_{it}) - w_{it} = kd_{it} - (1-k)w_{it} = kd_{it}\left(1 - \mathrm{e}^{-\ell^*_{it}}\right) \ , \tag{1}$$

where

$$\ell_{it} \equiv \ln \frac{d_{it}}{w_{it}} \ , \qquad \ell^*_{it} \equiv \ell_{it} + \text{lgt}\ k \ , \qquad \text{lgt}\ k \equiv \ln \frac{k}{1-k} \ .$$

The quantity $\ell^*_{it}$ is called Basel adjusted the log–leverage and $\ell^*_{it} > 0$ implies capital shortfall (1) is positive.

Basel II assumes $k = 0.08$ implying $\text{lgt}(k) = -2.44$, $\ell^*_{it} = \ell_{it} - 2.44$ and there is positive "Basel shortfall" if $\ell^*_{it} > 0$ or $\ell_{it} > 2.44$. As the log leverage $\ell_{it}$ increases to 2.44 the firm approaches the Basel II threshold.

Figure 2 displays Basel log–leverages for the four major and four minor Australian banks listed in Table 1 on the first trading day of each month from January 2003 through to December 2014. Note most banks are Basel compliant up to about 2008.

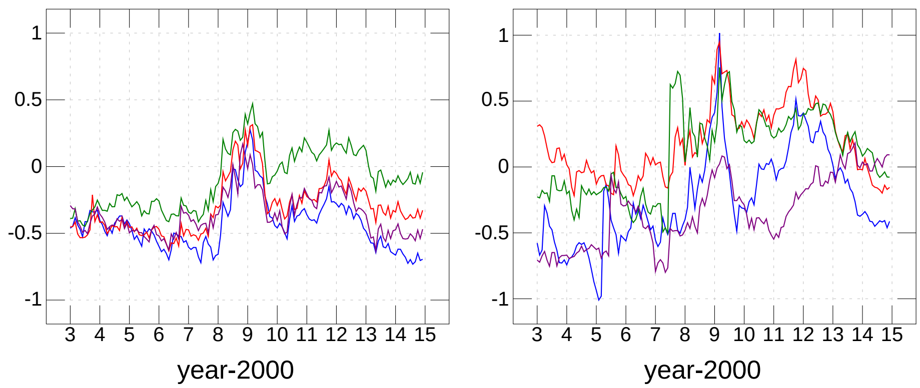


Figure 1: Basel log–leverages for four major (left panel) and four minor (right panel) Australian banks from 2003 through to end of 2014. Note all banks contravene the Basel II threshold (corresponding to Basel log-leverage of 0) on or around the early part of 2009.

## 3. Future capital shortfall

Financial institutions and regulators are concerned with future Basel compliance and understanding the likelihood of a future Basel breach. Future compliance depends on the future return on equity. Consider firm $i$ at time $t+h$ where $h > 0$. Then

$$\ell_{i,t+h} = \ln \frac{d_{i,t+h}}{w_{i,t+h}} = \ell_{it} - \nu_{it} \ , \qquad \nu_{it} \equiv \ln \frac{w_{i,t+h}}{w_{it}} \ ,$$

where it is assumed $d_{i,t+h} = d_{it}$. Hence $\nu_{it}$ is the return on equity over $(t, t+h)$. The future return is unknown at time $t$ but may be its distribution may be well understood.

In terms of the Basel leverage $\ell_{it}^*$ at time $t$, the Basel II shortfall at time $t+h$ is,

$$d_{it}k(1-\mathrm{e}^{\nu_{it}^*}) \ , \qquad \nu_{it}^* \equiv \nu_{it} - \ell_{it}^* = \nu_{it} - \ln\frac{d_{it}}{w_{it}} - \operatorname{lgt}\ k \ . \tag{2}$$

The Basel II limit is reached at $t+h$ if $\nu_{it} = \ell_{it}^*$ ($\nu_{it}^* = 0$) and breached if $\nu_{it} < \ell_{it}^*$ ($\nu_{it}^* < 0$). A breach is unlikely if $\ell_{it}^*$ is low i.e. either the log–leverage is low, or the Basel constant $k$ is low.

## 4. Systemic risk – SRISK

Brownlees and Engle (2015) define the systemic risk (SRISK) of a group of firms at time $t$ as

$$k\sum_i d_{it}|1-\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}^*})|^+ \ , \quad |x|^+ \equiv \max(0,x) \ , \quad \nu_{it}^* \equiv \nu_{it} - \ln\frac{d_{it}}{w_{it}} - \operatorname{lgt}\ k \ , \tag{3}$$

where $k$ is the Basel constant, $d_{it}$ and $\nu_{it}^*$ are, respectively, the debt and the Basel and log–leverage adjusted future return of firm $i$ at time $t$, and $\mathrm{E}_\phi$ denotes expectation conditional on a stress event $\phi$: in their case a severe general market downturn. The contribution of firm $i$ is thus a put on the conditional expected value of the Basel II shortfall of firm $i$ at $t+h$.

The proportionate contribution of firm $i$ to (3) is called the systemic risk of firm $i$:

$$\mathrm{SRISK}_i \equiv \frac{|1-\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}^*})|^+}{\mathcal{E}\{|1-\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}^*})|^+\}} \ , \tag{4}$$

where $\mathcal{E}$ denotes debt weighted averaging. A large $\mathrm{SRISK}_i$ indicates a systemically important firm. The expression (4) depends on $k$ only through its logit. The values of the "puts" $|1-\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}^*})|^+$ in (4) are known at time $t$: there is no uncertainty except for possible estimation uncertainty in the stressed expected value.

If the stressed distribution of $\nu_{it}^*$ is concentrated on the negative axis then $\mathrm{e}^{\nu_{it}^*} < 0$ and

$$|1-\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}^*})|^+ = \mathrm{E}_\phi(|1-\mathrm{e}^{\nu_{it}^*}|^+) \ .$$

If, under stress, the probability of $\nu_{it}^* > 0$ is positive then the last relation only holds approximately. The SRISK contribution of a firm is zero whenever

$$\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}^*}) \equiv \mathrm{e}^{-\ell_{it}^*}\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}}) \le 1 \ , \qquad \ln\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}}) \le \ell_{it}^* \equiv \ln\frac{d_{it}}{w_{it}} + \operatorname{lgt}(k) \ . \tag{5}$$

This inequality often holds even if the expectation is severely stressed. In particular if the stressed distribution of $\nu_{it}$ is normal with mean $\mu$ and variance $\sigma^2$ then

$$\ln\mathrm{E}_\phi(\mathrm{e}^{\nu_{it}}) = h\mu + h^2\frac{\sigma^2}{2} \ ,$$

and (5) is equivalent to

$$\mu + h\frac{\sigma^2}{2} \leq \frac{\ell^*_{it}}{h} \ ,$$

indicating a Basel breach is unlikely if $\ell^*_{it}$ is very negative equivalent to $\ell_{it}$ well below 2.44, the Basel limit.

The system SRISK is zero if all firm the SRISK contributions are zero that is if (5) holds for every $i$. This would be usual in the Australian case.

An improvement is where firms with capital surplus are included, since they, especially those with large surpluses, are systemically important but in a positive way by dampening the impact from distressed firms. Second, it is appropriate to consider the departure from the unconditional expectation to differentiate firms which only experience capital shortfall during a market downturn as opposed to those who already have large shortfalls in average conditions. This leads to the revised computation

$$\frac{1-\mathrm{E}_\phi(\mathrm{e}^{\nu^*_{it}}) - \{1-\mathrm{E}(\mathrm{e}^{\nu^*_{it}})\}}{\mathcal{E}\{|1-\mathrm{E}_\phi(\mathrm{e}^{\nu^*_{it}})|^+ - |1-\mathrm{E}_\phi(\mathrm{e}^{\nu^*_{it}})|^+\}} = \frac{\mathrm{E}(\mathrm{e}^{\nu^*_{it}}) - \mathrm{E}_\phi(\mathrm{e}^{\nu^*_{it}})}{\mathcal{E}\{|1-\mathrm{E}_\phi(\mathrm{e}^{\nu^*_{it}})|^+ - |1-\mathrm{E}_\phi(\mathrm{e}^{\nu^*_{it}})|^+\}}$$

The revised contribution of firm $i$ to total systemic risk is hence

$$\frac{\mathrm{SRISK}_i - \mathrm{E}(c_i)}{\mathrm{SRISK}_m - \mathrm{E}(c_m)} = \frac{w_i}{w_m}\frac{\mathrm{E}(r_i) - \mathrm{E}(r_i|u_m < a)}{\mathrm{E}(r_m) - \mathrm{E}(r_m|u_m < a)} \ ,$$

the product of the equity of firm $i$ relative to total market equity and return movement of firm $i$ in a market downturn relative to overall market movement.

## 5. Improved systemic risk measurement

For firm $i$ define the Basel put and Basel risk as

$$p_{it} \equiv k|1-\mathrm{e}^{\nu^*_{it}}|^+ \ , \qquad \mathrm{E}(p_{it}) \geq 0 \ , \tag{6}$$

where E denotes normal or unstressed expectation. The Basel risk in firm $i$ depends only $k$, the present leverage, and the lower tail of the $\nu_{it}$ distribution. The put value (6) is stated in terms of unit debt and increases with $k$, both directly from multiplication by $k$ and indirectly through its logit, making a positive put outcome more likely.

Default put options similar to (6) have been discussed in the insurance literature as critical to an evaluation of a firm: see for example Merton (1977), Doherty and Garven (1986), Cummins (1988), Myers and Read (2001) and Sherris (2006).

The total Basel put value and hence Basel risk for firm $i$ is $d_{it}\mathrm{E}(p_{it})$. This is the cost of insuring firm $i$ does not breach the Basel standard. In practice it may be appropriate to discount $p_{it}$ in (6) by the interest rate over the period $(t, t+h)$ and value the put with risk neutral rather than normal expectation.

The put $p_{it}$ is tradable since, given $k$, it relies only on the present leverage and the future return $\nu_{it}$ on equity $w_{it}$. Market participants can value the put in the same way as any other put and use the contract to diversify risk. Firms can buy puts in the market to hedge Basel default risk.

However regulators will value the puts $p_{it}$ using a stressed, rather than risk neutral, expectation. Stressed expectations correspond closely to conditional expectation given a systemically important event or events. Regulators are concerned with the possibility of Basel breaches if such events occur and hence have an overbearing interest in monitoring stressed expectations. In the extreme regulators can charge firms all or a proportion of their stressed put values if there are implicit government bailout guarantees.

In terms of the put values $\mathrm{E}(p_{it})$, the system Basel risk is defined as the debt weighted average of the individual firm's put values:

$$\mathcal{E}\{\mathrm{E}(p_{it})\} = \mathrm{E}\{\mathcal{E}(p_{it})\} \ , \tag{7}$$

since both E and $\mathcal{E}$ are linear. The system Basel risk is stated per unit of system debt. The dollar value of Basel risk is the Basel risk multiplied by the total system debt:

$$d_t\mathcal{E}\{\mathrm{E}(p_{it})\} = \sum_i d_{it}\mathrm{E}(p_{it}) \ , \qquad d_t \equiv \sum_i d_{it} \ .$$

Thus the monetary Basel risk is a weighted average of firm specific monetary Basel risks.

## 6. Systemic stress as stressed expectation

In the above development E denotes unstressed expectation: that is expectation that does not assume a stressful scenario. A stressed expectation, denoted $\mathrm{E}_\phi$, is a linear combination of expectations, each assuming some level of stress

$$\mathrm{E}_\phi(p_{it}) \equiv \mathrm{E}\{\phi\mathrm{E}(p_{it}|\phi)\} = \mathrm{E}(\phi p_{it}) = \mathrm{E}(p_{it}) + \mathrm{cov}\{\mathrm{E}(p_{it}|\phi), \phi\} \ . \tag{8}$$

Here $\phi \geq 0$ with $\mathrm{E}(\phi) = 1$ is thought of as a stress factor or "stressor," defining events impacting $p_{it}$ and serving as a basis for stress testing. Thus $\phi$–stressing changes the expected value $\mathrm{E}(p_{it})$ by the covariance between the conditional expectation $\mathrm{E}(p_{it}|\phi)$ and the stressor $\phi$.

The conditioning variable $\phi$ downplay or highlight different scenarios. The conditioning events are scenarios of interest and probabilistic weighting is according to the level of interest. The first equality in (8) from the definition of conditional probability (Whittle, 2000). The second equality follows from

$$\mathrm{cov}(p_{it}, \phi) = \mathrm{cov}\{\mathrm{E}(p_{it}|\phi), \phi\} \ .$$

The covariance is zero whenever $p_{it}$ is independent of $\phi$ which occurs if $\phi = 1$.

To a Bayesian the stressed expectation (8) is the mean of $p_{it}$ in the posterior density

$$\int \phi f(p_{it}|\phi)f(\phi)\mathrm{d}\phi = \int \phi f(p_{it}, \phi)\mathrm{d}\phi = \mathrm{E}(\phi|p_{it})f(p_{it}) \ . \tag{9}$$

Note (9) is a density since $\phi \geq 0$ and $\mathrm{E}(\phi) = 1$. Any outcome $p_{it}$ such that $\mathrm{E}(\phi|p_{it})$ is large is amplified and vice versa. Further

$$\mathrm{E}_s(p_{it}) = \mathrm{E}\left\{p_{it}\frac{\phi f(p_{it})}{f(p_{it})}\right\} ,$$

that is the density $f(p_{it})$ of $p_{it}$ is replaced by the density $\phi f(p_{it})$. Hence more importance is given to those outcomes given more weight by $\phi$.

## 7. Systemic beta

Writing $\sigma_\phi$ as the standard deviation of $\phi$, define the systemic beta of firm $i$ with respect to stress $\phi$ as

$$\beta_{it} \equiv \frac{\mathrm{E}_\phi(p_{it}) - \mathrm{E}(p_{it})}{\sigma_\phi} = \frac{\mathrm{E}\{(\phi-1)p_{it}\}}{\sigma_\phi} = \frac{\mathrm{cov}\{\mathrm{E}(p_{it}|\phi), \phi\}}{\sigma_\phi} , \qquad (10)$$

where $\sigma_\phi$ is the standard deviation of $\phi$. Thus

$$\mathrm{E}_\phi(p_{it}) = \mathrm{E}(p_{it}) + \beta_{it}\sigma_\phi , \qquad \mathrm{E}(p_{it}|\phi) \approx \mathrm{E}(p_{it}) + \beta_{it}\frac{\phi - 1}{\sigma_\phi} . \qquad (11)$$

The second, approximate, relationship is suggested from a regression interpretation of $\mathrm{cov}(p_{it}, \phi)/\sigma_\phi^2$ as the regression coefficient of $p_{it}$ on $\phi$.

The formulas in (11) suggests two ways of thinking about the systemic beta's $\beta_{it}$. First, the left hand side equation shows systemic stress $\beta_{it}$ serves to shift the mean of put prices by, on average, $\sigma_\phi$. The shift $\beta_{it}\sigma_\phi$ is the market price of stress embedded in firm $i$ at time $t$. Firms with large shifts embody stress risky firms.

The second interpretation is based on the right hand side approximation in (11) showing the change in the expected Basel put price if $\sigma_\phi$ units of stress $\phi$ are applied. The quantity $(\phi - 1)/\sigma_\phi$ is thought of as a "stress factor." The stress factor has mean 0 and standard deviation 1 and is scaled by $\beta_{it}$ to yield the actual stress effect on $p_{it}$. Thus $\beta_{it}$ is thought of a usual finance type "beta" with respect to the given $\phi$. Stress is measured in standardised units. Units of stress take on different meaning depending on $\phi$ as discussed below. The distribution of $\phi$ determines the distribution of the stress factor. If the stress factor is normally distributed then a stress effect more than doubling of the put value occurs less than about 2.5% of the time.

Note the systemic beta's are stated in terms of additions to the normal put price $\mathrm{E}(p_{it}$. Thus in an increasingly dire financial situation $\mathrm{E}(p_{it})$ will increase. However stress $\beta_{it}$ measures the change in current put values under further $\phi$–stress. This varies from Brownlees and Engle (2015) where $\mathrm{SRISK}_i$ does not distinguish between the current, possibly high, put prices $\mathrm{E}(p_{it})$ and the further increment due to potential stress. Thus in an increasingly dire financial situation put prices are liable to increase on account of decreasing expected returns and increasing volatility. However our definition of systemic stress using systemic beta's measures the further effect on account of potential extra stress imposed onto the system over and above the already existing stress. The systemic beta's capture "marginal" effects.

The monetary stress of a firm is $d_{it}\beta_{it}$ and represents the change in the monetary value of the Basel put if stress, as captured with $\phi$, is applied.

## 8. Systemic beta for a group of firms

A measure of the total systemic stress is to consider the debt weighted put

$$p_t \equiv \mathcal{E}(p_{it}) \tag{12}$$

This "system" put (12) pays out zero if all firms meet the Basel requirement with payments increasing with the number of Basel breaches, the sizes of the breaches, and the relative debt sizes of the breaching firms.

In terms of (12), the systemic beta $\beta_t$ for the system is defined as

$$\beta_t \equiv \frac{\mathrm{E}_\phi(p_t) - \mathrm{E}(p_t)}{\sigma_\phi} = \frac{\mathrm{cov}[\mathrm{E}\{\mathcal{E}(p_{it})|\phi\}, \phi]}{\sigma_\phi} = \mathcal{E}(\beta_{it}). \tag{13}$$

Thus the system systemic beta is the debt weighted average of firm systemic beta's. The equalities in (13) follow from the fact that E and $\mathrm{E}_\phi$ and $\mathcal{E}$ are all linear.

The monetary value system stress at time $t$ is the sum of monetary firm stresses

$$d_t\beta_t = d_t\mathcal{E}(\beta_{it}) = \sum_i d_{it}\beta_{it} \ .$$

## 9. Determining systemic beta's via simulation

Firm specific stress $\beta_{it}$ and system stress can be determined using simulation. For the former

$$\beta_{it} \approx \frac{1}{\sigma_\phi S}\sum_{s=1}^{S}(\phi^s - 1)p_{it}^s \ , \tag{14}$$

where $S$ is the simulation effort. The approximation is increasingly accurate as $S$ becomes large. In this simulation, the $(\phi^s, p_{it}^s)$ are pairs of stress and put values generated from a model: for example $\phi^s$ may be zero unless there is a market drop below the $\alpha$–percentile in which case it is $1/\alpha$. In this latter case the simulation (14) averages over all simulated put values $p_{it}^s$ paired with an at least $\alpha$–drop in the market.

## 10. Systemic beta's based on market return

The setup is illustrated with examples generalising and fine–tuning the approach of Engle.

*10.1. Market return*

Suppose $\phi = \mathrm{e}^{-\lambda\nu_{mt}} \approx 1 - \lambda\nu_{mt}$ where $\lambda$ is a parameter. Then if $\nu_{mt}$ is normal with mean $h\mu_{mt}$ and variance $h\sigma^2_{mt}$ then

$$\beta_{it} = \frac{\mathrm{cov}(p_{it}, \phi)}{\sigma_\phi} \approx \frac{-\mathrm{cov}(p_{it}, \nu_{mt})}{\sqrt{\mathrm{e}^{h(2\mu_{mt}+\sigma^2_{mt})}\left(\mathrm{e}^{h\sigma^2_{mt}} - 1\right)}} \ ,$$

*10.2. Market return below a percentile threshold*

Suppose the stress event is a market return in the bottom $\alpha$–tail of the market return distribution. Then $\phi(u) = 1/\alpha$ for $u < \alpha$ and 0 otherwise and

$$\mathrm{E}_\phi(p_{it}) \equiv \mathrm{E}\{\phi\mathrm{E}(p_{it}|\phi)\} = \frac{1}{\alpha}\int_0^\alpha \mathrm{E}(p_{it}|u)\mathrm{d}u = \mathrm{E}(p_{it}|\nu_{mt} < c_t) \ .$$

where $c_t$ cuts out $\alpha$ probability in the lower tail of the $\nu_{mt}$ distribution. Then

$$\beta_{it} \approx \frac{\mathrm{e}^{(\mathrm{lgt}\ \alpha)/2}}{S/\alpha} \sum_{u^s_{mt}<1/\alpha} p^s_{it} \ ,$$

where the estimate of $\sigma_\phi$ implicit in the final expression is established using a direct calculation. In the right hand side $s = 1, \ldots, S$ denotes simulations. The approximation decreases with the simulation effort $S$. The effective simulation effort is $S/\alpha$ and hence $\alpha$ small requires a large effort.

A fixed $\alpha$ cutoff implies the actual threshold depends on the distribution and hence on volatility and time. The actual threshold drop in the market $c_t = F^-_{mt}(\alpha)$, where $F^-_{mt}$ is the inverse distribution function of $\nu_{mt}$. The volatility associated with the latter distribution is approximately $\sqrt{h}\sigma_{mt}$ where $\sigma_{mt}$ is current market volatility. Regulators and practitioners are well versed in working with VaR type calculations and hence a varying VaR type cutoff combining market volatility and stress is closely aligned to current practice.

*10.3. Expected worst market return in independent copies*

If $\phi(u) = n(1-u)^n$ then $\phi(u)\mathrm{d}u = \mathrm{d}\{1 - (1-u)^n\}$ and

$$\mathrm{E}_\phi(p_{it}) \equiv \mathrm{E}\{\phi\mathrm{E}(p_{it}|\phi)\} = \int_0^1 \mathrm{E}(p_{it}|u)\mathrm{d}\{1 - (1-u)^n\} \ .$$

If $u$ is the market return percentile then $1 - (1-u)^n$ is the distribution of the worst percentile outcome in $n$ identical trials and hence the stressed expectation is that of the expected put price given the worst percentile market return in $n$ identical trials. Further $\sigma_\phi = (n-1)/\sqrt{2n-1}$ and

$$\beta_{it} \approx \frac{n\sqrt{2n-1}}{(n-1)S} \sum_s (1 - u^s_{mt})^{n-1} p^s_{it} \ .$$

Simulated returns $\nu^s_{mt}$ in the upper tail of the distribution have percentiles $u^s_{mt} \approx 1$ and hence for these returns the first factor in the sum is small if not negligable: the associated simulated put is heavily downweighed.

Note the contrast with the previous example where the bottom $\alpha$ proportion of simulated market returns are selected as the stressed sample. With the current specification for $\phi$, every simulated value contributes to the stress computation, albeit with different weights.

*10.4. Expected worst market return given a tail event*

The above two situations can be combined. Suppose $\phi(u) = c(0.05-u)^{19}$ for $u \leq 0.05$ and 0 otherwise and where $c$ is such that $\phi(u)$ integrates to 1. Then

$$\mathrm{E}(p_{it}) \approx \frac{1}{S/c} \sum_{u^s_{mt}<1/20} \left(u^s_{mt} - \frac{1}{20}\right)^{20} p^s_{it} \ .$$

Similar to the first example, the stressed sample picks up the bottom 5% of market returns. The bottom 5% of market returns is further stressed by progressively downweighing returns as the percentile approaches 1/20. Large simulation effort $S$ is required for a reasonable approximation since $S/c$ is the effective simulation size and $c$ is small.

*10.5. Stressors as weighted linear combinations of tail events*

Brownlees and Engle (2015) define a systemic event as a market based downturn greater than a certain threshold and a systemic event depends on the chosen threshold. This arbitrariness is partially sidestepped by choosing a decreasing function $\phi = \phi(u_{mt})$ on the unit interval and noting

$$\mathrm{E}\{\phi \mathrm{E}(p_{it}|\phi)\} = -\mathrm{E}\{u\phi'(u)\mathrm{E}(p_{it}|u_{mt} \leq u)\} \ . \tag{15}$$

Thus a stressed expectation, stressed with a decreasing function of the market return, is equivalent to taking a weighted average of conditional lower tail expectations. This circumvents an explicit choice for the market threshold, enabling a mixture of thresholds. The equivalence of the left and right hand sides of (15) is established with, writing $\mathrm{E}_v$ as expectation given $u_{mt} = v$, (*** Weihao can you check this****)

$$\int_0^1 \phi'(u) \int_0^u \mathrm{E}_v(\nu_{it}) \mathrm{d}v \mathrm{d}u = \int_0^1 \mathrm{E}_v(\nu_{it}) \int_v^1 \phi'(u) \mathrm{d}u \mathrm{d}v \ .$$

*10.6. Copula stressors*

Stress can be based on a vector of variables $x$ with distribution $F(x) = u$ and marginal densities $f(x_j)$. Suppose $c(u)$ is the copula density of $x$ implying the density of $x$ is $c(u)\prod_j f(x_j)$. If $\Phi(u)$ is another copula with density $\phi(u)$ consider

$$\mathrm{E}_\phi(p_{it}) \equiv \int \mathrm{E}(p_{it}|x)\phi(u)c(u) \prod_j \{f(x_j)\} \mathrm{d}x \ . \tag{16}$$

If $\phi(u) = 1$ then $\mathrm{E}_\phi(p_{it}) = \mathrm{E}(p_{it})$, the ordinary expectation. The density $\phi(u)$ magnifies potentially stressful situation such as where all component of $x$ are highly tail dependent. In the extreme case $\Phi(u) = \min_j(u_j)$ making the variables in $x$ comonotonic.

The copula density stressor $\phi(u)$ can be combined with marginal stressors written as functions of $x_j$ or percentiles $u_j$. If the latter then $\phi(u)\prod\phi(u_j)$ is the total stressor, corresponding to a density on the unit hypercube with non–uniform marginals $\phi(u_j) \neq 1$.

Copula stress effects are simulated as follows. Suppose $x$ is a vector $\nu_{mt}$ of market factors over the period $(t, t+h)$. Simulations of a joint model are used to derive vectors $(p^s_{it}, \nu^s_{mt})$, $s = 1, \ldots, S$. The $\nu^s_{mt}$ are converted to percentiles $u^s$ and scalars

$$\phi^s \equiv \phi(u^s)\prod_j \phi(u^s_j) \ , \qquad s = 1, \ldots, S \ .$$

The stress beta is then estimated as in (14).

Marginal distributions of $r_m$ and $r_n$ may be altered or unchanged. If unchanged then

$$f(r_n) = \int f(r_m, r_n)\mathrm{d}r_m = \int f_\phi(r_m, r_n)\mathrm{d}r_m$$

or

$$1 = \int f(r_m)c(u_m, u_n)\mathrm{d}r_m = \int f(r_m)\phi(r_m, r_n)c(u_m, u_n)\mathrm{d}r_m$$

since $f(r_m, r_n) = f(r_m)f(r_n)c(u_m, u_n)$.

### *10.7. Scenario based stress testing*

Stress events captured with $\phi$ and used in (10) can be defined with respect any events, including a discrete number of scenarios labelled $k = 1, 2, \ldots$. In this case

$$\mathrm{E}_\phi(p_{it}) = \sum_k \pi_k\phi_k\mathrm{E}(p_{it}|k) \ , \qquad \sum_k \pi_k\phi_k = \mathrm{E}(\phi) = 1,$$

where $\phi_k$ is the weight assigned to scenario $k$. The $\pi_k\phi_k \geq 0$ are modified probabilities which weigh different scenarios according to level of interest. For example in standard stress testing the $\phi_k$ are chosen such that significant weight is given to scenarios $k$ causing high firm distress. The “real world” probabilities $\pi_k$ are usually ignored by simply choosing $\pi_k\phi_k$ to be of required magnitude.

In the discrete case $\beta_{it}$ compares the expected value under an average of distress scenarios to the average without distress. The quantity $\sigma_\phi$ is now of limited relevance, signalling the volatility in the distress probabilities.

## 11. Risk weighted asset valuation

Risk weighted asset calculations recognise that some assets are more susceptible to devaluation in times of stress. In a firm valuation these more risky assets are valued less and the resulting leverage is used to assess Basel compliance.

The broad approach fits into the present framework. Suppose total assets of firm $i$ at time $t$ are

$$a_{it} \equiv d_{it} + w_{it} = \sum_j a_{ijt} \ ,$$

where $a_{ijt}$ is the value of the class $j$ assets at time $t$. Similarly the total value assets at time $t+h$ is, denoting $\nu_{ijt}$ as the return on asset class $j$ over the period $(t, t+h)$,

$$a_{i,t+h} = a_{it} \sum_j \frac{\mathrm{e}^{\nu_{ijt}} a_{ijt}}{a_{it}} = a_{it} \mathcal{E}_a(\mathrm{e}^{\nu_{ijt}}) \ ,$$

where $\mathcal{E}_a$ denotes asset weighted averaging using asset values at time $t$. The log of the debt to equity ratio at time $t+h$ is thus, assuming debt is constant over the period,

$$\ell_{i,t+h} = -\ln \frac{a_{i,t+h} - d_{it}}{d_{it}} = -\ln\{(1 + \mathrm{e}^{-\ell_{it}})\mathcal{E}_a(\mathrm{e}^{\nu_{ijt}}) - 1\} \ ,$$

and the Basel put equals, using the previous notation $\ell^*_{it} \equiv \ell_{it} + \mathrm{lgt}(k)$,

$$p_{it} = k|1 - \mathrm{e}^{\ell^*_{i,t+h}}|^+ = k\left|1 - \frac{\mathrm{e}^{\mathrm{lgt}(k)}}{(1 + \mathrm{e}^{-\ell_{it}})\mathcal{E}_a(\mathrm{e}^{\nu_{ijt}}) - 1}\right|^+ \ . \qquad (17)$$

A return sample size $S$ is assembled by simulating asset specific forward returns $\nu^s_{ijt}$ and stress factor $\phi^s$ and combining to arrive at stressed put prices. Initially assume the stress is a market return in the bottom $\alpha$–tail of the market return distribution. Then realisations with market return in the lower tail are used to compute an average of put values (17). The average approaches $\mathrm{E}_\phi(p_{it})$ as $S$ becomes large. The average put value across all $S$ is subtracted from the stressed sample average and divided by $\sigma_\phi$ to arrive at the estimate of $\beta_{it}$.

For more general $\phi$ it is convenient to estimate $\beta_{it}$ using (14). Thus given $\phi$, simulated asset returns $\nu^s_{ijt}$ are used to compute the corresponding simulated put value $p^s_{it}$ as in (17) and weighted market percentile $\phi^s \equiv \phi(u^s_{mt})$. These are then combined as in (14). If required, the stressed expectation is estimated using

$$\mathrm{E}_\phi(p_{it}) \approx \mu_{it} + \sigma_\phi \beta_{it} \ , \qquad \mu_{it} \approx \frac{1}{S} \sum_s p^s_{it} \ .$$

The above framework formalises risk weighted asset methodology used in capital setting. Easily (dis)stressed assets have a return distribution sensitive to market downturn. This implies, in a downturn, $\mathrm{e}^{\nu_{ijt}}$ is likely to be much less than 1 and the asset value $a_{ijt}$ is downweighed in the computation of $\mathcal{E}(\mathrm{e}^{\nu_{ijt}})$. In turn the put price is increased. The responsiveness of the put price depends on the proportion $a_{ijt}/a_{it}$ of assets in class $j$ held by firm $i$ at time $t$. Thus the $\mathrm{e}^{\nu_{ijt}}$ can be viewed as risk weights where the averaging over put values corresponds to an implicit average risk weight. The implicit weights can be backed out to give risk weighted assets

$$a_{it}\mathcal{E}_a\left(\frac{1}{S}\sum_s \phi^s \mathrm{e}^{\nu_{ijt}^s}\right) \approx \mathrm{E}_\phi\{\mathcal{E}_a(\mathrm{e}^{\nu_{ijt}})\} \ . \tag{18}$$

The expression in brackets in the left is the risk weight for asset $j$. Assets whose return $\nu_{ijt}$ tends to be very negative under stress $\phi$ are has high risk and low weight. An asset whose value does not vary has $\nu_{ijt} \equiv 0$ and a risk weight of 1 since on average $\phi^s$ is 1.

The difference

$$k - \frac{1}{S}\sum_s \phi^s \mathcal{E}_a(\mathrm{e}^{\nu_{ijt}^s}) \ ,$$

is the proportionate increase capital shortfall on account of risk weighting and as a proportion of total assets.

## 12. Application to Australian financial institutions

In this study we consider financial data for the 8 Australian banks.

Table 1: Four major and four minor Australian banks

| | |
|---|---|
| cba | Commonwealth Bank of Australia |
| anz | Australia & New Zealand Bank |
| nab | National Australia Bank |
| wbc | Westpac Banking Corporation |
| mqg | Macquarie Banking Group |
| boq | Bank of Queensland |
| ben | Bendigo and Adelaide Bank |
| aba | Auswide Bank |

The analysis is based on daily return data is collected from 3 April 2000 through to 1 December 2014 for a total of 3848 trading days. Prices, adjusted for dividends are plotted in Figure 2.

At each of the first trading day in the months from January 2003 through to December 2014, all previous returns were used estimate a TARCH-DCC model described below. For each first trading day of the month, the fitted model is then used to, simulate the forward return distribution over the next 22 trading days, corresponding to approximately one month. The details of the fitted models are described in the next subsection.

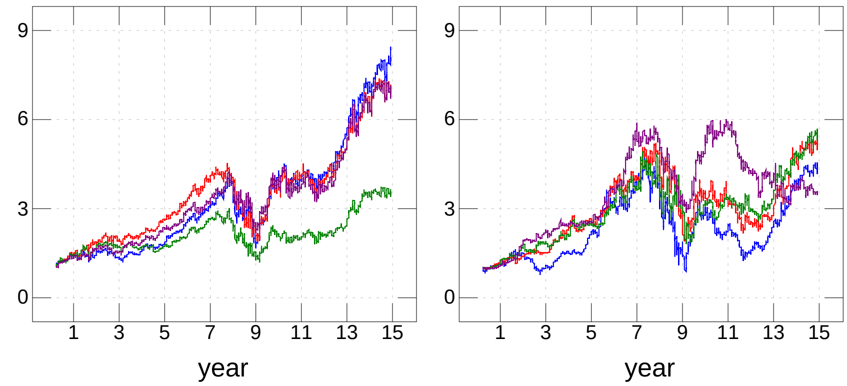


Figure 2: Basel log–leverages for four major (left panel) and four minor (right panel) Australian banks from April 2000 through to December 2014. Note all banks suffer falls starting the middle of 2008.

### *12.1. TARCH-DCC modelling*

For notational convenience the $i$ subscript is dropped and denote the daily (log) return as

$$r_t = \mu + \sigma_t \epsilon_t \ , \qquad \epsilon_t \sim (0,1) \ , \qquad \epsilon_t^- \equiv I(\epsilon_t < 0) = I(r_t < \mu) \ , \tag{19}$$

Thus returns are assumed to have an unconditionally constant mean. The notation $\epsilon_t^-$ indicates whether there is a negative return: $I$ denoting the indicator function. The volatility is modelled with the TARCH model

$$\sigma_{t+1}^2 = \omega + \sigma_t^2\{\beta + (\alpha + \gamma\epsilon_t^-)\epsilon_t^2\}. \tag{20}$$

Hence the response of the variance to $\epsilon_t^2$ is increased by $\gamma$ if $\epsilon_t < 0$ compared to the response if $\epsilon_t > 0$. This is a simple threshold GARCH model: called TARCH. In the fits it is assumed the mean $\mu$ does not vary with time $t$ and the single lag in both the $\sigma_t^2$ and $\epsilon_t$ is sufficient to capture the dynamics of volatility.

The model (19) and (20) is fit for each security $i$ including the market, $i = m$. The correlations between securities and the market are also modelled using positive definite recursions (Engle, 2002)

$$(Q_{t+1} - S) = \alpha(\eta_t\eta_t' - S) + \beta(Q_t - S) \ , \qquad \eta_t \equiv (\epsilon_{it}, \epsilon_{mt})',$$

A Dynamic Conditional Correlation (DCC) model is estimated for firm and market returns. Write

$$\mathrm{cov}(r_t|t) = V_t R_t V_t' \ , \qquad R_t \equiv \mathrm{cov}(\varepsilon_t|t) \ , \qquad \varepsilon_t = V_t^{-1} r_t \ ,$$

where $R_t$ and $V_t$ are the correlation and diagonal volatility matrix, respectively, at time $t$. Volatilities in $V_t$ are estimated using univariate threshold GARCH models. The estimate of $R_t$ corresponds to the correlations in covariance matrix $C_t$ assuming

$$(C_t - C) = \alpha(C_{t-1} - C) + \beta(\varepsilon_t \varepsilon_t' - C) \ , \qquad C \equiv \operatorname{cov}(\varepsilon_t) \ .$$

where $C$ is the long run covariance matrix.

*12.2. Forward return simulation*

The forward simulations are implemented similar to Brownlees and Engle (2015). Given the latest available volatility and correlation estimates the filter recursions are moved forward in time fed with innovations randomly chosen from past standardised innovations. Thus the innovations are chosen to have the same marginal distributions as applicable in the past. The random choices are such that the same market return innovations are used in each bivariate analysis.

Figure Figure 3 displays 6000 forward simulations for the joint CBA and market return on each of two dates, the first trading days in January 2009 and December 2014.

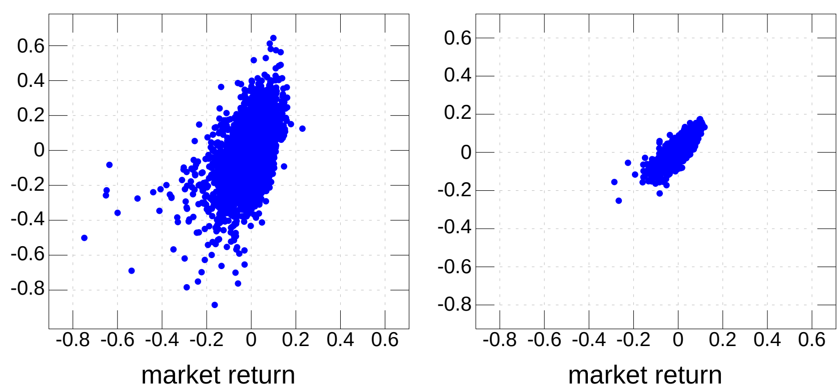


Figure 3: Six thousand simulated bivariate 1 month forward returns for CBA and the market at start of January 2009 (left panel) and December 2014 (right panel). Note scales in both panels are the same, the relatively large volatility in the left panel, and the left skew in the marginal distributions.

*12.3. Basel beta risk*

Figure Figure 4 displays Basel beta's over the years from 2003 through to the end of 2014. The employed stressor is the expected worst outcome in 12 independent copies:

$$\phi(u) = 12(1-u)^{11} \ , \qquad \beta_{it} = \frac{\mathrm{E}\{\min(p_{it}^1, \ldots, p_{it}^{12})\} - \mathrm{E}(p_{it})}{\sigma_\phi}$$

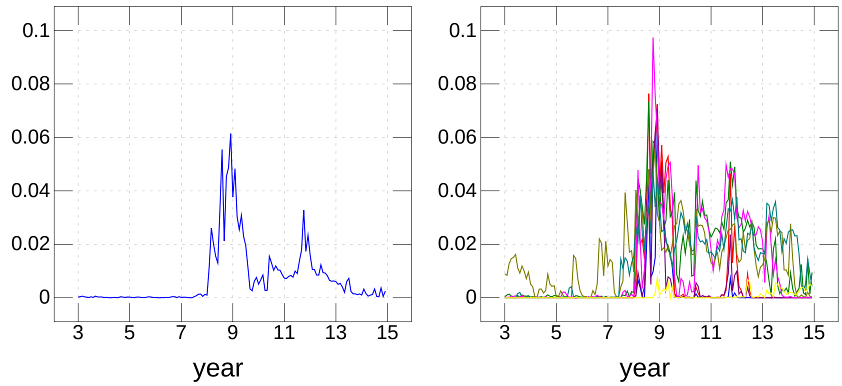


Figure 4: Basel beta's. The left panel displays the overall beta. The right panel ...

*12.4. Stressors*

Two different stressors are used:

## 13. Standardised SRISK and layer dependence

Standardising SRISK by subtracting the unconditional expected capital shortfall and dividing by the same assuming maximum systemic risk yields a quantity which is independent of initial debt and equity. In addition standardised SRISK reflects the local dependence between an individual firm and the market at the selected threshold of the systemic event. Hence varying the threshold yields the dependence structure of individual firm and market returns from benign events all the way to tail events.

The unconditional expected capital shortfall for firm $i$ is $\mathrm{E}(c_i) = kd_i - (1-k)w_i\{1+\mathrm{E}(r_i)\}$, suggesting the standardised SRISK

$$\mathrm{SRISK}_i^* \equiv \frac{\mathrm{SRISK}_i - \mathrm{E}(c_i)}{\max(\mathrm{SRISK}_i) - \mathrm{E}(c_i)} = \frac{\mathrm{E}(r_i) - \mathrm{E}(r_i|u_m < a)}{\mathrm{E}(r_i) - \mathrm{E}(r_i|u_i < a)}$$

$$= \frac{\mathrm{cov}\{r_i, \overline{I}_a(u_m)\}}{\mathrm{cov}\{r_i, \overline{I}_a(u_i)\}} = \frac{\mathrm{cov}\{r_i, I_a(u_m)\}}{\mathrm{cov}\{r_i, I_a(u_i)\}}$$

where $\overline{I}_a(z) \equiv (z < a)$ and $I_a(z) \equiv (z > a)$ are indicators. Maximum SRISK occurs if firm return is comonotonic with the market return or $u_i = u_m$, $u_i$ being the percentile rank of $r_i$. Standardised SRISK is hence layer dependence on the original scale, and mainly lies between 0 and 1 indicating independence and comonotonicity, respectively. Varying $a$ reveals the local dependence between firm and market returns at different thresholds, with $a \to 0$ yielding the lower extreme tail dependence in a market crash of maximum severity.

Performing the same standardising on $\widetilde{\text{SRISK}}_i$ which is a weighted average of SRISK for firm $i$ across various thresholds yields

$$\widetilde{\text{SRISK}}_i^* = \frac{\text{cov}\{r_i, \phi(u_m)\}}{\text{cov}\{r_i, \phi(u_i)\}} \ .$$

### 14. Comparing SRISK for different firms (not sure)

Suppose of interest is whether $\beta_{it}$ varies similarly across firms as $\tau$ varies where $\tau$ is the threshold, $k$, or some other parameter used to compute $\beta_{it}$. High correlation implies high systemic risk since firms are simultaneously affected.

Define $\epsilon_{it} \equiv (\phi - 1)p_{it}$ and $\epsilon_t$ as the vector with components $\epsilon_{it}$. Then $\text{E}(\epsilon_{it}|\tau) = \sigma_\phi(\tau)\beta_{it}(\tau)$, the stress in firm $i$ at the given parameter setting $\tau$ and

$$\text{cov}\{\text{E}(\epsilon_{it}|\tau)\} = \text{cov}(\epsilon_t) - \text{E}\{\text{cov}(\epsilon_{it}|\tau)\} \ ,$$

is the covariance between stresses as $\tau$, the stress parameter varies. Thus the covariances and correlations between firms as stress parameters vary can be computed from the covariability between the $\epsilon_{it}$ and the ...

### 15. Standardised SRISK and layer dependence

Standardising SRISK by subtracting the unconditional expected capital shortfall and dividing by the same assuming maximum systemic risk yields a quantity which is independent of initial debt and equity. In addition standardised SRISK reflects the local dependence between an individual firm and the market at the selected threshold of the systemic event. Hence varying the threshold yields the dependence structure of individual firm and market returns from benign events all the way to tail events.

The unconditional expected capital shortfall for firm $i$ is $\text{E}(c_i) = kd_i - (1 - k)w_i\{1 + \text{E}(r_i)\}$, suggesting the standardised SRISK

$$\begin{aligned} \text{SRISK}_i^* &\equiv \frac{\text{SRISK}_i - \text{E}(c_i)}{\max(\text{SRISK}_i) - \text{E}(c_i)} = \frac{\text{E}(r_i) - \text{E}(r_i|u_m < a)}{\text{E}(r_i) - \text{E}(r_i|u_i < a)} \\ &= \frac{\text{cov}\{r_i, \overline{I}_a(u_m)\}}{\text{cov}\{r_i, \overline{I}_a(u_i)\}} = \frac{\text{cov}\{r_i, I_a(u_m)\}}{\text{cov}\{r_i, I_a(u_i)\}} \end{aligned}$$

where $\overline{I}_a(z) \equiv (z < a)$ and $I_a(z) \equiv (z > a)$ are indicators. Maximum SRISK occurs if firm return is comonotonic with the market return or $u_i = u_m$, $u_i$ being the percentile rank of $r_i$. Standardised SRISK is hence layer dependence on the original scale, and mainly lies between 0 and 1 indicating independence and comonotonicity, respectively. Varying $a$ reveals the local dependence between firm and market returns at different thresholds, with $a \to 0$ yielding the lower extreme tail dependence in a market crash of maximum severity.

Performing the same standardising on $\widetilde{\text{SRISK}}_i$ which is a weighted average of SRISK for firm $i$ across various thresholds yields

$$\widetilde{\text{SRISK}}_i^* = \frac{\text{cov}\{r_i, \phi(u_m)\}}{\text{cov}\{r_i, \phi(u_i)\}} \ .$$

*15.1. Contagion effects*

Definition (7) can be generalized to capture contagion effects. Consider

$$\mathcal{E}^j\{\mathrm{E}^j(p_{it})\} = \mathrm{E}^j\{\mathcal{E}^j(p_{it})\} \ , \tag{21}$$

where $\mathrm{E}^j$ denotes the stressed conditional expectation, conditioning on a firm $j$ breach, and $\mathcal{E}^j$ is an debt weighted average excluding firm $j$. Firm $j$ is systemically important if a Basel breach in firm $j$ leads to large increase in system systemic risk

$$c_{ij} \equiv \frac{\mathrm{E}^j\{\mathcal{E}^j(p_{it})\}}{\mathrm{E}\{\mathcal{E}(p_{it})\}} - 1 \ .$$

*15.2. Diversification effects*

Suppose debt and equity are aggregated across firms to yield $d_t$ and $w_t$. The system log leverage is

$$\ell_t = \ln \frac{d_t}{w_t} = \ln \sum_i \frac{w_{it}}{w_t}\frac{d_{it}}{w_{it}} = \ln \mathcal{E}_w(\mathrm{e}^{\ell_{it}}) \ ,$$

where $\mathcal{E}_w$ denotes an equity weighted average. A system wide breach of the Basel II limit occurs at $t+h$ if $\nu_t < \ell_t^* \equiv \ell_t - 2.44$ where $\nu_t$ is the rate of return on total equity $w_t$:

$$\nu_t = \ln \frac{w_{t+h}}{w_t} = \ln \mathcal{E}_w\left(\frac{w_{i,t+h}}{w_{it}}\right) = \ln \mathcal{E}_w(\mathrm{e}^{\nu_{it}}) \ . \tag{22}$$

Similar to before define a put for the system and its (stressed) expected value

$$p_t \equiv k|1 - \mathrm{e}^{\nu_t - \ell_t^*}|^+ \ , \qquad \mathrm{E}_\phi(p_t) \le \mathcal{E}\{\mathrm{E}_\phi(p_{it})\} \ .$$

The inequality is a result of diversification effects: low liquidity in one firm is implicitly offset by high liquidity in other firms.

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