Systemic risk and contagion effects in Australian financial institutions and sectors

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1. Simplified notation

Suppose the system is currently at time t and the capital shortfall for firm i at time $t+h$ is

$$c_i \equiv kd_i - (1-k)w_i(1+r_i)$$

where k is the prudential requirement, d_i and w_i are debt and equity at time t, and r_i is the return on equity from time t to $t+h$. The only random quantity in the above definition is r_i. The total capital shortfall in the system is

$$c_m = \sum_i c_i = k\sum_i d_i - (1-k)\sum_i w_i(1+r_i) = kd_m - (1-k)w_m(1+r_m)$$

where

$$d_m \equiv \sum_i d_i \ , \qquad w_i \equiv \sum_i w_i \ , \qquad r_m \equiv \frac{\sum_i w_i r_i}{\sum_i w_i}$$

are the overall debt and equity for the market at time t and the market return on equity from time t to $t+h$. Note it is assumed that capital surpluses in certain firms are used to offset shortfalls in other firms, that is a diversification benefit is allowed in the system.

The systemic risk contribution by firm i at time $t+h$ is the expected shortfall in systemic event $r_m < a$

$$\text{SRISK}_i = \text{E}(c_i|r_m < a) = kd_i - (1-k)w_i\{1+\text{E}(r_i|r_m < a)\} \ ,$$

and is computed from the joint probability distribution of (r_i, r_m). Replace the systemic event with one having a probabilistic interpretation: $u_m < a$ where $0 < a < 1$ and u_m is the percentile rank of r_m. The refined systemic event $u < a$ captures the worst a of possible market returns and has probability a. In addition the threshold a is simply the percentile rank of the original threshold. Hence the revised systemic risk of firm i is

$$\text{SRISK}_i \equiv \text{E}(c_i|u_m < a) = kd_i - (1-k)w_i\{1+\text{E}(r_i|u_m < a)\} \ .$$

In addition the aggregate systemic risk is

$$\mathrm{SRISK}_m \equiv \sum_i \mathrm{SRISK}_i = \sum_i \mathrm{E}(c_i|u_m < a) = \mathrm{E}(c_m|u_m < a)$$
$$= kd_m - (1-k)w_m\{1 + \mathrm{E}(r_m|u_m < a)\} \ .$$

Note the original definition of aggregate systemic risk is the sum of positive systemic risks of individual firms

$$\sum_i (\mathrm{SRISK}_i)_+$$

and does not result in the above simplification. The original definition assumes no diversification: capital surpluses cannot cover shortfalls. The systemic risk of firm i is positive, indicating capital shortfall, if and only if

$$\mathrm{E}(r_i|u_m < a) < \frac{kd_i}{(1-k)w_i} - 1$$

2. Systemic risk

If d_{it} and w_{it} are the debt and equity of a firm i at time t then the capital shortfall at time t is

$$kd_{it} - (1-k)w_{it} = kd_{it}\left(1 - \mathrm{e}^{-\ell^*_{it}}\right) \ , \tag{1}$$

where

$$\ell^*_{it} \equiv \ell_{it} + \ln\frac{k}{1-k} \ , \qquad \ell_{it} \equiv \ln\frac{d_{it}}{w_{it}} \ .$$

The quantity ℓ^*_{it} is called Basel adjusted the log–leverage and $\ell^*_{it} > 0$ implies capital shortfall (1) is positive.

Basel II assumes $k = 0.08$ implying $\ell^*_{it} = \ell_{it} - 2.44$ and there is positive "Basel shortfall" if $\ell_{it} > 2.44$. As the log leverage ℓ_{it} increases to 2.44 the firm approaches the Basel II threshold.

2.1. Future returns

Consider firm i at time $t+h$ where $h > 0$. Then

$$\ell_{i,t+h} = \ln\frac{d_{i,t+h}}{w_{i,t+h}} = \ell_{it} - \nu_t \ , \qquad \nu_{it} \equiv \ln\frac{w_{i,t+h}}{w_{it}} \ ,$$

where it is assumed $d_{i,t+h} = d_{it}$. Hence ν_{it} is the return on equity over $(t, t+h)$. In terms of the Basel leverage ℓ^*_{it} at time t, the Basel II shortfall at time $t+h$ is,

$$d_{it}k(1 - \mathrm{e}^{\nu^*_{it}}) \ , \qquad \nu^*_{it} \equiv \nu_{it} - \ell^*_{it} = \nu_{it} - \ln\frac{d_{it}}{w_{it}} - \ln\frac{k}{1+k} \ . \tag{2}$$

The Basel II limit is reached at $t+h$ if $\nu_{it} = \ell^*_{it}$ and breached if $\nu_{it} < \ell^*_{it}$. A breach is unlikely if ℓ^*_{it} is low.

2.2. Engle systemic risk

Brownlees and Engle (2015) defines the systemic risk (SRISK) of a firm i as the expected Basel II shortfall at $t+h$,

$$d_{it}\tilde{\mathrm{E}}\{k(1-\mathrm{e}^{\nu_{it}^*})\} = d_{it}k\{1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})\} \ , \tag{3}$$

where the expectation $\tilde{\mathrm{E}}$ is a "stressed" expectation i.e. an expectation conditional on a stress in the system such as a major market downturn.

The systemic risk of firm i and the system systemic risk are defined as

$$d_{it}k\{1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})\} \ , \qquad k\sum_i d_{it}|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+ \ , \qquad |x|^+ \equiv \max(0,x) \ , \tag{4}$$

respectively. Finally the contribution of firm i to total systemic risk is defined as the positive expected capital shortfall relative to the same when aggregated across all firms:

$$\frac{d_{it}|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+}{\sum_i d_{it}|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+} = \frac{|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+}{\mathcal{E}\{|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+\}} \ ,$$

where $\mathcal{E}$ denotes a debt–weighted average. A large proportion indicates a systemically important firm.

The following comments appear relevant:

1. The SRISK contribution of a firm is zero whenever
$$\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}}) \leq \mathrm{e}^{\ell_{it}^*} \ . \tag{5}$$
This inequality generally holds even if the expectation is severely stressed.
2. If ν_{it} is normal then
$$\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}}) = \mathrm{e}^{h\mu_{it}+h^2\sigma_{it}^2/2}$$
where the mean and standard deviation are the stressed versions. In terms of this notation (5) is equivalent to
$$\mu_{it} + h\frac{\sigma_{it}^2}{2} \leq \ell_{it}^* \ ,$$
indicating a Basel breach is unlikely unless ℓ_{it} is near 2.44, the Basel limit.
3. The system SRISK is zero if all firm the SRISK contributions are zero that is if (5) holds for every i. This would be usual in the Australian case.

An improvement is where firms with capital surplus are included, since they, especially those with large surpluses, are systemically important but in a positive way by dampening the impact from distressed firms. Second, it is appropriate to consider the departure from the unconditional expectation to differentiate firms which only experience capital shortfall during a market downturn as opposed to those who already have large shortfalls in average conditions. This leads to the revised computation

$$\frac{1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*}) - \{1-\mathrm{E}(\mathrm{e}^{\nu_{it}^*})\}}{\mathcal{E}\{|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+ - |1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+\}} = \frac{\mathrm{E}(\mathrm{e}^{\nu_{it}^*}) - \tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})}{\mathcal{E}\{|1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+ - |1-\tilde{\mathrm{E}}(\mathrm{e}^{\nu_{it}^*})|^+\}}$$

The revised contribution of firm i to total systemic risk is hence

$$\frac{\text{SRISK}_i - \text{E}(c_i)}{\text{SRISK}_m - \text{E}(c_m)} = \frac{w_i}{w_m} \frac{\text{E}(r_i) - \text{E}(r_i|u_m < a)}{\text{E}(r_m) - \text{E}(r_m|u_m < a)} \ ,$$

the product of the equity of firm i relative to total market equity and return movement of firm i in a market downturn relative to overall market movement.

2.3. Improved systemic risk measurement

For firm i define the Basel put and Basel risk as

$$p_{it} \equiv k|1 - \text{e}^{\nu^*_{it}}|^+ \ , \qquad \text{E}(p_{it}) > 0 \ , \tag{6}$$

where E denotes normal or unstressed expectation. The Basel risk in firm i depends only k, the present leverage and the lower tail of the ν_{it} distribution. The put value (6) is stated in terms of unit debt and increases with k, both directly from multiplication by k and indirectly through its logit, making a positive put outcome more likely.

Default put options similar to (6) have been discussed in the insurance literature as critical to an evaluation of a firm: see for example Merton (1977), Doherty and Garven (1986), Cummins (1988), Myers and Read (2001) and Sherris (2006).

The total Basel put value and hence Basel risk for firm i is $d_{it}\text{E}(p_{it})$. This is the cost of insuring firm i does not breach the Basel standard. In practice it may be appropriate to discount p_{it} in (6) by the interest rate over the period $(t, t+h)$ and value the put with risk neutral rather than normal expectation.

The put p_{it} is tradable since, given k, it relies only on the present leverage and the future return ν_{it} on equity w_{it}. Market participants can value the put in the same way as any other put and use the contract to diversify risk. Firms can buy puts in the market to hedge Basel default risk. Regulators will value puts using a stressed, rather than risk neutral, expectation and regulators can monitor stressed put values for systemic risk. Regulators can charge firms their put value if there are implicit government bailout guarantees.

In terms of the put values $\text{E}(p_{it})$, the system Basel risk is defined as the debt weighted average of the individual firm's put values:

$$\mathcal{E}\{\text{E}(p_{it})\} = \text{E}\{\mathcal{E}(p_{it})\} \ . \tag{7}$$

The standardised system Basel risk is stated per unit of system debt. Further

$$d_t\mathcal{E}\{\text{E}(p_{it})\} = \sum_i d_{it}\text{E}(p_{it}) \ , \qquad d_t \equiv \sum_i d_{it} \ ,$$

shows, on the left, the system Basel risk and, on the right, the weighted sum of firm Basel risks.

2.4. Stressed expectations

In the above development E denotes unstressed expectation: that is expectation that does not assume a stressful scenario. A stressed expectation, denoted $\tilde{\mathrm{E}}$, is linear combination of expectations that do assume stress:

$$\tilde{\mathrm{E}}(p_{it}) \equiv \mathrm{E}\{\phi \mathrm{E}(p_{it}|\phi)\} = \mathrm{E}(\phi p_{it}) = \mathrm{E}(p_{it}) + \mathrm{cov}(p_{it}, \phi) \ . \tag{8}$$

Here $\phi \geq 0$ with $\mathrm{E}(\phi) = 1$ is thought of as a "stressor," defining events impacting put values and serving as a basis for stress testing. In practice the conditioning events ϕ downplay or highlight different scenarios. The conditioning events are scenarios of interest and probabilistic weighting is according to the level of interest.

Writing σ_ϕ as the standard deviation of ϕ, define the Basel stress of firm i with respect to stress ϕ as

$$\beta_{it} \equiv \frac{\tilde{\mathrm{E}}(p_{it}) - \mathrm{E}(p_{it})}{\sigma_\phi} = \frac{\mathrm{E}\{(\phi - 1)p_{it}\}}{\sigma_\phi} = \frac{\mathrm{cov}(p_{it}, \phi)}{\sigma_\phi} \ , \tag{9}$$

where σ_ϕ is the standard deviation of ϕ.

Stress β_{it} captures the change in the expected put price when σ_ϕ units of stress ϕ are applied. Hence the change in stress is measured in standardised units. Units of stress take on different meaning depending on the specific context as discussed below.

The quantity $(\phi - 1)/\sigma_\phi$ is thought of a "stress factor." The stress factor has mean 0 and standard deviation 1 and serves to scale the put value p_{it} to yield a stress effect. The distribution of ϕ determines the distribution of the stress factor. If the stress factor is normally distributed then a stress effect more than doubling of the put value occurs less than about 2.5% of the time.

Stress effects can be determined by simulation:

$$\beta_{it} \approx \frac{1}{\sigma_\phi S} \sum_{s=1}^{S} (\phi^s - 1) p_{it}^s \ ,$$

where S is the simulation effort.

Stress events implicit in (9) and captured with ϕ can be defined with respect any events. This article is concerned with stresses caused by systemic events such as a severe market downturn i.e. a large negative value for the forward return ν_{mt}. In this context the aim is to compute $\tilde{\mathrm{E}}(p_{it})$ given stress in ν_{it} defined as an extreme percentile u_{mt} of ν_{mt}.

The monetary Basel stress of firm i at time t is $d_{it}\beta_{it}$. Note the Basel stress is stated in terms of unit increases in stress ϕ. Further it measures the impact of stress. Thus in an increasingly dire financial situation it is expected that put prices p_{it} will increase. However stress β_{it} measures the change in current put values under further ϕ–stress.

2.5. System stress

Systemic events are expected to affect all firms i. A measure of the total effect is to consider effect on the debt weighted put

$$p_t \equiv \mathcal{E}(p_{it}) \ , \qquad \tilde{\mathrm{E}}(p_t) - \mathrm{E}(p_t) = \mathrm{cov}\{\mathcal{E}(p_{it}), \phi\} = \mathcal{E}\{\mathrm{cov}(p_{it}, \phi)\} \ .$$

The "system" put p_t pays out zero if all firms meet the Basel requirement with payments increasing with the number of Basel breaches, the sizes of the breaches, and the relative debt sizes of the breaching firms.

Define the system systemic risk as

$$\beta_t \equiv \frac{\tilde{\mathrm{E}}(p_t) - \mathrm{E}(p_t)}{\sigma_\phi} = \mathcal{E}(\beta_{it}) \ , \tag{10}$$

The monetary Basel system stress at time t is the sum of monetary firm stresses

$$d_t \beta_t = d_t \mathcal{E}(\beta_{it}) = \sum_i d_{it} \beta_{it} \ .$$

3. Standardised SRISK and layer dependence

Standardising SRISK by subtracting the unconditional expected capital shortfall and dividing by the same assuming maximum systemic risk yields a quantity which is independent of initial debt and equity. In addition standardised SRISK reflects the local dependence between an individual firm and the market at the selected threshold of the systemic event. Hence varying the threshold yields the dependence structure of individual firm and market returns from benign events all the way to tail events.

The unconditional expected capital shortfall for firm i is $\mathrm{E}(c_i) = kd_i - (1-k)w_i\{1 + \mathrm{E}(r_i)\}$, suggesting the standardised SRISK

$$\mathrm{SRISK}_i^* \equiv \frac{\mathrm{SRISK}_i - \mathrm{E}(c_i)}{\max(\mathrm{SRISK}_i) - \mathrm{E}(c_i)} = \frac{\mathrm{E}(r_i) - \mathrm{E}(r_i | u_m < a)}{\mathrm{E}(r_i) - \mathrm{E}(r_i | u_i < a)}$$

$$= \frac{\mathrm{cov}\{r_i, \overline{I}_a(u_m)\}}{\mathrm{cov}\{r_i, \overline{I}_a(u_i)\}} = \frac{\mathrm{cov}\{r_i, I_a(u_m)\}}{\mathrm{cov}\{r_i, I_a(u_i)\}}$$

where $\overline{I}_a(z) \equiv (z < a)$ and $I_a(z) \equiv (z > a)$ are indicators. Maximum SRISK occurs if firm return is comonotonic with the market return or $u_i = u_m$, u_i being the percentile rank of r_i. Standardised SRISK is hence layer dependence on the original scale, and mainly lies between 0 and 1 indicating independence and comonotonicity, respectively. Varying a reveals the local dependence between firm and market returns at different thresholds, with $a \to 0$ yielding the lower extreme tail dependence in a market crash of maximum severity.

Performing the same standardising on $\widetilde{\mathrm{SRISK}}_i$ which is a weighted average of SRISK for firm i across various thresholds yields

$$\widetilde{\mathrm{SRISK}}_i^* = \frac{\mathrm{cov}\{r_i, \phi(u_m)\}}{\mathrm{cov}\{r_i, \phi(u_i)\}} \ .$$

The setup is illustrated with two examples generalising the approach of Engle.

3.0.1. Market return below a percentile threshold

Suppose the stress event is a market return in the bottom α–tail of the market return distribution. Then $\phi(u) = 1/\alpha$ for $u < \alpha$ and 0 otherwise and

$$\sigma_\phi = \sqrt{\int_0^\alpha \frac{\mathrm{d}u}{\alpha^2} - 1} = \sqrt{\frac{1-\alpha}{\alpha}} = \mathrm{e}^{-(\mathrm{logit}\ \alpha)/2} \ .$$

Then

$$\beta_{it} = \frac{\tilde{\mathrm{E}}(p_{it}) - \mu_{it}}{\sqrt{(1-\alpha)/\alpha}} = \frac{\alpha \mathrm{E}\{I(u_{mt} < \alpha)p_{it}\}}{\mathrm{e}^{-(\mathrm{logit}\ \alpha)/2}} \approx \frac{\mathrm{e}^{(\mathrm{logit}\ \alpha)/2}}{S/\alpha} \sum_{u^s_{mt} < 1/\alpha} p^s_{it} \ ,$$

where $s = 1, \ldots, S$ denotes simulations and I the indicator function. The approximation decreases with the simulation effort S. The effective simulation effort is S/α and hence α small requires a large S.

A fixed percentage threshold means the actual threshold depends on the distribution and hence on volatility and time.

3.0.2. Expected worst market return in independent copies

Assume $\phi(u) = n(1-u)^{n-1}$ where u is the market return percentile. Then $\mathrm{E}\{\phi(u)\} = 1$ and $\sigma_\phi = xx$ and $\tilde{\mathrm{E}}$ is computed assuming the worst h day return in n independent and identical copies of the current situation, and

$$\beta_{it} = \frac{n\mathrm{E}\{(1-u)^{n-1}p_{it}\}}{\sigma_\phi} \approx \frac{1}{xxS/n} \sum_s (1-u^s_{mt})^{n-1} p^s_{it} \ .$$

Simulated returns ν^s_{mt} in the upper tail of the distribution have percentiles $u^s_{mt} \approx 1$ and hence for these returns the second factor in the sum is small if not negligable: the associated simulated put is heavily downweighed.

Note the contrast with the previous example where the bottom α proportion of simulated market returns are selected as the stressed sample. In the latter situation every simulated value contributes to the expectation, albeit with vastly different weights.

3.0.3. Worst market return given a tail event

The above two situations can be combined. Suppose $\phi(u) = c(0.05-u)^{19}$ for $u \le 0.05$ and 0 otherwise and where c is such that $\phi(u)$ integrates to 1. Then

$$\mathrm{E}(p_{it}) \approx \frac{1}{S/c} \sum_{u^s_{mt} < 1/20} \left(u^s_{mt} - \frac{1}{20}\right)^{20} p^s_{it} \ .$$

Similar to the first example, the stressed sample picks up the bottom 5% of market returns. The bottom 5% of market returns is further stressed by progressively downweighing returns as the percentile approaches 1/20. Enormous simulation effort S is required for a reasonable approximation since S/c is the effective simulation size.

3.1. Comparing SRISK for different firms (not sure)

Suppose of interest is whether β_{it} varies similarly or different across firms as τ varies where τ is the threshold, k or some other parameter used to compute β_{it}. High correlation implies high systemic risk since firms are simultaneously affected.

Define $\epsilon_{it} \equiv (\phi - 1)p_{it}$ and ϵ_t as the vector with components ϵ_{it}. Then $\mathrm{E}(\epsilon_{it}|\tau) = \sigma_\phi(\tau)\beta_{it}(\tau)$, the stress in firm i at the given parameter setting τ and

$$\mathrm{cov}(\epsilon_t) = \mathrm{cov}\{\mathrm{E}(\epsilon_{it}|\tau)\} + \mathrm{E}\{\mathrm{cov}(\epsilon_{it}|\tau)\} \ ,$$

is the covariance between stresses as τ, the stress parameter varies. Thus the covariances and correlations between firms as stress parameters vary can be computed from the covariability between the ϵ_{it} and the ...

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where $\overline{I}_a(z) \equiv (z < a)$ and $I_a(z) \equiv (z > a)$ are indicators. Maximum SRISK occurs if firm return is comonotonic with the market return or $u_i = u_m$, u_i being the percentile rank of r_i. Standardised SRISK is hence layer dependence on the original scale, and mainly lies between 0 and 1 indicating independence and comonotonicity, respectively. Varying a reveals the local dependence between firm and market returns at different thresholds, with $a \to 0$ yielding the lower extreme tail dependence in a market crash of maximum severity.

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4.1. Contagion effects

Definition (7) can be generalized to capture contagion effects. Consider

$$\mathcal{E}^j\{\mathrm{E}^j(p_{it})\} = \mathrm{E}^j\{\mathcal{E}^j(p_{it})\} \ , \tag{11}$$

where E^j denotes the stressed conditional expectation, conditioning on a firm j breach, and $\mathcal{E}^j$ is an debt weighted average excluding firm j. Firm j is systemically important if a Basel breach in firm j leads to large increase in system systemic risk

$$c_{ij} \equiv \frac{\mathrm{E}^j\{\mathcal{E}^j(p_{it})\}}{\mathrm{E}\{\mathcal{E}(p_{it})\}} - 1 \ .$$

4.2. Diversification effects

Suppose debt and equity are aggregated across firms to yield d_t and w_t. The system log leverage is

$$\ell_t = \ln \frac{d_t}{w_t} = \ln \sum_i \frac{w_{it}}{w_t}\frac{d_{it}}{w_{it}} = \ln \mathcal{E}_w(\mathrm{e}^{\ell_{it}}) \ ,$$

where $\mathcal{E}_w$ denotes an equity weighted average. A system wide breach of the Basel II limit occurs at $t+h$ if $\nu_t < \ell_t^* \equiv \ell_t - 2.44$ where ν_t is the rate of return on total equity w_t:

$$\nu_t = \ln \frac{w_{t+h}}{w_t} = \ln \mathcal{E}_w\left(\frac{w_{i,t+h}}{w_{it}}\right) = \ln \mathcal{E}_w(\mathrm{e}^{\nu_{it}}) \ . \tag{12}$$

Similar to before define a put for the system and its (stressed) expected value

$$p_t \equiv k|1 - \mathrm{e}^{\nu_t - \ell_t^*}|^+ \ , \qquad \tilde{\mathrm{E}}(p_t) \le \mathcal{E}\{\tilde{\mathrm{E}}(p_{it})\} \ .$$

The inequality is a result of diversification effects: low liquidity in one firm is implicitly offset by high liquidity in other firms.

4.3. Risk weighted assets

Suppose total assets of a company are d_t+w_t. Total assets are often replaced by risk weighted assets implying

$$w_t = d_t\{\mathcal{E}_\pi(a_{jt}) - 1\} \ , \qquad \ln \frac{d_t}{w_t} = -\ln\{\mathcal{E}_\pi(a_{jt}) - 1\}$$

where j runs over the various asset classes and $\mathcal{E}_\pi$ denotes a risk weighted average of assets: the weights in this average typically do not add to 1.

In h periods time, supposing debt does not change, the change in log–leverage is

$$\ln\{\mathcal{E}_\pi(a_{j,t+h}) - 1\} - \ln\{\mathcal{E}_\pi(a_{jt}) - 1\}$$

Suppose the asset portfollow grows at rate $\mathcal{E}(\nu_{jt})$.

$d_{it} + \mathrm{e}^{-v_{it}} w_{it}$ where $v_{it} > 0$ is the implicit discounting on equity used to arrive at risk weighted assets. Using risk weighted assets implies the modified leverage

$$\ln\left\{\frac{d_{it}}{\mathrm{e}^{-v_{it}} w_{it}}\right\} = \ell_{it} + v_{it} \ .$$

In terms of this setup the systemic risk in firm i is then $\mathrm{E}(p_{it})$ computed with

$$p_t \equiv k|1 - \mathrm{e}^{\nu_t - \ell_t^* - v_{it}}|^+$$

Since $v_{it} \geq 0$ the put is more valuable. Note

$$v_{it} = -\ln\left\{\frac{\mathcal{E}_\pi(a_{it})}{w_{it}} - \mathrm{e}^{\ell_{it}}\right\}$$

5. Alternative market stressors

If $\phi = \phi(u_{mt})$ with ϕ decreasing on $[0,1]$ then stress is modelled in terms of market return and with larger stress if u_{mt} is smaller. The impact on $\tilde{\mathrm{E}}(p_{it})$ is larger if ν_{it} and ν_{mt} are highly correlated and

$$\tilde{\mathrm{E}}(p_{it}) = -\mathrm{E}\{u\phi'(u)\mathrm{E}(p_{it}|u_{mt} \leq u)\} \ , \tag{13}$$

and hence if the stressor is market return, then the stressed expectation can be written as a stressed expectation of conditional tail expectations. This circumvents an explicit choice for the market threshold, enabling a mixture of thresholds.

Brownlees and Engle (2015) define a systemic event as a market based downturn greater than a certain threshold and hence is dependent on the chosen threshold. Explicit choice of the threshold is avoided by choosing a weighted average of percentile thresholds

The left hand side expresses stressed expectation in terms of a weighted average $w(u)$ of tail expectations across different percentiles while the right hand side is the form in (8). The equivalence of the left and right hand sides of (13) is established with

$$\int_0^1 \frac{w(u)}{u}\left\{\int_0^u \mathrm{E}(\nu_{it}|u_{mt} = v)\}\mathrm{d}v\right\}\mathrm{d}u = \int_0^1 \mathrm{E}(\nu_{it}|u_{mt} = v)\int_v^1 \frac{w(u)}{u}\mathrm{d}u\mathrm{d}v \ ,$$

which reduces to the right hand side of (13) provided $w(u) = -u\phi'(u)$ implying $\mathrm{E}\{\phi(u)\} = 1$. If $w(u) \geq 0$ then ϕ is a decreasing function.

(Not sure if the next paragraph is still relevant. The issue is that all our risk stuff is encapsulated in the put price and change in put price) Applying the same weighted averaging to SRISK yields

$$\widetilde{\mathrm{SRISK}}_i \equiv \int_0^1 \mathrm{SRISK}_i(a)w(a)\mathrm{d}a = \mathrm{E}\{c_i\phi(u_m)\}$$
$$= kd_i - (1-k)w_i[1 + \mathrm{E}\{r_i\phi(u_m)\}] \ ,$$

which is the aversion adjusted expected capital shortfall.

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