

# Systemic risk and contagion effects in Australian financial institutions and sectors

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## 1. Literature review

The starting point for the proposed research is the recent literature and the CIFR targeted areas and APRA aims and functions. This recent literature includes the following Adrian and Brunnermeier (2011), Acharya et al. (2012), Acharya et al. (2012) and Brownlees and Engle (2010). The proposed research aims to extend and apply these techniques particularly in relation to the entities regulated by APRA. Thus our broad aim is to develop, implement and bring to bear recent developments in stress testing on the aims of APRA and the CIFR targeted research areas detailed above.

## 2. Improved measures of contagion and systematic risk

CoVaR<sub>q</sub> as proposed in Adrian and Brunnermeier (2011) is a basis for proposed measures contagion, exposure and systemic risk. It suffers from a number of drawbacks:

- Couched in terms of VaR<sub>q</sub> containing the scale of the original measurements. It is worthwhile to have measures and techniques robust to scale.
- Conditioning on VaR<sub>0.5</sub> is undesirable and relatively intractable. In our proposal we reference stress with respect to the unconditional VaR<sub>q</sub>. This permits a more transparent analysis and estimation.
- Our proposed approach separates out scale effects and interdependence effects and aims to relates these separately to external variables including shocks and drivers of systemic risk. Thus VaR<sub>q</sub> movements due to scale are disentangled from movements due to codependence with separate driver responses.

### 3. Significance of the project and policy implications

Understanding the impact of external shocks and their propagation through the financial system is vital for managing and remediating systemic risk. Effective regulation is dependent upon the development of a robust and reliable set of appropriate risk measures. We propose new measures of systemic risk that relate marginal and joint distributions separately to external drivers. This allows for more cogent and coherent stress testing as it includes the estimation of contagion effects, exposure effects and systemic risk across related entities and different financial sectors. Improved stress testing, estimation of risk effects and transmission of shocks through the financial system will make for more cogent prudential policy, prudential margin setting and better identify sources of risk to the financial system.

### 4. Percentile sensitivity and contagion

#### 4.1. Theorem

Suppose  $x$  is a random vector with marginal distributions<sup>1</sup>

$$F(x) \equiv \{F(x_1), \dots, F(x_p)\} \equiv (u_1, \dots, u_p) \equiv u .$$

Further suppose  $0 \leq q \leq 1$  is given and  $Q(x)$  is the vector of  $q$ -quantiles

$$F\{Q(x)\} = q\mathbf{1} = Q(u) ,$$

where  $\mathbf{1}$  is a vector of  $p$  ones. Define the stress vector with respect to  $x_j$  as<sup>2</sup>

$$\frac{dQ(x)}{dx_j} \equiv Q(x|u_j > q) - Q(x) , \quad (1)$$

where  $Q(x|u_j > q)$  is the vector of  $q$ -quantiles of  $x$  given  $u_j > q$ . Then if  $Q(x)$  is linear in  $q$  then

$$\frac{dQ(x_i)}{dQ(x_j)} \equiv \frac{dQ(x_i)/dx_j}{dQ(x_j)/dx_j} = \frac{f_j}{f_i} \frac{q_{ij}}{q(1-q)} , \quad q_{ij} = C_{ij}(q + q_{ij}, q) - q^2 , \quad (2)$$

where  $f_i$  and  $f_j$  are the densities of  $x_i$  and  $x_j$  evaluated at  $Q(x_i)$  and  $Q(x_j)$  and  $C_{ij}$  is the copula of  $(u_i, u_j)$ . Further if

$$s_{ij} \equiv \frac{q_{ij}}{q(1-q)} ,$$

then  $-1 \leq s_{ij} \leq 1$  with  $s_{ij} = \pm 1$  if  $x_i$  and  $x_j$  are comonotonic and countermonotonic, respectively. If  $x_i$  and  $x_j$  are independent then  $s_{ij} = 0$ . If  $u_i$  and  $u_j$  are exchangeable then  $s_{ji} = s_{ij}$ .

<sup>1</sup>To economise on notation, write  $F_j(x) \equiv F_j(x_j) \equiv F(x_j)$  and

<sup>2</sup>In Adrian and Brunnermeier (2011)  $\Delta CoVar_q \equiv q_{y|x=q_x} - q_y$ . Variable  $y$  is generally the “financial system” and hence considered is the change in the  $VaR_q$  of the system when institution  $x$  stressed, with stress interpreted as  $x = q_x$ . On page 10 of their paper they incorrectly state “... *CoVaR* conditions on the event that [an] institution is at its *VaR* level, which occurs with probability  $q$ .”

#### 4.2. Proof

By definition

$$\frac{dQ(u_i)}{du_j} \equiv Q(u_i|u_j > q) - q \equiv q_{ij}$$

and  $q_{ij}$  is such that

$$q = \frac{P(u_i \leq q + q_{ij}, u_j > q)}{1 - q} = \frac{q + q_{ij} - C_{ij}(q + q_{ij}, q)}{1 - q}, \quad (3)$$

Rearranging yields the second equation in (2). Now

$$\frac{dQ(x_i)}{dx_j} \equiv F_i^-(q + q_{ij}) - Q(x_i) \approx (F_i^-)'(q)q_{ij} = \frac{q_{ij}}{f_i}, \quad (4)$$

where  $'$  denotes differentiation. The approximation follows from a first order Taylor expansion and is exact if the quantile is linear in  $q$ . Similarly

$$\frac{dQ(x_j)}{dx_j} \equiv F_j^-\{q + q(1 - q)\} - Q(x_j) \approx \frac{q(1 - q)}{f_j}. \quad (5)$$

Dividing (4) by (5) yields the first equation in (2) and completes the proof.

#### 4.3. Discussion

The critical result is that sensitivities factor into contributions from the ratios of the marginal densities and quantities calculated from the pairwise copulas. The via the implicit equation for  $q_{ij}$  in (2), solved using a root finding algorithm. The quantities  $q_{ij}$  are implicitly defined from the pairwise copulas. In summary the “sensitivity” matrix

$$S \equiv \frac{dQ(x)}{dQ'(x)},$$

where  $'$  denotes transposition has ones on the diagonal quantities between  $\pm 1$  off the diagonal. The matrix  $S$  is called the sensitivity matrix and displays the sensitivity of the  $q$ -quantile of each component of  $x$  to stress in the same or other components.

Since  $f_i$  and  $f_j$  are the densities evaluated at  $Q(x_i)$  and  $Q(x_j)$  it follows that the ratio  $f_i/f_j$  can be replaced by the ratio of the hazards. This is useful in cases were it is practical to model the hazard rather than the density.

#### 4.4. Implementation

Figure 1 displays the empirical copulas calculated from four weekly closing stock prices labelled anz, cba, mcq and wbc for  $n = 761$  weeks from 2000 April 12 through to 2014 October 29. The empirical copulas are calculated by converting each observation to an empirical percentile and plotting the same against each of the other series percentiles.

Figure 1: Pairwise copulas of bank stocks cba, anz, mqg, wbc, and the overall bank index. Red lines plot sensitivity  $s_{ij}$  as a function of  $q$ .

To estimate  $q_{ij}$  at a particular  $q$ , the second equation in (2) is iterated<sup>3</sup> starting from  $q_{ij} = 0$  where copulas are estimated as

$$\hat{C}_{ij}(u_i, u_j) \equiv \hat{E}\{(p_{ik} \leq u_i)(p_{jk} \leq u_j)\} .$$

Here  $\hat{E}$  computes the empirical mean over the cases  $k = 1, \dots, n$  and  $p_{ik}$  and  $p_{jk}$  are the empirical percentiles of case  $k$  of  $x_i$  and  $x_j$ , respectively.

#### 4.5. Generalisations

Similar results apply when  $Q(x)$  is replaced by other risk measures such as  $R(x) \equiv E\{xr(u)\}$  where  $r$  is a given function which acts componentwise and  $xr(u)$  denotes componentwise multiplication. For example if  $r(u) = mu^{m-1}$  then  $R(x) = E\{\max(x^1, \dots, x^m)\}$  where  $x^1, \dots, x^m$  are  $m$  independent copies of  $x$ . In this case the risk measure is the expected worst outcome in  $m$  independent trials.

With  $R(x)$  the analogue of (1) is

$$\frac{dR(x)}{dx_j} \equiv R(x|u_j > r_j) - R(x_j) , \quad r_j \equiv P\{x_j \leq R(x_j)\} . \quad (6)$$

This differs from (1) in that  $r_j$  replaces  $q = P\{x_j \leq Q(x_j)\}$ . If the components of  $R(x)$  are linear in the  $r_i$  then

$$\frac{dR(x_i)}{dR(x_j)} = \frac{f_j}{f_i} \frac{q_{ij}}{q(1-q)} , \quad q_{ij} = C_{ij}(r_i + q_{ij}, r_j) - r_i r_j , \quad (7)$$

where  $f_i$  and  $f_j$  are the  $x_i$  and  $x_j$  densities at the  $r_i$  and  $r_j$  quantiles, respectively.

## 5. Financial sensitivity and contagion

Suppose the singular value decomposition of the matrix  $S$  of sensitivities at a particular  $q$  is  $S = UDV'$ . Here  $D$  a diagonal matrix of singular values, arranged in decreasing order along the diagonal and  $U$  and  $V$  are orthonormal:  $U'U = V'V = I$ . If the series are independent then  $U = D = V = I$ . If the series are comonotonic then  $U = V = (1, 0)$  while  $D = \text{diag}(1, 0, \dots, 0)$ . Hence in the comonotonic case  $S = 11'$ .

So the best thing is to compare

$$Q = s_{i1}c_{j1} + \dots + s_{ip}c_{jp}$$

If  $Q = I$  then  $S = C = I$ . If  $Q = 11'$ , a matrix of ones, then  $S = \sqrt{p}(1, 0)$  and  $C = (c, 0)$  both have rank 1: here  $s$  and  $c$  are the leading columns of  $S$  and  $C$ , respectively and  $0$  a matrix of zeros.

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<sup>3</sup>The second equation in (2) is a contraction mapping and hence has a fixed point.

The approximation  $Q \approx sc'$  is exact if  $Q$  has rank 1 implying  $Q = 11'$ . The approximation  $Q \approx sc'$  is inadequate if  $Q = I$ . Inadequacy can be judged from the largest singular value in  $D$ , as a proportion of  $p$ , the number of variables or rows in  $Q$ .

$$Q_* \equiv \frac{1}{p-1}(Q - I) = UDV' , \quad s \equiv Q_*1 , \quad c \equiv Q'_*1 .$$

$$Q = I + s1' + UDV' , \quad s \equiv \frac{1}{p-1}(Q - I)1$$

Then  $s$  is the vector of average  $\text{VaR}_q$  sensitivity of each variable to all others. Further  $c$  is the average  $\text{VaR}_q$  impact of each variable on all others. The matrices  $U$ ,  $D$  and  $V$  define the singular value (svd) decomposition of  $Q_*$ , arranged so that diagonal matrix  $D$  has the singular values on the diagonal in descending order. If  $b = d_1 u_1$  and  $c = v_1$  where  $u_1$ ,  $d_1$  and  $v_1$  are the first column, top diagonal entry, and first column of  $U$ ,  $D$  and  $V$  respectively, then for two variables  $y \neq x$  in  $Q$ ,

$$\Delta_x q_y = s_y + b_y c_x + \epsilon_{yx}$$

This states that percentage sensitivities are, apart from the “error”  $\epsilon_{yx}$ , an average sensitivity plus a scaled response to the contagious effect of the  $x$  variable. The contagious effects contained in  $c$  are estimated by maximising the explanation of  $Q$ .

The vector  $1'Q$  sums the changes in  $\text{VaR}_q$  when each of the column variables is stressed, and writes this as a proportion of the change in the variable being stressed. These proportional sums, subtracting 1 and divided by  $p - 1$  where  $p$  is the number of variables, measures the average contagion of each variable on all others:  $c' = (p - 1)^{-1}1'(Q - I)$  or  $c = (p - 1)^{-1}(Q - I)'1$

Alternatively the vector  $Q1$  sums the changes in  $\text{VaR}_q$  of each row variable when all the column variables are stressed. Again it is appropriate to remove the effect of a variable on itself and consider the average over the remaining variables:  $s \equiv (p - 1)^{-1}(Q - I)1$ . If  $Q_* \equiv (p - 1)^{-1}(Q - I)$  then  $s = Q_*1$  and  $c = Q'_*1$  are the vectors of sensitivities and contagions, respectively. If  $Q = I$  then  $s = c = 0$ . If  $Q = 11'$  then  $s = c = 1$ .

## 6. Systemic risk and causal chains

A rank one approximation to the matrix  $Q$  is  $Q \approx sc'$ . Vector  $c$  is an index of the contagious impact of each of variables on the others while  $s$  measures the sensitivity of each variable to each of the others. The vectors  $s$  and  $c$  are derived from the singular value decomposition  $Q = UDV'$  where  $U'U = V'V = 1$  and where  $s$  and  $c$  are the first column of  $UD$  and  $V$  respectively, assuming the svd is organised so that the singular values in the diagonal matrix  $D$  are organised from largest to smallest. The appropriateness of the summarisation  $sc'$  is measured with  $\text{tr}(Q - sc')$ .

If the variables are independent then  $Q = I$  and both  $s$  and  $c$  equal a column of the identity matrix with  $sc'$  a matrix of 0's except in a single diagonal position

where it is 1. Then all but one variable has a contagion effect and only that variable is sensitive to the contagion provided by the variable.

If the variables are comonotonic then  $Q = 11'$  and  $s = 1$  and  $c = 1$  where 1 denotes a vector of ones. Thus the rank 1 approximation is exact and each variable is equally contagious and equally sensitive.

Note that

$$Q^n \approx (s'c)^{n-1}sc' = ,$$

If the random variables are independent the  $Q - I = 0$  and  $a = b = k = 0$  and there is no error in the first order svd approximation. If the random variables are comonotonic then  $Q = 11'$  and  $Q - I$  has ones everywhere except on the diagonal where it is zero. The vector of row means is then  $p^{-1}(p - 1)1$

Systemic risk in the system is measured with  $b'k = d_1(u_1'v_1)$ . In the case of comonotonic random variables  $Q = 11'$ ,  $a = 1$  and  $x$ , where  $p$  is the number of variables,  $d_1 = 1$ .

Furthermore we may define quantities such as  $u^- \equiv \text{VaR}_q(v|u \leq q)$  measuring the impact of a non distressed state in  $v$ . For brevity we do not dwell on these constructs in this writeup although the ramifications and potential uses of these constructs will be investigated in the research.

## 7. Literature

We propose a measure for systemic risk: CoVaR, the value at risk (VaR) of financial institutions conditional on other institutions being in distress. We define an institution's (marginal) contribution to systemic risk as the difference between CoVaR and the financial system's VaR. From our estimates of CoVaR for characteristic-sorted portfolios of publicly traded financial institutions, we quantify the extent to which characteristics such as leverage, size, and maturity mismatch predict systemic risk contribution. We argue for macro-prudential regulation based on the degree to which such characteristics forecast systemic risk contribution.

## 8. Econometric implementation

The above development sets out our proposed broad framework for linking bivariate copulas and marginals to external variables and shocks study the impact of the same on stresses within the system and the contagious effects of crises. Proposed econometric analysis will implement and extend Brownlees and Engle (2010).

## 9. Data

We will employ publicly available data as published by APRA and other regulators.

## References

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