Copulas: summary and proofs

Write F as the probability distribution of random vector x and F_* as the corresponding vector of marginal distributions: $F_*(x)$ evaluates the marginal densities of F at the respective components of x. If h is a vector of increasing component maps then $F \circ h$ is the distribution of $h^-(x)$ and $(F \circ h)_* = F_* \circ h$.

A copula is a probability distribution C on the unit hypercube with uniform marginals: $C_* = I$. Define the copula generated by F as $C_F \equiv F \circ F_*^-$ and the distribution generated by C from F_* as $F_C \equiv C \circ F_*$.

1. $(C_F)_* = (F \circ F_*^-)_* = I$ hence C_F is a copula, and $C_C = C \circ C_*^- = C$
2. $(F_C)_* = (C \circ F_*)_* = C_* \circ F_* = F_*$ and $F = C_F \circ F_*$
3. $x \sim F_C$ implies $F_*(x) \sim C$ while $u \sim C_F$ implies $F_*^-(u) \sim F$
4. $F_{C_F} = F \circ F_*^- \circ F_* = F$ and $C_{F_C} \equiv F_C \circ (F_C)_*^- = C \circ F_* \circ F_*^- = C$
5. **Monotonic transform**: $C_{F\circ h} = (F \circ h) \circ (F \circ h)_*^- = F \circ h \circ h^- \circ F_*^- = C_F$
6. **Copula density**: C_F has density $c_F \equiv (f \circ F_*^-)/(\Pi \circ f_* \circ F_*^-)$ where f and f_* are the densities corresponding to F and F_*, respectively.
7. **Conditional**: $c(u^i|u_i) = c(u^i, u_i)/c(u_i) = c(u)$; $C(u^i|v_i) = \int_0^{u^i} c(v)\mathrm{d}v^i$
8. **Independence**: If $F = \Pi \circ F_*$ then $C_F = \Pi \circ F_* \circ F_*^- = \Pi$. Further $f = \Pi \circ f_*$ and $f \circ F_*^- = \Pi \circ f_* \circ F_*^-$ and $c_F = 1$.
9. **Perfect dependence**: $x = h(1\epsilon)$, $\epsilon \sim G$ is univariate and h as above. Then $P(x \leq z) = P\{1\epsilon \leq h^-(z)\} = (\min \circ G_! \circ h^-)(z)$ where $G_!$ applies G to each component. Hence $F = \min \circ G_! \circ h^-$, $F_* = G_! \circ h^-$ and $C_F = \min$.
10. **Gaussian copula** is $C_\Phi = \Phi \circ \Phi_*^-$ where Φ is the Gaussian distribution with standard normal marginals. The meta–Gaussian model is $x = (F_*^- \circ \Phi_*)(\epsilon)$, $\epsilon \sim N(0, R)$ where R is the covariance and correlation matrix of Φ.
11. **Meta–gamma** model is $x = (F_*^- \circ \Gamma_*)(\Psi\epsilon)$, $\epsilon \sim \Gamma(\mu, \nu)$ where $\Psi \equiv \{1, \mathrm{diag}(\psi)\}$, ψ a vector of parameters, and Γ_* the vector of marginal distributions associated with $\Psi\epsilon$.
12. **Survival copula** is $C \circ 1_-$ with $1_-(u) = 1 - u$ and $1_-^- = 1_-$
13. **Levy copula** is L defined on the positive orthont such that $L_* = I$
14. $\mathrm{d}F(x) \equiv \mathrm{d}(C \circ F_*)(x) = c(u)(\Pi \circ \mathrm{d}F_*)(x)$. Top down aversion adjustment is $\mathrm{d}F^\downarrow(x) = \phi(u_+)c(u)(\Pi \circ \mathrm{d}F_*)(x)$ and $f^\downarrow(x_i) = \mathrm{E}\{\phi(u_+)|u_i\}f(x_i)$, u_+ wrt $C \circ F_*$. Bottom up is $\mathrm{d}F^\uparrow(x) = (\Pi \circ \phi)(u)c(u)(\Pi \circ \mathrm{d}F_*)(x)$ implying $\mathrm{d}F^\downarrow(x)/\mathrm{d}F^\uparrow(x) = \phi(u_+)/(\Pi \circ \phi)(u)$, $f^\uparrow(x_i) = \phi(u_i)f(x_i)$

15. The correlation matrix of $u \sim C$ is called Spearman's rho. The means and variances of each component of u are 1/2 and 1/12, respectively. Hence Spearman's rho correlation matrix is

$$\frac{1}{1/12}\left\{\mathrm{E}(uu') - \frac{1}{4}11'\right\} = 12\int uu'\mathrm{d}C(u) - 3 = 12\int C(u)\mathrm{d}u - 3 \ .$$

16. Kendall's tau is one third of Spearman's rho calculated with copula $C^2\Pi^{-1}$. To show this note the total differential of the last expression is

$$2C\Pi^{-1}\mathrm{d}C - C^2\Pi^{-2}\mathrm{d}\Pi$$

To show this note that $\mathrm{d}C(u) = c(u)\mathrm{d}u$ where $\mathrm{d}u$ is the product of the individual differential elements. Thus $\mathrm{d}C^2 = 2C\mathrm{d}C =$

and

this suppose $\hat{C}$ is such that $C = (\hat{C}\Pi)^{1/2}$. Then

$$\Pi\mathrm{d}C^2(u) = 2C(u)\Pi\mathrm{d}C(u) \qquad \Rightarrow \qquad \mathrm{d}C^2(u) = 2C(u)\Pi^{-}(u)\mathrm{d}C(u) \ .$$

$$\begin{aligned} C\mathrm{d}C &= (\hat{C}\Pi)^{1/2}\frac{\hat{C}^{-1/2}\Pi^{1/2}\mathrm{d}\hat{C} + \hat{C}^{1/2}\Pi^{-1/2}\mathrm{d}\Pi}{2} = \frac{\Pi\mathrm{d}\hat{C} + \hat{C}\mathrm{d}\Pi}{2} \\ &= \hat{C}\mathrm{d}\Pi = C^2\Pi^{-1}\mathrm{d}\Pi \ . \end{aligned}$$

Hence for the bivariate case Kendall's tau is

$$\tau = 4\int C(u,v)\mathrm{d}C(u,v) - 1 = 4\int \frac{C^2(u,v)}{uv}\mathrm{d}u\mathrm{d}v - 1 = 4\frac{\rho+3}{12} - 1 = \frac{\rho}{3}$$

where ρ is Spearman's rho calculated with $C^2(u,v)/(uv)$. Note the marginal distribution is $C^2(u,1)/u = u^2/u = u$ and similarly for v and hence C/Π is a copula. If $C = \Pi$ then $C^2/\Pi = \Pi$. Hence $\tau \neq \rho$ even if $C = \Pi$.

Conormal distributions

A distribution N is conormal if $N_* = \Phi_*$ where Φ is the joint normal with standard normal marginals. Thus N_* computes standard normal probabilities. The conormal defined from F is $N_F \equiv F \circ P_*^-$ where $P_* = \Phi_*^- \circ F_*$. The joint N_F is conormal since $(N_F)_* = \Phi_*$. Further $F = N_F \circ P_*$ and hence any joint can be written as a conormal composed with a function of marginals. Note $P_*(x)$ computes z–scores from the percentiles $F_*(x)$.

The joint defined by conormal N and F_* is $F_N \equiv N \circ P_*$. Note $(F_N)_* = F_*$ and detailed calculations show $N_{F_N} = N$ and $F_{N_F} = F$. If $F = \Phi$ then $N_F = F = F_N$. Hence the joint normal with standard normal marginals is the "identity."

Any joint normal is of the form $\Phi \circ z$ where $z(x)$ are z–scores. Since $(F \circ z)_* = F_* \circ z$, the conormal defined by $\Phi \circ z$ is

$$N_{\Phi \circ z} \equiv \Phi \circ z \circ (\Phi \circ z)_*^- \circ N_* = \Phi \circ \Phi_*^- \circ N_* = \Phi \ , \qquad N_{\Phi \circ z} \circ P_* = \Phi \circ \Phi_*^- \circ F_* \ .$$

Thus using a conormal defined from a joint normal is equivalent to using a Gaussian copula. The joint defined by N and $(\Phi \circ z)_*$ is

$$(\Phi \circ z)_N = N \circ \Phi_*^- \circ (\Phi \circ z)_* = N \circ z \ ,$$

the conormal applied to the distribution of $z(x)$.