Layer dependence as a measure of local dependence

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Abstract

This paper introduces, analyses and illustrates a new measure of local dependence called "layer dependence." Layer dependence accurately captures varying dependence in the joint distribution. Layer dependence satisfies coherence properties similar to $\rho + s$, such as lying between -1 and 1, with -1, 0 and 1 resulting from countermonotonicity, independence and comonotonicity, respectively. Taking a weighted average of layer dependence values across the joint distribution yields Spearman's correlation and alternative measures of overall dependence. Copulas fitted using layer dependence are tailored to past data and easily incorporate expert opinion on the dependence structure.

Keywords: Local dependence; rank dependence; conditional tail expectation; Spearman's correlation; concordance.

1. Local dependence and layer dependence

Dependence between two variables generally varies with percentile. For example extreme movements in two stock markets are likely be highly related whereas minor fluctuations may be relatively independent. Natural catastrophes create significant insurance losses for several classes of business at the same time, while attritional losses between various classes are weakly dependent.

Local dependence measures aim to capture the dependence structure of a bivariate distribution. This contrasts with measures of overall dependence such as Pearson correlation, Spearman's ρ and Kendall's τ (Embrechts et al., 2002). Local dependence measures include the univariate tail concentration (Venter, 2002), correlation curve (Bjerve and Doksum, 1993), and bivariate measures by Bairamov et al. (2003), Jones (1996) and Holland and Wang (1987).

This paper introduces, illustrates and analyzes an alternate local dependence measure called "layer dependence." Layer dependence is the covariance between a random variable and a single "layer" of another. Layer dependence is also the

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"gap" between upper and lower conditional tail expectations. Random variables are replaced with their percentile rank transforms and layer dependence is calculated entirely from the copula underlying the joint distribution. Hence of interest is rank dependence rather than dependence between random variables in their original scale, as the latter is often distorted by marginal distributions.

Layer dependence satisfies "coherence" properties similar to linear correlation: between -1 and 1, constant and equal to -1, 0 and 1 for countermonotonic, independent and comonotonic random variables, sign switching when ranking order reverses, and taking on higher values when dependence is stronger. Taking a weighted average of layer dependences across the joint distribution yields Spearman's ρ and alternative coherent measures of overall dependence.

Layer dependence provides a more appropriate and accurate measure of local dependence compared to existing measures. Higher dispersion from various points of the 45^o line reduces layer dependence and vice versa. Calculating layer dependence at the first instance from past data or parametric copula extracts essential and interpretable information – the dependence structure. For a parametric copula, the implication of its parametric form and parameters on the dependence structure is not always apparent. Similar problems apply when past data is scarce.

Layer dependence offers an alternative approach to copula modeling. First compute layer dependence values from past data, and apply parametric smoothing. Further adjust, if necessary, to incorporate expert opinion. A copula is then fitted to refined layer dependence values. The fitted copula overcomes the inflexibility of parametric copulas to closely capture the dependence structure in past data, whilst avoiding uncertainties of empirical copulas at the other extreme.

The remaining paper is structured as follows. Section 2 defines, illustrates and analyzes layer dependence. Section **??** calculates layer dependence for common copulas, and proposes using layer dependence to summarise the dependence structure of any bivariate copula. Section 6 formalises coherence properties of layer dependence. Further properties of layer dependence, such as tail dependence, are discussed in §7. Section 4 expresses layer dependence as a negative function of the proportion of discordant points and their average dispersion. Section 8 computes a weighted average of layer dependences to yield Spearman's ρ and alternative measures of overall dependence. Section 9 formulates layer dependence in terms of observed random variables instead of their percentile ranks. Existing local dependence measures are assessed in §12. Section 13 applies layer dependence to copula modeling, and uses historical stock returns as an illustration. Section 14 concludes.

2. Layer dependence

Suppose u and v denote percentile ranks of continuous random variables x and y. Then (u, v) has uniform marginals and the joint distribution, $C(u, v)$, is a copula.

This paper defines and discusses a new form of local dependence between u and v, called "layer dependence." Layer dependence exploits the decomposition

$$u = \int_0^1 (u > \alpha)\mathrm{d}\alpha = \int_0^u \mathrm{d}\alpha \ , \tag{1}$$

where $\mathrm{d}\alpha$ is the "α–layer" of u and $(u > a)$ is the indicator function: 1 if $u > \alpha$ and 0 otherwise. Given α, the derivative of α–layer with respect to u is the Dirac-delta function $(u = \alpha)$: infinite when $u = \alpha$ and 0 otherwise. Hence α–layers reflect movements in u at α, ignoring other movements. Thus (1) implies u is formed from infinitely many layers, each layer capturing the variability of u at a point.

The α–layer dependence between v and the α–layer of u is defined as

$$\ell_\alpha \equiv \frac{\mathrm{cov}\{v, (u > \alpha)\}}{\mathrm{cov}\{u, (u > \alpha)\}} = \frac{\mathrm{cor}\{v, (u > \alpha)\}}{\mathrm{cor}\{u, (u > \alpha)\}} \ , \quad 0 \le \alpha \le 1 \ , \tag{2}$$

where cov and cor denote covariance and correlation, respectively, calculated with respect to C. Denominators in (2) are independent of C and imply $\ell_\alpha = 1$ if $u = v$ and $\ell_\alpha = -1$ if $u = 1 - v$. Hence ℓ_α measures dependence between v and u at α. Further $-1 \le \ell_\alpha \le 1$. Independence implies $\ell_\alpha = 0$.

Expanding covariances in (2) yields

$$\ell_\alpha = 2\left\{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha)\right\} \ , \tag{3}$$

where E calculates expectations with respect to C. Hence ℓ_α is proportional to the "gap" between upper and lower conditional tail expectations of v, with tails defined relative to α. Given α, a larger gap between values of v over $u > \alpha$ and $u \le \alpha$ implies stronger dependence between v and u at α. Independence at α implies zero gap, or $\mathrm{E}(v|u > \alpha) = \mathrm{E}(v|u \le \alpha)$.

Using (1) and (2), Spearman's correlation between u and v is

$$\rho_S \equiv \mathrm{cor}(u, v) = \frac{\mathrm{cov}(u, v)}{\mathrm{var}(u)} = \frac{\int_0^1 \mathrm{cov}\{v, (u > \alpha)\}\mathrm{d}\alpha}{1/12} = \mathcal{E}(\ell_\alpha) \ . \tag{4}$$

where the expectation $\mathcal{E}$ is calculated with respect to the density $6\alpha(1 - \alpha)$ on $[0, 1]$. Weights $6\alpha(1 - \alpha)$ integrate to 1, are a minimum 0 at $\alpha = 0$ and 1, and increase symmetrically to a maximum of 1.5 at $\alpha = 0.5$. Varying the weights in $\mathcal{E}(\ell_\alpha)$ emphasises different ℓ_α and areas of the relationship. For example using weights $n\alpha^{n-1}$ where $n > 0$ gives increasing weight to the upper tail. This is further discussed in section 8.

3. Layer dependence curves for different copulas

The nine panels in Figure 1 display (u, v) scatterplots of symmetric copulas, and associated layer dependence ℓ_α curves. Each copula has $\rho_S = 0.6$. Each

ℓ_α reflects the dependence structure between u and v. Given α, ℓ_α is larger if points are more clustered around (α, α) and vice versa and as formalised in section 4.

The nine panels highlight the inadequacies of using Spearman's correlation ρ_S to measure overall dependence, particularly in the tails. In contrast, layer dependence ℓ_α captures dependence structure across the percentiles.

4. Layer dependence, discordance and dispersion

If (u, v) is exchangeable, $C(u, v) = C(v, u)$, then ℓ_α measures the lack of discordance and dispersion at α:

$$\ell_\alpha = 1 - 2(1 + \gamma_\alpha)\delta_\alpha \ , \tag{5}$$

where

$$\gamma_\alpha \equiv \text{cor}\{(u \le \alpha), (v > \alpha)\} = \text{cor}\{(u > \alpha), (v \le \alpha)\} \ ,$$
$$\delta_\alpha \equiv \text{E}\left\{(|u - v|)|(u - \alpha)(v - \alpha) < 0\right\} \ .$$

Equation (5) is proved below. The correlation $-1 \le \gamma_\alpha \le 1$ measures the tendency for (u, v) to be discordant at α: $u - \alpha$ and $v - \alpha$ are of opposite sign. In addition $0 \le \delta_\alpha \le 1$ measures average dispersion between values of u and v discordant at α.

Proof of (5) follows from

$$\gamma_\alpha = -\frac{\text{cov}\{(u \le \alpha), (v \le \alpha)\}}{\text{var}\{(u \le \alpha)\}} = \frac{\alpha^2 - C(\alpha, \alpha)}{\alpha(1 - \alpha)} \ ,$$

and

$$\begin{aligned}
\delta_\alpha &= 2\text{E}\left\{(u - v)(u > v)|(u - \alpha)(v - \alpha) < 0\right\} \\
&= \frac{2\text{E}\{(u - v)(u > v)(u > \alpha)(v \le \alpha)\}}{2\text{E}\{(u > \alpha)(v \le \alpha)\}} = \frac{\text{E}\{(u - v)(u > \alpha)(v \le \alpha)\}}{\alpha - C(\alpha, \alpha)} \\
&= \frac{\text{E}\{(u - v)(u > \alpha)\} - \text{E}\{(u - v)(u > \alpha)(v > \alpha)\}}{\alpha - C(\alpha, \alpha)} = \frac{\text{E}\{(u - v)(u > \alpha)\}}{\alpha - C(\alpha, \alpha)} \ .
\end{aligned}$$

Substituting the above expressions for γ_α and δ_α into the right hand side of (5) yields ℓ_α, completing the proof.

Result (5) explains the behaviour of ℓ_α in Figure 1. Layer dependence ℓ_α is large if there are few discordant pairs at α, and discordant pairs at α are close to the $45°$ degree line. Vice versa for small ℓ_α. If $\ell_\alpha = 1$ then $\gamma_\alpha = 0$ or $\delta_\alpha = 0$, implying u and v are simultaneously below or above α and $u = v$ for discordant pairs. If $\ell_\alpha = 1$ in an interval, then $u = v$ in the interval.

Figure 2 illustrates (5) using two copulas with $\ell_\alpha = 1$ and $\ell_\alpha = 0.86$ at $\alpha = 0.75$. When ℓ_α is maximum at 1, there is no discordance or dispersion at α, and $\gamma_\alpha = -1$ and $\delta_\alpha = 0$. As ℓ_α decreases from 1, the number of discordant pairs and their dispersion increases.

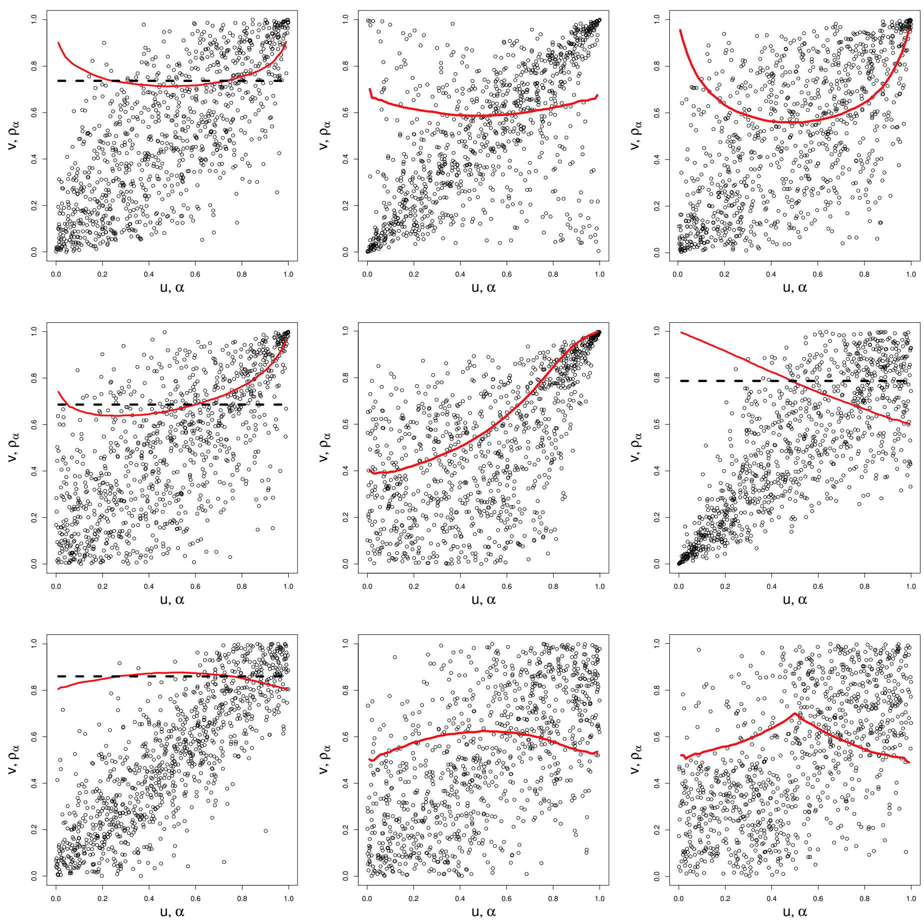

Figure 1: Copulas with different level dependence ℓ_α (red lines) but the same $\rho_S = 0.6$. What are the dashed lines in some of the panels?

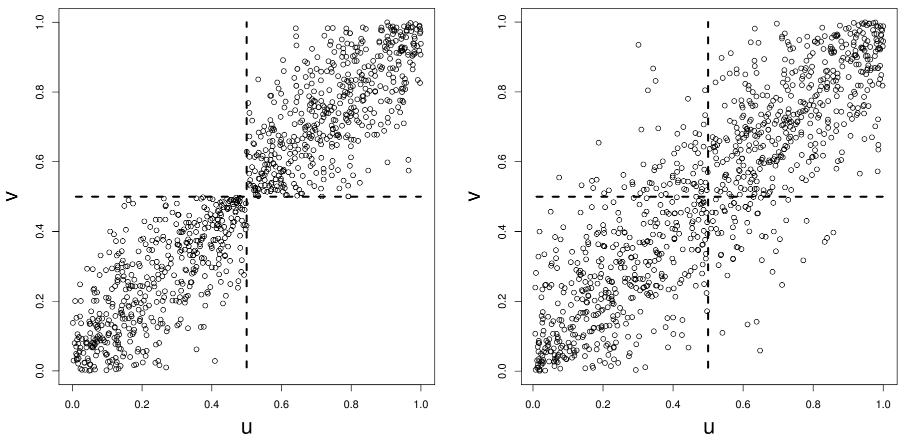

Figure 2: The left and right panel show $\ell_{0.75} = 1$ and $\ell_{0.75} = 0.86$, respectively. In the left panel, $\gamma_{0.75} = -1$ and $\delta_{0.75} = 0$. In the right panel, $\gamma_{0.75} = -0.65$ and $\delta_{0.75} = 0.21$. Why is the wrong picture here? Please load up to github.

5. Simulating a copula given layer dependence

Write $\tau_\alpha \equiv \mathrm{E}(v|u > \alpha)$. Then

$$\tau'_\alpha \equiv \frac{\mathrm{dE}(v|u > \alpha)}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{\mathrm{E}\{v(u > \alpha)\}}{1-\alpha} = \frac{\tau_\alpha - \mu_\alpha}{1-\alpha} \ , \quad \mu_\alpha \equiv \mathrm{E}(v|u = \alpha) \ .$$

Hence $\mu_\alpha = \tau_\alpha - (1-\alpha)\tau'_\alpha$ and τ_α is monotonic if μ_α is monotonic.

Suppose $\tau_\alpha = (1 + \alpha\ell_\alpha)/2$ is given for $\alpha = i/n$, $i = 1, \ldots, n$ with values denoted $\tau_1, \tau_1, \ldots, \tau_n$. Then $C(u, v)$ with level dependence ℓ_α is simulated using

$$v_i = F\{F^{-}(\mu_i) + \sigma\epsilon_i\} \ , \ \mu_i = \tau_i - (n-i)(\tau_i - \tau_{i-1}) \ , \ \epsilon_i \sim N(0,1) \ , \ i = 1, \ldots, n \ ,$$

with $\tau_0 = 1/2$ and σ chosen so that Spearman's ρ for the each sample is expected to match ρ_S implied by ℓ_α as in (4). Here F is an appropriate distribution such as the normal or logit: $F^{-}(\mu_i) = \ln\{\mu_i/(1-\mu_i)\}$. In the logic case setting $\sigma = 1$ appears to ensure a match between the theoretical and empirical ρ_S. To enforce the $v_1, \ldots, v_n$ are empirically uniform the final transformation F can be replaced by $\hat{\mathrm{P}}$, computing empirical percentiles.

6. Coherence properties of layer dependence

Layer dependence ℓ_α satisfies five "coherence" properties. These properties are extensions of properties applying to Spearman's correlation ρ_S.

- **Bounds**: Layer dependence lies between -1 and 1. Hence layer dependence is bounded in the same way as ρ_S.
- **Perfect dependence**: Constant layer dependence of -1 or 1 are equivalent to countermonotonicity and comonotonicity. Thus $\ell_\alpha = -1$ for all α if and only if $v = 1 - u$ while $\ell_\alpha = 1$ for all α if and only if $v = u$.
- **Independence**: If $C(u, v) = uv$ then $\ell_\alpha \equiv 0$. The converse is not true – zero layer dependence does not imply independence as shown by the following counterexample. Assume $v = u$ and $v = 1 - u$ with equal probability. Then $\mathrm{E}(v|u = t) = 0.5$ for all $0 \le t \le 1$ implying $\mathrm{E}(v|u > \alpha) = \mathrm{E}(v|u \le \alpha) = 0.5$. Hence $\ell_\alpha = 0$. However u and v are not independent.
- **Symmetry**: Ranking either u or v in the opposite direction switches the sign of layer dependence. Changing the ranking order of both variables preserves the sign of layer dependence.
- **Ordering**: Higher correlation order (Dhaene et al., 2009) leads to higher layer dependence. Consider bivariate uniform (u^*, v^*) exceeding (u, v) in correlation order: $C^*(a, b) \ge C(a, b)$ for all $0 \le a, b \le 1$, where C^* is the joint distribution of (u^*, v^*). Then $\ell^*_\alpha \ge \ell_\alpha$, $0 \le \alpha \le 1$ where ℓ^*_α denotes the α–layer dependence of (u^*, v^*). Hence greater dependence leads to higher layer dependence.

Perfect dependence, independence, symmetry and ordering properties follow from the definition of layer dependence in (2). Constant layer dependence of one implies $\mathrm{E}(v|u > \alpha) = (\alpha+1)/2$ and $\mathrm{E}(v|u = \alpha) = \alpha$, for all $0 \le \alpha \le 1$, hence $v = u$. Similarly constant layer dependence of minus one implies $v = 1-u$. The ordering property holds since higher correlation order implies larger covariances (Dhaene et al., 2009). Prove the bounds property by combining ordering and perfect dependence properties, and noting countermonotonicity and comonotonicity represent minimum and maximum correlation order, respectively.

7. Further properties of layer dependence

This section considers further properties of level dependence and relates the same to other measures.

7.1. One-sided conditional tail expectations

Since $\mathrm{E}(v) = \alpha \mathrm{E}(v|u \le \alpha) + (1-\alpha)\mathrm{E}(v|u > \alpha)$ it is straightforward to show

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u)} = \frac{\mathrm{E}(v|u \le \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u \le \alpha) - \mathrm{E}(u)} \ ,$$

the gap between upper or lower conditional tail expectations of v and the unconditional expectation. As before denominators are scaling factors ensuring $\ell_\alpha = 1$ if u and v are comonotonic.

7.2. Tail dependence

Joe (1997) considers coefficients of lower and upper tail dependence in terms of the extreme tail probabilities

$$\lim_{\alpha\to 0} \mathrm{E}\{(v \le \alpha)|u \le \alpha\}\ , \quad \lim_{\alpha\to 1} \mathrm{E}\{(v > \alpha)|u > \alpha\}\ .$$

Coefficient values of one indicate perfect positive tail dependence, and occur if and only if u and v converge simultaneously to zero (lower tail) or one (upper tail). Perfect negative tail dependence definitions are achieved by replacing $(v \le \alpha)$ and $(v > \alpha)$ by $(v > 1-\alpha)$ and $(v \le 1-\alpha)$, respectively.

Tail dependence models dependence at percentiles 0 and 1. Perfect tail dependence occurs when catastrophic events such as bank failure or market crash manifests in both series. Layer dependence models perfect tail dependence by assuming both variables simultaneously attain their minimum or maximum. Coefficients of tail dependence for copulas are discussed in Joe (1997).

For upper tail dependence. From (3), $\ell_1 = 2\mathrm{E}(v|u=1)-1$. Hence $\ell_1 = 1$ if and only if $\mathrm{E}(v|u=1) = 1$, that is $\ell_1 = 1$ if and only if $u = 1$ implies $v = 1$. Similarly $\ell_1 = -1$ if and only if $u = 1$ implies $v = 0$. Similar remarks apply to the lower tail: $\ell_0 = 1$ if and only if $u = 0$ implies $v = 0$ while $\ell_0 = -1$ if and only if $u = 0$ implies $v = 1$. Hence perfect negative tail dependence implies one variable approaches minimum when the other approaches maximum.

7.3. Layer dependence does not uniquely characterise a copula

The layer dependence function ℓ_α does not uniquely characterise $C(u,v)$. For example suppose $v = u$ and $v = 1-u$ with equal probability, implying $\mathrm{E}(v|u \le \alpha) = \mathrm{E}(v|u > \alpha) = 0.5$ for all α. (Maybe work out copula??). Then $\ell_\alpha = 0$ for all α. The same ℓ_α results if u and v are independent: $C(u,v) = uv$. Layer dependence only captures first-order conditional tail expectations.

7.4. Exchangeability

If u and v are exchangeable, layer dependence in (2) is invariant when u and v are switched. Archimedean copulas (McNeil et al., 2005) are exchangeable. If exchangeability fails then layer dependence differs when layers of v instead of u are applied. An analogous property applies to least squares regression: regressing v on u is the same as regressing u on v, if the joint distribution is exchangeable. This paper assumes exchangeability unless stated otherwise.

7.5. Preservation of convex combination

Suppose (u,v) and (u^*,v^*) are bivariate uniform with layer dependence ℓ_α and ℓ^*_α, respectively. Then $\pi\ell_\alpha + (1-\pi)\ell^*_\alpha$ is the layer dependence of $x(u,v) + (1-x)(u^*,v^*)$ where $x \sim B(\pi)$. The proof is straightforward using (2). This result generalises to multiple and continuous convex combinations of bivariate uniform random variables (Is "bivariate" correct here – maybe should be vector).

7.6. Copula integration

Note $\mathrm{cov}\{(u>\alpha),(v>\beta)\} = C(\alpha,\beta)-\alpha\beta$ and apply (1) to v, to derive

$$\ell_\alpha = \frac{\int_0^1 \mathrm{cov}\{(u>\alpha),(v>\beta)\}\mathrm{d}\beta}{\alpha(1-\alpha)/2} = \frac{2\int_0^1 C(\alpha,\beta)\mathrm{d}\beta-\alpha}{\alpha(1-\alpha)} = \frac{2\int_0^1 C(\beta|\alpha)\mathrm{d}\beta-1}{1-\alpha} .$$

Thus ℓ_α scales $C(\beta|\alpha)$ to ensure ℓ_α lies between ± 1. The integral reduces the dimension of the copula from two to one.

For Archimedean copula (McNeil et al., 2005)

$$\int_0^1 C(\alpha,\beta)\mathrm{d}\beta = \int_0^1 \psi^- \left\{\psi(\alpha)+\psi(\beta)\right\}\mathrm{d}\beta ,$$

where ψ is the generator function and ψ^- its inverse. In this cases closed form expressions for ℓ_α or $\mathrm{E}(v|u>\alpha)$ do not exist. This argues against the use of Archimedean copulas.

8. Alternatives to Spearman's rho

Spearman's ρ is, from (4), an expectation of ℓ_α using probability density $w_\alpha = 6\alpha(1-\alpha)$. Other overall dependence measures are arrived at by using other probability densities:

$$\rho_W \equiv \int_0^1 w_\alpha \ell_\alpha \mathrm{d}\alpha = \int_0^1 w_\alpha \frac{\mathrm{cov}\{v,(u>\alpha)\}}{\mathrm{cov}\{u,(u>\alpha)\}}\mathrm{d}\alpha = \mathrm{cov}\{W(u),v\} ,$$

where

$$W(u) \equiv 2\int_0^u \frac{w_\alpha}{\alpha(1-\alpha)}\mathrm{d}\alpha .$$

With $w_\alpha = 6\alpha(1-\alpha)$ the weights are symmetric and highest at the median, and decrease to zero towards both tails. Hence ρ_S emphasizes dependence at moderate percentiles. In addition dependence at percentiles below and above the median are treated symmetrically. Thus ρ_W is the covariance between v and a transformation $W(u)$ of u. For ρ_S in (4), the importance function is $w_\alpha = 6\alpha(1-\alpha)$ and $W(u) \propto u$.

Since ρ_W is a weighted average of ℓ_α, coherence properties of layer dependence in §6 apply to ρ_W: bounded by ± 1 and constant values of -1, 0 and 1 under countermonotonicity, independence and comonotonicity, respectively, sign reversal when ranking order switches, and higher values for stronger dependence.

The following importance functions yield alternative overall dependence measures in addition to ρ_S:

- Suppose dependence over all percentiles are equally important. Then w_α is the uniform density and

$$\rho_W = 2\mathrm{cov}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} = \sqrt{\frac{2}{3}}\ \mathrm{cor}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} ,$$

  a multiple of linear correlation between v and the logit of u. This dependence measure exceeds ρ_S when tail dependence is pronounced, since ρ_W places more importance on tail dependence.

- If $w_\alpha = 3\alpha^2$ then dependence at higher percentiles are considered more important modelling concern for simultaneous large events, such as large insurance losses. Then

$$\rho_W = 6\mathrm{cov}\left\{v, \log\left(\frac{\mathrm{e}^{-u}}{1-u}\right)\right\} = \sqrt{3}\mathrm{cor}\left\{v, -\log(1-u)\right\} - \frac{\rho_S}{2} .$$

- If dependence over percentiles below the median is more important then for example $w_\alpha = 3(1-\alpha)^2$ yielding

$$\rho_W = 6\mathrm{cov}\left\{v, \log\left(u\mathrm{e}^{-u}\right)\right\} = \frac{\rho_S}{2} - \sqrt{3}\mathrm{cor}(v, \log u) .$$

- Suppose w_α is derived from V_α, the inverse of a marginal distribution function with derivative V'_α such that

$$w_\alpha = \frac{\mathrm{cov}\{u,(u>\alpha)\}V'_\alpha}{\int_0^1 \mathrm{cov}\{u,(u>\alpha)\}V'_\alpha \mathrm{d}\alpha} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(V_u, u)} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(x, u)} ,$$

  where $x = V_u$. This yields the Gini correlation (**?**)

$$\rho_W = \frac{\mathrm{cov}(V_u, v)}{\mathrm{cov}(V_u, u)} = \frac{\mathrm{cov}\{x, G(y)\}}{\mathrm{cov}\{x, F(x)\}} ,$$

  where $F \equiv V^-$ and G are distribution functions of observed random variables x and y, respectively. In this example weight function depends on the marginal distribution of V_u. More skewed distributions place higher importance on upper tail dependence.

9. Layer dependence for non–uniform random variables

Level dependence can be defined for non–uniform random variables and taking a weighted average of layer dependence between random variables yields Pearson correlation. Similar to Pearson correlation, layer dependence between non uniform random variables depends on the marginal distributions.

Suppose $y \geq 0$ has marginal distribution G and percentile ranks $v = G(y)$. Further let u be the percentile rank of another variable x. Analogous to (2), define

$$\ell^*_\alpha \equiv \frac{\mathrm{cov}\{G^-(v), (u > \alpha)\}}{\mathrm{cov}\{G^-(u), (u > \alpha)\}} \;, \quad 0 \leq \alpha \leq 1 \;, \tag{6}$$

The denominator ensures $\ell^*_\alpha = 1$ if x and y are perfectly dependent or comonotonic. Calculation of ℓ^*_α involves the marginal distribution G of y, similar to Pearson correlation. In comparison original layer dependence ℓ_α does not involve marginal distributions, similar to ρ_S.

layer dependence ℓ^*_α partially satisfies coherence properties of ℓ_α in §6. If x and y are independent or comonotonic then $\ell^*_\alpha = 0$ and 1, respectively, for all α. If x and y are countermonotonic then $v = 1 - u$ and

$$\ell^*_\alpha = \frac{\mathrm{cov}\{G^-(1-u), (u > \alpha)\}}{\mathrm{cov}\{G^-(u), (u > \alpha)\}} \leq 0 \;,$$

reducing to -1 if G^- is symmetric: $G^-(1-u) - G(0.5) = G(0.5) - G(u)$. Layer dependence ℓ^*_α also preserves correlation order: if x and y increase in correlation order in the sense of Dhaene et al. (2009) then ℓ^*_α is higher for all α, since increase in correlation order translates to increase in covariance values. Layer dependence ℓ^*_α is not symmetric since $\ell^*_\alpha \neq -1$ in general if x and y are countermonotonic but $\ell^*_\alpha = 1$ under comonotonicity.

9.1. Gap between conditional tail expectations

Manipulating layer dependence ℓ^*_α yields

$$\ell^*_\alpha = \frac{\mathrm{E}(y|x > V_\alpha) - \mathrm{E}(y|x \leq V_\alpha)}{\mathrm{E}(y^*|x > V_\alpha) - \mathrm{E}(y^*|x \leq V_\alpha)} = \frac{\mathrm{E}(y|u > \alpha) - \mathrm{E}(y|u \leq \alpha)}{\mathrm{E}(y^*|u > \alpha) - \mathrm{E}(y^*|u \leq \alpha)} \;,$$

the gap between upper and lower conditional tail expectations of y. A similar expression for original layer dependence ℓ_α exists in (3), where conditional tail expectations of percentile rank v are evaluated instead of y.

Using the income-education example in §2, the population is again divided into two segments depending on whether education is below or above the α-percentile. Average actual income is measured in each segment rather than average income ranking. The calculated gap in average actual income is divided by the maximum gap, or the difference between average actual income above and below the α-percentile.

9.2. Link to Pearson correlation

Taking the following weighted average of ℓ^*_α yields scaled Pearson correlation between x and y:

$$\int_0^1 \ell^*_\alpha \left[\frac{\mathrm{cov}\{y^*, (u > \alpha)\} V'_\alpha}{\mathrm{cov}(y^*, x)}\right] \mathrm{d}\alpha = \frac{\mathrm{cov}\left\{y, \int_0^1 (u > \alpha) V'_\alpha \mathrm{d}\alpha\right\}}{\mathrm{cov}(y^*, x)} = \frac{\mathrm{cov}(y, V_u)}{\mathrm{cov}(y^*, x)}$$

$$= \frac{\text{cor}(y,x)}{\text{cor}(y^*,x)} \ , \qquad \int_0^1 \frac{\text{cov}\{y^*,(u>\alpha)\}V'_\alpha}{\text{cov}(y^*,x)}\text{d}\alpha = \frac{\text{cov}(y^*,V_u)}{\text{cov}(y^*,x)} = 1 \ ,$$

where weights integrate to one. The denominator $\text{cor}(y^*,x)$ in scaled Pearson correlation ensures unity if x and y are comonotonic. Similarly a weighted average of layer dependence ℓ_α over all α, using quadratic weights, yields ρ_S as shown in (4).

9.3. Relationship between two layer dependences

layer dependences between original random variables and between their percentile ranks are generated by a common "dependence generating function"

$$g_{\alpha,\beta} \equiv \frac{\text{cov}\{(u>\alpha, v>\beta)\}}{\text{cov}\{(u>\alpha, u>\beta)\}} = \frac{C(\alpha,\beta)-\alpha\beta}{\min(\alpha,\beta)-\alpha\beta} \ , \quad 0 \le \alpha,\beta \le 1 \ ,$$

where C is the copula underlying (u,v). The dependence generating function $g_{\alpha,\beta}$ measures dependence between α-layer of x and β-layer of v. In particular $g_{\alpha,\beta}=1$ for all α and β if u and v are perfectly dependent, and $g_{\alpha,\beta}=0$ if u and v are independent. Negative dependence yields negative $g_{\alpha,\beta}$.

Taking a weighted average of $g_{\alpha,\beta}$ over all β yields layer dependence, between percentile ranks and between observed random variables. Layer dependence ℓ_α is generated by the weighted integral

$$\ell_\alpha = \int_0^1 w_{\beta,1} g_{\alpha,\beta}\text{d}\beta \ , \quad w_{\beta,1} = \frac{\min(\alpha,\beta)-\alpha\beta}{\int_0^1\{\min(\alpha,\beta)-\alpha\beta\}\text{d}\beta} = \frac{\min(\alpha,\beta)-\alpha\beta}{0.5\alpha(1-\alpha)}$$

and layer dependence ℓ^*_α is generated by a different weighted integral

$$\ell^*_\alpha = \int_0^1 w_{\beta,2} g_{\alpha,\beta}\text{d}G^-(\beta) \ , \quad w_{\beta,2} = \frac{\min(\alpha,\beta)-\alpha\beta}{\int_0^1\{\min(\alpha,\beta)-\alpha\beta\}\text{d}G^-(\beta)}$$

$$= \frac{\min(\alpha,\beta)-\alpha\beta}{\text{cov}\{G^-(u),(u>\alpha)\}} \ .$$

For ℓ^*_α, weights attached to $g_{\alpha,\beta}$ are proportional to the derivative $(G^-)'$. Hence $g_{\alpha,\beta}$ is weighted more heavily over values of β where the derivative $\{G^-(\beta)\}'$ is large, such as in the tail of a right skewed distribution. In comparison weights for layer dependence ℓ_α are "marginal free."

The dependence generating function captures complete dependence information due to its direct relationship with the copula C. Correlations $\text{cor}(u,v)$ and $\text{cor}(x,y)$ show completely summarised dependence information. Layer dependences ℓ_α and ℓ^*_α balances the extremes and shows dependence information on a single dimension.

9.4. Illustration using Gumbel copula

The four panels Figure 3 calculate layer dependence curves between observed random variables, assuming combinations of symmetric and right skewed beta marginal distributions for x and y. A Gumbel copula identical to Figure 1 is assumed. Original layer dependence is included for comparison. Layer dependence retains its overall shape when observed random variables are used instead of percentile ranks.

10. Measuring dependence asymmetry

Rewrite layer dependence as

$$\ell_\alpha = \mathrm{cov}\left\{v, \frac{(u > \alpha)}{0.5\alpha(1-\alpha)}\right\} \ , \quad \mathrm{cov}\{u, (u > \alpha)\} = 0.5\alpha(1-\alpha) \ .$$

Measures of dependence asymmetry involves taking a difference between "upper tail" and "lower tail" dependence. Upper tail dependence is a weighted average of layer dependence from percentiles $0.5 \leq z \leq 1$ to 1:

$$\begin{aligned} \ell^+ = \int_z^1 w_\alpha \ell_\alpha \mathrm{d}\alpha &= \mathrm{cov}\left\{v, \int_z^1 \frac{(u > \alpha)}{0.5\alpha(1-\alpha)} w_\alpha \mathrm{d}\alpha\right\} \\ &= \mathrm{cov}\left\{v, (u > z) \int_z^u \frac{2w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha\right\} \end{aligned}$$

where w_α is a weighting function symmetric about $\alpha = 0.5$. Similarly lower tail dependence is a weighted average of layer dependence from percentiles 0 to $1 - z$:

$$\begin{aligned} \ell^- = \int_0^{1-z} w_\alpha \ell_\alpha \mathrm{d}\alpha &= \mathrm{cov}\left\{v, (u \leq 1-z) \int_0^u \frac{2w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha\right\} \\ &+ \mathrm{cov}\left\{v, (u > 1-z)\right\} \int_0^{1-z} \frac{2w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha \ , \end{aligned}$$

noting the overall weight on layer dependence assigned to ℓ^- is equal to that for ℓ^+, since w_α is symmetric. The measure of dependence asymmetry is hence

$$.\ell^+ - \ell^- = \mathrm{cov}\left\{v, (u > z) \int_z^u \frac{2w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha\right\}$$

$$-\mathrm{cov}\left\{v, (u \leq 1-z) \int_0^u \frac{2w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha\right\} - \mathrm{cov}\left\{v, (u > 1-z)\right\} \int_0^{1-z} \frac{2w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha \ .$$

A positive measure indicates $\ell^+ > \ell^-$ hence upper tail dependence is more prominent than lower tail dependence, and vice versa. The measure is zero for copulas with symmetric dependence structures, such as the Gaussian copula.

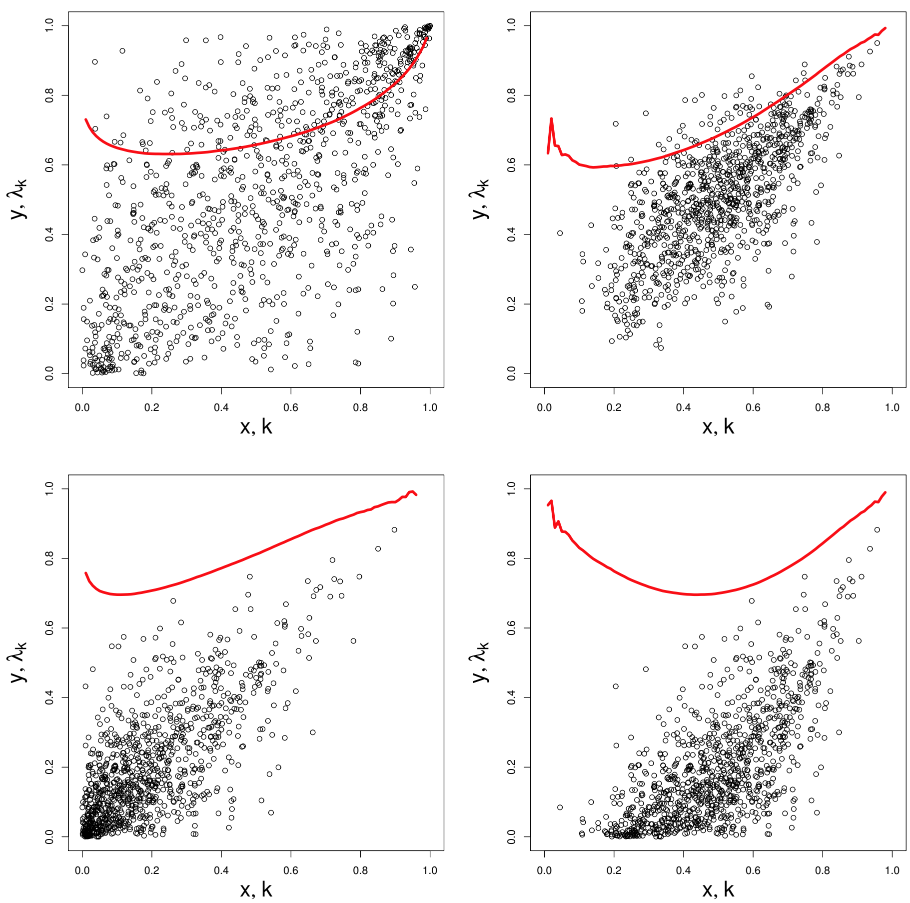

Figure 3: layer dependence curves between x and y (red line) for a Gumbel copula. The top left panel assumes uniform x and y, yielding original layer dependence. The top right panel assumes symmetric x and y. The bottom left panel assumes right skewed x and y. The bottom right panel assumes symmetric x and right skewed y.

11. Generating factor copulas given layer dependence

This section describes an algorithm to model a copula satisfying a given layer dependence function. The given layer dependence function may be estimated from past data, and possibly incorporate parametric smoothing and expert opinion. Non-linear regression copula models are assumed where layer dependence is controlled by the probability distribution of the systematic component relative to the noise component.

A non-linear regression copula model of (u, v) is

$$v = \mathrm{P}\left\{S(u) + \epsilon\right\} \ , \quad u \sim U(0,1) \ , \quad \epsilon \sim N(0,1) \tag{7}$$

where u and ϵ are independent and S is an increasing function. P represents the probability integral transform, hence (u, v) is bivariate uniform. Call $S(u)$ and ϵ systematic and noise components of the copula model, respectively. Specifying S completes the copula model.

Intuitively, the volatility pattern of $S(u)$ controls layer dependence between u and v. Measure the volatility of $S(u)$ at u as the derivative $S'(u)$, the gap between adjacent percentiles. If $S(u)$ has high volatility in the upper tail then $S(u)$ dominates v for large values of u, yielding strong layer dependence between u and v at high layers. Vice versa for the lower tail. If $S(u)$ has low volatility at all percentiles then v is dominated by noise, resulting in weak dependence. Lastly if $S^- = c\Phi^-$ for some constant c, where Φ^- is the inverse distribution function of the standard normal, then systematic and noise components are both normally distributed, yielding a Gaussian copula.

The following derives S to satisfy a "target" layer dependence function ℓ_α. S is specified either non-parametrically or parametrically. A non-parametric S is specified over a large number of points in the unit interval, while a parametric S is restricted to a class of functions. Once S is specified, the copula model (7) is simulated by generating large samples of u and ϵ and then calculating a sample of v as the empirical distribution function of simulated $S(u) + \epsilon$.

A non-parametric S is derived iteratively as follows. First generate large samples of u and ϵ. Without loss of generality, assume the u-sample is ordered and "error free": $[u_1, \ldots, u_n]$ where $u_i = i/n$. Generate the ϵ-sample $[\epsilon_1, \ldots, \epsilon_n]$ independently of the u-sample. The aim is to derive an S-sample $[s_1, \ldots, s_n]$ where $s_i = S(u_i) = S(i/n)$ such that the resulting layer dependence function is ℓ_α. Initialise the S-sample by for example setting $s_i^1 = c\Phi^-(u_i)$ where c is a constant selected to achieve equal ρ_S as ℓ_α. Repeat the following steps for $t = 1, 2, \ldots$, until convergence:

1. At iteration t, update the v-sample by setting $v_i^t = \mathrm{R}(s_i^t + \epsilon_i)$ where R computes percentile ranks lying in the unit interval.

2. Compute the "fitted" layer dependence function $\hat{\ell}_\alpha$ for the current (u, v)-sample, at all values of α in the u-sample.

3. Compute first order differences of the S-sample: $[d_1^t, \dots d_{n-1}^t]$ where $d_i^t = s_{i+1}^t - s_i^t$, representing volatility of the systematic component.

4. Update first order differences based on the corresponding gap between target and fitted layer dependence functions: $d_i^{t+1} = d_i^t \times (\ell_{i/n}/\hat{\ell}_{i/n})^a$ where a is the adjustment sensitivity, say 2.

5. Update the S-sample by combining updated first order differences: $s_{i+1}^{t+1} = s_i^{t+1} + d_i^{t+1}$. The first value of the S-sample remains unchanged: $s_1^{t+1} = s_1^t$.

At points where the target layer dependence exceeds fitted layer dependence, volatility of S at the same point is increased so that the systematic component increases its dominance over the noise component. Vice versa where target layer dependence falls below fitted layer dependence. Therefore S is iteratively "reshaped" depending on the gap between target and fitted layer dependence until the gap is satisfactorily small.

A parametric S is derived by first restricting S to a class of increasing functions, for example $S = G_\theta^-$ where θ is a set of parameters. Given θ, a (u, v)-sample is simulated yielding a fitted layer dependence function $\hat{\ell}_\alpha$. The optimal S is based on the value of θ minimising the gap between target layer and fitted dependence functions. The gap may be formulated for example as the "mean square error" $\sum(\ell_\alpha - \hat{\ell}_\alpha)^2$ where the sum applies to a large range of values of α in the unit interval.

Non-parametric S generally achieves superior fit to the target layer dependence function, compared to parametric S. In addition the copula model (7) generally does not permit closed form applications and hence simulation is required. In this case non-parametric S performs equally, if not better, than parametric S.

12. Comparison with existing local dependence measures

The following compares layer dependence with existing local dependence measures: tail concentration (Venter, 2002), correlation curve (Bjerve and Doksum, 1993), and bivariate measures by Bairamov et al. (2003), Jones (1996) and Holland and Wang (1987). Calculations are applied to Gaussian, Gumbel, Clayton and Frank copulas in Figure 1. Layer dependence is more reflective of local dependence (dispersion and concordance between points), and satisfies all five coherence properties in §6. In addition there is a direct relationship between layer dependence and ρ_S in (4).

12.1. Tail concentration

Venter (2002) defines tail concentration at $0 \le \alpha \le 1$ by combining lower and upper conditional tail probabilities relative to α:

$$\tau_\alpha \equiv (\alpha \le 0.5)\mathrm{P}(v \le \alpha | u \le \alpha) + (\alpha > 0.5)\mathrm{P}(v > \alpha | u > \alpha) \; . \tag{8}$$

Higher tail concentration implies u and v are more likely to fall in the same lower tail ($\alpha \leq 0.5$) or upper tail ($\alpha > 0.5$). Tail concentration partially satisfies coherence properties described in §6, and partially reflects local dependence. Layer dependence refines tail concentration through standardisation and reflecting average dispersion between discordant points. Further discussion of tail concentration is shown below.

Rewrite tail concentration in terms of the copula C of (u, v):

$$\tau_\alpha = (\alpha \leq 0.5)\frac{C(\alpha,\alpha)}{\alpha} + (\alpha > 0.5)\frac{1-2\alpha+C(\alpha,\alpha)}{1-\alpha} \ ,$$

therefore tail concentration depends only on the diagonal section of the copula, and is hence symmetric in u and v. Independence implies $\tau_\alpha = \alpha(\alpha \leq 0.5) + (1-\alpha)(\alpha > 0.5)$, and comonotonicity and countermonotonicity yield $\tau_\alpha = 1$ and $\tau_\alpha = 0$, respectively. Hence tail concentration does not satisfy independence and perfect dependence (countermonotonicity) coherence properties. Symmetry is also not satisfied since tail concentration is non-negative. The ordering property is satisfied since tail concentration increases with the copula C. Lastly $0 \leq \tau_\alpha \leq 1$ since τ_α is a probability, hence the bounds property holds.

Standardising tail concentration improves its coherence and reflection of local dependence. The latter is demonstrated via an illustration below. Standardisation involves subtracting the value assuming independence, and dividing the result by its maximum value:

$$\tau_\alpha^* \equiv \frac{\tau_\alpha - \{\alpha(\alpha \leq 0.5) + (1-\alpha)(\alpha > 0.5)\}}{1 - \{\alpha(\alpha \leq 0.5) + (1-\alpha)(\alpha > 0.5)\}} = \frac{C(\alpha,\alpha) - \alpha^2}{\alpha(1-\alpha)} = \gamma_\alpha \ .$$

Standardised tail concentration $\tau_\alpha^* = 0$ if u and v (independence property), and is negative if u and v are negatively dependent. Standardised tail concentration is also equal to the measure of concordance γ_α defined in (**??**).

From (5), layer dependence combines standardised tail concentration $\tau_\alpha^* = \gamma_\alpha$ and average dispersion between discordant points δ_α. Hence layer dependence refines tail concentration in two ways:

- layer dependence includes standardisation to achieve coherence. Standardisation also yields a more accurate reflection of local dependence.
- layer dependence reflects average dispersion between discordant u and v, yielding additional accuracy in local dependence measurement.

Figure 4 demonstrates the above refinements by comparing tail concentration, its standardised value and layer dependence using identical copulas as Figure 1. Tail concentration, without standardisation, does not always trace local dependence. For example tail concentration decreases at the upper tail of the Gumbel copula, despite upper tail dependence. Similarly for the Clayton copula. Standardisation corrects these inconsistencies. Layer dependence further refines the calculation by reflecting dispersion between discordant points. For example layer dependence increases to one at the upper tail of the Gumbel copula and lower tail of the Clayton copula, whereas standardised tail concentration does not increase completely to one, despite the presence of tail dependence. In addition standardised tail concentration decreases significantly at both tails of Guassian and Frank copulas although there is no increased dispersion of (u, v) in those areas.

12.2. Correlation curve

Bjerve and Doksum (1993) defines correlation curve based on the conditional distribution of v given u. Correlation curve has similar values and coherence properties as layer dependence. However calculated values of correlation curve are volatile due to reliance on a "pointwise" conditional distribution.

From Bjerve and Doksum (1993), the correlation curve of v with respect to u at α is defined as

$$c_\alpha \equiv \frac{\sigma\mu'_\alpha}{\sqrt{(\sigma\mu'_\alpha)^2 + \sigma^2_\alpha}} = \mathrm{sign}(\mu'_\alpha) \times \frac{1}{\sqrt{1+s^2_\alpha}} \ , \quad 0 \le \alpha \le 1 \ ,$$

$$s^2_\alpha \equiv \left(\frac{\sigma_\alpha}{\mu'_\alpha \sigma}\right)^2 \ , \quad \sigma^2 \equiv \mathrm{var}(v) \ , \quad \sigma^2_\alpha \equiv \mathrm{var}(v|u=\alpha) \ , \quad \mu_\alpha \equiv \mathrm{E}(v|u=\alpha) \ .$$

where μ_α and σ^2_α are the conditional mean and variance of v at $u = \alpha$, respectively, and $\sigma^2 = 1/12$ is the unconditional variance of v. In addition μ'_α is the derivative of μ_α with respect α. Bjerve and Doksum (1993) also generalises the correlation curve by replacing mean and variance with other location and scale measures such as median and interquartile range, respectively.

The correlation curve c_α at any α is affected by two factors: the derivative μ'_α and the ratio σ^2_α/σ^2. The former is the sensitivity of the conditional mean of v to changes in $u = \alpha$. Higher sensitivity implies stronger local dependence between u and v, increasing c_α. The second is the conditional variance of v relative to the unconditional variance. A higher ratio implies values of v conditional on $u = \alpha$ are more variable, hence the conditional mean of v at $u = \alpha$ is a less satisfactory predictor of v. The result is lower local dependence and c_α.

Correlation curve satisfies similar coherence properties as layer dependence. In particular, for all α, $-1 \le c_\alpha \le 1$ and $c_\alpha = -1$, 0 and 1 if u and v are countermonotonic, independent and comonotonic, respectively. Replacing u or v with its complement reverse the sign of c_α. In addition c_α increases when u and v are more "regression dependent." Bjerve and Doksum (1993) further discusses properties of correlation curves.

Figure 5 graphs correlation curves using identical copulas as Figure 1. Layer dependence is included as a comparison. Correlation curve varies similarly as layer dependence in all four copulas. However values of correlation curve are volatile, despite a large sample size of 10 million. In comparison calculations of

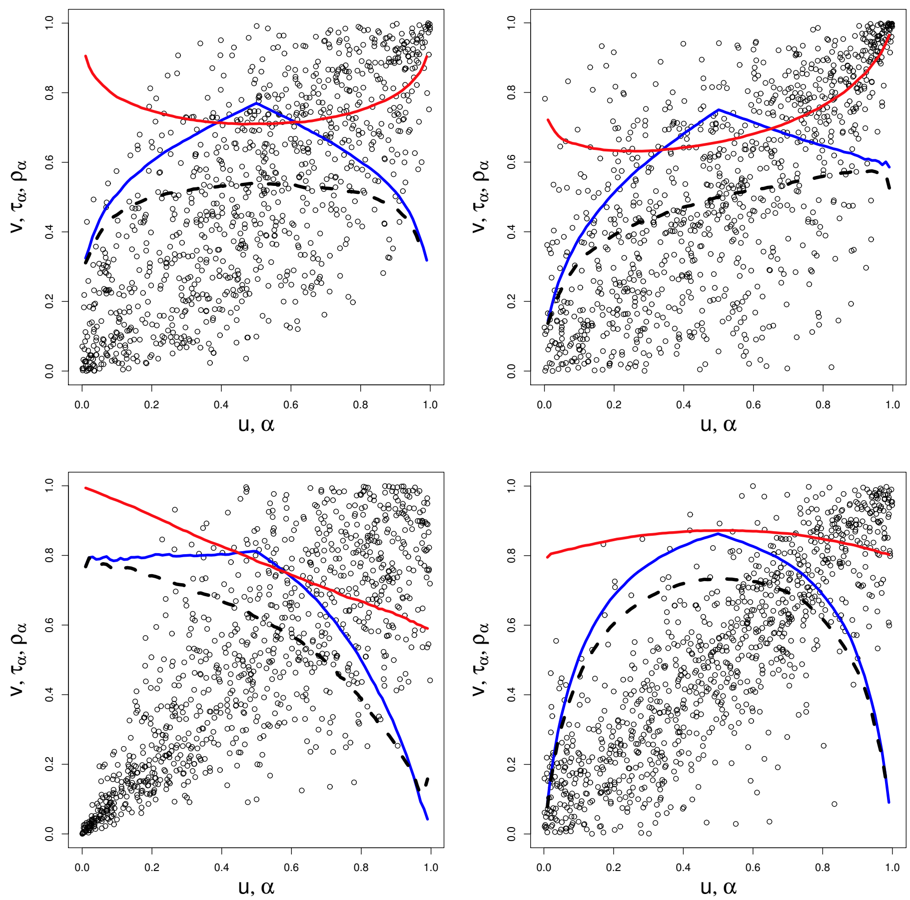

Figure 4: Calculation of tail concentration (blue line) for a Gaussian copula (top left), Gumbel copula (top right), Clayton copula (bottom left) and Frank copula (bottom right). Standardised values (dotted line) and layer dependence (red line) are also shown in each panel.

layer dependence in Figure 1 utilises a 100000-sample. The volatile correlation curve is due to involvement of conditional means, the derivative, and conditional variances at specific values of u.

12.3. Bivariate local dependence measures

Bairamov et al. (2003) defines a bivariate local dependence function, measuring dependence at various values of (u, v), by generalising $\rho + S$ using first and second order conditional expectations. Jones (1996) and Holland and Wang (1987) also define a bivariate local dependence function, based on partial derivatives of the log joint density function. Bivariate local dependence functions maintain the dimension of a bivariate joint distribution, and are less graphically interpretable than univariate local dependence functions such as layer dependence, tail concentration and correlation curve. In addition whilst the local dependence by Bairamov et al. (2003) is constrained in $[-1, 1]$, the local dependence function by Jones (1996) and Holland and Wang (1987) is unconstrained and is $-\infty$ and ∞ for countermonotonic and comonotonic variables, respectively.

13. Copula fitting using layer dependence

The literature suggests several approaches on fitting a copula to past data. If marginal distributions are unknown, a parametric approach typically involves selecting parametric copula and marginal distributions, and estimating parameters by maximising joint likelihood (Denuit et al., 2005). A semi-parametric approach replaces marginal distributions in joint likelihood with empirical values (Oakes, 1989). The choice of parametric copula may be restricted to a specific class of copulas. Genest and Rivest (1993) suggests an approach to select the generator function of Archimedean copulas (McNeil et al., 2005). Alternatively a visual assessment of data may suggest an appropriate parametric copula with similar dependence structure, for example the Gumbel copula if upper tail dependence is present and Clayton copula if lower tail dependence is present. At the other extreme of copula fitting is to use the empirical copula, when the volume of past data is sufficiently large. (**?**) discusses a semi-parametric approach for multivariate copulas, based on vine copulas.

This paper proposes a copula fitting approach based on layer dependence:

- Calculate layer dependence curve with desired granularity from percentile rank data.
- Fit a parametric curve to calculated layer dependence curve.
- Refine layer dependence using expert knowledge, such as existence and layer of tail dependence.
- Fit a copula to refined layer dependence curve.

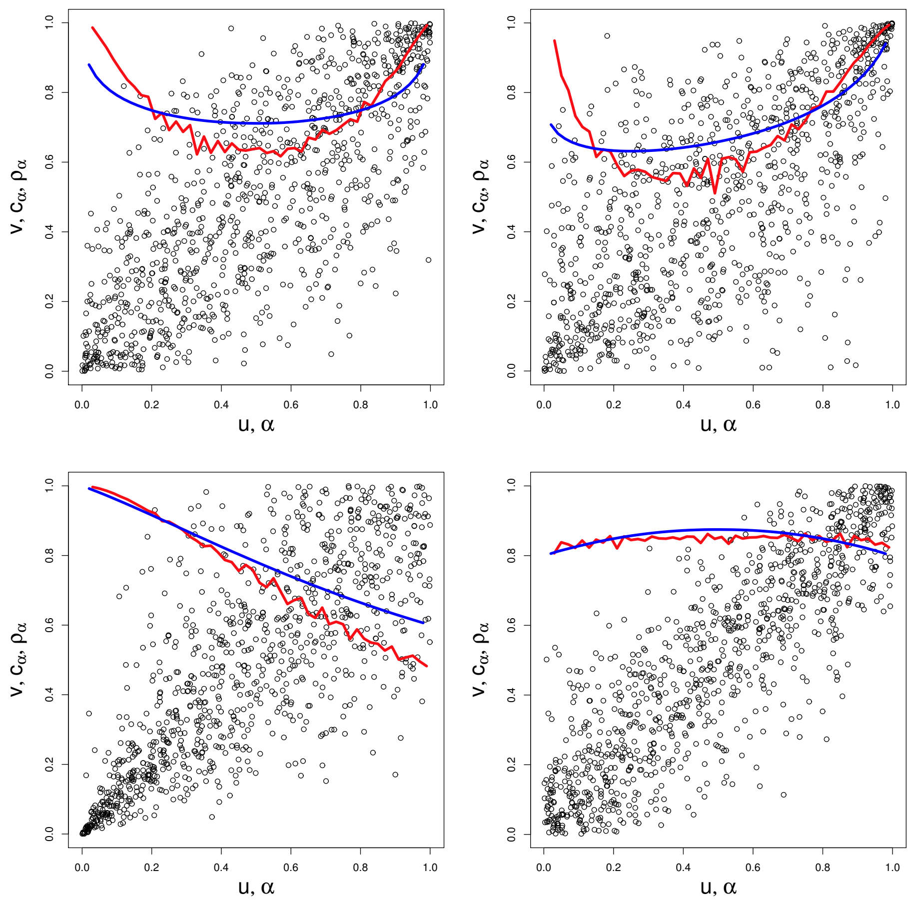

Figure 5: Calculation of correlation curve (red line) for a Gaussian copula (top left), Gumbel copula (top right), Clayton copula (bottom left) and Frank copula (bottom right). Layer dependence (blue line) is also shown in each panel.

The layer dependence approach offers two advantages over parametric copulas. Firstly layer dependence reflects the dependence structure exhibited in past data, whereas parametric copulas, with their relatively restricted dependence structures, may not properly fit past data. Secondly, layer dependence accommodates expert knowledge of local dependence, whereas parametric copulas permit limited changes to the dependence structure once a parametric form is selected.

In comparison to empirical copulas, layer dependence is more robust and less affected by data inadequacies. Layer dependence summarises past data into conditional tail means, and a parametric curve is fitted to calculated values. Hence layer dependence captures advantages of parametric and empirical copulas – the fitted copula utilises a smooth layer dependence curve, and the dependence structure underlying the fitted copula mirrors past data.

13.1. Illustration using stock market returns

The following fits a copula to historical NASDAQ and FTSE daily returns using layer dependence. The dependence structure of the fitted copula mirrors past data, in particular overall and tail dependence.

The left panel in Figure 6 plots NASDAQ and FTSE daily percentile rank returns from 1991 and 2013. Layer dependence calculated at every integer percentile and ρ_S are shown in the same panel. Spearman's ρ_S is 0.4, indicating moderate dependence between NASDAQ and FTSE daily returns. Layer dependence increases towards both tails, hence major positive and negative corrections in NASDAQ and FTSE are likely to occur simultaneously. Lastly layer dependence appears to be symmetric about the 50th percentile.

The right panel in Figure 6 graphs layer dependence refined from empirical values displayed in the left panel, and a copula fitted to refined layer dependence. Refined layer dependences closely trace empirical values, including the tails. In addition ρ_S underlying the fitted copula is approximately equal to the empirical value. Lastly the fitted copula is visually close to past data.

The following outlines the approach to obtain refined layer dependence and the corresponding fitted copula. Derive refined layer dependence by fitting a symmetric polynomial to empirical values:

$$\hat{\ell}_\alpha = a + b(\alpha - 0.5)^2 + c(\alpha - 0.5)^4 \ ,$$

where a, b and c are constants. Least squares estimation by varying a, b and c yields a refined layer dependence curve

$$\hat{\ell}_\alpha = 0.36 + 0.46(\alpha - 0.5)^2 + 1.73(\alpha - 0.5)^4 \ , \quad 0 \le \alpha \le 1 \ .$$

The copula fitted to the refined layer dependence curve assumes the following model for (u, v):

$$(u, v) = (z \le c) \times \{(u_r, v_r) \times c\} + (z > c) \times \{(c, c) + (u_r, v_r) \times (1 - c)\} \ ,$$

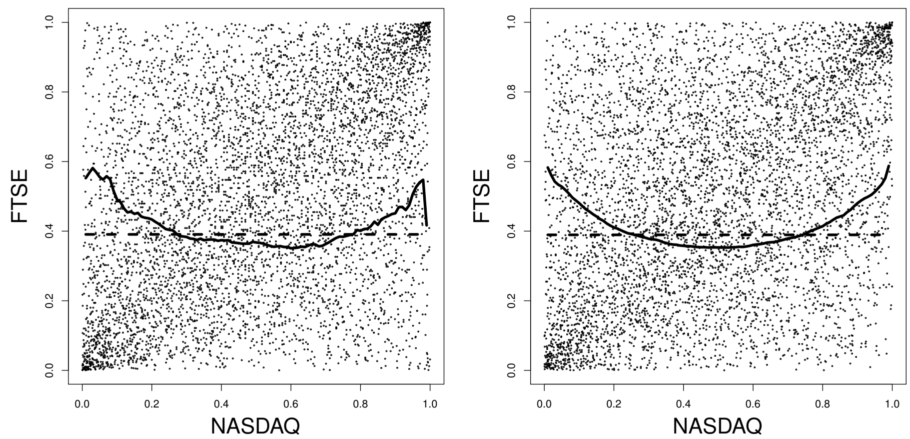

Figure 6: The left panel shows daily percentile rank returns from NASDAQ vs FTSE from 1991 to 2013, with calculated layer dependence and ρ_S. The right panel shows layer dependence refined from empirical values and a fitted copula. Spearman's ρ_S underlying the fitted copula is also shown.

$$0 \le c \le 1 \ , \quad c \sim H \tag{9}$$

where (u_r, v_r) is bivariate uniform with constant layer dependence r, and c varies randomly over the unit interval with distribution function H. Given c, u and v are perfectly concordant relative to c (simultaneously below or above c). Values of u and v below or above c are dependent, with constant layer dependence r.

The model (9) achieves layer dependence $\hat{\ell}_\alpha$ by setting $r = 0.2$ and letting c follow the mixed probability distribution

$$\mathrm{P}(c > t) = 0.95 - 4.27t + 16.44t^2 - 32.61t^3 + 32.51t^4 - 13.02t^5 \ ,$$

$$\mathrm{P}(c = 0) = 0.05 \ , \quad 0 \le t \le 1 \ .$$

To simulate (u, v) from (9), first simulate c using its probability distribution specified above. Then simulate (u_r, v_r) where $r = 0.2$, in turn yielding (u, v). The approach to simulate (u_r, v_r) is shown in §**??**.

14. Conclusion

layer dependence accurately captures dependence structures in bivariate copulas, and satisfies coherence properties. Taking weighted averages of layer dependence curves yields ρ_S and alternative overall dependence measures.

Using layer dependence in copula fitting captures dependence structures in past data, whilst flexibly accommodating expert opinion. Layer dependence achieves a balance between parametric approaches (smooth fit, low flexibility) and empirical approaches (volatile fit, high flexibility).

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