Layer dependence as a measure of local dependence

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Abstract

This paper introduces, analyses and illustrates a new measure of local dependence called "layer dependence." Layer dependence accurately captures varying dependence in the joint distribution. Layer dependence satisfies coherence properties similar to Spearman's correlation, such as lying between -1 and 1, with -1, 0 and 1 resulting from countermonotonicity, independence and comonotonicity, respectively. Taking a weighted average of layer dependence values across the joint distribution yields Spearman's correlation and alternative measures of overall dependence. Copulas fitted using layer dependence are tailored to past data and easily incorporate expert opinion on the dependence structure.

Keywords: Local dependence; rank dependence; conditional tail expectation; Spearman's correlation; concordance.

1. Local dependence and layer dependence

Dependence between two variables generally varies with percentile. For example extreme movements in two stock markets are likely be highly related whereas minor fluctuations may be relatively independent. Natural catastrophes create significant insurance losses for several classes of business at the same time, while attritional losses between various classes are weakly dependent.

Local dependence measures aim to capture the dependence structure of a bivariate distribution. This contrasts with measures of overall dependence such as Pearson correlation, Spearman's ρ and Kendall's τ (Embrechts et al., 2002). Local dependence measures include the univariate tail concentration (Venter, 2002), correlation curve (Bjerve and Doksum, 1993), and bivariate measures by Bairamov et al. (2003), Jones (1996) and Holland and Wang (1987).

This paper introduces, illustrates and analyzes an alternate local dependence measure called "layer dependence." Layer dependence is the covariance between a random variable and a single "layer" of another. Layer dependence is also the

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"gap" between upper and lower conditional tail expectations. Random variables are replaced with their percentile rank transforms and layer dependence is calculated entirely from the copula underlying the joint distribution. Hence of interest is rank dependence rather than dependence between random variables in their original scale, as the latter is often distorted by marginal distributions.

Layer dependence satisfies "coherence" properties similar to linear correlation: between -1 and 1, constant and equal to -1, 0 and 1 for countermonotonic, independent and comonotonic random variables, sign switching when ranking order reverses, and taking on higher values when dependence is stronger. Taking a weighted average of layer dependence values across the joint distribution yields Spearman's ρ and alternative coherent measures of overall dependence.

Layer dependence provides a more appropriate and accurate measure of local dependence compared to existing measures. Higher dispersion from various points of the 45° line reduces layer dependence and vice versa. Calculating layer dependence at the first instance from past data or parametric copula extracts essential and interpretable information – the dependence structure. For a parametric copula, the implication of its parametric form and parameters on the dependence structure is not always apparent. Similar problems apply when past data is scarce.

Layer dependence offers an alternative approach to copula modeling. First compute layer dependence values from past data, and apply parametric smoothing. Further adjust, if necessary, to incorporate expert opinion. A copula is then fitted to refined layer dependence values. The fitted copula overcomes the inflexibility of parametric copulas to closely capture the dependence structure in past data, whilst avoiding uncertainties of empirical copulas at the other extreme.

The remaining paper is structured as follows. Section 2 defines and analyzes layer dependence. Section 3 illustrates layer dependence for several copulas. Section 4 explains the behaviour of layer dependence by decomposing it into a negative function of discordance and dispersion. Section 5 describes coherence properties of layer dependence. Links to existing literature are highlighted in section 6. Further properties and expressions of layer dependence are discussed in sections 7 and 8 respectively. Section 9 suggests approaches to simulate a copula given the layer dependence curve, and section 10 applies layer dependence to copula modeling, using historical stock returns as an illustration. Section 11 discusses further applications of layer dependence, such as forming measures of overall dependence and dependence asymmetry. Section 12 formulates layer dependence in terms of observed random variables instead of their percentile ranks. Section 13 concludes.

2. Layer dependence

Suppose u and v are percentile ranks of continuous random variables x and y. Then (u, v) has standard uniform marginals and its joint distribution C is a copula (Nelson, 1999).

This paper defines and discusses a new form of local dependence between u and v, called "layer dependence." Of interest is rank dependence rather than dependence between x and y which is influenced by marginal distributions of x and y. Layer dependence exploits the decomposition

$$u = \int_0^1 (u > \alpha)\mathrm{d}\alpha = \int_0^u \mathrm{d}\alpha \ , \tag{1}$$

where $(u > \alpha)\mathrm{d}\alpha$ is the "α–layer" of u and $(u > a)$ is the indicator function: 1 if $u > \alpha$ and 0 otherwise. Given α, the derivative of α–layer with respect to u is 1 when $u = \alpha$ and 0 otherwise. This implies α–layer reflects movements in u at α, and ignores movements elsewhere. Thus u, from (1), is formed from infinitely many layers, each layer capturing the variability of u at a different point.

The α–layer dependence between v and the α–layer of u is defined in terms of covariance calculations

$$\ell_\alpha \equiv \frac{\mathrm{cov}\{v, (u > \alpha)\}}{\mathrm{cov}\{u, (u > \alpha)\}} = \frac{\mathrm{cor}\{v, (u > \alpha)\}}{\mathrm{cor}\{u, (u > \alpha)\}} \ , \quad 0 \le \alpha \le 1 \ , \tag{2}$$

where cov and cor calculate covariance and correlation, respectively, using C. Denominators in (2) are independent of C and forces $\ell_\alpha = 1$ if $u = v$ and $\ell_\alpha = -1$ if $u = 1 - v$. Hence α–layer dependence ℓ_α measures dependence between v and u at α. Further $-1 \le \ell_\alpha \le 1$. Independence implies $\ell_\alpha = 0$. Coherence properties of ℓ_α are formalised in section 5.

Expanding covariance terms in (2) and manipulating yields

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u|u \le \alpha)} = 2\left\{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v|u \le \alpha)\right\} \ , \tag{3}$$

where E calculates expectations with respect to C. The first expression in (3) is the expected change in v relative to the expected change in u when u cross α. The latter is 0.5 regardless of α, yielding the second expression in (3). Hence large ℓ_α implies v is sensitive to movements in u across α, indicating strong dependence between v and u at α. When $\ell_\alpha = 0$, v is unchanged on average when u crosses α, hence u and v are independent at α.

Using (1) and (2), write Spearman's correlation between u and v as

$$\rho_S \equiv \mathrm{cor}(u, v) = \frac{\mathrm{cov}(u, v)}{\mathrm{var}(u)} = \frac{\int_0^1 \mathrm{cov}\{v, (u > \alpha)\}\mathrm{d}\alpha}{1/12} = \mathcal{E}(\ell_\alpha) \ . \tag{4}$$

where the expectation $\mathcal{E}$ is calculated over $0 \le \alpha \le 1$ using density $6\alpha(1 - \alpha)$. Hence Spearman's correlation averages α–layer dependence using a certain density on α. The density integrates to 1, has minimum 0 at $\alpha = 0$ and 1, and increases symmetrically to 1.5 at $\alpha = 0.5$. Varying the density leads to different emphasis on different areas of the relationship between u and v, and yields alternative measures of overall dependence. For example using the density $n\alpha^{n-1}$ where $n > 0$ puts increasing weight on upper tail dependence. This is further discussed in section 11.

3. Layer dependence curves for various copulas

The nine panels in Figure 1 display (u, v) scatterplots of symmetric copulas, and their layer dependence curves: ℓ_α over $0 \le \alpha \le 1$. Each copula has Spearman's correlation $\rho_S = 0.6$.

Each ℓ_α curve reflects the dependence structure between u and v. Given α, ℓ_α is larger if points are more clustered around (α, α) and vice versa, as formalised in section 4. In addition, ℓ_α increases to 1 in the tails where points converge to the 45° degree line indicating perfect dependence.

The nine panels in Figure 1 highlight the inadequacies of using Spearman's correlation to measure overall dependence, particularly in the tails. In contrast, layer dependence curves capture the dependence structure of each copula.

4. Layer dependence, discordance and dispersion

If (u, v) is exchangeable, $C(u, v) = C(v, u)$, then layer dependence ℓ_α measures the lack of discordance and dispersion at α:

$$\ell_\alpha = 1 - 2(1 + \gamma_\alpha)\delta_\alpha \; , \tag{5}$$

where

$$\gamma_\alpha \equiv \mathrm{cor}\{(u \le \alpha), (v > \alpha)\} = \mathrm{cor}\{(u > \alpha), (v \le \alpha)\} \; ,$$
$$\delta_\alpha \equiv \mathrm{E}\left\{(|u - v|) | (u - \alpha)(v - \alpha) < 0\right\} \; .$$

A proof of (5) is shown below. The correlation $-1 \le \gamma_\alpha \le 1$ measures the tendency for (u, v) to be discordant at α: opposite signs on $u - \alpha$ and $v - \alpha$, and the expectation $0 \le \delta_\alpha \le 1$ measures dispersion between values of u and v discordant at α.

The proof of (5) follows from

$$\gamma_\alpha = -\frac{\mathrm{cov}\{(u \le \alpha), (v \le \alpha)\}}{\mathrm{var}\{(u \le \alpha)\}} = \frac{\alpha^2 - C(\alpha, \alpha)}{\alpha(1 - \alpha)} \; ,$$

and

$$\begin{aligned}
\delta_\alpha &= 2\mathrm{E}\left\{(u - v)(u > v) | (u - \alpha)(v - \alpha) < 0\right\} \\
&= \frac{2\mathrm{E}\{(u - v)(u > v)(u > \alpha)(v \le \alpha)\}}{2\mathrm{E}\{(u > \alpha)(v \le \alpha)\}} = \frac{\mathrm{E}\{(u - v)(u > \alpha)(v \le \alpha)\}}{\alpha - C(\alpha, \alpha)} \\
&= \frac{\mathrm{E}\{(u - v)(u > \alpha)\} - \mathrm{E}\{(u - v)(u > \alpha)(v > \alpha)\}}{\alpha - C(\alpha, \alpha)} = \frac{\mathrm{E}\{(u - v)(u > \alpha)\}}{\alpha - C(\alpha, \alpha)} \; .
\end{aligned}$$

Substituting the above expressions for γ_α and δ_α into the right hand side of (5) yields the expression for ℓ_α in (3), completing the proof.

Result (5) explains the behaviour of layer dependence curves in Figure 1. Layer dependence ℓ_α is larger if there are fewer discordant pairs at α, and

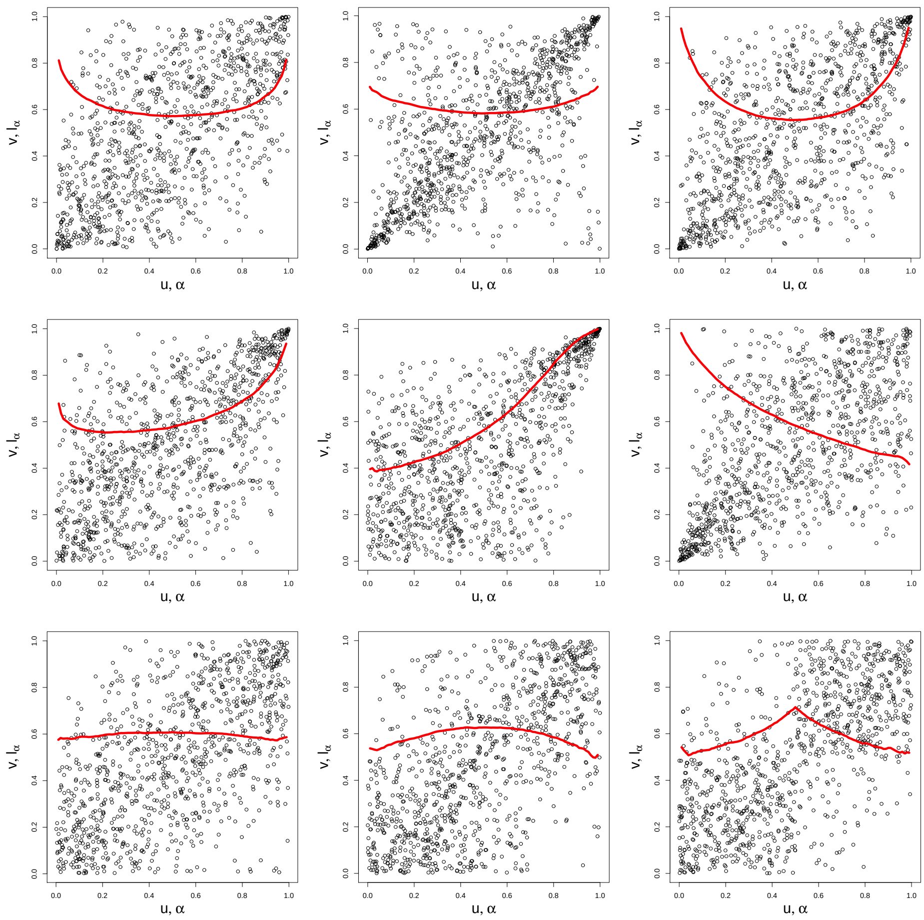

Figure 1: Copulas with different layer dependence curves ℓ_α over 0‘$\alpha \leq 1$ (red curves) but the same $\rho_S = 0.6$.

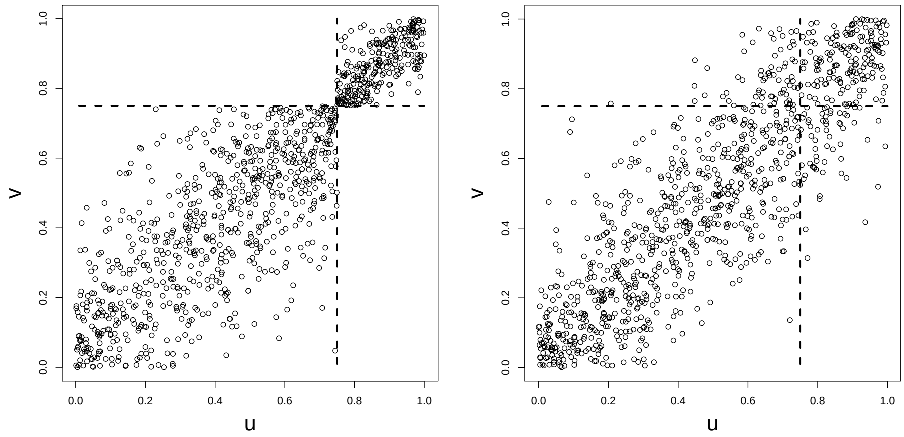

Figure 2: The left and right panel show $\ell_{0.75} = 1$ and $\ell_{0.75} = 0.86$, respectively. In the left panel, $\gamma_{0.75} = -1$ and $\delta_{0.75} = 0$. In the right panel, $\gamma_{0.75} = -0.65$ and $\delta_{0.75} = 0.21$.

discordant pairs at α are closer to the 45° degree line. The former indicates smaller γ_α and the latter indicates smaller δ_α. Vice versa for small ℓ_α. If $\ell_\alpha = 1$ then $\gamma_\alpha = -1$ or $\delta_\alpha = 0$, implying u and v are simultaneously below or above α and $u = v$ for discordant pairs. If $\ell_\alpha = 1$ in an interval, then $u = v$ in the same interval.

Figure 2 illustrates the relationship between layer dependence, discordance and dispersion in (5) using two copulas with $\ell_\alpha = 1$ and $\ell_\alpha = 0.86$ at $\alpha = 0.75$. When $\ell_\alpha = 1$, there is no discordance or dispersion at α. As ℓ_α decreases from 1, the number of discordant pairs and their dispersion increases.

5. Coherence properties of layer dependence

Layer dependence ℓ_α satisfies five "coherence" properties. These properties are extensions of properties applying to Spearman's correlation ρ_S.

- **Bounds**: Layer dependence lies between -1 and 1: $-1 \leq \ell_\alpha \leq 1$ for all α. Hence layer dependence is bounded in the same way as ρ_S.
- **Perfect dependence**: Constant layer dependence of -1 or 1 are equivalent to countermonotonicity and comonotonicity. Thus $\ell_\alpha = -1$ for all α if and only if $v = 1 - u$ while $\ell_\alpha = 1$ for all α if and only if $v = u$.
- **Independence**: If u and v are independent then $\ell_\alpha \equiv 0$. The converse is not true – zero layer dependence does not imply independence as shown by the following counterexample. Assume $v = u$ and $v = 1 - u$ with equal probability. Then $\mathrm{E}(v|u = t) = 0.5$ for all $0 \leq t \leq 1$ implying

$E(v|u > \alpha) = E(v|u \le \alpha) = 0.5$. Hence $\ell_\alpha = 0$. However u and v are not independent.

- **Symmetry**: Ranking either u or v in the opposite direction switches the sign of layer dependence. Changing the ranking order of both variables preserves the sign of layer dependence.
- **Ordering**: Higher correlation order (Dhaene et al., 2009) leads to higher layer dependence. Consider bivariate uniform (u^*, v^*) exceeding (u, v) in correlation order: $C^*(a, b) \ge C(a, b)$ for all $0 \le a, b \le 1$, where C^* is the joint distribution of (u^*, v^*). Then $\ell^*_\alpha \ge \ell_\alpha$, $0 \le \alpha \le 1$ where ℓ^*_α denotes the α–layer dependence of (u^*, v^*). Hence greater dependence leads to higher layer dependence.

Independence, symmetry and ordering properties follow from the definition of layer dependence in (2). From (3), constant layer dependence of one implies $E(v|u > \alpha) = (\alpha + 1)/2$ and $E(v|u = \alpha) = \alpha$, for all $0 \le \alpha \le 1$, hence $v = u$. Similarly constant layer dependence of minus one implies $v = 1 - u$. The ordering property holds since higher correlation order implies larger covariances (Dhaene et al., 2009). Prove the bounds property by combining ordering and perfect dependence properties, and noting countermonotonicity and comonotonicity represent minimum and maximum correlation order, respectively.

6. Links to the literature

6.1. Tail dependence

Tail dependence models dependence at percentiles 0 and 1. Perfect tail dependence occurs when catastrophic events such as bank failure or market crash manifests in both series. Layer dependence models perfect tail dependence by assuming both variables simultaneously attain minimum or maximum. Coefficients of tail dependence discussed in Joe (1997) yield an identical interpretation.

Joe (1997) defines coefficients of lower and upper tail dependence in terms of the extreme tail probabilities

$$\lim_{\alpha \to 0} E\{(v \le \alpha)|u \le \alpha\} \ , \quad \lim_{\alpha \to 1} E\{(v > \alpha)|u > \alpha\} \ .$$

Coefficient values of one indicate perfect positive tail dependence, and occur if and only if u and v converge simultaneously to zero (lower tail) or one (upper tail). Negative tail dependence definitions replace $(v \le \alpha)$ and $(v > \alpha)$ in the above expressions by $(v > 1 - \alpha)$ and $(v \le 1 - \alpha)$, respectively.

Layer dependence provides a consistent view of perfect tail dependence as Joe (1997). For the upper tail, from (3), $\ell_1 = 2E(v|u = 1) - 1$. Hence $\ell_1 = 1$ if and only if $E(v|u = 1) = 1$, that is $\ell_1 = 1$ if and only if $u = 1$ implies $v = 1$. Similarly $\ell_1 = -1$ if and only if $u = 1$ implies $v = 0$. Similar remarks apply to the lower tail: $\ell_0 = 1$ if and only if $u = 0$ implies $v = 0$ while $\ell_0 = -1$ if and only if $u = 0$ implies $v = 1$. Hence perfect tail dependence implies variables simultaneously achieve their extreme values.

6.2. Tail concentration

Tail concentration (Venter, 2002) is a local dependence measure formed by combining lower and upper conditional tail probabilities at α:

$$\tau_\alpha \equiv (\alpha \le 0.5)\mathrm{P}(v \le \alpha|u \le \alpha) + (\alpha > 0.5)\mathrm{P}(v > \alpha|u > \alpha) \ .$$

Higher tail concentration τ_α implies u and v are more likely to fall in the same lower tail ($\alpha \le 0.5$) or upper tail ($\alpha > 0.5$). Rewrite tail concentration in terms of the copula C of (u, v):

$$\tau_\alpha = (\alpha \le 0.5)\frac{C(\alpha,\alpha)}{\alpha} + (\alpha > 0.5)\frac{1 - 2\alpha + C(\alpha,\alpha)}{1-\alpha} \ .$$

Standardising tail concentration yields the opposite of the discordance component γ_α forming layer dependence in (5):

$$\tau^*_\alpha \equiv \frac{\tau_\alpha - \{\alpha(\alpha \le 0.5) + (1-\alpha)(\alpha > 0.5)\}}{1 - \{\alpha(\alpha \le 0.5) + (1-\alpha)(\alpha > 0.5)\}} = \frac{C(\alpha,\alpha) - \alpha^2}{\alpha(1-\alpha)} = -\gamma_\alpha \ ,$$

implying $\ell_\alpha = 1 - 2\delta_\alpha(1 - \tau^*_\alpha)$ where δ_α, as defined in section 4, is the average dispersion between points (u, v) discordant at α. Standardisation subtracts the value under independence: $\tau_\alpha = \alpha(\alpha \le 0.5) + (1-\alpha)(\alpha > 0.5)$, and divides the result by the same where u and v are comonotonic: $\tau_\alpha = 1$.

Therefore layer dependence refines tail concentration by including additional information on average dispersion between discordant points.

7. Further properties of layer dependence

7.1. Layer dependence does not uniquely characterise a copula

Layer dependence curves ℓ_α over $0 \le \alpha \le 1$ do not uniquely characterise the copula C. For example suppose $v = u$ and $v = 1 - u$ with equal probability, implying $\mathrm{E}(v|u \le \alpha) = \mathrm{E}(v|u > \alpha) = 0.5$ for all α. Then $\ell_\alpha = 0$, from (3). The copula is $C(u, v) = 0.5\{\min(u, v) + \max(u + v - 1, 0)\}$. The same ℓ_α results if u and v are independent: $C(u, v) = uv$.

Non-uniqueness arises as layer dependence only captures first-order conditional tail expectations, based on (3).

7.2. Layer dependence under non-exchangeable copula

If u and v are exchangeable, layer dependence in (2) is invariant when u and v are switched. Archimedean copulas (McNeil et al., 2005) are exchangeable. If exchangeability fails then layer dependence differs when layers of v instead of u are applied, that is dependence between v and α–layer of u differs from dependence between u and α–layer of v. An analogous property applies to least squares regression: regressing v on u is the same as regressing u on v, if the joint distribution is exchangeable. This paper assumes exchangeability unless stated otherwise.

7.3. Layer dependence preserves convex combination

Suppose (u, v) and (u^*, v^*) are both bivariate uniform with layer dependence ℓ_α and ℓ^*_α, respectively. Then $\pi\ell_\alpha+(1-\pi)\ell^*_\alpha$ is the layer dependence of $x(u, v)+(1-x)(u^*, v^*)$ where $x \sim B(\pi)$. The proof is straightforward using (3). This result generalises to multiple and continuous convex combinations of a vector of bivariate uniform random variables.

8. Other expressions for layer dependence

8.1. One-sided conditional tail expectations

Since $\mathrm{E}(v) = \alpha\mathrm{E}(v|u \le \alpha) + (1-\alpha)\mathrm{E}(v|u > \alpha)$ it is straightforward to show

$$\ell_\alpha = \frac{\mathrm{E}(v|u > \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u > \alpha) - \mathrm{E}(u)} = \frac{\mathrm{E}(v|u \le \alpha) - \mathrm{E}(v)}{\mathrm{E}(u|u \le \alpha) - \mathrm{E}(u)} \ ,$$

the gap between upper or lower conditional tail expectations of v and the unconditional expectation. As before denominators are scaling factors ensuring $\ell_\alpha = 1$ if u and v are comonotonic and -1 if countermonotonic.

8.2. Copula integration

Note $\mathrm{cov}\{(u > \alpha), (v > \beta)\} = C(\alpha, \beta) - \alpha\beta$ and apply (1) to v, to derive

$$\ell_\alpha = \frac{\int_0^1 \mathrm{cov}\{(u > \alpha), (v > \beta)\}\mathrm{d}\beta}{\alpha(1-\alpha)/2} = \frac{2\int_0^1 C(\alpha, \beta)\mathrm{d}\beta - \alpha}{\alpha(1-\alpha)} = \frac{2\int_0^1 C(\beta|\alpha)\mathrm{d}\beta - 1}{1-\alpha} \ ,$$

where $C(\beta|\alpha) = \mathrm{P}(v \le \beta|u \le \alpha)$ is the conditional copula. Thus ℓ_α integrates copulas and conditional copulas to reduce their dimension from two and one, and scales the result to ensure it lies between ± 1.

With Archimedean copulas (McNeil et al., 2005) $C(\alpha, \beta) = \psi^- \{\psi(\alpha) + \psi(\beta)\}$ where ψ is the generator function and ψ^- its inverse. In this case closed form expressions for the integrals and hence ℓ_α or $\mathrm{E}(v|u > \alpha)$ do not exist. This argues against the use of Archimedean copulas.

9. Simulating a copula given layer dependence

The following are two approaches to simulate a copula given layer dependence ℓ_α for all $0 \le \alpha \le 1$. The copula is not unique to ℓ_α since ℓ_α, from (3), relates to first order conditional expectations rather than the entire conditional distribution associated with the copula.

9.1. Factor model

The first approach assumes a factor copula model

$$u = G_{s+\epsilon}(s + \epsilon_u) \ , \quad v = G_{s+\epsilon}(s + \epsilon_v) \ , \quad s \sim G_s \ , \quad \epsilon_u, \epsilon_v \sim N(0,1) \ ,$$

where s, ϵ_u and ϵ_v are independent. G_s and $G_{s+\epsilon}$ are distributions of s and $s + \epsilon_u$, respectively, and $G_{s+\epsilon}$ depends on G_s. Both u and v are uniform and the joint distribution of (u, v) is exchangeable. The common factor s generates and controls layer dependence between u and v.

The factor copula model generates positive layer dependence. Negative layer dependence is created by first simulating the corresponding positive layer dependence, and then taking complements of either percentile rank.

To generate a sample of (u, v) from the factor copula model with layer dependence ℓ_α, $0 \le \alpha \le 1$, choose a large integer n and simulate $\epsilon_{u,i}, \epsilon_{v,i} \sim N(0,1)$ for $i = 1, \ldots, n$. Initialise s_i by for example setting it equal to a normal sample with standard deviation chosen such that $s_i + \epsilon_{u,i}$ and $s_i + \epsilon_{v,i}$ have Spearman's correlation consistent with ℓ_α. Then repeat:

1. Compute "fitted" $\hat{\ell}_{j/n}$ between (u_i, v_i) for $j = 1, \ldots, n$, where u_i and v_i are calculated from the percentile ranks of $s_i + \epsilon_{u,i}$ and $s_i + \epsilon_{v,i}$, respectively.
2. Keep s_1 unchanged and update $s_2, \ldots, s_n$ according to
$$s_{i+1} \leftarrow s_i + \left(\frac{\ell_{i/n}}{\hat{\ell}_{i/n}} \right)^a (s_{i+1} - s_i) \ , \quad a > 0 \ ,$$
in turn yielding an updated sample (u_i, v_i).
3. Go to 1 unless $\|\ell_{i/n} - \hat{\ell}_{i/n}\|$ is less than some pre-specified small amount.

Step 1 can be simplified by computing $\hat{\ell}_{j/n}$ over a fewer number of points and then fitting a parametric curve to computed values. The resulting sample (u_i, v_i) for $i = 1, \ldots, n$ once the iteration is complete has layer dependence $\hat{\ell}_\alpha \approx \ell_\alpha$.

9.2. Regression model

The second approach writes $\tau_\alpha \equiv \mathrm{E}(v|u > \alpha) = (1 + \alpha\ell_\alpha)/2$. Then

$$\tau'_\alpha \equiv \frac{\mathrm{dE}(v|u > \alpha)}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{\mathrm{E}\{v(u > \alpha)\}}{1 - \alpha} = \frac{\tau_\alpha - \mu_\alpha}{1 - \alpha} \ , \quad \mu_\alpha \equiv \mathrm{E}(v|u = \alpha) \ ,$$

where μ_α is the regression curve of v on u. Hence $\mu_\alpha = \tau_\alpha - (1 - \alpha)\tau'_\alpha$ and τ_α is monotonic if μ_α is monotonic.

Suppose ℓ_α and hence τ_α is given for $\alpha = i/n$, $i = 1, \ldots, n$ with values denoted $\tau_1, \tau_1, \ldots, \tau_n$. Then $C(u, v)$ with layer dependence ℓ_α is simulated using

$$v_i = F\{F^-(\mu_i) + \sigma\epsilon_i\} \ , \ \mu_i = \tau_i - (n - i)(\tau_i - \tau_{i-1}) \ , \ \epsilon_i \sim N(0,1) \ , \ i = 1, \ldots, n \ ,$$

with $\tau_0 = 1/2$ and σ chosen so that Spearman's ρ for the each sample is expected to match ρ_S implied by ℓ_α as in (4). Here F is an appropriate distribution such as the normal or logit: $F^{-}(\mu_i) = \ln\{\mu_i/(1-\mu_i)\}$. In the logic case setting $\sigma = 1$ appears to ensure a match between the theoretical and empirical ρ_S. To enforce the $v_1, \ldots, v_n$ are empirically uniform the final transformation F can be replaced by $\hat{\mathrm{P}}$, computing empirical percentiles.

10. Copula fitting using layer dependence

10.1. Approach

This paper proposes a copula fitting approach based on layer dependence:

- Calculate layer dependence curve with desired granularity from percentile rank data.
- Smooth calculated layer dependence curve either parametrically or semi-parametrically.
- Refine layer dependence using expert knowledge, such as existence and layer of tail dependence.
- Fit a copula to refined layer dependence curve using methods described in section 9.

The layer dependence approach offers two advantages over fitting parametric copulas. Firstly layer dependence reflects the dependence structure exhibited in past data, whereas parametric copulas, with their relatively restricted dependence structures, may not properly fit past data. Secondly, layer dependence accommodates expert knowledge of local dependence, whereas parametric copulas permit limited changes to the dependence structure once a parametric form is selected.

In comparison to empirical copulas, layer dependence is more robust and less affected by data inadequacies. Layer dependence summarises past data into conditional tail means, and a parametric curve is fitted to calculated values. Hence layer dependence captures advantages of parametric and empirical copulas – the fitted copula utilises a smooth layer dependence curve, and the dependence structure underlying the fitted copula mirrors past data.

10.2. Illustration using stock market returns

The following fits a copula to historical NASDAQ and FTSE daily returns using layer dependence. The dependence structure of the fitted copula mirrors past data, in particular overall and tail dependence.

The left panel in Figure 3 plots NASDAQ and FTSE daily percentile rank returns from 1991 and 2013. Layer dependence calculated at every integer

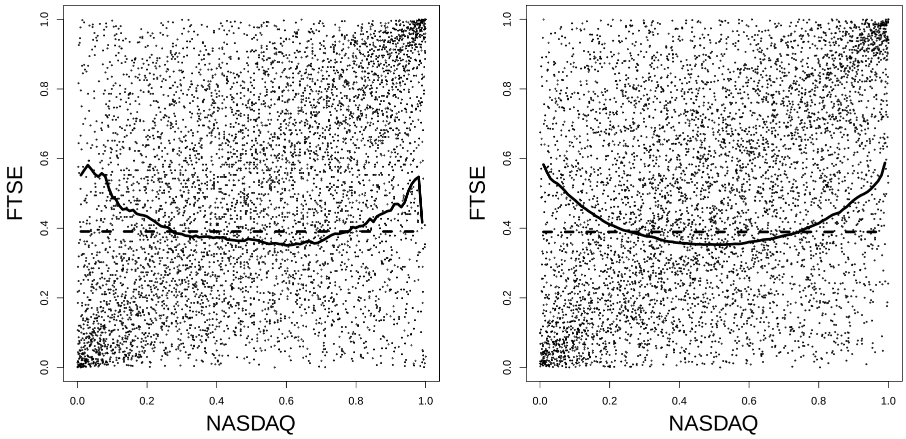

Figure 3: The left panel shows daily percentile rank returns from NASDAQ vs FTSE from 1991 to 2013, with calculated layer dependence and ρ_S. The right panel shows layer dependence refined from empirical values and a fitted copula. Spearman's ρ_S underlying the fitted copula is also shown.

percentile and ρ_S are shown in the same panel. Spearman's ρ_S is 0.4, indicating moderate dependence between NASDAQ and FTSE daily returns. Layer dependence increases towards both tails, hence major positive and negative corrections in NASDAQ and FTSE are likely to occur simultaneously. Lastly layer dependence appears to be symmetric about the 50th percentile.

The right panel in Figure 3 graphs layer dependence refined from empirical values displayed in the left panel, and a copula fitted to refined layer dependence using the factor model. Refined layer dependences closely trace empirical values, including the tails. In addition ρ_S underlying the fitted copula is approximately equal to the empirical value. Lastly the fitted copula is visually close to past data.

The following outlines the approach to obtain refined layer dependence and the corresponding fitted copula. Derive refined layer dependence by fitting a symmetric polynomial to empirical values:

$$\hat{\ell}_\alpha = a + b(\alpha - 0.5)^2 + c(\alpha - 0.5)^4 \ ,$$

where a, b and c are constants. Least squares estimation by varying a, b and c yields a refined layer dependence curve

$$\hat{\ell}_\alpha = 0.36 + 0.46(\alpha - 0.5)^2 + 1.73(\alpha - 0.5)^4 \ , \quad 0 \le \alpha \le 1 \ .$$

11. Additional applications of layer dependence

The following describes two additional applications of layer dependence. The first is formulate measures of overall rank dependence apart from Spearman's correlation. The second is measuring dependence asymmetry.

11.1. Alternative measures of overall dependence

Spearman's correlation is, from (4), an expectation of ℓ_α over α using probability density $6\alpha(1-\alpha)$. Alternative rank dependence measures are derived by using a different probability density w_α:

$$\rho_W \equiv \int_0^1 w_\alpha \ell_\alpha \mathrm{d}\alpha = \int_0^1 w_\alpha \frac{\mathrm{cov}\{v,(u>\alpha)\}}{\mathrm{cov}\{u,(u>\alpha)\}} \mathrm{d}\alpha = \mathrm{cov}\{W(u), v\} \ ,$$

where

$$W(u) \equiv 2\int_0^u \frac{w_\alpha}{\alpha(1-\alpha)} \mathrm{d}\alpha \ ,$$

is the weighted cumulative of w_α. Hence ρ_W is the covariance between v and a transformation $W(u)$ of u. For Spearman's correlation ρ_S, $w_\alpha = 6\alpha(1-\alpha)$ and $W(u) \propto u$.

The density w_α specifies "importance" of dependence at percentile α. With Spearman's correlation, $w_\alpha = 6\alpha(1-\alpha)$: importance is symmetric and highest at the median, and decreases to zero at both tails. Hence ρ_S emphasizes dependence at moderate percentiles and diminishes dependence in the tails. Alternative densities are described below.

Since ρ_W is a weighted average of ℓ_α, coherence properties of layer dependence described in section 5 apply to ρ_W: bounded by ± 1, constant values of -1, 0 and 1 under countermonotonicity, independence and comonotonicity, respectively, sign reversal when ranking order switches, and higher values for stronger dependence. Identical properties apply to Spearman's correlation.

The following densities w_α yield alternative rank dependence measures:

- Suppose dependence over all percentiles are equally important. Then w_α is the uniform density and

  $$\rho_W = 2\mathrm{cov}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} = \sqrt{\frac{2}{3}}\ \mathrm{cor}\left\{v, \log\left(\frac{u}{1-u}\right)\right\} \ ,$$

  a multiple of correlation between v and the logit of u. If tail dependence is pronounced, $\rho_W > \rho_S$ since ρ_W weights tail dependence more heavily compared to ρ_S.

- If $w_\alpha = 3\alpha^2$ then dependence at higher percentiles are considered more important. This density is applicable when upper tail dependence exists and has significant impact, for example the simultaneous occurrence of large insurance losses. Then

$$\rho_W = 6\mathrm{cov}\left\{v, \log\left(\frac{\mathrm{e}^{-u}}{1-u}\right)\right\} = \sqrt{3}\mathrm{cor}\left\{v, -\log(1-u)\right\} - \frac{\rho_S}{2} \ .$$

  If dependence is higher over percentiles above the median then $\rho_W > \rho_S$.

- If dependence over percentiles below the median is more important then for example $w_\alpha = 3(1-\alpha)^2$ yielding

$$\rho_W = 6\mathrm{cov}\left\{v, \log\left(u\mathrm{e}^{-u}\right)\right\} = \frac{\rho_S}{2} - \sqrt{3}\mathrm{cor}(v, \log u) \ .$$

- Suppose w_α is derived from V_α, the inverse of a marginal distribution function with derivative V'_α such that

$$w_\alpha = \frac{\mathrm{cov}\{u, (u>\alpha)\}V'_\alpha}{\int_0^1 \mathrm{cov}\{u, (u>\alpha)\}V'_\alpha \mathrm{d}\alpha} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(V_u, u)} = \frac{\alpha(1-\alpha)V'_\alpha}{\mathrm{cov}(x, u)} \ ,$$

  where $x = V_u$. This yields the Gini correlation (Schechtman and Yitzhaki, 1999)

$$\rho_W = \frac{\mathrm{cov}(V_u, v)}{\mathrm{cov}(V_u, u)} = \frac{\mathrm{cov}\{x, G(y)\}}{\mathrm{cov}\{x, F(x)\}} \ ,$$

  where $F \equiv V^-$ and G are distribution functions of x and y, respectively. In this example the density depends on the marginal distribution of x. More skewed distributions lead to greater emphasis on upper tail dependence.

11.2. Measuring dependence asymmetry

Dependence asymmetry is the difference between dependence in upper tail compared to the lower tail. Dependence asymmetry is important for example when modeling the sum of two random variables. Asymmetric dependence, where upper tail dependence is stronger for example, increases the right skewness of the sum since large outcome of both random variables are more likely to occur simultaneously.

Measure dependence asymmetry as $\ell^+ - \ell^-$ where

$$\ell^+ \equiv \int_z^1 w_\alpha \ell_\alpha \mathrm{d}\alpha \ , \quad \ell^- = \int_0^{1-z} w_\alpha \ell_\alpha \mathrm{d}\alpha \ , \quad 0.5 \le z \le 1 \ ,$$

and w_α is a weighting function symmetric about $\alpha = 0.5$. Hence dependence in the upper tail is the weighted average of layer dependence over percentiles above z, and dependence in the lower tail averages over percentiles below $1-z$. For example the Gaussian or t copulas with symmetric dependence have $\ell^+ = \ell^-$. For the Clayton copula, $\ell^- > \ell^+$ while $\ell^+ > \ell^-$ for the Gumbel copula.

Substituting the expression for ℓ_α in (2) yields

$$\ell^+ = \mathrm{cov}\left\{v, \int_z^1 \frac{(u>\alpha)}{0.5\alpha(1-\alpha)} w_\alpha \mathrm{d}\alpha\right\} = \mathrm{cov}\left\{v, (u>z)\int_z^u \frac{2w_\alpha}{\alpha(1-\alpha)}\mathrm{d}\alpha\right\}$$

and similarly

$$\ell^- = \mathrm{cov}\left\{v, \int_0^{1-z} \frac{(u>\alpha)}{0.5\alpha(1-\alpha)} w_\alpha \mathrm{d}\alpha\right\} = \mathrm{cov}\left\{v, (u\le 1-z)\int_0^u \frac{2w_\alpha}{\alpha(1-\alpha)}\mathrm{d}\alpha\right\}$$
$$+\mathrm{cov}\left\{v, (u>1-z)\right\}\int_0^{1-z} \frac{2w_\alpha}{\alpha(1-\alpha)}\mathrm{d}\alpha \ .$$

12. Layer dependence between non–uniform random variables

Consider random variables x and y. Analogous to (2), define layer dependence between x and y as the scaled covariance between y and k–layer of x:

$$\ell_k^* \equiv \frac{\mathrm{cov}\{y,(x>k)\}}{\mathrm{cov}\{y^*,(x>k)\}} = \frac{\mathrm{cor}\{y,(x>k)\}}{\mathrm{cor}\{y^*,(x>k)\}} \ ,$$

where k lies in the support of x, and y^* has identical marginal distribution as y and is comonotonic with x. The denominator hence ensures $\ell_k^* = 1$ for all k when x and y are comonotonic.

Layer dependence ℓ_k^* partially satisfies coherence properties of ℓ_α in section 5. If x and y are independent or comonotonic then $\ell_k^* = 0$ and 1, respectively, for all k. If x and y are countermonotonic then write $x = F^-(u)$ and $y = G^-(1-u)$ where F^- and G^- are inverse distribution functions of x and y, respectively. Layer dependence in this case is

$$\ell_k^* = \frac{\mathrm{cov}\{G^-(1-u),(u>\alpha)\}}{\mathrm{cov}\{G^-(u),(u>\alpha)\}} \le 0 \ , \quad \alpha = F(k) \ ,$$

reducing to -1 if G^- is symmetric: $G^-(1-u) - G(0.5) = G(0.5) - G(u)$. Layer dependence ℓ_k^* also preserves correlation order: if x and y increase in correlation order in the sense of Dhaene et al. (2009) then ℓ_k^* is higher for all k, since increase in correlation order translates to increase in covariance values. Layer dependence ℓ_k^* is symmetric if the distribution of y is symmetric. Lastly averaging ℓ_k^* yields a link to correlation between x and y:

$$\int \frac{\mathrm{cov}\{y^*,(x>k)\}}{\mathrm{cov}(x,y^*)}\ell_k^* \mathrm{d}k = \frac{\mathrm{cov}(x,y)}{\mathrm{cov}(x,y^*)} = \frac{\mathrm{cor}(x,y)}{\mathrm{cor}(x,y^*)}$$

applying the result in (1).

The four panels Figure 4 calculate layer dependence curves between x and y with combinations of symmetric and right skewed beta marginal distributions. A Gumbel copula is assumed. Layer dependence between percentile ranks is included for comparison. Layer dependence retains its overall shape when observed random variables are used instead of percentile ranks.

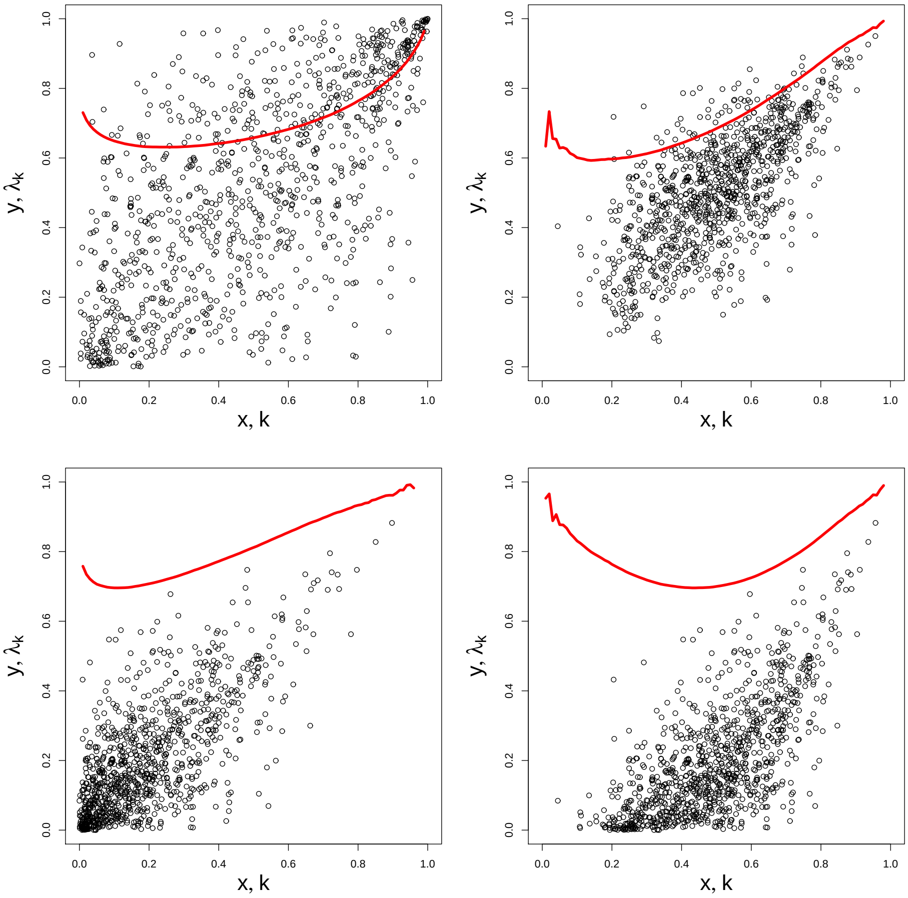

Figure 4: layer dependence curves between x and y (red line) for a Gumbel copula. The top left panel assumes uniform x and y, yielding original layer dependence. The top right panel assumes symmetric x and y. The bottom left panel assumes right skewed x and y. The bottom right panel assumes symmetric x and right skewed y.

13. Conclusion

Layer dependence accurately captures dependence structures in bivariate copulas, and satisfies coherence properties. Taking weighted averages of layer dependence curves yields Spearman's correlation and alternative overall dependence measures.

Using layer dependence in copula fitting captures dependence structures in past data, whilst flexibly accommodating expert opinion. Layer dependence achieves a balance between parametric approaches (smooth fit, low flexibility) and empirical approaches (volatile fit, high flexibility).

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