1. Regression dependence model

Write the PG model as

$$z = \alpha(q) + \beta q + h(q)\epsilon$$

where q, z and ϵ are standard normal distributed, and ϵ is independent of q and z. A sample of (q, z) is provided and the aim is to estimate α, β and h. These unknowns are subject to the following conditions:

- Assume $\alpha(q)$ and q are uncorrelated:

$$\mathrm{cov}\{\alpha(q), q\} = \mathrm{E}\{\alpha(q)q\} = 0 \ .$$

- The expected value of z is 0:

$$\mathrm{E}\{\alpha(q)\} + \beta\mathrm{E}(q) + \mathrm{E}\{h(q)\}\mathrm{E}(\epsilon) = \mathrm{E}\{\alpha(q)\} = 0 \ .$$

- The variance of z is 1:

$$\begin{aligned} \mathrm{cov}(z, z) &= \mathrm{cov}\{\alpha(q) + \beta q + h(q)\epsilon, \alpha(q) + \beta q + h(q)\epsilon\} \\ &= \mathrm{E}\{\alpha^2(q)\} + \beta^2 + \mathrm{E}\{h^2(q)\} = 1 \ , \end{aligned}$$

  noting $\mathrm{cov}\{\alpha(q), h(q)\epsilon\}$ and $\mathrm{cov}\{q, h(q)\epsilon\}$ are both 0.

Based on the above conditions, the correlation between q and z is β:

$$\rho \equiv \mathrm{cor}(q, z) = \mathrm{cov}(q, z) = \mathrm{cov}\{q, \alpha(q) + \beta q + h(q)\epsilon\} = \beta$$

The local correlation between q and z is

$$\rho_q = \left[1 + \left\{\frac{h(q)}{\alpha'(q) + \beta}\right\}^2\right]^{-0.5}$$

noting $\mathrm{E}(z|q) = \alpha(q) + \beta q$ and $\mathrm{cov}(z|q) = h^2(q)$.

The aim is to obtain estimates of α, β and h using a sample of (q, z), subject to the above conditions. For example β can be set equal to the empirical correlation between q and z. Then estimate α and h either parametrically or non-parametrically using least squares. The smoothness of α and h is user-selected. Obtaining α, β and h yields a dependence model on the (q, z) scale and estimates of overall and local correlation.

References