1. Systemic risk

Suppose a firm at time t has debt D_t and equity W_t. Then at t, $D_t + W_t$ is the total claim on the firm. Regulated firms are generally required to hold a proportion k of claims $k(D_t + W_t)$ as a prudential margin. If $k(D_t + W_t) > W_t$ then, prudentially speaking, the firm has capital shortfall $k(D_t + W_t) - W_t$. Negative capital shortfall indicates positive working capital.

Assume $D_t = D_{t+1} = \cdots = D_{t+h}$ where h is integer, and

$$W_{t+h} = (1 + i_{t+1} + \cdots + i_{t+h})W_t = (1 + r_t)W_t \ ,$$

serving to define the firm return r_t. Note r_t is the h period arithmetic return up to time $t + h$. Then the capital shortfall at time $t + h$ is

$$\begin{aligned} S_t &\equiv k(D_{t+h} + W_{t+h}) - W_{t+h} = kD_t - (1-k)(1+r_t)W_t \\ &= \{k\ell_t - (1-k)r_t - 1\} W_t \ , \qquad \ell_t \equiv \frac{D_t + W_t}{W_t} \ , \end{aligned} \tag{1}$$

where ℓ_t is the leverage ratio.

A systemic event or "crisis" occurs in the interval $(t+1, t+h)$ if the market return over this period is less than threshold τ. The systemic risk of the firm is the conditional expectation at time t of capital shortfall at time $t + h$ given a crisis:

$$\rho_t \equiv \mathrm{E}^c(S_t) = \{k\ell_t - (1-k)\mathrm{E}^c(r_t) - 1\}W_t \ . \tag{2}$$

The condition $D_{t+h} = D_t$ or less strongly $\mathrm{E}^c_t(D_{t+h}) = D_t$ indicates the value of debt does not change if there is a crisis. Evaluating the systemic risk in a firm requires the evaluation or estimation of the "crisis" expectation $\mathrm{E}^c(r_t)$ also called the long run marginal expected shortfall (LRMES).

Systemic risk (2) is a point estimate of capital shortfall (1). The "crisis" probability distribution of r_t, denoted $F^c(r_t)$ is the conditional probability of r_t given a crisis. In a crisis,

$$\mathrm{P}\left[S_t \leq \left\{k\ell_t - (1-k)F^{c-}(q) - 1\right\} W_t\right] = q \ ,$$

where F^{c-} is the inverse of the crisis distribution F^c.

Given a number of firms, ρ_t is thought of as a vector with components ρ_{it} corresponding to each firm i. Aggregate systemic risk is then $1'\rho_t$. The proportion of systemic risk in firm j is $\rho^+_{it}/(1'\rho_t)$ where ρ^+_{it} is the positive part of ρ_{it}.

An econometric models links returns r_{it} to the market return r_{mt} enabling the calculation of the crisis expectations $\mathrm{E}^c(r_{it})$. Given a joint distribution for (r_{it}, r_{mt}) the distribution is simulated and

$$\mathrm{E}^c(r_{it}) \approx \frac{\hat{\mathrm{E}}\{(r_{mt} \leq \tau)r_{it}\}}{\hat{\mathrm{E}}\{(r_{mt} \leq \tau)\}} = \frac{\hat{\mathrm{E}}\{(u_{mt} \leq q)F^-_{it}(u_{it})\}}{q} \ , \tag{3}$$

where the r values in the middle expressions denote simulated values and $\hat{\mathrm{E}}$, averages over the simulated values. The u_{it} and u_{mt} are the percentiles of r_{it} and r_{mt}, respectively.

The right hand expression in (3) suggests using a copula to model the joint distributions (r_{it}, r_{mt}). Simulate (u_{it}, u_{mt}), from the copula, discarding pairs with $u_{mt} > F(q)$. For each accepted (u_{it}, u_{mt}) determine the actual return using the respective marginal distributions. Issues to be addressed are:

- Are the marginal distributions time invariant? Could have situation where the marginal distribution evolves over time and/or is state dependent
- Is the copula time invariant? Could have situation where the copula is state dependent.

The first step here is to examine the marginal distributions of the returns and the empirical copula of returns.

2. Empirical studies

Brownlees and Engle (2015) use data on 95 large US financial firms for the period 2000-2012: daily equity returns and market capitalization from CRSP; quarterly book values for equity and debt from Compustat. A Dynamic Conditional Correlation (DCC) model is estimated for firm and market returns with threshold GARCH volatilities. Write

$$\mathrm{cov}(r_t|t) = V_t R_t V_t' \ , \qquad R_t \equiv \mathrm{cov}(\varepsilon_t|t) \ , \qquad \varepsilon_t = V_t^{-1} r_t \ ,$$

where V_t is the diagonal matrix of volatilities at time t and R_t is the correlation matrix at time t. Then volatilities are estimated using univariate threshold GARCH models. The estimate of R_t is the correlation matrix corresponding to fitted covariance matrix C_t, fitted using the model

$$(C_t - C) = \alpha(C_{t-1} - C) + \beta(\varepsilon_t \varepsilon_t' - C) \ , \qquad C \equiv \mathrm{cov}(\varepsilon_t) \ .$$

Brownlees and Engle (2015) find the above systemic risk measure a) successfully identifies systemically risky firms during the GFC; b) predicts capital injections by Fed Reserve; and c) aggregate systemic risk provides early warning of declines in industrial production and higher unemployment.

Comment. With the DCC model both volatilities and correlations have a time dependent structure. These are modelled separately. An alternative approach is where

2.1. Zeta score modelling

The zeta score of an observation x_i is $\zeta_t \equiv (\Phi^{-} \circ \mathrm{P}) r_t$ where P denotes the empirical percentile and Φ the standard normal distribution. By construction the ζ_i are normally distributed and hence should be particularly amenable to modelling.

How is time dependence modelled with a copula? One way is to plot percentiles $(u_{it}, u_{i,t-1})$ for each i or $(\zeta_{it}, \zeta_{i,t-1})$

Engle et al. (2014) use data on 196 large financial institutions in Europe: banks, insurance companies, financial services firms and real estate firms; daily equity returns and market capitalization from Datastream; quarterly book values for equity and debt from Compustat; world and Europe equity indexes from MSCI

- Period: 1990-2012
- Econometric methods:
  - estimate multi-factor, time varying model of returns using Dynamical Conditional Betas, Engle (2014)
  - model volatility of errors using univariate asymmetric GARCH model
  - use skewed t distribution for marginals of the innovations and a t copula for the dependence structure between the innovations
  - estimation proceeds recursively from an international model [World and European indexes]; then country models [add in respective country index and use the parameters from the international model] then firm model [include firm returns and use parameters from country model]
  - directly estimate LRMES
    * simulate forward 125 days returns from model estimates
    * use $S = 50,000$ draws to compute

$$\text{LRMES}_{jt} = \frac{\sum_{s=1}^{S} R^s_{jt+1:t+h} \times I(R^s_{mt+1:t+h} < -40\%)}{\sum_{s=1}^{S} I(R^s_{mt+1:t+h} < -40\%)}$$

where $I(x) = 1$ if true and 0 otherwise. where $I(x) = 1$ if true and 0 otherwise.

- Results:
  - rank firms, firm types and countries by SRISK: banks contribute 83% insurance companies 15% of systemic risk in Europe; highest countries - France and UK contribute 52%
  - aggregate SRISK Granger-causes industrial production and business confidence index
  - SRISK is positively related to changes in 3-month interbank rate but not significantly related to stock market return and volatility.

3. CoVaR

Adrian and Brunnermeier (2011) propose CoVaR which is the VaR of the financial system m when financial institution j is in distress which is operationally defined as its VaR(q) level.

$$\Pr(r_t^m \leq \text{CoVaR}_{qt}^{m|j} | r_t^j = \text{VaR}_{qt}^j) = q.$$

Increased risk to system m when financial institution j is in distress

$$\Delta\text{CoVaR}_{qt} = \text{CoVaR}_{qt}^{m|j} - \text{CoVaR}_{qt}^{m|b^j}$$

where b^j denotes the benchmark state for j and equals its median return.

Girardi and Ergün (2013) suggest to condition on when financial institution j is at its VaR level at best:

$$\Pr(r_t^m \leq \text{CoVaR}_{qt}^{m|j} | r_t^j \leq \text{VaR}_{qt}^j) = q.$$

They define the benchmark state b^j as a one standard deviation around the mean

$$\mu_t^j - \sigma_t^j \leq r_t^j \leq \mu_t^j + \sigma_t^j.$$

and measure the percentage systemic risk contribution of j by

$$\Delta\text{CoVaR}_{q,t}^{m|j} = 100 \times (\text{CoVaR}_{qt}^{m|j} - \text{CoVaR}_{qt}^{m|b^j})/\text{CoVaR}_{qt}^{m|b^j}.$$

Girardi and Ergün (2013) argue their measure improves on Adrian and Brunnermeier (2011) as it

- is more general and relevant as it takes into account severity of tail losses
- facilitates back-testing of CoVaR using standard back-tests for VaR.

Mainik and Schaanning (2012) also show the Girardi and Ergün (2013) measure is better

- it is consistent with respect to the dependence parameter unlike the Adrian and Brunnermeier (2011) measure which is not, e.g.in a bivariate Gaussian model, the Adrian and Brunnermeier (2011) measure is decreasing in the correlation!
- consistency property also extends to Conditional Expected Shortfall [CoES].

3.1. Adrian and Brunnermeier (2011)

- Data: weekly equity returns of 357 US bank holding companies from CRSP; quarterly balance sheet data from Compustat; state variables: VIX, liquidity spread; change in T-bill rate; change in slope of yield curve; change in credit spread; controls - market equity return; return on real estate sector.

- Period: 1986-2010
- Econometric methods:
  - use quantile regressions of asset returns to estimate $CoVaR$
  - add lagged state variables for conditional results
  - construct out-of-sample forward ΔCoVaR by projecting ΔCoVaR on lagged size, leverage, maturity mismatch and industry dummies
- Results:
  - VaR and ΔCoVaR are weakly related in the cross section but strongly related over time
  - ΔCoVaR larger for firms with higher leverage, more maturity mismatch and larger size
  - strong negative correlation between contemporaneous ΔCoVaR and forward ΔCoVaR

3.2. Girardi and Ergün (2013)

- Data: estimate $CoVaR$ and conduct back-tests using daily equity returns of 74 large US financial firms
- Period: 2000-20087
- Econometric methods:
  - bivariate GARCH-DCC models assuming (i) bivariate Gaussian (ii) bivariate skewed-t distributions
  - 3-step procedure: 1. estimate individual VaR using univariate GARCH; 2. estimate bivariate density for returns using bivariate GARCH; 3. numerically solve CoVaRs

$$\int_{-\infty}^{\mathrm{CoVaR}_{qt}^{m|j}} \int_{-\infty}^{\mathrm{VaR}_{qt}^{j}} pdf_t(x,y)dydx = q^2$$

$$\int_{-\infty}^{\mathrm{CoVaR}_{qt}^{m|b^j}} \int_{\mu_t^j-\sigma_t^j}^{\mu_t^j+\sigma_t^j} pdf_t(x,y)dydx = p_t^j q.$$

- Results:
  - backtests favour skewed-t distribution
  - rank of contribution to systemic risk: depositary institutions [largest]; broker-dealers; insurance companies; non-depositary institutions [smallest]
  - VaR and ΔCoVaR are weakly related cross sectionally and over time
  - all industry groups showed substantial increase in pre crisis ΔCoVaR
  - ΔCoVaR is positively related to size and equity beta, and to leverage during down markets.

4. CATFIN

Allen et al. (2012) create a macroindex of systemic risk designated CATFIN which is the arithmetic average of three VaR measures using the Generalized Pareto distribution [GPD], skewed generalized error distribution [SGED] and a nonparametric approach being the relevant quantile of the empirical distribution.

$$\mathrm{VaR}_{\mathrm{GPD}} = \mu + \left(\frac{\sigma}{\xi}\right)\left[\left(\frac{\alpha N}{n}\right)^{-\xi} - 1\right]$$

where μ, σ, ξ are location, scale and shape parameters. α is loss probability level, n is number of extremes and N is total data points.

Obtain $\mathrm{VaR}_{\mathrm{SGED}}$ by solving numerically

$$\int_{-\infty}^{\mathrm{VaR}_{\mathrm{SGED}(\alpha)}} f_{\mu,\sigma,\kappa,\lambda}(z)dz = \alpha$$

where f is SGED probability density function; κ controls height and tails of the density and λ is skewness.

Allen et al. (2012) also create analogous measures for expected shortfall but report that the predictability results are similar.

4.1. Allen et al. (2012)

- Data: monthly returns and market capitalization for US financial companies; Chicago Fed National Activity Index [CFNAI] and alternative macro-economic performance indicators; collect similar stock price data for EU and Asia and GDP growth rates
- Period: 1973-2009
- Econometric methods:
  - use maximum likelihood to estimate GPD and SGED parameters from monthly returns
  - estimate predictive auto-regressions of CFNAI using CATFIN as a predictor plus other control variables
- Results:
  - CATFIN is negatively related to the future CFNAI for 1- to 6-month ahead forecast horizons
  - this predictability comes from banks
  - results are robust to alternative measures of economic performance
  - regional CATFIN also has predictive power for GDP growth in EU and Asia

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