Copula–Based Models for Financial Time Series

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Abstract This paper presents an overview of the literature on applications of copulas in the modelling of financial time series. Copulas have been used both in multivariate time series analysis, where they are used to characterize the (conditional) cross-sectional dependence between individual time series, and in univariate time series analysis, where they are used to characterize the dependence between a sequence of observations of a scalar time series process. The paper includes a broad, brief, review of the many applications of copulas in finance and economics.

1 Introduction

The central importance of risk in financial decision-making directly implies the importance of dependence in decisions involving more than one risky asset. For example, the variance of the return on a portfolio of risky assets depends on the variances of the individual assets and also on the linear correlation between the assets in the portfolio. More generally, the distribution of the return on a portfolio will depend on the univariate distributions of the individual assets in the portfolio and on the dependence between each of the assets, which is captured by a function called a 'copula'.

The number of papers on copula theory in finance and economics has grown enormously in recent years. One of the most influential of the 'early' papers on copulas in finance is that of Embrechts, McNeil and Straumann (2002), which was circulated as a working paper in 1999. Since then, scores of papers have been written, exploring the uses of copulas in finance, macroeconomics, and

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microeconomics, as well as developing the estimation and evaluation theory required for these applications. Nelsen (2006) and Joe (1997) provide detailed and readable introductions to copulas and their statistical and mathematical foundations, while Cherubini, *et al.* (2004) focus primarily on applications of copulas in mathematical finance and derivatives pricing. In this survey I focus on financial time series applications of copulas.

A copula is a function that links together univariate distribution functions to form a multivariate distribution function. If all of the variables are continuously distributed,[1] then their copula is simply a multivariate distribution function with *Uniform*$(0, 1)$ univariate marginal distributions. Consider a vector random variable, $\mathbf{X} = [X_1, X_2, ..., X_n]'$, with joint distribution $\mathbf{F}$ and marginal distributions $F_1, F_2, ..., F_n$. Sklar's (1959) theorem provides the mapping from the individual distribution functions to the joint distribution function:

$$\mathbf{F}(\mathbf{x}) = \mathbf{C}(F_1(x_1), F_2(x_2), ..., F_n(x_n)), \quad \forall\, \mathbf{x} \in \mathbb{R}^n. \tag{1}$$

From any multivariate distribution, $\mathbf{F}$, we can extract the marginal distributions, F_i, and the copula, $\mathbf{C}$. And, more useful for time series modelling, given any set of marginal distributions $(F_1, F_2, ..., F_n)$ and any copula $\mathbf{C}$, equation (1) can be used to obtain a joint distribution with the given marginal distributions. An important feature of this result is that the marginal distributions do not need to be in any way similar to each other, nor is the choice of copula constrained by the choice of marginal distributions. This flexibility makes copulas a potentially useful tool for building econometric models.

Since each marginal distribution, F_i, contains all of the univariate information on the individual variable X_i, while the joint distribution $\mathbf{F}$ contains all univariate *and* multivariate information, it is clear that the information contained in the copula $\mathbf{C}$ must be all of the dependence information between the X_i's[2]. It is for this reason that copulas are sometimes known as 'dependence functions', see Galambos (1978). Note that if we define U_i as the 'probability integral transform' of X_i, i.e. $U_i \equiv F_i(X_i)$, then $U_i \sim Uniform(0, 1)$, see Fisher (1932), Casella and Berger (1990) and Diebold, *et al.* (1998). Further, it can be shown that $\mathbf{U} = [U_1, U_2, ..., U_n]' \sim \mathbf{C}$, the copula of $\mathbf{X}$.

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[1] Almost all applications of copulas in finance and economics assume that that variables of interest are continuously distributed. Notable exceptions to this include Heinen and Rengifo (2003) and Grammig, *et al.* (2004). The main complication that arises when considering marginal distributions that are not continuous is that the copula is then only uniquely defined on the Cartesian product of supports of the marginal distributions. Obtaining a copula that is defined on $\mathbb{R}^n$ requires an interpolation method. See Denuit and Lambert (2005) for one such method.

[2] It is worth noting that some dependence measures of interest in finance, and elsewhere, depend on both the copula *and* the marginal distributions; standard linear correlation is the leading example. Depending on one's orientation, and the application at hand, this is either a drawback of such dependence measures or a drawback of copula theory.

If the joint distribution function is n-times differentiable, then taking the n^{th} cross-partial derivative of equation (1) we obtain:

$$\begin{aligned}\mathbf{f}(\mathbf{x}) &\equiv \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n}\mathbf{F}(\mathbf{x}) \\ &= \prod_{i=1}^{n} f_i(x_i) \cdot \frac{\partial^n}{\partial u_1 \partial u_2 \cdots \partial u_n}\mathbf{C}(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \\ &\equiv \prod_{i=1}^{n} f_i(x_i) \cdot \mathbf{c}(F_1(x_1), F_2(x_2), ..., F_n(x_n)), \end{aligned} \tag{2}$$

and so the joint density is equal to the product of the marginal densities and the 'copula density', denoted $\mathbf{c}$. This of course also implies that the joint log-likelihood is simply the sum of univariate log-likelihoods and the 'copula log-likelihood', which is useful in the estimation of copula-based models:

$$\log \mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \log f_i(x_i) + \log \mathbf{c}(F_1(x_1), F_2(x_2), ..., F_n(x_n)). \tag{3}$$

The decomposition of a joint distribution into its marginal distributions and copula allows the researcher a great deal of flexibility in specifying a model for the joint distribution. This is clearly an advantage when the shape and goodness-of-fit of the model for the joint distribution is of primary interest. In situations where the researcher has accumulated knowledge about the distributions of the *individual* variables and wants to use that in constructing a joint distribution, copulas also have a valuable role. In other situations, for example when the researcher is primarily focussed on the conditional mean and/or conditional variance of a vector of variables, copulas may not be the 'right tool for the job', and more standard vector autoregressive models and/or multivariate GARCH models, see Silvennoinen and Teräsvirta (2008), may be more appropriate. For a lively discussion of the value of copulas in statistical modelling of dependence, see Mikosch (2006) and the associated discussion (in particular that of Embrechts, Joe, and Genest and Rémillard) and rejoinder.

To illustrate the potential of copulas for modelling financial time series, I show in Figure 1 some bivariate densities constructed using Sklar's theorem. All have $F_1 = F_2 = N(0,1)$, while I vary $\mathbf{C}$ across different parametric copulas,[3] constraining the linear correlation to be 0.5 in all cases. The upper left plot shows the familiar elliptical contours of the bivariate Normal density

[3] The Normal and Student's t copulas are extracted from bivariate Normal and Student's t distributions. The Clayton and Gumbel copulas are discussed in Nelsen (2006), equations 4.2.1 and 4.2.4 respectively. The symmetrised Joe-Clayton (SJC) copula was introduced in Patton (2006a) and is parameterised by the upper and lower tail dependence coeffficients, τ^U and τ^L. The mixed Normal copula is an equally-weighted mixture of two Normal copulas with parameters ρ_1 and ρ_2 respectively.

(with Normal marginals and a Normal copula), while the other plots show some of the flexibility that various copula models can provide. To quantify

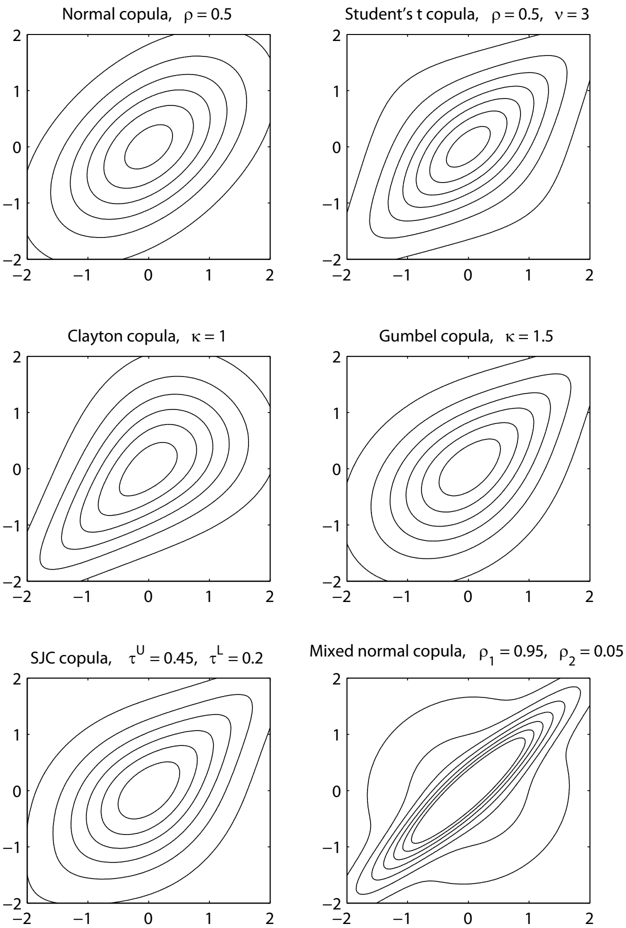

Fig. 1 Iso-probability contour plots for joint distributions with $N(0,1)$ marginal distributions and linear correlation coefficients of 0.5.

the different dependence structures provided by each copula, we can compare the dependence measures implied by each of these distributions, see Table 1. 'Quantile dependence' is related to a measure due to Coles, *et al.* (1999), and measures the probability of two variables both lying above or below a given quantile of their univariate distributions. It is defined as $\tau(q) = C(q,q)/q$ for $q \leq 1/2$ and $\tau(q) = \{1 - 2q + C(q,q)\}/(1-q)$ for $q > 1/2$. Lower and upper tail dependence can be defined as the limits of the quantile dependence measures: $\lim_{q \to 0} \tau(q) = \tau^L$ and $\lim_{q \to 1} \tau(q) = \tau^U$, if the limits exist, which they do for the six copulas presented here.

Figure 1 and Table 1 show the variety of joint densities that may be constructed using copulas, even when we impose that both margins are Normal and that the correlation coefficient is 0.5. In many financial applications differences in, for example, lower tail dependence will have important implications. For example, if two assets have the Student's t copula rather than the Normal copula, then the probability of both asset returns lying below their lower 5% quantile (i.e., their 5% Value-at-Risk, see Embrechts, *et al.* 2008, and Christoffersen, 2008) is 0.37 rather than 0.24, meaning that a portfolio of these two assets will exhibit more extreme returns than identical assets with a Normal copula.

Table 1: Measures of dependence for joint distributions with various copulas

				Tail Dependence		5% Quantile Dependence	
Copula	Parameter(s)		Linear Correlation	Upper	Lower	Upper	Lower
Normal(ρ)	0.5	–	0.50	0.00	0.00	0.24	0.24
Student's $t(\rho, \nu)$	0.5	3	$0.50^{\dagger}$	0.31	0.31	$0.37^{\dagger}$	$0.37^{\dagger}$
Clayton(κ)	1	–	$0.50^{\dagger}$	0.00	0.50	0.10	0.51
Gumbel(κ)	1.5	–	$0.50^{\dagger}$	0.41	0.00	0.44	0.17
SJC(τ^U, τ^L)	0.45	0.20	$0.50^{\dagger}$	0.45	0.20	0.46	0.27
Mixed Normal(ρ_1, ρ_2)	0.95	0.05	0.50	0.00	0.00	0.40	0.40

Figures marked with '$\dagger$' are based on simulations or numerical quadrature.

2 Copula–Based Models for Time Series

The application of copulas to time series modelling currently has two distinct branches. The first is the application to multivariate time series, where the focus is in modelling the joint distribution of some random vector, $\mathbf{X}_t = [X_{1t}, X_{2t}, ..., X_{nt}]'$, conditional on some information set $\mathcal{F}_{t-1}$. (The information set is usually $\mathcal{F}_{t-1} = \sigma(\mathbf{X}_{t-j}; j \geq 1)$, though this need not necessarily be the case.) This is an extension of some of the early applications of copulas in statistical modelling where the random vector of interest could be assumed to be independent and identically distributed (iid), see Clayton (1978) and Cook and Johnson (1981) for example. This application leads directly to the consideration of time-varying copulas.

The second application in time series is to consider the copula of a sequence of observations of a univariate time series, for example, to consider the joint distribution of $[X_t, X_{t+1}, ..., X_{t+n}]'$. This application leads us to consider Markov processes and general nonlinear time series models. We discuss each of these branches of time series applications of copulas below.

2.1 Copula–based models for multivariate time series

In this sub-section we consider the extension required to consider the conditional distribution of $\mathbf{X}_t$ given some information set $\mathcal{F}_{t-1}$. Patton (2006a) defined a "conditional copula" as a multivariate distribution of (possibly correlated) variables that are each distributed as *Uniform*$(0, 1)$ conditional on $\mathcal{F}_{t-1}$. With this definition, it is then possible to consider an extension of Sklar's theorem to the time series case:

$$\begin{aligned} &\mathbf{F}_t(\mathbf{x}|\mathcal{F}_{t-1}) \qquad (4)\\ &= \mathbf{C}_t(F_{1,t}(x_1|\mathcal{F}_{t-1}), F_{2,t}(x_2|\mathcal{F}_{t-1}), ..., F_{n,t}(x_n|\mathcal{F}_{t-1})|\mathcal{F}_{t-1}), \quad \forall\, \mathbf{x} \in \mathbb{R}^n , \end{aligned}$$

where $X_i|\mathcal{F}_{t-1} \sim F_{i,t}$ and $\mathbf{C}_t$ is the conditional copula of $\mathbf{X}_t$ given $\mathcal{F}_{t-1}$.

The key complication introduced when applying Sklar's theorem to conditional distributions is that the conditioning set, $\mathcal{F}_{t-1}$, must be the same for all marginal distributions and the copula. Fermanian and Wegkamp (2004) and Fermanian and Scaillet (2005) consider the implications of a failure to use the same information set, and define a 'conditional *pseudo* copula' to help study this case[4]. Failure to use the same information set for all components on the right-hand side of equation (4) will generally imply that the function on the left-hand side of equation (4) is *not* a valid conditional joint distribution function. See Patton (2006a) for an example of this failure.

It is often the case in financial applications, however, that some of the information contained in $\mathcal{F}_{t-1}$ is not relevant for all variables. For example, it might be that each variable depends on its own first lag, but not on the lags of any other variable. Define $\mathcal{F}_{i,t-1}$ as the smallest subset of $\mathcal{F}_{t-1}$ such that $X_{it}|\mathcal{F}_{i,t-1} \overset{\mathcal{D}}{=} X_{it}|\mathcal{F}_{t-1}$. With this it is possible to construct each marginal distribution model using only $\mathcal{F}_{i,t-1}$, which will likely differ across margins, and then use $\mathcal{F}_{t-1}$ for the copula, to obtain a valid conditional joint distribution. However, it must be stressed that in general the same information set must be used across all marginal distribution models and the copula model,

[4] The 'pseudo-copula' of Fermanian and Wegkamp (2004) is not to be confused with the 'quasi-copula' of Alsina, *et al.* (1993) and Genest, *et al.* (1999), which is used to characterize operations on distribution functions that cannot correspond to an operation on random variables.

before possibly reducing each of these models by eliminating variables that are not significant/important[5].

The consideration of conditional copulas leads naturally to the question of whether these exhibit significant changes through time. Conditional correlations between financial asset returns are known to fluctuate through time, see Andersen, *et al.* (2006) and Bauwens *et al.* (2006) for example, and so it is important to also allow for time-varying conditional copulas. Patton (2002, 2006a) allows for time variation in the conditional copula by allowing the parameter(s) of a given copula to vary through time in a manner analogous to a GARCH model for conditional variance (Engle (1982) and Bollerslev (1986)). Jondeau and Rockinger (2006) employ a similar strategy. Rodriguez (2007), on the other hand, considers a regime switching model for conditional copulas, in the spirit of Hamilton (1989). Chollete (2005), Garcia and Tsafack (2007), and Okimoto (2006) employ a similar modelling approach, with the latter author finding that the copula of equity returns during the low mean-high variance state is significantly asymmetric (with non-zero lower tail dependence) while the high mean-low volatility state has a more symmetric copula. Panchenko (2005b) considers a semi-parametric copula-based model of up to five assets, building on Chen and Fan (2006b), discussed below, where the marginal distributions are estimated nonparametrically and the conditional copula is specified to be Normal, with a correlation matrix that evolves according to the DCC specification of Engle (2002). Lee and Long (2005) combine copulas with multivariate GARCH models in an innovative way: they use copulas to construct flexible distributions for the residuals from a multivariate GARCH model, employing the GARCH model to capture the time-varying correlation, and the copula to capture any dependence remaining between the conditionally uncorrelated standardised residuals.

It is worth noting that, for some of the more complicated models above, it can be difficult to establish sufficient conditions for stationarity, which is generally required for standard estimation methods to apply, as discussed in Section 2.3 below. Results for general classes of *univariate* nonlinear processes are presented in Carrasco and Chen (2002) and Meitz and Saikkonen (2004), however similar results for the multivariate case are not yet available.

2.2 Copula–based models for univariate time series

In addition to describing the cross-sectional dependence between two or more time series, copulas can also be used to describe the dependence between observations from a given univariate time series, for example, by captur-

[5] For example, in Patton (2006a) I study the conditional joint distribution of the returns on the Deutsche mark/U.S. dollar and Japanese Yen/U.S. dollar exchange rates. In that application Granger-causality tests indicated that the marginal distributions depended only on lags of the "own" variable; lags of other variables were not significant.

ing the dependence between $[X_t, X_{t+1}, ..., X_{t+n}]'$. If the copula is invariant through time and satisfies a constraint on its multivariate marginals[6], and the marginal distributions are identical and also invariant through time, then this describes a stationary Markov process. The main benefit of this approach to univariate time series modelling is that the researcher is able to specify the unconditional (marginal) distribution of X_t separately from the time series dependence of X_t. For example, the six joint distributions plotted in Figure 1 could be used to generate a stationary first-order Markov process, with the marginal distribution of X_t being $N(0,1)$, and with various copulas describing the dependence between X_t and X_{t+1}. In Figure 2 I plot the conditional mean of X_{t+1} given $X_t = x$, along with the conditional mean ± 1.65 times the conditional standard deviation of X_{t+1} given $X_t = x$, for each of the six distributions from Figure 1. In the upper left panel is the familiar case of joint normality: a linear conditional mean and constant conditional variance. The other five panels generally display non-linear conditional mean and variance functions. In Figure 3 I plot the density of X_{t+1} conditional on $X_t = -2,\ 0$, and 2. Now in the upper left panel we see the familiar figure of Normal conditional densities, while in the other panels the conditional densities are non-Normal. Amongst other things, the figures for the Student's t and mixed Normal copulas emphasise that radial symmetry of the joint distribution (i.e., symmetry around both the main diagonal and the off-diagonal) is not sufficient for symmetry of the conditional marginal densities.

Darsow, *et al.* (1992) study first-order Markov processes based on copulas. They provide a condition equivalent to the Chapman-Kolmogorov equations for a stochastic process that focusses solely on the copulas of the variables in the process. Furthermore, the authors are able to provide a necessary and sufficient condition for a stochastic process to be Markov by placing conditions on the multivariate copulas of variables in the process (in contrast with the Chapman-Kolmogorov equations which are necessary but not sufficient conditions). Ibragimov (2005, 2006) extends the work of Darsow, *et al.* (1992) to higher-order Markov chains and provides several useful results, and a new class of copulas. Beare (2007) studies the weak dependence properties of Markov chains through the properties of their copulas and, amongst other things, shows that tail dependence in the copula may result in the Markov chain not satisfying standard mixing conditions. Gagliardini and Gouriéroux (2007b) propose and study copula-based time series models for durations, generalising the autoregressive conditional duration model of Engle and Russell (1998).

[6] For example, if $n = 3$, then it is required that the marginal joint distribution of the first and second arguments is identical to that of the second and third arguments. Similar conditions are required for $n > 3$.

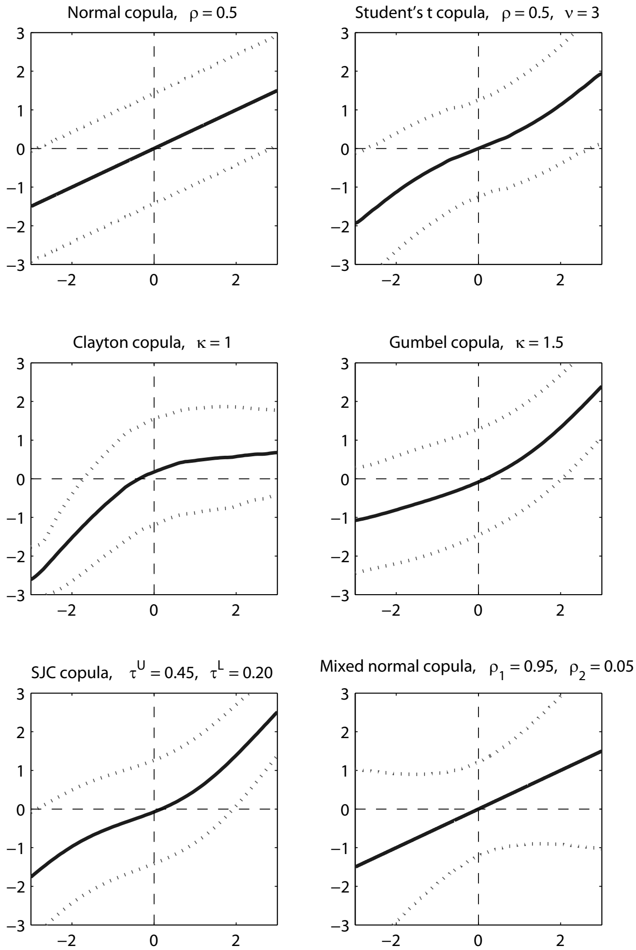

Fig. 2 Conditional mean ± 1.64 times the conditional standard deviation, for joint distributions with $N(0,1)$ marginal distributions and linear correlation coefficients of 0.5.

2.3 Estimation and evaluation of copula–based models for time series

The estimation of copula-based models for multivariate time series can be done in a variety of ways. For fully parametric models (the conditional marginal distributions and the conditional copula are all assumed known up to a finite-dimensional parameter) maximum likelihood (ML) is the ob-

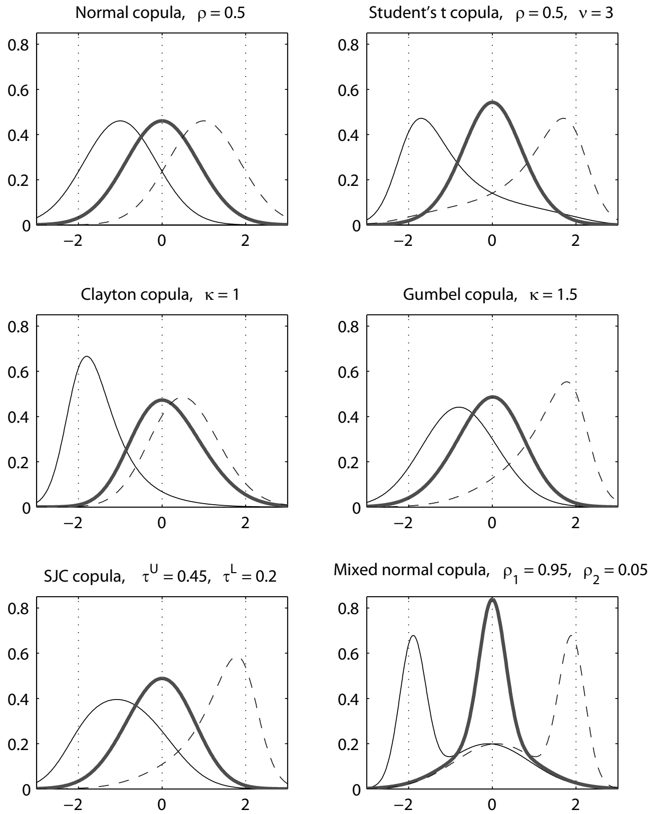

Fig. 3 Conditional densities of Y given $X = -2$ (thin line), $X = 0$ (thick line), $X = +2$ (dashed line), for joint distributions with $N(0, 1)$ marginal distributions and linear correlation coefficients of 0.5.

vious approach. If the model is such that the parameters of the marginal distributions can be separated from each other and from those of the copula, then multi-stage ML estimation is an option. This method, sometimes known as the "inference functions for margins" (IFM) method, see Joe and Xu (1996) and Joe (1997, Chapter 10), involves estimating the parameters of the marginal distributions via univariate ML, and then estimating the parameter of the copula conditional on the estimated parameters for the marginal distributions. This estimation method has the benefit of being computationally tractable, at the cost of a loss of full efficiency. The theory for this estimation method for *iid* data is presented in Shih and Louis (1995) and Joe and Xu (1996). Patton (2006b) presents the theory for time series data, drawing on the work on Newey and McFadden (1994) and White (1994), and some simulation results that motivate multi-stage estimation.

Fully nonparametric estimation methods for copula models in the *iid* case were studied by Genest and Rivest (1993) and Capéraà, *et al.,* (1997), amongst others. Fully nonparametric estimation of copulas for time series data was studied by Fermanian and Scaillet (2003) and Ibragimov (2005).

An important benefit of using copulas to construct multivariate models is that the models used in the marginal distributions need not be of the same type as the model used for the copula. One exciting possibility that this allows is non- or semi-parametric estimation of the marginal distributions, combined with parametric estimation of the copula. Such a model avoids the 'curse of dimensionality' by only estimating the one-dimensional marginal distributions nonparametrically, and then estimating the (multi-dimensional) copula parametrically. The theory for this estimator in the *iid* case is presented in Genest, *et al.* (1995) and Shih and Louis (1995). Theory for the time series case is presented in Chen and Fan (2006b) and Chen, *et al.* (2006). Chen and Fan (2006b) also consider the important case that the copula model may be mis-specified. Gagliardini and Gouriéroux (2007a) consider copula specifications that are semi-parametric, while Sancetta and Satchell (2004) consider semi-nonparametric copula models.

The estimation of fully parametric copula-based univariate time series models is discussed in Joe (1997, Chapter 8). Chen and Fan (2006a) consider the estimation of semi-parametric copula-based univariate time series models, where the unconditional distribution is estimated nonparametrically and the copula is estimated parametrically. The work of Ibragimov (2006) and Beare (2007) on conditions for some form of mixing to hold are also relevant here.

The evaluation of a given model is important in any econometric application, and copula-based modelling is of course no exception. The evaluation of copula-based models takes two broad approaches. The first approach evaluates the copula-based multivariate density model in its entirety, and thus requires methods for evaluating multivariate density models, see Diebold, *et al.* (1999) and Corradi and Swanson (2005). In the second approach one seeks to evaluate solely the copula model, treating the marginal distribution models as nuisance parameters. Fermanian (2005) and Scaillet (2007) consider such an approach for models based on *iid* data, while Malevergne and Sornette (2003) and Panchenko (2005a) consider tests for time series models. Genest *et al.* (2007) provide an extensive review of goodness-of-fit tests for copulas, focussing on the *iid* case, and present the results of a simulation study of the size and power of several tests.

Comparisons between a set of competing copula-based models can be done either via economic criteria, such in some of the papers reviewed in the next section, or statistical criteria. For the latter, likelihood ratio tests (either nested or, more commonly, non-nested, see Vuong (1989) and Rivers and Vuong (2002) for example) can often be used. Alternatively, information criteria, such as the Akaike or Schwarz's Bayesian Information Criteria (AIC, BIC) can be used to penalise models with more parameters.

3 Applications of Copulas in Finance and Economics

The primary motivation for the use of copulas in finance comes from the growing body of empirical evidence that the dependence between many important asset returns is non-normal. One prominent example of non-normal dependence is where two asset returns exhibit greater correlation during market downturns than during market upturns. Evidence against the univariate normality of asset returns has a long history, starting with Mills (1927), but evidence against 'copula normality' has accumulated only more recently. Erb, *et al.* (1994), Longin and Solnik (2001) and Ang and Chen (2002), Ang and Bekaert (2002), Bae, *et al.* (2003) all document, without drawing on copula theory, evidence that asset returns exhibit non-normal dependence, that is, dependence that is not consistent with a Normal copula. This evidence has wide-ranging implications for financial decision-making, in risk management, multivariate option pricing, portfolio decisions, credit risk, and studies of 'contagion' between financial markets. In the remainder of this section I discuss some of the research done in these areas.

The first area of application of copulas in finance was risk management. Just as 'fat tails' or excess kurtosis in the distribution of a single random variable increases the likelihood of extreme events, the presence of non-zero tail dependence increases the likelihood of *joint* extreme events. As illustrated in Table 1, even copulas that are constrained to generate the same degree of linear correlation can exhibit very different dependence in or near the tails. The focus of risk managers on Value-at-Risk (VaR), and other measures designed to estimate the probability of 'large' losses, makes the presence of non-normal dependence of great potential concern. Cherubini and Luciano (2001), Embrechts, *et al.* (2003) and Embrechts and Höing (2006) study the VaR of portfolios using copula methods. Hull and White (1998) is an early paper on VaR for collections of non-normal variables. Rosenberg and Schuermann (2006) use copulas to consider 'integrated' risk management problems, where market, credit and operational risks must be considered jointly. McNeil, *et al.* (2005) and Alexander (2008) provide clear and detailed textbook treatments of copulas and risk management.

In derivatives markets non-normal dependence has key pricing, and therefore trading, implications. Any contract with two or more 'underlying' assets will generally have a price that is affected by both the strength and the shape of the dependence between the assets. A simple such contract is one that pays £1 if all underlying assets have prices above a certain threshold on the contract maturity date; another common contract is one that has a pay-off based on the minimum (or the maximum) of the prices of the underlying assets on the contract maturity date. Even derivatives with just a single underlying asset may require copula methods if the risk of default by the counter-party to the contract is considered economically significant: these are so-called “vulnerable options”. A recent book by Cherubini, *et al.* (2004) considers derivative pricing using copulas in great detail, and they provide an interesting introduction to copulas based on option pricing, as an alternative to the more standard statistical introductions in Joe (1997) and Nelsen (2006) for example. Other papers that consider option pricing with copulas include Rosenberg (2003), Bennett and Kennedy (2004), van den Goorbergh, *et al.* (2005) and Salmon and Schleicher (2006). Other authors, see Taylor and Wang (2004) and Hurd, *et al.* (2005), have instead used observed derivatives prices to find the implied copula of the underlying assets.

The booming market in credit derivatives (credit default swaps and collateralised debt obligations, for example) and the fact that these assets routinely involve multiple underlying sources of risks has lead to great interest in copulas for credit risk applications. An early contribution is from Li (2000), who was first to use copulas in a credit risk application, and was more generally one of the first to apply copulas in finance. See also Frey and McNeil (2001), Schönbucher and Schubert (2001) and Giesecke (2004) for applications to default risk. Duffie (2004) argues that copulas are too restrictive for certain credit risk applications.

One of the most obvious places where the dependence between risky assets impacts on financial decisions, and indeed was the example used at the start of this survey, is in portfolio decisions. Under quadratic utility and/or multivariate Normality (or more generally, multivariate ellipticality, see Chamberlain, 1983) the optimal portfolio weights depend only upon the first two moments of the assets under consideration, and so linear correlation adequately summarises the necessary dependence information required for an optimal portfolio decision. However when the joint distribution of asset returns is not elliptical, as the empirical literature cited above suggests, and when utility is not quadratic in wealth, the optimal portfolio weights will generally require a specification of the entire conditional distribution of returns. Patton (2004) considers a bivariate equity portfolio problem using copulas, and Garcia and Tsafack (2007) consider portfolio decisions involving four assets: stocks and bonds in two countries. The extension to consider portfolio decisions with larger numbers of assets remains an open problem.

The final broad topic that has received attention from finance researchers using copula methods is the study of financial ‘contagion’. Financial contagion is a phenomenon whereby crises, somehow defined, that occur in one market lead to problems in other markets *beyond* what would be expected on the basis of fundamental linkages between the markets. The Asian crisis of 1997 is one widely-cited example of possible contagion. The difficulty in contagion research is that a baseline level of dependence between the markets must be established before it can be asserted that the dependence increased during a period of crisis. The heavy focus on levels and changes in dependence has lead several researchers to apply copula methods in their study of contagion. Rodriguez (2007) was the first to apply copulas to contagion, which he studies with a Markov switching copula model. See Chollete, *et al.* (2005) and Arakelian and Dellaportas (2005) for alternative approaches.

Finally, there are a number of interesting papers using copulas in applications that do not fit into the broad categories discussed above. Bouyé and Salmon (2002) use copulas for quantile regressions, Breymann, *et al.* (2003) study the copulas of financial assets using intra-daily data, sampled at different frequencies, Daul, *et al.* (2003) and Demarta and McNeil (2005) study the Student's t copula and some useful extensions, Heinen and Rengifo (2003) use copulas to model multivariate time series of counts, Smith (2003) uses copulas to model sample selection, related to earlier work touching on copulas for this problem by Lee (1983), Bonhomme and Robin (2004) use copulas to model a large panel of earnings data, Bartram, *et al.* (2006) use a time-varying conditional copula model to study financial market integration between seventeen European stock market indices, Granger, *et al.* (2006) use copulas to provide a definition of a 'common factor in distribution', Hu (2006) uses mixtures of copulas to separate the degree of dependence from the 'shape' of dependence, and Brendstrup and Paarsch (2007) use copulas in a semiparametric study of auctions.

4 Conclusions and Areas for Future Research

In this survey I have briefly discussed some of the extensions of standard copula theory that are required for their application to time series modelling, and reviewed the existing literature on copula-based models of financial time series. This is a fast-growing field and the list of references will no doubt need updating in the near future.

In reviewing the extant literature on copulas for finance a number of topics stand out as possible avenues for future research. The most obvious, and perhaps difficult, is the extension of copula-based multivariate time series models to high dimensions. Existing models are not well-designed for higher-dimension applications; what is needed is a flexible yet parsimonious way of characterising high dimension copulas. A similar problem was faced in the multivariate ARCH literature in the mid-1990s, see Bauwens, *et al.* (2006). Two popular approaches to solve that problem are factor-based ARCH models and extensions, see Alexander and Chibumba (1998) and van der Weide (2002) for example, and the DCC model of Engle (2002) and its extensions, see Cappiello, *et al.* (2006) for example. Perhaps similar approaches will prove fruitful in high-dimensional copula modelling.

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