Systems and Signals 414 Practical 2: Using the DFT

Aim: Understand the use of the DFT in discrete-time signal analysis.

Hand in: Please hand in this notebook as a PDF file on sunlearn by Sunday, 4 March at 23:55. To save your notebook to a PDF, you can print the notebook from your browser and then choose to `Save as PDF`. (If you are doing the practical on a machine with LaTeX, you can also select `File → Download as → PDF via LaTeX (.pdf)` directly in the notebook). After exporting your notebook, upload the PDF by clicking on `Practical 2 submission` on sunlearn and following the steps. You may submit your work multiple times; only the last submission will be marked. **No late submissions will be accepted.**

Task: Do the following assignment using Jupyter. Document the task indicating your methodology, theoretical results, numerical results and discussions as necessary. Your graphs should have labeled axes with the correct units indicated. If you get stuck with a Numpy or Scipy function, go look up the usage at https://docs.scipy.org (https://docs.scipy.org). Also take a look at the provided coding examples.

Preamble code and helper functions:

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In [1]:
```
#All the necessary imports

#%matplotlib notebook
%matplotlib inline

import pylab as pl
import numpy as np
from scipy import signal
import IPython.display

#Nicer matplotlib dimensions
pl.rcParams['figure.figsize'] = (9,3)

#A helper-function to setup a proper plot
def setup_plot(title, y_label='', x_label='', newfig=True):
    if newfig:
        pl.figure()
    pl.margins(*(pl.array(pl.margins())+0.05))
    pl.title(title)
    pl.ylabel(y_label)
    pl.xlabel(x_label)

#A helper function to plot an additional x-axis at the top
def add_top_axis(x_values, x_label=''):
    _, b = pl.rcParams['figure.figsize']
    pl.gca().title.set_position([.5, 1.18])
    ax1 = pl.gca()
    ax2 = ax1.twiny()
    ax2.margins(*(pl.array(ax1.margins())))
    ln, = ax2.plot(x_values, np.zeros_like(x_values)) # Cre
ate a dummy plot
    ln.remove()
    ax2.set_xlabel(x_label)

#Download yesterday.wav from courses.ee.sun.ac.za and retur
n it as a numpy array
def download_and_load_audio(url, mono=True, factor_of_2_len
gth=True):
    import os
    import urllib
    import scipy.io
    from scipy.io import wavfile

    filename = os.path.split(url)[-1]
    #Download if path does not already exist
    if not os.path.isfile(filename):
        urllib.request.urlretrieve(url, filename)
    sample_frequency, signal_array = wavfile.read(filename)
    #Normalise signal and return
    if mono and len(signal_array.shape)==2:
        signal_array = np.sum(signal_array, axis=1)
    signal_array = signal_array/np.max([np.max(signal_array
), -np.min(signal_array)])
    if factor_of_2_length:
        signal_array = signal_array[:2**np.floor(np.log2(le
```

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Questions

The audio for today is the solo from Baker Street (http://www.youtube.com/watch?v=Fo6aKnRnBxM), with Rick Sanchez as backup singer.

Use proper Markdown headings (by changing the cell type) to differentiate between Question 1, Question 2, etc. and label your graphs properly.

Question 1

Download the audio file `bakerickstreet_noisy.wav` from sunlearn. Then use the provided function, `download_and_load_audio` to read the audio waveform file into a NumPy array. We consider this array as the discrete-time signal $x[n]$.

Load and listen to the signal using `IPython.display.Audio`. Note the high-pitch sinusoidal noise contained in the signal.

1.1) Find two frequency f_1 and f_2 (in Hz) for the two interfering sinusoidal signals in $x[n]$. Do so by first calculating $X[k]$ (using `np.fft.fft`), then plotting $|X[k]|$ against the f_ω axis, and finally reading the noisy f_ω responses off the graph.
Note: Audio in general is non-stationary, but here we are looking for sinusoids (which are stationary), so we can just take the DFT of the entire signal.
NB: To generate interactive graphs with zoom-in functionality for better interpretation; replace `%matplotlib inline` in the preample block with `%matplotlib notebook`, restart the kernel, and rerun the necessary blocks. Afterwards, change the plotting settings back to `%matplotlib inline` in a likewise manner before handing in the exported PDF.

1.2) Consider the noise contained in signal $x[n]$ as the signal $q[n]$ such that

$$q(t) = \sin(f_1 \cdot 2\pi t) + \sin(f_2 \cdot 2\pi t)$$

$$q[n] = q(t) \text{ sampled at } F_s = 10\text{kHz}.$$

Now generate 5 seconds worth of $q[n]$ (derive your timing info from F_s) and play it using `IPython.display.Audio`. Does it sound similar to the noise found in signal $x[n]$?

1.3) Plot the magnitude response $|Q[k]|$ against the f_ω axis. Are there a similarity between $Q[k]$ and $X[k]$?

1.4) Generate the signal $y[n]$ by filtering out the the interfering sinusoidal noise from $x[n]$. Do so by applying the following difference equation to $x[n]$ (same procedure as Practical 0):

$$\begin{aligned} y[i] = (&x[i] - 1.18x[i-1] + 2.00x[i-2] - 1.18x[i-3] \quad + x[i-4] \\ &+1.13y[i-1] - 1.84y[i-2] + 1.04y[i-3] - 0.85y[i-4]) \end{aligned}$$

For any out-of-bound access to indices (that is for $i < 0$ and for $i >= N$), assume an array value of 0. Please keep $y[n]$ the same length as $x[n]$, since the length was specifically chosen to be a factor of 2 for FFT optimisation.
Helpful hint and tip: In Python, when indexing an array/list with a negative number, such as `arr[-a]` for $a = 1, 2, \ldots$, it returns the value at index `N-1-i`. Therefore, to avoid boundary checking when applying the difference equation, we can append 4 zeros to $x[n]$ (using numpy: x

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Question 2:

For the signal $x(t)$ and $x[n]$, we have

$x(t) = \sin(4000 \cdot 2\pi t)$, and

$x[n] = x(t)$ sampled at $F_s = 12$kHz for 5 periods of $x(t)$.

2.1) Sketch (by hand) the magnitude spectrum $|X(f)|$ of $x(t)$. Then sketch (by hand) the magnitude spectrum $|X[k]|$ of $x[n]$, assuming we take the DFT window over all 5 periods.

2.2) Stem the signal $x[n]$ against the sample time axis and stem the magnitude spectra $|X[k]|$ against the sample frequency axis.

2.3) Generate $\tilde{x}[n]$ as a zero-padded $x[n]$ of 10 times the original length, and stem $|\tilde{X}[k]|$ against the k axis.

2.4) Compose $\hat{X}[k]$ as the DFT of $x[n]$, but zero-padded to 10 times the original size by adding zeros to the center of the $X[k]$ array. Determine and stem the IDFT $\hat{x}[n]$ and the original $x[n]$ as two different plots.
Remember: Where should the zeros go when zero-padding $X[k]$?

Question 3:

For the signal $p(t)$ and $p[n]$, we have

$p(t) = \cos(900 \cdot 2\pi t) + 0.15\cos(800 \cdot 2\pi t).$, and

$p[n] = p(t)$ sampled at $F_s = 2$kHz for 50 samples.

3.1) Stem the 50-point DFT magnitude spectrum $|P[k]|$ of $p[n]$ against the f_ω axis.

3.2) Estimate the frequencies present in $p[n]$ from the plot in 3.1. Why is it difficult?

3.3) Generate $\tilde{p}[n]$ as a zero-padded $p[n]$ of 100 times the original length, and plot (not stem) $|\tilde{P}[k]|$ against the f_ω axis.

3.4) Generate $\hat{p}[n]$ by applying a Hamming-window to $p[n]$ and zero-padding the signal to be of 100 times the original length. Plot (not stem) $|\hat{P}[k]|$ against the f_ω axis.

3.5) Comment on the differences between 3.2, 3.3, and 3.4 with regard to the ease of determining the frequencies.

Bonus Question

What is the the sequel to *Roy: A life well lived*?

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Additional Question

Although you do not need to hand in this question, testweek is coming, so keep fit!

A.1) Consider the following sequence $X[k] = \{0, 1, 1\}$, what is the solution for $x[n]$, the IDFT of $X[k]$, as a discrete-time sinusoid.

A.2) Obtain $\tilde{X}[k]$ by zero-padding $X[k]$ appropriately to be 100 samples of length. Where would you add the zeros? What would be the solution for $\tilde{x}[n]$, the IDFT of $\tilde{X}[k]$, as a discrete-time sinusoid?

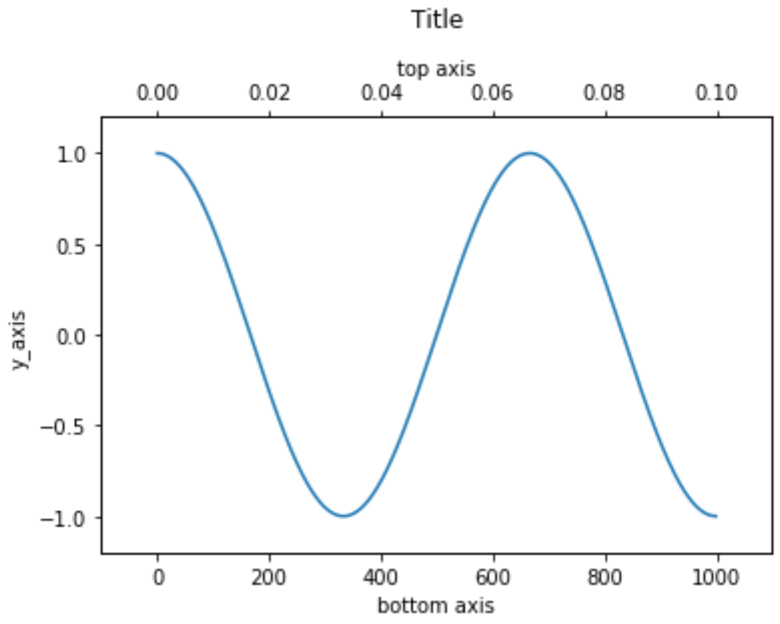
A.3) Generate the signals in 4.1 and 4.2 in Python; stem $\tilde{X}[k]$ and plot $\tilde{x}[n]$.

Coding examples

Plotting an additional axis

In [2]:
```
#Normal plot with matplotlib plot and setup_plot helper
setup_plot('Title', 'y_axis', 'bottom axis')
pl.plot(np.arange(1000), np.cos(np.linspace(0, 3*np.pi, 1000)));

#Add an extra x-axis at the top with 1e-4 scale
add_top_axis(np.arange(1000)*1e-4, 'top axis')
```



Note the difference between linspace and arange:

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In [3]:
```
#Every second element from 0 upto (but excluding) 10
print('a)', np.arange(0,10,1))

#5 elements from 0 upto (but excluding) 10
print('b)', np.linspace(0,10,10,False))

#5 elements from 0 upto 10 (avoid!!!)
print('c)', np.linspace(0,10,10))
```

```
a) [0 1 2 3 4 5 6 7 8 9]
b) [0. 1. 2. 3. 4. 5. 6. 7. 8. 9.]
c) [ 0.          1.11111111  2.22222222  3.33333333  4.4444
4444  5.55555556
  6.66666667  7.77777778  8.88888889 10.        ]
```

Audio player with Ipython.lib.display.Audio

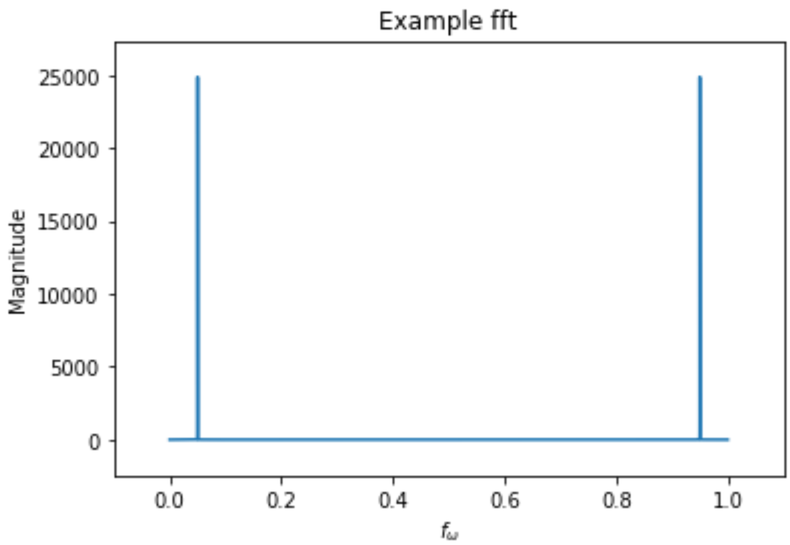
In [4]:
```
example_signal = np.sin(2*np.pi*500*np.linspace(0,5,50000))
IPython.lib.display.Audio(rate=10000,
                          data=example_signal)
```

Out[4]:

Plotting the spectrum of a signal

In [5]:
```
#Plotting the spectrum of a signal
example_fft = np.fft.fft(example_signal)
example_fw_axis = np.linspace(0, 1, len(example_fft), False
)

setup_plot('Example fft', 'Magnitude', '$f_\omega$')
pl.plot(example_fw_axis, np.abs(example_fft));
```



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Answer space:

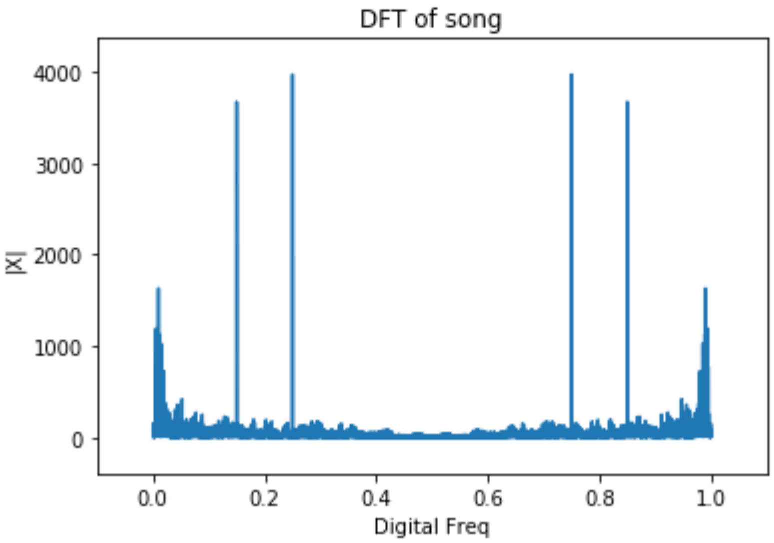
Question 1

In [6]:
```
rate, adata = download_and_load_audio("bakerickstreet_noisy
.wav")
IPython.lib.display.Audio(rate=rate, data=adata)
```

Out[6]:

In [7]:
```
#1.1
X = np.fft.fft(adata)
fw = np.linspace(0, 1, X.size, False)
setup_plot('DFT of song', '|X|', 'Digital Freq')
pl.plot(fw, np.abs(X))
```

Out[7]: [<matplotlib.lines.Line2D at 0x7f9250247ac8>]



f1 = 0.15 and f2 = 0.25

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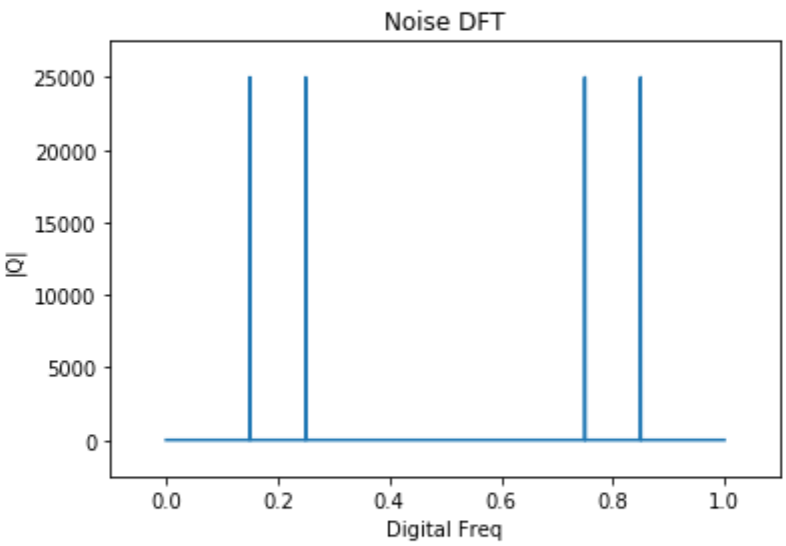
In [8]:
```
#1.2
f1 = 0.15 * rate
f2 = 0.25 * rate
fs = 10000

time = np.linspace(0, 5, 5*fs, False)
q = np.sin(2 * np.pi * time * f1) + np.sin(2 * np.pi * time
* f2)

IPython.lib.display.Audio(rate=fs, data=q)
```

Out[8]:

In [9]:
```
#1.3
Q = np.fft.fft(q)
fw = np.linspace(0, 1, Q.size, False)

setup_plot('Noise DFT', '|Q|', 'Digital Freq')
pl.plot(fw, np.abs(Q))
```

Out[9]: [<matplotlib.lines.Line2D at 0x7f9250247710>]



The impulses are the same as in X just without the other peaks in the signal

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In [10]:

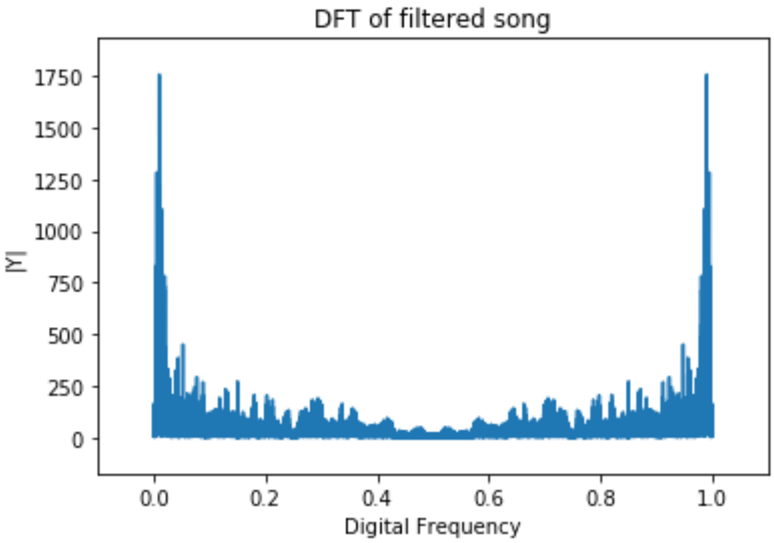
```
#1.4
x = np.array(adata)
y = np.append(x, np.zeros(4))

for i in range(adata.size):
    y[i] += -1.18*x[i-1] +2*x[i-2] -1.18*x[i-3] + x[i-4] +
1.13*y[i-1] -1.84*y[i-2] +1.04*y[i-3] -0.85*y[i-4]

y = y[:-4]
fwy = np.linspace(0, 1, y.size, False)

setup_plot('DFT of filtered song', '|Y|', 'Digital Frequenc
y' )
pl.plot(fwy, np.abs(np.fft.fft(y)))
IPython.lib.display.Audio(rate = rate, data = y)
```

Out[10]:

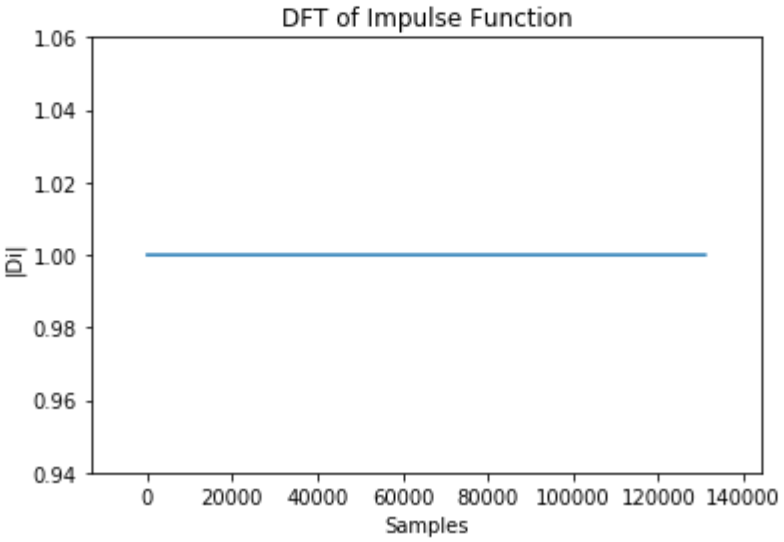


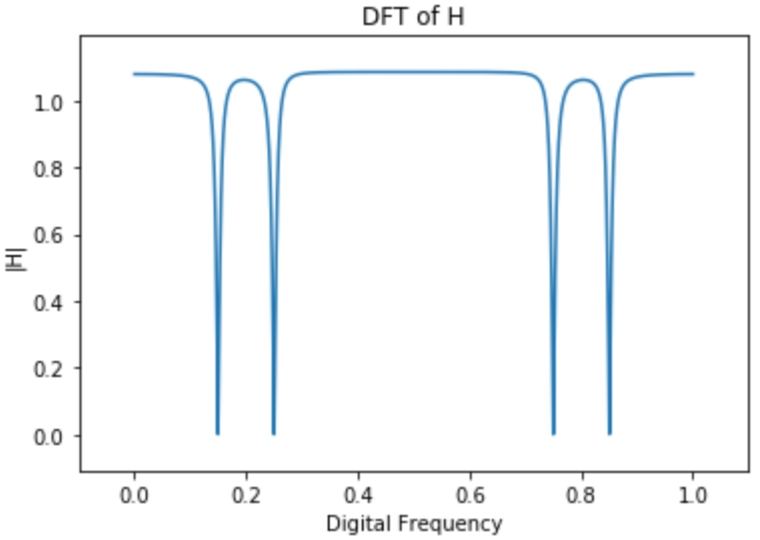
In [11]:
```
#1.5
di = np.append(np.ones(1), np.zeros(adata.size +3))
Di = np.abs(np.fft.fft(di))
setup_plot('DFT of Impulse Function', '|Di|', 'Samples')
pl.plot(Di);

h = np.array(di)
for i in range(adata.size):
     h[i] += -1.18*di[i-1] +2*di[i-2] -1.18*di[i-3] + di[i-
4] + 1.13*h[i-1] -1.84*h[i-2] +1.04*h[i-3] -0.85*h[i-4]

h = h[:-4]
fwh = np.linspace(0, 1, h.size, False)
setup_plot('DFT of H', '|H|', 'Digital Frequency' )
pl.plot(fwh, np.abs(np.fft.fft(h)))
```

Out[11]: [<matplotlib.lines.Line2D at 0x7f92505c6e80>]





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Question 2

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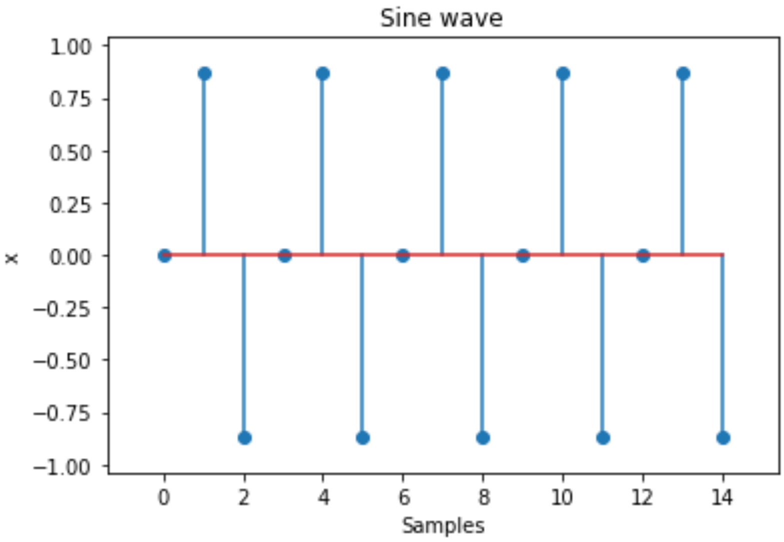
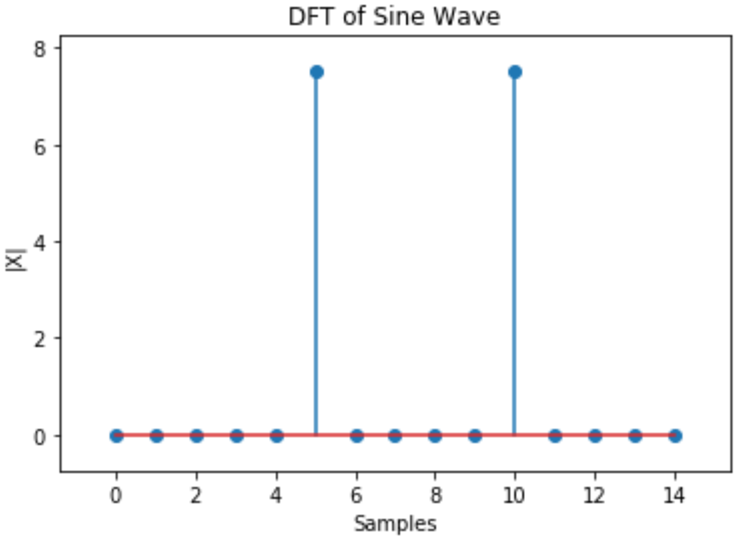
In [12]:
```
#2.1 - Answered on paper
#2.2
f = 4000
fs = 12000
samples = 5

t = np.linspace(0, samples/f, int(samples*fs/f), False)
x = np.sin(f * 2 * np.pi* t)
X = np.abs(np.fft.fft(x))

setup_plot('Sine wave', 'x', 'Samples')
pl.stem(x)

setup_plot('DFT of Sine Wave', '|X|', 'Samples')
pl.stem(X)
```

Out[12]: <Container object of 3 artists>





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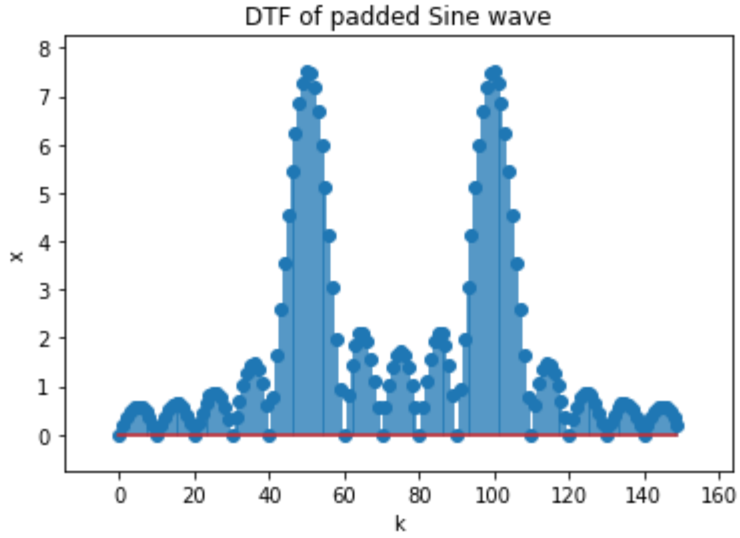
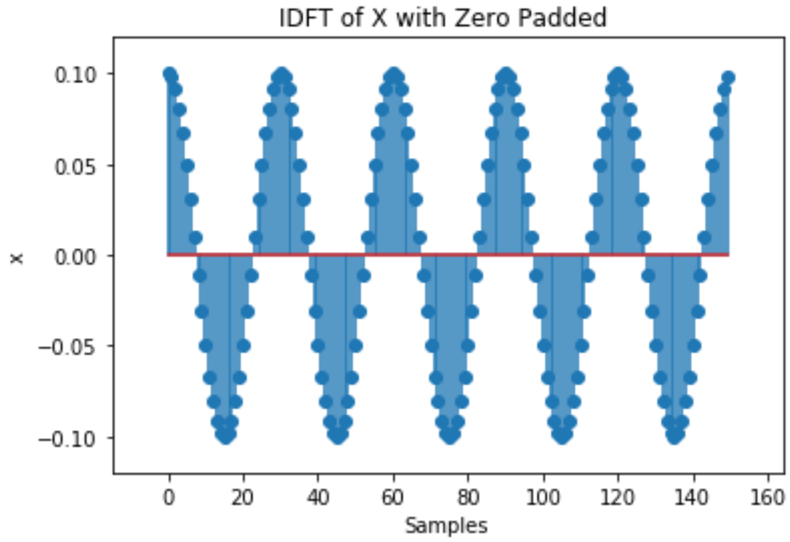
In [13]:
```
#2.3
t2 = np.append(t, np.zeros(int(9*samples*fs/f)))
x2 = np.sin(f * 2 * np.pi * t2)
X2 = np.abs(np.fft.fft(x2))
setup_plot('DTF of padded Sine wave', 'x', 'k')
pl.stem(X2)
```

Out[13]: <Container object of 3 artists>



In [14]:
```
#2.4
p1 = int(X.size/2)
p2 = X[p1:]
X = np.append(X[:p1], np.zeros(X.size * 9))
X = np.append(X, p2)

xt = np.real(np.fft.ifft(X))
setup_plot('IDFT of X with Zero Padded ', 'x', 'Samples')
pl.stem(xt);
```



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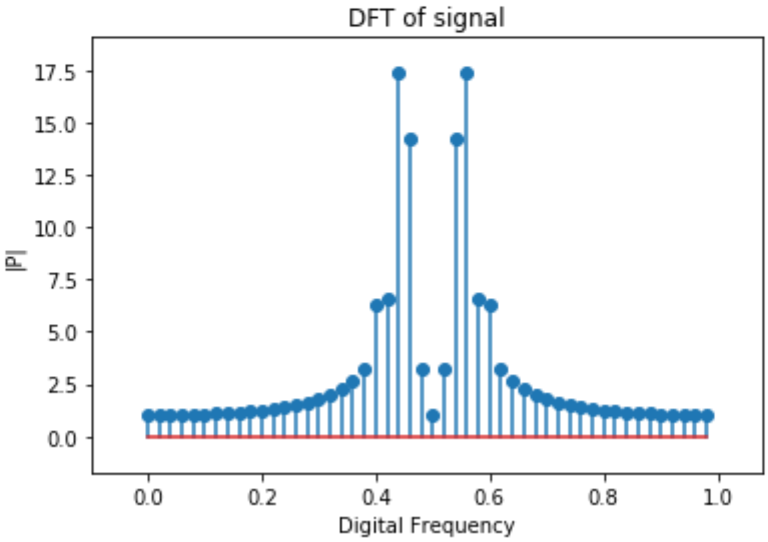
Question 3

In [15]:
```
#3.1
fs = 2000
samples = 50
fw = np.linspace(0, 1, samples, False)
t = np.linspace(0, samples/fs, samples, False)
p = np.cos(1800 * np.pi * t) + 0.15*np.cos(1600* np.pi * t)
P = np.abs(np.fft.fft(p))

setup_plot('DFT of signal', '|P|', 'Digital Frequency')
pl.stem(fw, P)
```

Out[15]: <Container object of 3 artists>



3.2 - The low sampling resoltion makes it dificult to get an exact value, however 0.45 and 0.55 would be approximate.

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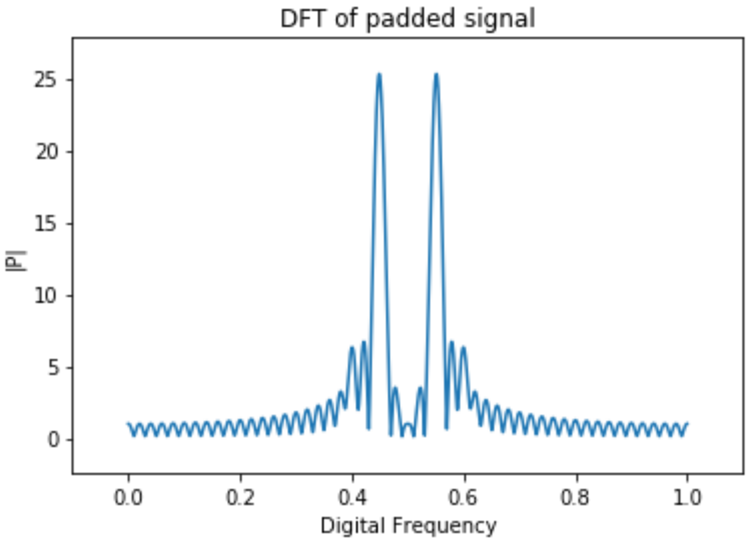
In [17]:
```
#3.3
p2 = np.append(p, np.zeros(99*samples))
P2 = np.abs(np.fft.fft(p2))
fw2 = np.linspace(0, 1, 100*samples, False)

setup_plot('DFT of padded signal', '|P|', 'Digital Frequenc
y')
pl.plot(fw2, P2)
```

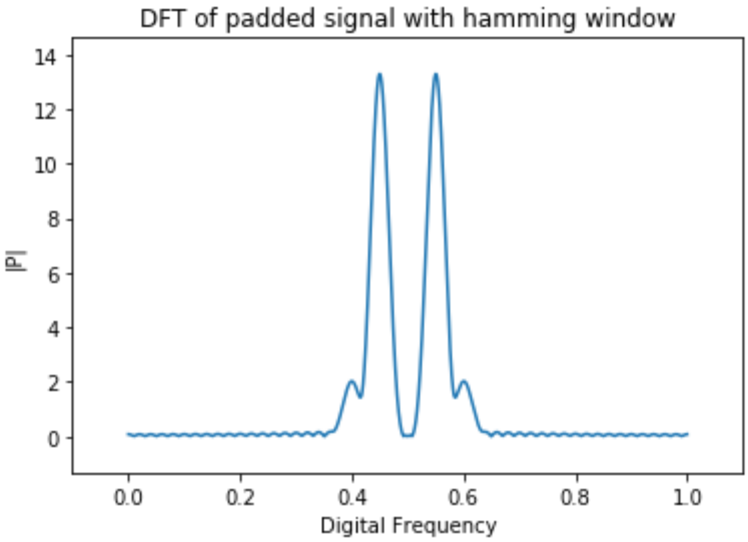
Out[17]: [<matplotlib.lines.Line2D at 0x7f9247080f28>]



In [18]:
```
#3.4
ham = np.hamming(p.size)
window = ham*p
window = np.append(window, np.zeros(99 * samples))

P3 = np.abs(np.fft.fft(window))
setup_plot('DFT of padded signal with hamming window', '|P|
', 'Digital Frequency')
pl.plot(fw2, P3)
```

Out[18]: [<matplotlib.lines.Line2D at 0x7f924de6f518>]



3.5 - In 3.2 there is not enough detail, in 3.3 it is easier, however noise is still present. in 3.4 it is the easiest to determine due to its resolution and low noise

Bonus

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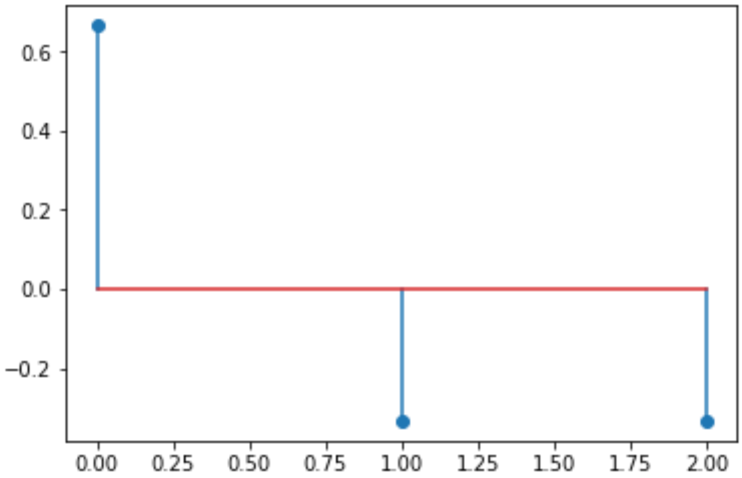
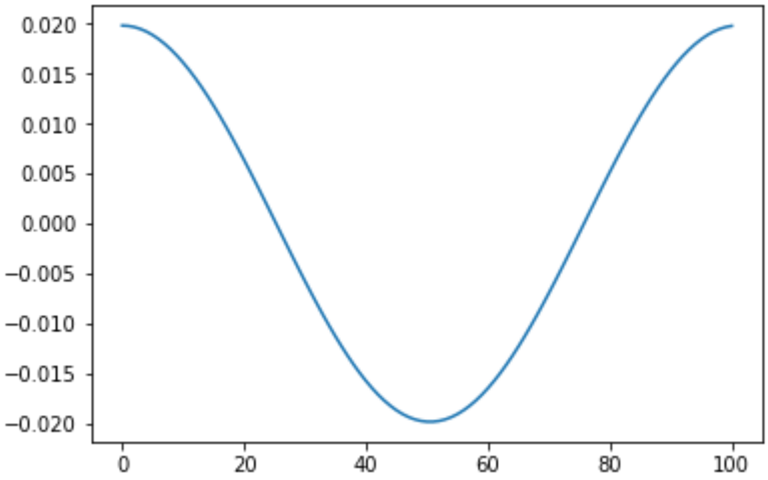
In [19]:
```
X = [0, 1, 1]
pl.stem(np.real(np.fft.ifft(X)))
```

Out[19]: <Container object of 3 artists>



In [20]:
```
X = np.array([0, 1])
X = np.append(X, np.zeros(98))
X = np.append(X, [1])
pl.plot(np.real(np.fft.ifft(X)));
```



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