Mechanics Lecture Syllabus

Pieter Kok, September 2024

Table of Contents

Lecture Plan

Week 1: Where do forces come from?

Key concept: There are two types of forces, conservative and non-conservative. Conservative forces conserve the energy of the body they act on, and they can be described as the gradient of a potential.

Weekly problems: Construct the potential energy $V(x)$ in a variety of physical situations in 1D and take its derivative; work as integral over force.

Week 2: Newton's second law

Key concept: Equating Newton's second law, $F = ma$, to a physical force F leads to a differential equation that describes the motion of the body of mass m. This is called the *Equation of Motion* of the body.

Weekly problems: Finding v and a from $x(t)$ for several examples, as well as finding $x(t)$ from given v and/or a (integration constants!). Calculating F from various examples of $V(x)$.
Tutorial: Equations of motion in a periodic potential.

Week 3: Vectors

Key concept: Position, velocity, and acceleration are *vectors* in three dimensions. We need to understand the mathematics of vectors in order to calculate the equations of motion of a body in three dimensions.

Weekly problems: Calculate time derivatives of vectors, calculate centre of mass.
Tutorial: Tangent and Normal Vectors of a Particle Trajectory.
Homework: Motion of a Loudspeaker Cone.

Week 4: Motion and forces in 2 and 3 dimensions

Key concept: Multiple bodies with forces between them have a centre of mass that has its own equation of motion.

Weekly problems: Summing vectors to zero, trigonometry, force balance.
Tutorial: A Mass Sliding up a Ramp.

Week 5: Conservation of energy and momentum

Key concept: Conservation of momentum leads to the laws of collision. If energy is also conserved, the collision is *elastic*, otherwise it is *inelastic*.

Weekly problems: Collision problems (elastic & inelastic) in 1D and 2D.
Tutorial: Collision of Two Particles.
Homework: Centre of Mass Motion.

Week 6: Applications
In this lecture we will consider a few applications of the basic concepts of mechanics that we have covered so far, and show the broad applicability of the theory.
Weekly problems: ODE/vector problems
Tutorial: Escape Velocity.

Week 7: Reading Week Revision of kinematics, forces and potentials, vectors, differential equations, work and energy.

Week 8: Polar coordinates and circular motion

Key concept: Rotational motion is described more easily in polar coordinates. The cross product with the angular velocity ω then becomes a convenient shortcut for the time derivative in uniform circular motion.

Weekly problems: Varying basis vectors with position in polar coordinates, versus constant basis vectors in Cartesian coordinates; sketching paths in polar coordinates, with tangents decomposed in $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$. Time derivatives of *vectors* in polar coordinates.
Tutorial: The Archimedean Spiral.

Week 9: Angular momentum

Key concept: Angular momentum is the amount of motion associated with rotations. It is conserved in systems that experience only central forces, like gravity or electrical forces in atoms. It can be used to analyse scattering problems and orbital mechanics.

Weekly problems: Calculate $\mathbf{r} \times \mathbf{p}$ for several physical situations and explore the importance of the choice of origin.
Tutorial: The Capture Radius of the Moon.
Homework: Exoplanet Detection.

Week 10: Rotating rigid bodies

Key concept: The shape of a body has influence on its resistance against changes in rotational motion. This captured by the momentum of inertia. It is the rotational analog of mass as the resistance against changes in linear motion.

Weekly problems: Calculate the moment of inertia for a finite point mass distribution and explore parallel axes theorem; moment of inertia of a thin disk, different shapes rolling downhill.
Tutorial: The Physical Pendulum.

Week 11: Applications of angular momentum

Applications of angular momentum, kinetic energy of rotating bodies, scattering problems.
Weekly problems: Prove parallel axes theorem; moment of inertia of a thin disk, different shapes rolling downhill.
Tutorial: Energy Generation in Accretion Disks.
Homework: Rolling Motion Down a Ramp.

Week 12: Capstone lecture: Kepler's laws (not examined)

Key concept: The inverse-square-law of gravitation leads to Kepler's laws of orbital motion. To show this, we make use of conservation of energy and angular momentum.

Mathematical Model Building

A key learning outcome of this module is knowing how to approach mathematical model building in physics, which is the translation of a real-world problem—almost always idealised—into mathematical form. This mathematical model then becomes the basis for a quantitative analysis, either by substituting numbers given in the problem or by implementing the model on a computer.

Along with experimentation proficiency, mathematical model building is the core skill of a physicist, and arguably the most difficult to master. It requires familiarity with a good amount of advanced mathematics, knowledge of the general physical principles and laws, and a good deal of creativity. Luckily, there is a method to help you along:

1. **Imagine the problem**. Make a sketch—this can be quite abstract. It is supposed to help you organise your thoughts about the problem. Give mathematical symbols to the quantities that you think are relevant to the problem. This establishes the *notation* for this problem in which every quantity is represented by a *unique* symbol.

2. Identify the **physical principles and laws** that are relevant to the problem. Write down any formulas or equations that apply. Make sure to match your notation to the question. This likely means you will have to change the notation of familiar formulas. This is not only perfectly acceptable, it is *expected*.

3. Identify what the problem is asking for; what **physical quantity** do you need to find a formula for? Write down the relevant formulas or equations that involve this quantity, again matching the notation to the question. If you need new quantities, give them a new unique symbol; *do not reuse symbols!*

4. Are there any **approximations** that you can make? Most physical problems are too complicated to solve exactly, and we will need to make approximations. They are often stated in the question, such as point masses or frictionless motion, but other approximations may be made along the way.

5. **Relate the desired quantities from part 3 to the physical laws and principles of part 2**. Try to look ahead where your maths will lead you to judge whether this is the right path. Work through the maths (substitutions, derivatives, integrations) to get the mathematical expression you are after. *Resist the urge to substitute numbers*, unless they lead to significant simplifications of your mathematical expressions.

6. **Test your final answer** for extremal values to make sure that your answer behaves in the expected ways. If it doesn't, use it to go back to step 1-5 and re-evaluate your thought processes. Most students completely skip this step, but testing your final answer will deepen your understanding of the problem and help hunt down errors in your analysis earlier.

As with everything, the more you practice this, the better you will get at it. Many of the questions in the weekly problem sets and the tutorial and homework questions are structured such that following the above approach will lead to the right answer. This approach works not only for the mechanics part of this course, but for all problems that require some mathematical modelling throughout your degree.

A note on the delivery of this part of the module:
You are required to attend the lectures and tutorials (we keep attendance records). Lectures are also recorded, but I recommend not rewatching them. These lecture notes should be sufficient for the background preparation, although good textbooks (like Kleppner and Kolenkow) can be a valuable addition.

Most of your efforts should go into the weekly problem sets, which range from fairly straightforward to quite challenging. Full solutions will be provided the week after the problems are released. To get the most out of these problems, you should seriously try them when they are released, and revisit them again after the solutions are released. When you see the problems without solutions, it forces you to think about them deeply. This prepares you for the solutions, resulting in a much more profound learning experience. To help motivate you, the content of the problem sets are part of the examinable material.

Lecture 1 Where do forces come from?

Key concept: There are two types of forces, conservative and non-conservative. Conservative forces conserve the energy of the body they act on, and they can be described as the gradient of a potential.

What is mechanics? It's not just about projectile motion and pulleys, although there is plenty of that. Mechanics is a way of understanding the behaviour of physical systems. Anything that produces a force can be analysed using mechanics, and that includes gravity, electric and magnetic forces, and even many-body systems like gasses[1]. Only with the development of quantum mechanics did physics truly depart from the framework for understanding Nature that mechanics provided. Relativity, and electric and magnetic fields, can and have all been incorporated into a classical mechanical description.

Mechanics was first cast in the language of forces by Newton in the 1680s, and that was a radical concept at the time, even mystical to some. It was subsequently developed by many others. Laplace introduced differential equations into mechanics in the 1770s, and Hamilton applied the concept of energy to mechanics in the 1820s, developing the language of mechanics as we use it today. Lagrange and Euler developed the calculus of variations for mechanics in the 1750s, which is used equally often in modern physics as an alternative formulation to Hamilton's.

So what is mechanics? It is a mathematical framework describing the motion of physical systems. It relates the positions and velocities of objects to forces and potentials, and tells us how to calculate how systems behave. It would not be a stretch to claim that the entirety of classical physics falls under the umbrella of mechanics, even though areas like optics and electromagnetism do not simply reduce to the positions and velocities of particles. In this course, we will consider a slightly more modest scope of mechanics as a discipline. We will treat only one, or a few objects at a time, excluding many-body physics. We will not talk about fields, but leave that for second semester. Instead, we bring together the mathematical tools that underpin mechanics.

In this course we will develop the following concepts:

1. Position, velocity and acceleration as mathematical functions of time,
2. forces as vectors and Newton's laws of motion,
3. the relation between force, potential, and work,

Most of the course is about developing these techniques, but in the second half we consider the special case of rotating and orbiting bodies. They are incredibly common in physics, and require new concepts such as angular momentum and the moment of inertia. We conclude with the application that started it all for Newton, namely orbital mechanics and the demonstration that Kepler's laws follow from his inverse-square law of gravity and his three laws of motion.

First, we will ask a simple question: where do forces come from? Let's consider some simple examples that you are already familiar with.

Uniform gravitational acceleration

When I drop a ball from a (modest) height h, it experiences a force that is directed towards the ground, $F = mg$, with m the mass of the ball and $g = 9.81$ m s^{-2} the gravitational acceleration. The ball started out at rest, and accelerates towards the ground. What makes the ball 'want' to leave the higher position and move towards the lower position? One answer is that the Earth attracts the ball.

[1]In *statistical mechanics* we consider the average motion of molecules, obeying the laws of mechanics.

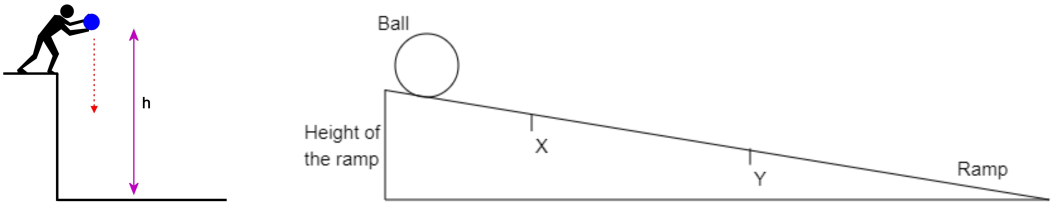


Figure 1 – Left: dropping an object from a height h to the ground. Right: letting a ball roll from the same height down a ramp.

Another, more fruitful answer is that the ball starts out in a position with high potential energy, and it moves towards a position with low potential energy. The force on the ball is in the direction of decreasing potential energy. When we choose the zero point of the potential energy at ground level, the potential energy at height h is given by

$$V_{\mathrm{gr}} = mgh\,. \tag{1.1}$$

The force $F = mg$ is in the direction in which the potential energy *decreases*.

Next, imagine we let the ball roll down a ramp (see Fig 1). If the ramp is very shallow, the force on the ball is quite a bit weaker than when we let it drop straight down. As a result, the ball won't pick up speed as fast as when we let it drop straight down. The force is proportional to the *change* in the potential energy.

Hooke's law

Imagine a spring with a block of mass m attached to the end. We arrange it horizontally, where the block can slide without friction on a surface. This way we can ignore the role of gravity. The other end of the spring is secured to a fixed point. There is an equilibrium position $x = 0$, where the block is placed such that the spring is neither compressed nor extended. We need to apply a force to move the block, because it will compress or extend the spring. The direction of the compressing/extending force is always *away* from the equilibrium position $x = 0$. Similarly, from Newton's third law, the spring exerts a force on the block that is always *towards* $x = 0$. It is called a *restoring* force, and because it always points towards $x = 0$, it has a different sign when it is compressing then when it is extending. Moreover, the further the spring is extended, the stronger the restoring force F on the block:

$$F = -kx\,, \tag{1.2}$$

where k is the spring constant. This is of course Hooke's law. The minus sign is very important. It makes the force on the block a restoring force.

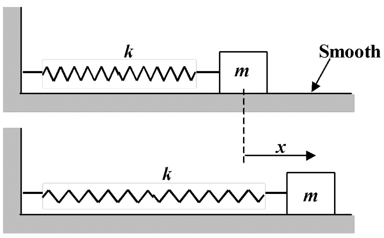


Figure 2 – A black of mass m attached to a spring with spring constant k. The spring is in the equilibrium position in the top figure, with the block position $x = 0$.

We can also understand this force as a result of a potential. When we compress or extend the spring by an amount x, the potential energy of the system is

$$V_{\text{spring}} = \frac{1}{2}kx^2 . \tag{1.3}$$

It does not matter whether we compress the spring or extend it, the potential energy will be the same, since $(-x)^2 = x^2$. The restoring force on the block is in the direction for which the potential decreases, i.e., in the direction of $x = 0$. So again, the force is given by the *change* in potential with the distance x. This is given by a derivative of the potential with respect to x. Since the force is in the direction of decreasing potential, it has to be the negative derivative:

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x} . \tag{1.4}$$

When we substitute the potential V_{spring}, you see that we recover $F = -kx$. Note that all the signs for F are correct, both for positive and negative x.

Note that we can add a constant V_0 to the potential, and it will not contribute to the force, since the derivative of a constant is zero. This means that we can pick the zero point of our potential wherever we want, without observable consequences for the force. We already used this in the first example, when we chose $V = 0$ at ground level. We tend to pick the zero point of V in such a way that it makes our analysis simplest.

Newton's law of gravity

Newton's famous insight was that the force that pulls apples to Earth is the same that makes the moon orbit the Earth. The constant gravitational acceleration g is only an approximation for when the drop height h is small compared to the radius of the Earth. The force on a mass m due to another mass M obeys Newton's inverse-square law of gravity:

$$F = -\frac{GmM}{r^2} , \tag{1.5}$$

where $G = 6.67408 \times 10^{-11}\ \mathrm{m}^3\ \mathrm{kg}^{-1}\ \mathrm{s}^{-2}$ is Newton's constant of gravitation. We included a minus sign, because the force of gravity is attractive. The force lies in the direction along the axis that connects the two masses.

We can also cast this problem in terms of potential energy. The closer that the mass m is to mass M, the lower its potential energy. We define $V = 0$ for the situation where the two masses are infinitely far apart, and we can write the potential as

$$V_{\text{gravity}} = -\frac{GmM}{r} . \tag{1.6}$$

The direction in which the potential changes is r. So if we want to find the force, we need to take the derivative in the r direction:

$$F(r) = -\frac{\mathrm{d}V_{\text{gravity}}}{\mathrm{d}r} = -\frac{GmM}{r^2} , \tag{1.7}$$

which agrees with equation (1.5). Note that we take the derivative with respect to r, not x. In general we will have to be careful about the direction in which the potential has its steepest descent, because that is where the force will be directed. The steepest descent is exactly the direction a ball will roll on a map with uneven elevation (for example rolling down a hill).

Electrostatic force between two charges

As a final example of forces between objects, consider two charges, q_1 and q_2 a distance d apart. The force on charge q_1 due to charge q_2 is given by

$$F_{\text{electric}} = \frac{q_1 q_2}{4\pi\epsilon_0 d^2}\,, \tag{1.8}$$

where ϵ_0 is the permittivity of free space. Even though we wrote a distance d, this is essentially again an inverse-square law, and it makes sense to use the coordinate r instead of d. The force is again directed along the axis connecting the two charges, and to a large degree the connection with the electric potential energy is very similar to that of the two masses in Newton's law of gravity:

$$V_{\text{electric}} = \frac{q_1 q_2}{4\pi\epsilon_0 r}\,. \tag{1.9}$$

The one notable difference is the lack of the minus sign. In gravity, all the masses are positive numbers, whereas charges can be both positive and negative. This potential says that when two charges have different signs, the overall potential is negative and they attract each other. When the charges have equal sign the potential is positive and the charges repel. You can verify that we obtain the correct electrostatic force when we take the negative derivative with respect to r:

$$F(r) = -\frac{\mathrm{d}V_{\text{electric}}}{\mathrm{d}r}\,. \tag{1.10}$$

Again, the force is in the direction of the steepest descent.

There are a number of things to note here:

1. The force is in the direction of the steepest descent, and therefore force is a vector while the potential is a scalar. We have not shown this systematically in our equations so far, but it is important to keep in mind that we deal with vectors.
2. The sign of the force is important; this is just the one-dimensional aspect of the fact that a force is a vector. Don't ignore it!
3. Not all forces can be written as derivatives of a potential, most notably friction forces don't have a potential, and magnetic forces have a much more complicated potential that we will not discuss here. These forces are still important, but we will have to deal with them in other ways.

Lecture 2 Newton's second law

Key concept: Equating Newton's second law, $F = ma$, to a physical force F leads to a differential equation that describes the motion of the body of mass m. This is called the *Equation of Motion* of the body.

Newton's second law says that $F = ma$: the force F on a body with mass m causes an acceleration a of that body. We will explore how this law allows us to describe a range of motion, but in order to do that systematically, we have to introduce the mathematical notions of position, velocity, and acceleration. Afterwards, we will link this force to the forces we discussed in the first lecture.

2.1 Position, velocity and acceleration as functions of time

The position of a particle in one dimension as a function of time, $x(t)$, the velocity of the particle $v(t)$, and its acceleration $a(t)$ are related by time derivatives:

$$v(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \qquad \text{and} \qquad a(t) = \frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} . \tag{2.1}$$

This means that the velocity is the *rate of change* of position, and the acceleration is the *rate of change* of velocity. At any point in time the particle may be completely described by its position, its velocity, and the energy potential it moves in. Given a particular potential, when we know the position of a particle *and* its velocity at time t, we will know where the particle will be at time $t + \mathrm{d}t$, namely

$$x(t + \mathrm{d}t) = x(t) + v(t)\,\mathrm{d}t . \tag{2.2}$$

The force on the particle, or the potential it moves in, will determine what velocity $v(t + \mathrm{d}t)$ the particle will have.

In order to find the position of a particle when the velocity or acceleration is given, we have to integrate:

$$x(t) = x(t_0) + \int_{t_0}^{t} v(t')\,\mathrm{d}t' \qquad \text{and} \qquad v(t) = v(t_0) + \int_{t_0}^{t} a(t')\,\mathrm{d}t' . \tag{2.3}$$

Note that we need to add constants $x(t_0)$ and $v(t_0)$ to determine the starting value of the position and velocity. These constants disappear when we take the derivative with respect to t. Also note the change in integration variable t' to distinguish it from the integration limit t. You should verify that when you calculate the time derivatives of $x(t)$ and $v(t)$ in these integral forms, you retrieve $v(t)$ and $a(t)$.

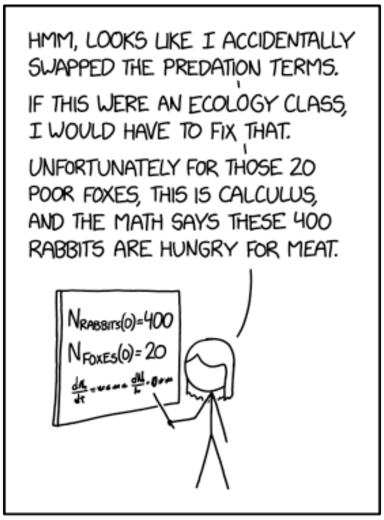
2.2 Equation of motion for problems in one dimension

We can now create the mathematical models that describe the motion of the particles in the examples from the previous lecture. Newton's second law tells us that $F = ma$, and for conservative forces we established that the force is equal to the space derivative of a potential V. If we write the acceleration as the second time derivative of position, we have the general relation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\mathrm{d}V}{\mathrm{d}x} . \tag{2.4}$$

We will look at a number of applications of this rule in the remainder of this lecture.

Let's start with the simplest case, when there is no force acting on the particle: $F = 0$. Using Newton's second law, we find that $F = ma = 0$, which means that the acceleration a is equal to



EVERY BROKEN MATHEMATICAL MODEL IS JUST A GLIMPSE INTO A TERRIFYING ALTERNATE UNIVERSE.

zero (we are not considering massless particles in this course). Qualitatively, you know what it means for a particle to have zero acceleration: it has a constant velocity. However, in preparation for the more involved examples, we will see how we treat this mathematically. In order to find the velocity of a particle, we have to integrate the acceleration over time. From equation (2.3) we find

$$v(t) = v(t_0) + \int_{t_0}^{t} a(t')\,\mathrm{d}t' = v(t_0)\,, \tag{2.5}$$

since $a(t') = 0$. In other words, the velocity $v(t)$ at any time t is equal to the velocity $v(t_0)$ at some time t_0. That is just a fancy way of saying that the velocity is constant, and its value is $v(t_0)$.

We can go further and integrate the velocity to get the position as a function of time using equation (2.3) again:

$$x(t) = x(t_0) + \int_{t_0}^{t} x(t')\,\mathrm{d}t' = x(t_0) + v(t_0)\int_{t_0}^{t} \mathrm{d}t' = x(t_0) + v(t_0)(t - t_0)\,. \tag{2.6}$$

Often we use the special value $t_0 = 0$, and the constants $x(t_0)$ and $v(t_0)$ are given the special symbols x_0 and v_0, respectively. With these substitutions, we obtain the very familiar result

$$x(t) = x_0 + v_0 t\,. \tag{2.7}$$

This is called *inertial motion*: in the absence of forces, a particle will move with a constant velocity. You recognise this as Newton's first law of motion: "a uniformly moving body continues to move uniformly unless acted on by a force".

As soon as a force acts on our particle, it no longer moves uniformly, that is, with constant velocity. Let's consider the next simplest case, where a constant force is acting on our particle: $F = k$, where k is a constant. Again, we use Newton's second law to write $F = ma = k$ and integrate twice to find the velocity and the position of the particle, using equation (2.3):

$$\begin{aligned} v(t) &= v(t_0) + \int_{t_0}^{t} a(t')\,\mathrm{d}t' = v(t_0) + \frac{k}{m}\int_{t_0}^{t} \mathrm{d}t' \\ &= v(t_0) + \frac{k}{m}(t - t_0) = v(t_0) + a_0(t - t_0)\,, \end{aligned} \tag{2.8}$$

where we have written $a_0 = k/m$ to tidy up our equations. Next, we integrate again to obtain the position of the particle:

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} v(t')\,\mathrm{d}t' \\ &= x(t_0) + \int_{t_0}^{t} \left[v(t_0) + a_0(t' - t_0)\right]\,\mathrm{d}t' \\ &= x(t_0) + v(t_0)(t' - t_0) + \frac{1}{2}a_0(t' - t_0)^2 \,. \end{aligned} \tag{2.9}$$

Using again the special values $t_0 = 0$, x_0, and v_0, these are easily recognised as the kinematic equations ("suvat"):

$$x(t) = x_0 + v_0 t + \frac{1}{2}a_0 t^2 \qquad \text{and} \qquad v(t) = v_0 + a_0 t \,. \tag{2.10}$$

Hence, the familiar equations for uniformly accelerated motion follow directly from Newton's second law and a constant force. The position $x(t)$ in equation (2.10) is not just a single equation, but rather a *family* of equations: we can choose any value of x_0 and v_0, and the given acceleration a_0 then determines the path of the particle for all times. The values x_0 and v_0 are called 'initial values', and you will have to deduce them from the problem you are trying to solve.

Example 1: We want to find the position as a function of time of an object with mass m sliding along a ramp without friction. The potential energy of the object along the ramp is $V = mgy$, with g the gravitational acceleration and y the vertical coordinate. The equation for a ramp sloping down towards the right is $y = -\tan\alpha\, x + h$, with α the slope of the ramp. Since we want the horizontal position $x(t)$, we substitute the ramp equation into V to obtain

$$V = -mg\tan\alpha\, x + mgh \,. \tag{2.11}$$

The force in the x direction on the object is the negative derivative of V with respect to x:

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} = mg\tan\alpha \,. \tag{2.12}$$

However, we want the force *along* the ramp, so we need to make an adjustment. From trigonometry, we can see that we need to include a factor $\cos\alpha$, which means that $F = mg\tan\alpha\cos\alpha = mg\sin\alpha$. Setting this force equal to $F = ma$, we obtain

$$ma = mg\sin\alpha \,, \tag{2.13}$$

and we have motion with a constant acceleration $a_0 = g\sin\alpha$. Equation (2.10) then gives the position as

$$x(t) = x_0 + v_0 t + \frac{g\sin\alpha}{2}\, t^2 \,. \tag{2.14}$$

If the starting point of the object is $x = 0$, the vertical coordinate at the start is $y = -\sin\alpha \cdot 0 + h = h$. Assuming that the object starts at rest, $v_0 = 0$, the equations of motion become

$$x(t) = \frac{1}{2}g\sin\alpha\, t^2 \qquad \text{and} \qquad v(t) = g\sin\alpha\, t \,. \tag{2.15}$$

At the bottom of the ramp, $y = 0$, the horizontal position is $x = h/\sin\alpha$. From this we can find the time it takes for the object to slide to the bottom:

$$x(t_{\text{bottom}}) = \frac{h}{\sin\alpha} = \frac{1}{2} g \sin\alpha \, t_{\text{bottom}}^2 \qquad \Longrightarrow \qquad t_{\text{bottom}} = \sqrt{\frac{2h}{g\sin^2\alpha}} \,. \tag{2.16}$$

You should check that this expression makes sense for various values of α, especially $\alpha = 90°$ and $\alpha = 0°$. Doing this check after every calculation helps you find errors that you may have made.

We now have a systematic method for finding the equations of motion of a particle moving in the presence of a conservative force: We determine the potential energy (which is often easier than finding the force directly), take the derivative to find the force acting on the particle, and equate this to $F = ma$. Let's see how this works when we have a quadratic potential $V = \frac{1}{2}kx^2$, for example a block with mass m on a spring with spring constant k and displacement from the equilibrium position x. The force on the block due to the spring is given by

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} = -kx \,. \tag{2.17}$$

This is Hooke's law. Equating this to $F = ma$, we obtain

$$ma = -kx \,. \tag{2.18}$$

At this point, we can write a as the second derivative of x with respect to time, and reorganise equation (2.18) as

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = 0 \,, \tag{2.19}$$

with $\omega^2 = k/m$. This is a differential equation in $x(t)$, an equation that cannot be solved algebraically but must be integrated. Our previous two examples were also differential equations in disguise:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = a_0 \,,$$

and we solved these via explicit integration. Equation (2.19) has a known solution:

$$x(t) = A\sin(\omega t) + B\cos(\omega t) \,, \tag{2.20}$$

with A, B constants that must be determined by the initial conditions of the problem, e.g., by considering what is the position and velocity of the block at $t = 0$. You will study these equations of motion in detail in the *Oscillations and Waves* part of this course.

As a final example of a potential energy that leads to equations of motion, consider the central potential energy

$$V(r) = -\frac{k}{r} \,. \tag{2.21}$$

One example is a body of mass m in the gravitational potential of another body of mass M, where $k = GMm$ and G is the gravitational constant. Another example is the electric potential energy between two charges q_1 and q_2, where $k = -q_1 q_2/4\pi\epsilon_0$ with ϵ_0 the electric permittivity. Just like in the example of the Lennard-Jones potential we use the radial coordinate r. The force on a body in this central potential is again found by differentiation:

$$F(r) = -\frac{dV}{dr} = -\frac{k}{r^2}\,. \tag{2.22}$$

This is the inverse-square force law that Newton identified as describing gravity. When $k > 0$ the force is negative, pointing to decreasing r. This is an *attractive* force. When $k < 0$ the force is positive and points towards increasing r. This is a *repulsive* force. When two charges have opposite sign, $k > 0$ and these charges attract each other. Like charges repel each other because in that case $k < 0$. Masses are always positive, so they always attract each other.

Using Newton's second law for the radial coordinate r, we find

$$m\frac{d^2r}{dt^2} = -\frac{k}{r^2}\,. \tag{2.23}$$

Unfortunately, it is mathematically hard to solve this differential equation. It is also an oversimplified problem, because by considering only r, we restricted ourselves to a one-dimensional problem. This was perfectly fine for the two atoms in the Lennard-Jones potential, because there is a meaningful oscillatory motion along the axis through the two atoms. However, for the $1/r$ potential, objects will be moving in orbits, which is motion in a *plane*, requiring at least two coordinates. In the next lecture we will explore the position, velocity and acceleration of particles as vectors, and develop the mathematics needed to describe motion in two and three dimensions. Even then, the problem of orbital motion is still far from trivial. But we can solve it in the sense that we will derive Kepler's three laws of orbital motion from the inverse-square force law. In order to achieve this, we will introduce a variety of concepts and conservation laws that form the backbone of not only mechanics, but also quantum mechanics, relativity and atomic physics.

Lecture 3 Vectors

Key concept: Position, velocity, and acceleration are *vectors* in three dimensions. We need to understand the mathematics of vectors in order to calculate the equations of motion of a body in three dimensions.

3.1 Vectors in three dimensions

Any object in three dimensions has a position that is given by three numbers. In cartesian coordinates, these are $x(t)$, $y(t)$, and $z(t)$ with respect to some origin (0,0,0). The position can be written as a time-dependent vector

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \equiv \begin{pmatrix} r_x(t) \\ r_y(t) \\ r_z(t) \end{pmatrix}, \tag{3.1}$$

and taking the time derivatives of each component yields the three-dimensional velocity and acceleration of the object:

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} \qquad \text{and} \qquad \mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \dot{v}_x(t) \\ \dot{v}_y(t) \\ \dot{v}_z(t) \end{pmatrix} = \begin{pmatrix} a_x(t) \\ a_y(t) \\ a_z(t) \end{pmatrix}, \tag{3.2}$$

where we use a subscript x, y, or z to denote the velocity and acceleration components in that spatial direction. We also use a dot to denote time derivatives.

Just like in the one-dimensional case, we can integrate velocity and acceleration with respect to time in order to find the position and velocity, respectively. We need to do this for each of the three components separately:

$$x(t) = x(t_0) + \int_{t_0}^{t} v_x(t')\,\mathrm{d}t', \quad y(t) = y(t_0) + \int_{t_0}^{t} v_y(t')\,\mathrm{d}t', \quad z(t) = z(t_0) + \int_{t_0}^{t} v_z(t')\,\mathrm{d}t'. \tag{3.3}$$

We can write this as

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \int_{t_0}^{t} \mathbf{v}(t')\,\mathrm{d}t'. \tag{3.4}$$

However, you should always remember that these are really three separate integrals. Similarly, we can write for the velocity

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \int_{t_0}^{t} \mathbf{a}(t')\,\mathrm{d}t'. \tag{3.5}$$

Another useful way in which we can write the position, velocity and acceleration in three dimensions is using the unit vectors $\hat{\mathrm{e}}_x$, $\hat{\mathrm{e}}_y$, and $\hat{\mathrm{e}}_z$:

$$\hat{\mathrm{e}}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{\mathrm{e}}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{\mathrm{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.6}$$

You have seen these before as $\hat{\mathrm{e}}_x = \hat{\mathbf{i}}$, $\hat{\mathrm{e}}_y = \hat{\mathbf{j}}$, and $\hat{\mathrm{e}}_z = \hat{\mathbf{k}}$, but we will use this 'new' notation that shows a more explicit link with the cartesian coordinates x, y, and z. This will be useful later when we employ other coordinate systems, like polar coordinates. With these unit vectors we can write the velocity and acceleration as

$$\mathbf{v} = \frac{\mathrm{d}x}{\mathrm{d}t}\,\hat{\mathrm{e}}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\,\hat{\mathrm{e}}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\,\hat{\mathrm{e}}_z \quad \text{and}$$
$$\mathbf{a} = \frac{\mathrm{d}v_x}{\mathrm{d}t}\,\hat{\mathrm{e}}_x + \frac{\mathrm{d}v_y}{\mathrm{d}t}\,\hat{\mathrm{e}}_y + \frac{\mathrm{d}v_z}{\mathrm{d}t}\,\hat{\mathrm{e}}_z = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,\hat{\mathrm{e}}_x + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\,\hat{\mathrm{e}}_y + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\,\hat{\mathrm{e}}_z\,. \tag{3.7}$$

3.2 Tangent vectors and normal vectors to a curve

Consider a curved path of a particle, shown in Fig. 3. At time t the particle is at position A with position vector $\mathbf{r}$, and at time t' later the particle is at position B with position vector $\mathbf{r}'$. Define $\Delta t = t' - t$ and $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$. The line segment along the curve is Δs, which is a little bit longer than $\Delta\mathbf{r}$ because it is not straight. The average velocity $\mathbf{v}_{\text{gem}}$ over the segment $\Delta\mathbf{r}$ is then defined as

$$\mathbf{v}_{\text{gem}} = \frac{\Delta\mathbf{r}}{\Delta t}\,. \tag{3.8}$$

In the limit where $\Delta t \to 0$, this becomes the time derivative $\mathbf{v} = \dot{\mathbf{r}}$, as expected:

$$\mathbf{v} = \lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,, \tag{3.9}$$

where we have written $\mathbf{r}' = \mathbf{r}(t+\Delta t)$. However, we can also see from Fig. 3 that in this limit the velocity $\mathbf{v}$ is the *tangent vector* to the curve. This is true for any continuous and differentiable path of a particle: the velocity at a point along the path is the tangent vector to the path at that point.

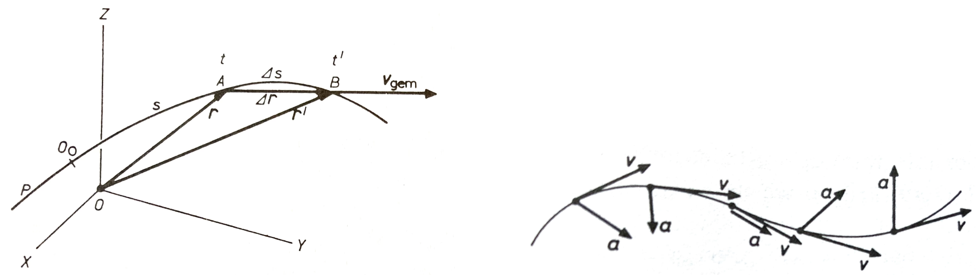


Figure 3 – Left: curved path in three dimensions. Right: velocity and acceleration along a curved path.

The situation is different for the acceleration. In order to have a curved path we need to bend the particle away from moving in a straight line. This requires an acceleration away from the tangent to the curve. An example is given in Fig. 3 (right). In the extreme case of a circular path the acceleration always points to the centre, and the tangential component is zero. We can use this to explore the more general case.

Consider a velocity $\mathbf{v} = v\hat{\mathrm{e}}_T$ at point A along the curve, where $\hat{\mathrm{e}}_T$ is the unit vector tangent to the curve at A. The magnitude of the velocity vector (the speed) is given by v. We can write the acceleration as

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v\hat{\mathrm{e}}_T) = \frac{\mathrm{d}v}{\mathrm{d}t}\hat{\mathrm{e}}_T + v\frac{\mathrm{d}\hat{\mathrm{e}}_T}{\mathrm{d}t}\,, \tag{3.10}$$

where we used the product rule for derivatives. The unit vector $\hat{\mathrm{e}}_T$ may change direction along the curve, so the derivative $\mathrm{d}\hat{\mathrm{e}}_T/\mathrm{d}t$ is generally not zero. In order to say a little bit more about this time dependence, we make the drawing in Fig. 4.

At the point A, we define the tangent and normal unit vectors

$$\hat{\mathrm{e}}_T = \hat{\mathrm{e}}_x\cos\phi + \hat{\mathrm{e}}_y\sin\phi \qquad \text{and} \qquad \hat{\mathrm{e}}_N = -\hat{\mathrm{e}}_x\sin\phi + \hat{\mathrm{e}}_y\cos\phi\,. \tag{3.11}$$

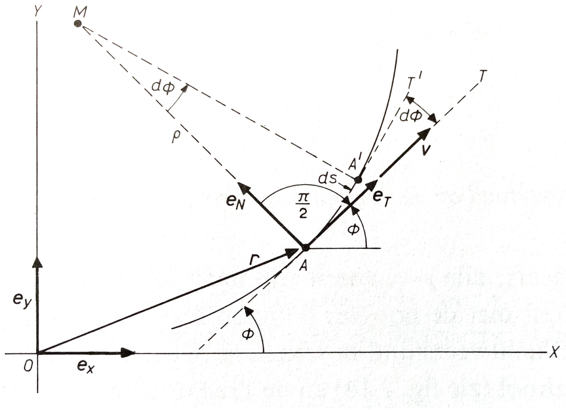


Figure 4 – The radius of curvature ρ.

We can take the time derivative of $\hat{e}_T$:

$$\frac{d\hat{e}_T}{dt} = \hat{e}_x \frac{d\cos\phi}{dt} + \hat{e}_y \frac{d\sin\phi}{dt} = -\hat{e}_x \sin\phi \frac{d\phi}{dt} + \hat{e}_y \cos\phi \frac{d\phi}{dt} . \tag{3.12}$$

This yields the very simple relationship

$$\frac{d\hat{e}_T}{dt} = \hat{e}_N \frac{d\phi}{dt} . \tag{3.13}$$

We can now relate the time derivative of ϕ to the *radius of curvature* ρ in Fig. 4 as follows:

$$\frac{d\phi}{dt} = \frac{d\phi}{ds}\frac{ds}{dt} = v\frac{d\phi}{ds} , \tag{3.14}$$

where s is again the segment of the curve from A to A'. From the triangle MAA' in the figure, you can see that $ds = \rho d\phi$, and therefore $d\phi/ds = 1/\rho$. Hence

$$\frac{d\phi}{dt} = \frac{v}{\rho} \qquad \text{and} \qquad \frac{d\hat{e}_T}{dt} = \frac{v}{\rho}\hat{e}_N . \tag{3.15}$$

Substituting this into the expression for the acceleration in equation (3.10), we obtain

$$\mathbf{a} = \frac{dv}{dt}\hat{e}_T + \frac{v^2}{\rho}\hat{e}_N . \tag{3.16}$$

The first term is the component of acceleration or deceleration in the direction of the motion (i.e., tangential), while the second term is the normal acceleration that changes the direction of travel of the particle. You can see that in the case of uniform circular motion, where the speed is constant but the direction is always changing, the acceleration reduces to the familiar v^2/R, where the radius of curvature ρ is equal to the radius of the circle R.

For arbitrary curved paths, the radius of curvature is constantly changing, which results in quite dramatic movements of the point M. How does M move as the path changes from an upwardly bent trajectory to a downward trajectory? The radius of curvature ρ is more a conceptual tool for understanding the tangential and normal components of **a** than a computational tool for solving problems.

Lecture 4 Motion and forces in 2 and 3 dimensions

Key concept: Multiple bodies with forces between them have a centre of mass that has its own equation of motion.

Now that we have the necessary tools to describe motion in two and three dimensions, we can write Newton's second law in vector form:

$$\mathbf{F} = m\mathbf{a}\,. \tag{4.1}$$

The acceleration $\mathbf{a}$ always points in the same direction as $\mathbf{F}$. In this lecture we will first explore Newton's first law in two and three dimensions, and see how this leads to the notion of force balance. Next we will introduce the forces between multiple bodies and the centre of mass. Finally, we will introduce the notion of momentum, and re-express Newton's second law in terms of changes in momentum.

4.1 Force balance

Newton's first law states that there is no *net* force on a body at rest or in uniform linear motion. This means that there may be multiple forces acting on the body, but they all sum to zero. In one dimension, this is easy to understand:

$$F_1 + F_2 + \ldots + F_N = 0\,. \tag{4.2}$$

All these forces lie along the same line. In three dimensions, this becomes a *vector* equation:

$$\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_N = 0\,, \tag{4.3}$$

which should be understood as

$$\begin{pmatrix} F_{1,x} \\ F_{1,y} \\ F_{1,z} \end{pmatrix} + \begin{pmatrix} F_{2,x} \\ F_{2,y} \\ F_{2,z} \end{pmatrix} + \ldots + \begin{pmatrix} F_{N,x} \\ F_{N,y} \\ F_{N,z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \equiv 0\,, \tag{4.4}$$

Therefore, *each component* should add up to zero:

$$\begin{aligned} F_{1,x} + F_{2,x} + \ldots + F_{N,x} &= 0 \\ F_{1,y} + F_{2,y} + \ldots + F_{N,y} &= 0 \\ F_{1,z} + F_{2,z} + \ldots + F_{N,z} &= 0\,. \end{aligned} \tag{4.5}$$

When you draw your diagrams, you will need to decompose your forces into their components.

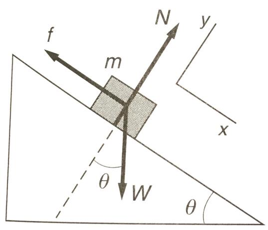


Figure 5 – Forces balance in two dimensions. From Kleppner & Kolenkow, CUP 2014.

As an example, consider a mass m on a frictionless ramp that is subject to a force f preventing it from sliding down (see Fig. 4.1). Gravity produces a downward force W, the weight, and the ramp produces a normal force N that prevents the mass from moving into the ramp. The normal force will always have exactly the magnitude to keep it on the ramp. We can choose our x and y coordinates any way we like, as long as they are perpendicular (orthogonal) to each other. In this problem it is convenient to take x down the ramp, and y perpendicular to the ramp. Since the mass is at rest, the three forces should balance, i.e., sum to zero:

$$\mathbf{f} + \mathbf{N} + \mathbf{W} = 0\,. \tag{4.6}$$

In vector form this is

$$\begin{pmatrix} -f \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ N \end{pmatrix} + \begin{pmatrix} W\sin\theta \\ -W\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0\,. \tag{4.7}$$

Setting the x and y components to zero leads to the two equations

$$-f + W\sin\theta = 0 \qquad \text{and} \qquad N - W\cos\theta = 0\,. \tag{4.8}$$

Since $W = mg$, and we want to know the values of the magnitudes f and N in terms of m and θ, we can rearrange

$$f = mg\sin\theta \qquad \text{and} \qquad N = mg\cos\theta\,. \tag{4.9}$$

The normal force N is due to Newton's third law, which states that a force on one body (the ramp) due to another (the mass) produces a force equal in magnitude and opposite in direction on the first body (the mass) due to the other (the ramp). Note the subtlety here that the normal force on the mass due to the ramp is *not* along $\mathbf{W}$, so it is the component perpendicular to the ramp that determines $\mathbf{N}$.

Example 1: Solve the problem in figure 4.1 using coordinates where the x direction is horizontal and the y direction is vertical.

Another important concept for vectors is the *point of engagement*, or the starting point of the arrow that represents the force. We treated the mass in figure as a point mass, and all forces therefore engage at the centre of gravity. However, for extended systems this can be different. For example, while the weight force (mg) always has the centre of gravity as the point of engagement, the normal force has the boundary between the block and the ramp surface as the point of engagement. Any friction force also has the point of engagement lying in the contact surface between the block and the ramp. We will explore the importance of this when we consider torque in a future lecture.

4.2 Newton's third law

In the example of the block on the ramp in the previous section, the two bodies exert a force on each other: the weight of the block is a force *on* the ramp, while the ramp exerts an equal and opposite force on the block, namely the normal force. This is a consequence of Newton's third law that forces *always* appear in pairs that are equal and opposite, but acting on different objects. Imagine that object 1 exerts a force $\mathbf{F}_{12}$ on object 2. The point of engagement of $\mathbf{F}_{12}$ is on object 2. The third law states that object 2 also exerts a force on object 1, $\mathbf{F}_{21}$ that is opposite and equal in magnitude to $\mathbf{F}_{12}$:

$$\mathbf{F}_{21} = -\mathbf{F}_{12}\,. \tag{4.10}$$

The two objects do not need to be touching, as is the case with the block and the ramp. The gravitational force on the Earth by the Sun is a vector $\mathbf{F}_\mathrm{g}$ acting on the Earth and pointing towards the Sun. Similarly, the Earth exerts an equal force on the Sun in opposite direction, $-\mathbf{F}_g$. The acceleration of the Earth $\mathbf{a}_E$ due to the Sun's gravity can be written as

$$\mathbf{a}_{\mathrm{E}} = \frac{\mathbf{F}_{\mathrm{g}}}{m_{\mathrm{E}}}, \tag{4.11}$$

where m_E is the mass of the Earth. Similarly, the acceleration of the Sun $\mathbf{a}_\odot$ due to the Sun's gravity can be written as

$$\mathbf{a}_{\odot} = \frac{\mathbf{F}_{\mathrm{g}}}{m_{\odot}}, \tag{4.12}$$

where $m_\odot$ is the mass of the Sun. Even though the forces are the same in magnitude, the acceleration of the Earth is much larger than that of the Sun due to the great difference in masses m_E and $m_\odot$.

Often we want to consider a system of N masses m_1, m_2, $\dots m_N$. This can be a galaxy of stars, the solar system, an atomic nucleus with orbiting electrons, or a formation of flying drones, to name a few examples. There may be forces between these objects. If there is a force $\mathbf{F}_{jk}$ on mass m_k due to mass m_j, then there is also a force $\mathbf{F}_{kj} = -\mathbf{F}_{jk}$ on mass m_j due to mass m_k due to Newton's third law.

4.3 Momentum

At this point we introduce the *momentum* $\mathbf{p}$ of a body with mass m and velocity $\mathbf{v}$ as $\mathbf{p} = m\mathbf{v}$. Momentum is a very useful concept when we want to quantify 'how much motion' there is in a body. It plays a more fundamental role dan acceleration in Newton's second law, and we can write instead

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{4.13}$$

The change in motion of a body is equal to the force exerted on that body. Equation (4.13) is equivalent to $\mathbf{F} = m\mathbf{a}$, but *only* when the mass is constant. There are however situations in which this is not the case.

Example 2: Consider a rocket travelling with (increasing) velocity $\mathbf{v}$ relative to some rest frame (e.g., the Earth). The rocket expels hot gas with a velocity $\mathbf{v}'$ relative to the same rest frame, such that the exhaust velocity relative to the rocket is $\mathbf{v}_e = \mathbf{v}' - \mathbf{v}$. Assume furthermore that $\mathbf{v}_e$ is constant. We are interested in the force $\mathbf{F}$ on the rocket by the exhaust gas, and the velocity of the rocket after some time t.

The momentum of the rocket at time t is $\mathbf{p}(t) = m\mathbf{v}(t)$, and a short time interval $\mathrm{d}t$ later, the momentum has become

$$\mathbf{p}(t + \mathrm{d}t) = (m - \mathrm{d}m)(\mathbf{v} + \mathrm{d}\mathbf{v}) + \mathrm{d}m\mathbf{v}', \tag{4.14}$$

where $\mathrm{d}m$ is the (positive) mass of hot gas expelled by the rocket. The first term in the right-hand side is the new momentum of the rocket, while the second term, $\mathrm{d}m\mathbf{v}'$, is the momentum of the expelled hot gas. Expanding the product in equation (4.14), we obtain

$$\mathbf{p}(t + \mathrm{d}t) = m\mathbf{v} + m\mathrm{d}\mathbf{v} - \mathbf{v}\mathrm{d}m + \mathrm{d}m\mathbf{v}' = m\mathbf{v} + m\mathrm{d}\mathbf{v} + \mathbf{v}_e\mathrm{d}m, \tag{4.15}$$

where we dropped the term that is quadratic in the infinitesimals, $\mathrm{d}m\mathrm{d}\mathbf{v}$, since they vanish in the final result (we nearly always drop products or powers of infinitesimals, since the vanish in the limit). The change in momentum is

$$\mathrm{d}\mathbf{p} = \mathbf{p}(t + \mathrm{d}t) - \mathbf{p}(t) = m\mathrm{d}\mathbf{v} + \mathbf{v}_e\mathrm{d}m. \tag{4.16}$$

Hence

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}_e\frac{\mathrm{d}m}{\mathrm{d}t} . \tag{4.17}$$

If there is an external force $\mathbf{F}$ on the rocket, such as Earth's gravity, then we have

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}_e\frac{\mathrm{d}m}{\mathrm{d}t} . \tag{4.18}$$

If the rocket accelerates in free space without external forces, then

$$\mathbf{F} = 0 \quad \longrightarrow \quad \mathbf{m}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{v}_e\frac{\mathrm{d}m}{\mathrm{d}t} = 0 , \tag{4.19}$$

and the momentum imparted on the hot gas balances the momentum gained by the rocket and its remaining fuel.

Let $\mathbf{F} = m\mathbf{g}$, the gravitational force close to the Earth's surface. Our equation of motion becomes

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{\mathbf{v}_e}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = \mathbf{g} . \tag{4.20}$$

Here $\mathbf{v}$ and $\mathbf{v}_e$ point upward, while $\mathbf{g}$ points downward. This then becomes a one-dimensional problem

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \frac{v_e}{m}\frac{\mathrm{d}m}{\mathrm{d}t} = -g , \tag{4.21}$$

where the positive direction is upward. Integrating this equation, we obtain

$$\int_{v_0}^{v} \mathrm{d}v + v_e \int_{m_0}^{m} \frac{\mathrm{d}m'}{m'} = -g\int_0^t \mathrm{d}t , \tag{4.22}$$

or

$$v = v_0 + v_e \ln\frac{m}{m_0} - gt . \tag{4.23}$$

If m is the final mass and t is the time in which all fuel has been used up, then v is the final speed of the rocket. Note that we have assumed that g does not change while the rocket is burning its fuel, which is typically not a very good approximation for very large rockets that keep burning their fuel until well into the upper atmosphere. We also ignored air resistance.

4.4 Centre of mass

The centre of mass position $\mathbf{R}$ of N particles is the sum of all the positions of the particles, weighed by their masses:

$$\mathbf{R} = \frac{\sum_{j=1}^{N} m_j\mathbf{r}_j}{\sum_{j=1}^{N} m_j} . \tag{4.24}$$

Here, the $\sum$ symbol means that we sum over all j from 1 to N. The positions of the particles with respect to the centre of mass are $\mathbf{r}'_j = \mathbf{r}_j - \mathbf{R}$, and if we use the $\mathbf{r}'_j$ for our description of the particles' motion, the centre of mass always lies at the origin. This is often very convenient in calculations.

The velocity of the centre of mass can be found directly by differentiating $\mathbf{R}$ with time:

$$\mathbf{V} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} = \frac{1}{M}\sum_{j=1}^{N} m_j \frac{\mathrm{d}\mathbf{r}_j}{\mathrm{d}t} = \frac{1}{M}\sum_{j=1}^{N} m_j \mathbf{v}_j = \frac{1}{M}\sum_j \mathbf{p}_j \, . \tag{4.25}$$

where $M = \sum_j m_j$ is the total, time independent, mass of the system. We can define the total momentum $\mathbf{P} = \sum_j \mathbf{p}_j$, such that

$$\mathbf{P} = M\mathbf{V} \, . \tag{4.26}$$

This means that the total momentum is the total mass times the velocity of the centre of mass. When two constituent masses in this system exchange momentum, for example due to a collision, one part gains $\Delta\mathbf{p}$, while the other part loses the same amount, $-\Delta\mathbf{p}$. This is ensured by Newton's third law. That means that the overall momentum $\mathbf{P}$ remains the same, even if the constituent parts are a swirling mess of interacting bodies. Only an *external* force $\mathbf{F}_{\mathrm{ext}}$ can change $\mathbf{P}$:

$$\mathbf{F}_{\mathrm{ext}} = \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} \, . \tag{4.27}$$

This is Newton's second law again. It means that we can solve the motion of complex bodies if we consider the position and momentum of the centre of mass. It is also why we can treat planets and stars as point objects.

Lecture 5 Conservation of energy and momentum

Key concept: Conservation of momentum leads to the laws of collision. If energy is also conserved, the collision is *elastic*, otherwise it is *inelastic*.

5.1 Conservation of energy

In the problem set of lecture 1 we saw that the work W done on an object by a conservative force F along a (one-dimensional) path is given by

$$W = \int_{x_0}^{x_1} F\,\mathrm{d}x = -\int_{x_0}^{x_1} \frac{\mathrm{d}V}{\mathrm{d}x}\,\mathrm{d}x = V(x_0) - V(x_1)\,. \tag{5.1}$$

Using Newton's second law, $F = \mathrm{d}p/\mathrm{d}t$, we can relate the potential change between two positions to the change in momentum:

$$\begin{aligned} V(x) - V(x_0) &= -\int_{x_0}^{x} \frac{\mathrm{d}p}{\mathrm{d}t}\mathrm{d}x = -\int_{p(x_0)}^{p(x)} v\,\mathrm{d}p = -\int_{p(x_0)}^{p(x)} \frac{p}{m}\,\mathrm{d}p = -\left[\frac{p^2}{2m}\right]_{p(x_0)}^{p(x)} \\ &= -\left(\frac{p(x)^2}{2m} - \frac{p(x_0)^2}{2m}\right)\,. \end{aligned} \tag{5.2}$$

You will recognise the right-hand side in the last line as the change of kinetic energy T of the object, and this means that without any further effects on the object, the potential energy is transformed into kinetic energy. As the potential energy decreases, or $V(x) < V(x_0)$, the kinetic energy increases and vice versa. Bringing everything to the left-had side of the equation, we end up with an expression for the conservation of energy:

$$\Delta V + \Delta T = 0\,, \tag{5.3}$$

where we denote changes by the greek letter Δ (pronounced "delta"). Note that this is a statement about *changes*. We can roll a marble down a ramp in the basement or in the penthouse (assuming constant g), and we will get the same results for the change in kinetic energy even though the potential energy of the marble in the penthouse is much greater than in the basement. Conservation of energy applies to the changes in energy in a process.

This is a special case of a broader conservation principle: the total energy in a closed system is conserved. It doesn't mean that energy is always in a useful form. For example, when a friction force slows down a block sliding down a ramp, the final velocity is lower than if all potential energy was converted into kinetic energy. The 'missing' energy has become the heat generated by the friction between the block and the ramp. It is typically not possible to recover this form of energy, but when we take it into account the total energy in the system is conserved.

When working with energy conservation in practical problems, we often compare the initial state of our physical system to its final state and do some simple book keeping. In the example above, the energy of the initial state, with the block at rest atop the ramp at height h, is

$$E_{\text{before}} = mgh\,, \tag{5.4}$$

where m is the mass of the block. The energy of the final state, at the bottom of the ramp, is

$$E_{\text{after}} = \frac{1}{2}mv^2 + Q\,, \tag{5.5}$$

where v is the final speed of the block and Q is the heat generated by the friction. When there is no friction, we can calculate v from h and m. If there is friction, we can determine the heat it generates by measuring v and using energy conservation: $E_{\text{after}} = E_{\text{before}}$, or

$$Q = mgh - \frac{1}{2}mv^2\,. \tag{5.6}$$

We will use this method when we discuss collisions a bit later on. It will be critical to determine whether there is any heat generated in a collision, so you can tell whether you can use energy conservation for the particles that are colliding.

5.2 Momentum conservation

Another important law of physics is the conservation of momentum: for any system that does not experience a net external force, its momentum is conserved. For a single particle this is simply Newton's second law:

$$\mathbf{F} = 0 \quad \rightarrow \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = 0\,. \tag{5.7}$$

However, we saw that this law also holds for composite bodies consisting of N components with mass m_j, centre of mass coordinate $\mathbf{R}$, and centre of mass momentum $\mathbf{P}$:

$$\mathbf{F}_{\mathrm{ext}} = 0 \quad \rightarrow \quad \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = 0\,. \tag{5.8}$$

One example is a supernova (we ignore the external gravitational force on the star towards the centre of the galaxy). In the rest frame of the star before the explosion, the momentum is $\mathbf{P} = 0$ since the velocity of the centre of mass $\mathbf{V}$ is zero. That does not mean all the parts of the star are at rest. Far from it! But all these small momenta sum up to zero:

$$\mathbf{P}_{\mathrm{before}} = \mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_N = \sum_{j=0}^{N} \mathbf{p}_j = 0\,, \tag{5.9}$$

where N is *very* large. After the explosion, all the parts have been blown away, so the momentum of the individual parts may be much larger than before: $\mathbf{p}'_j > \mathbf{p}_j$, but taken together, they still sum up to the same total momentum as before:

$$\mathbf{P}_{\mathrm{after}} = \mathbf{p}'_1 + \mathbf{p}'_2 + \cdots + \mathbf{p}'_M = \sum_{j=0}^{M} \mathbf{p}'_j = 0\,. \tag{5.10}$$

We may no longer have the same number of parts in the system after the explosion, so $M \neq N$ in general. Since this is a closed system, the energy is conserved as well, and the increased velocities of the constituent parts must come from somewhere. In a supernova, there is a lot of potential energy stored away in the atomic nuclei. When these nuclei fuse in a supernova explosion, this energy gets released and is turned into momentum.

There are also important situations where momentum is not conserved. For example, a ball bouncing off a wall changes its momentum: right before the bounce it was moving, say, to the right with $\mathbf{p}_{\mathrm{before}} = m\mathbf{v}$, and immediately after the bounce it is moving to the left with $\mathbf{p}_{\mathrm{after}} = -m\mathbf{v}$. The wall exerted a force on the ball that turned it around. The overall impulse $\mathbf{S}$ that turned the ball around is

$$\mathbf{S} = \mathbf{p}_{\mathrm{after}} - \mathbf{p}_{\mathrm{before}} = \mathbf{F}_{\mathrm{wall}}\Delta t\,, \tag{5.11}$$

where Δt is the time of contact between the ball and the wall, and $\mathbf{F}_{\mathrm{wall}}$ is the force exerted by the wall on the ball.

When we consider the ball and the wall together (assuming that there is no other system present, no Earth, no child throwing the ball), then momentum will be conserved, because we also have to consider the force of the ball exerted on the wall—Newton's third law—and the total net force on the combined system is zero. The total momentum before the bounce is $\mathbf{p}_{\text{before}} = m\mathbf{v}$, with the velocity of the wall before the bounce zero. After the bounce, the momentum is

$$\mathbf{p}_{\text{after}} = -m\mathbf{v} + M\mathbf{v}_{\text{w}} , \tag{5.12}$$

where M is the mass of the wall and $\mathbf{v}_{\text{w}}$ is the velocity of the wall after the bounce. M is typically much larger than the mass of the ball m. Momentum conservation $\mathbf{p}_{\text{after}} = \mathbf{p}_{\text{before}}$ means that the velocity of the wall becomes

$$-m\mathbf{v} + M\mathbf{v}_{\text{w}} = m\mathbf{v} , \qquad \to \qquad \mathbf{v}_{\text{w}} = \frac{2m}{M}\mathbf{v} , \tag{5.13}$$

which is tiny compared to $\mathbf{v}$ when $M \gg m$.

5.3 Collisions

The ball bouncing against a wall is a simple example of a collision problem. In this section we will develop the theory of collisions in more detail and for more general problems. We have to distinguish between two types of collisions, elastic and inelastic. Elastic collisions obey both energy and momentum conservation; the total kinetic energy before and after the collision remains unchanged. Inelastic collisions involve a change in kinetic energy, so that

$$T_{\text{before}} = T_{\text{after}} + Q . \tag{5.14}$$

Typically Q is a positive quantity, for example when some of the kinetic energy in the collision is converted to heat. However, Q can also be negative when the collision triggers a release of potential energy (as in the supernova explosion). For example, a ship colliding with a sea mine triggers a detonation that creates a lot more kinetic energy than the kinetic energy that went into the collision. This is sometimes called a *super-elastic* collision. Finally, when $Q = 0$ the collision is elastic.

Let's consider a simple elastic collision of a particle with mass m_1 with another particle of mass m_2. Furthermore, we assume that m_1 has an initial velocity of $\mathbf{v}_1$, and m_2 is at rest at the origin. We consider this a two-dimensional problem, so the collision will take place entirely in the xy-plane. Since the collision is elastic, both the momentum and the kinetic energy is conserved:

$$\mathbf{p}_{\text{before}} = m_1\mathbf{v}_1 \qquad \text{and} \qquad T_{\text{before}} = \frac{1}{2}m_1 v_1^2 . \tag{5.15}$$

After the collision, we find

$$\mathbf{p}_{\text{after}} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2 \qquad \text{and} \qquad T_{\text{after}} = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 , \tag{5.16}$$

where we used $\mathbf{u}_1$ and $\mathbf{u}_2$ for the velocities of m_1 and m_2 after the collision. To solve this, we want to work out the final velocities. However, we want to get rid of the vector quantities if we can, because they complicate things. We can write the term with $\mathbf{u}_2$ in terms of $\mathbf{u}_1$ and $\mathbf{v}_1$:

$$\mathbf{p}_{\text{after}} = \mathbf{p}_{\text{before}} : \qquad m_1\mathbf{u}_1 + m_2\mathbf{u}_2 = m_1\mathbf{v}_1 \qquad \to \qquad m_2\mathbf{u}_2 = m_1(\mathbf{v}_1 - \mathbf{u}_1) . \tag{5.17}$$

Taking the square of this equation yields

$$m_2^2 u_2^2 = m_1^2(\mathbf{v}_1 - \mathbf{u}_1)^2 = m_1^2(\mathbf{v}_1 \cdot \mathbf{v}_1 - 2\mathbf{v}_1 \cdot \mathbf{u}_1 + \mathbf{u}_1 \cdot \mathbf{u}_1) = m_1^2(v_1^2 + u_1^2) - 2m_1^2 v_1 u_1 \cos\theta , \tag{5.18}$$

where θ is the angle between $\mathbf{v}_1$ and $\mathbf{u}_1$, in other words, the angle with which particle 1 is deflected by the collision. Dividing this expression by m_2, we obtain

$$m_2 u_2^2 = \frac{m_1^2}{m_2}(v_1^2 + u_1^2) - 2\frac{m_1^2}{m_2} v_1 u_1 \cos\theta . \tag{5.19}$$

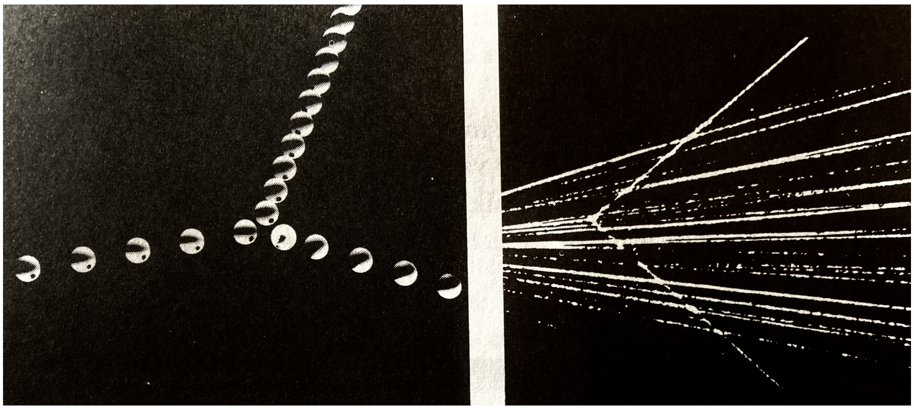


Figure 6 – Left: two equal-mass snooker balls make an angle of 90° after collision. Right: a collision of two alpha particles make an angle of 90° between their outgoing velocities.

Next, we consider the conservation of kinetic energy:

$$T_{\text{before}} = T_{\text{after}} : \qquad \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 \,. \tag{5.20}$$

We substitute $m_2 u_2^2$ from equation (5.19) into this expression and divide by m_1:

$$u_1^2 - v_1^2 + \frac{m_1}{m_2}\left(v_1^2 + u_1^2\right) - 2\frac{m_1}{m_2} v_1 u_1 \cos\theta = 0 \,. \tag{5.21}$$

When we look at the special case where $m_1 = m_2 = m$, you can prove that

$$\cos\theta = \frac{u_1}{v_1} \,. \tag{5.22}$$

Moreover, from $m\mathbf{v}_1 = m\mathbf{u}_1 + m\mathbf{u}_2$ we find

$$m^2 v_1^2 = (m\mathbf{u}_1 + m\mathbf{u}_2)^2 = m^2 u_1^2 + m^2 u_2^2 + 2m^2 \mathbf{u}_1 \cdot \mathbf{u}_2 \,. \tag{5.23}$$

Using equation (5.20), we see that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$, or the two velocities of the two masses are perpendicular to each other after the collision. Two examples of this are shown in Fig. 6.

Often a collision between two particles is most conveniently analysed in the centre of mass frame. In this case the momentum before and after the collision is zero, and we can write this simply as $\mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{p}'_1 = -\mathbf{p}'_2$, where the prime indicates the momentum after the collision. We can further use equation (5.14) to write

$$\frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} = \frac{p_1'^2}{2m'_1} + \frac{p_2'^2}{2m'_2} + Q \,, \tag{5.24}$$

where we allowed for the possibility that the masses change after the collision: $m_1 \to m'_1$ and $m_2 \to m'_2$. This can happen, for example, in particle accelerators where exotic particles are created from colliding protons. Using momentum conservation, we can express this entirely in terms of p_1 and p'_1:

$$\frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) p_1^2 = \frac{1}{2}\left(\frac{1}{m'_1} + \frac{1}{m'_2}\right) p_1'^2 + Q \,. \tag{5.25}$$

The quantities in brackets are the *reduced mass* for the two particles μ and μ' before and after the collision (see the problem set for lecture 4), so we can write

$$\frac{p_1^2}{2\mu} = \frac{{p'_1}^2}{2\mu'} + Q\,. \tag{5.26}$$

If the collision is elastic ($Q = 0$) and the masses do not change ($\mu = \mu'$), then the magnitude of the momentum in the centre of mass frame does not change: $p_1 = p'_1$. There still has been an exchange of momentum, because the direction of $\mathbf{p}'_1$ may be different from $\mathbf{p}_1$.

Lecture 6 Applications

In this lecture we will consider a few applications of the basic concepts of mechanics that we have covered so far, and show the broad applicability of the theory.

Example 1: The Lennard-Jones potential, shown in Fig. 7, describes the potential energy between two atoms,

$$V(r) = -\epsilon\left[2\left(\frac{r_{\rm m}}{r}\right)^6 - \left(\frac{r_{\rm m}}{r}\right)^{12}\right], \tag{6.1}$$

where r is the distance between the molecules, and $-\epsilon$ is the bottom of the potential well at a distance $r_{\rm m}$. This is a very complicated potential, but we can solve for the situation when the two atoms are roughly a distance $r_{\rm m}$ apart. In this case the two atoms are bound together since they are in the potential well.

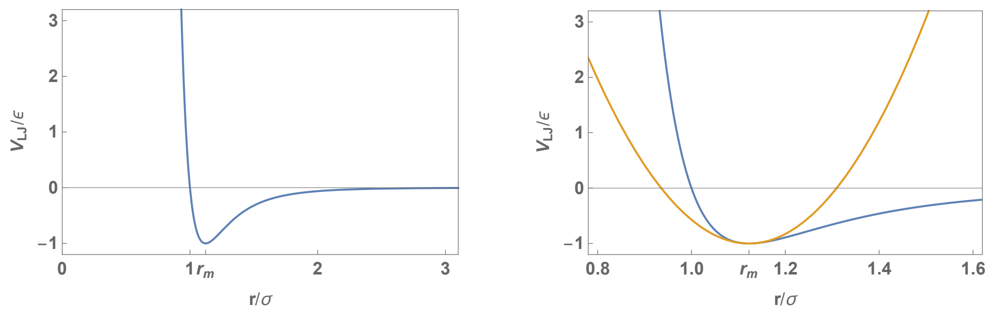


Figure 7 – The Lennard-Jones potential. Left: the potential with minimum at $r_{\rm m}$. Right: the parabolic approximation of the Lennard-Jones potential around $r_{\rm m}$ that leads to a linear restoring force.

A slight change in distance between them will increase the potential energy, and there will be a force restoring the atoms to a distance $r_{\rm m}$. We find the force from

$$F(r) = -\frac{{\rm d}V}{{\rm d}r} = -12\epsilon\left[\frac{r_{\rm m}^6}{r^7} - \frac{r_{\rm m}^{12}}{r^{13}}\right]. \tag{6.2}$$

This is still far too complicated to substitute into Newton's law, so we will approximate this force with a Taylor series around $r_{\rm m}$, where we are interested in the physical behaviour of the atoms:

$$F(r) = F(r_{\rm m}) + (r - r_{\rm m})\left.\frac{{\rm d}F}{{\rm d}r}\right|_{r=r_{\rm m}} + \frac{(r-r_{\rm m})^2}{2}\left.\frac{{\rm d}^2F}{{\rm d}r^2}\right|_{r=r_{\rm m}} + \dots \tag{6.3}$$

We can take the derivatives of F with respect to r and substitute $r = r_{\rm m}$:

$$F(r_{\rm m}) = 0 \qquad \text{and} \qquad \left.\frac{{\rm d}F}{{\rm d}r}\right|_{r=r_{\rm m}} = -12\epsilon\left(\frac{13r_{\rm m}^{12}}{r^{14}} - \frac{7r_{\rm m}^6}{r^8}\right)_{r=r_{\rm m}} = -\frac{72\epsilon}{r_{\rm m}^2} \tag{6.4}$$

We need the force only up to the first non-zero term, the linear term. This corresponds to the quadratic potential around $r_{\rm m}$:

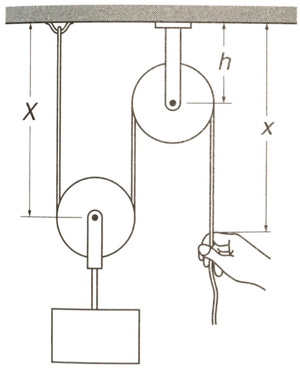
$$F(r) \simeq -(r - r_{\rm m})\frac{72\epsilon}{r_{\rm m}^2}. \tag{6.5}$$

We can define $k = 72\epsilon/r_{\rm m}^2$ and the radial displacement from the equilibrium position $x = r - r_{\rm m}$. The force is then in a form that we can use in Newton's second law:

$$F(x) \simeq -kx. \tag{6.6}$$

This is formally identical to Hooke's law, and the solutions to the differential equation are therefore given by equation (2.20). The two atoms thus oscillate harmonically towards and away from each other around the equilibrium separation r_{m}.

Example 2: Consider a system with two pulleys as shown in the figure below. Calculate the force needed to lift a load with mass m using this pulley system. You will need to consider the acceleration of the load for the two coordinates x and X.



We work out the acceleration $\ddot{x}$ of the hand, which is of course related to the force with which the hand pulls ($F_h = m\ddot{x}$), and compare it to the acceleration of the block $\ddot{X}$. Assuming that the length of the string is a constant l and the radius of the wheels is R, we know that

$$l = X + \pi R + (X - h) + \pi R + (x - h)\,.$$

Rearranging this, we obtain a relation between x and X:

$$x = l - 2(X + \pi R - h)\,.$$

Taking the second derivative to obtain the relationship between the accelerations of the hand and the mass, we find

$$\ddot{X} = -\frac{1}{2}\ddot{x}\,.$$

To raise the block a distance s, the hand must pull down the rope *twice* that distance downwards. Note the relative signs and the direction of pulling and lifting. To work out the force with which the hand must pull, we can use the work done on the block:

$$W = Fs = mgs\,.$$

By conservation of energy, the work done by the hand is equal to the work done on the block, and we find

$$W_h = F_h \times 2s = mgs \qquad \Rightarrow \qquad F_h = \frac{1}{2}mg\,.$$

In other words, we have to apply only *half* the force that we would need if we were to lift the block directly, but the trade-off is that we need to pull the rope *twice* the distance we want to move the block. Using more pulleys you can lift heavier objects with the same force.

Example 3: We want to calculate the terminal velocity of a sphere with mass m in a fluid under gravity. We can start with the exercise in problem set 2, where we looked at the drag force. Here, we have a drag force against the direction of the velocity v:

$$F_{\text{drag}} = -bv\,,$$

as well as the gravitational force $F_{\text{gr}} = mg$ due to the gravitational acceleration g. The net force experience by the sphere is The sum of these two forces,

$$F_{\text{net}} = F_{\text{gr}} + F_{\text{drag}} = mg - bv\,.$$

The positive direction is chosen downwards. We set this equal to $F_{\text{net}} = ma$, but instead of finding the position $x(t)$, we want to find a differential equation for the velocity $v(t)$. That is actually easier:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{b}{m}v + g\,. \tag{6.7}$$

To solve a differential equation, we often make a so-called *Ansatz* (German), or 'guess'. Here, we will try

$$v(t) = u + w\mathrm{e}^{-st}\,,$$

where we need to find u, w and s. First we calculate $\dot{v}(t)$:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -ws\,\mathrm{e}^{-st}\,,$$

and we substitute this back into the differential equation (6.7):

$$-ws\,\mathrm{e}^{-st} = -\frac{b}{m}\left(u + w\mathrm{e}^{-st}\right) + g\,.$$

This can only be a valid equation if the factors in front of the exponentials are the same, and the remaining constants sum to zero:

$$-ws = -\frac{b}{m}w \qquad \text{and} \qquad -\frac{b}{m}u + g = 0\,.$$

From this we find that

$$s = \frac{b}{m} \qquad \text{and} \qquad u = \frac{mg}{b}\,.$$

We do not know what w is yet. To find it, we need to use the initial condition for the differential equation, e.g., $v(t=0) = v_0$. The we find

$$v(t) = \frac{mg}{b} + w\,\mathrm{e}^{-bt/m}\,.$$

Hence

$$v(0) = \frac{mg}{b} + w\,\mathrm{e}^{0} = v_0 \qquad \to \qquad w = v_0 - \frac{mg}{b}\,.$$

This leads to the final result

$$v(t) = \frac{mg}{b}\left(1 - \mathrm{e}^{-bt/m}\right) + v_0\,\mathrm{e}^{-bt/m}\,. \tag{6.8}$$

When $g = 0$ we recover the result from problem set 2, as required. But more interestingly, we can find the terminal velocity by taking the limit of $t \to \infty$. The exponetials will tend to zero in this limit, so we have

$$v_{\text{terminal}} = \lim_{t\to\infty} v(t) = \frac{mg}{b}\,.$$

This is independent of the initial velocity v_0, but it does depend on b, which itself captures all the relevant information about the shape and density of the object, as well as the viscosity of the fluid or gas.

Lecture Reading Week Revision

You should be comfortable with the following topics:

Kinematics

1. Position of a moving particle is a three-dimensional vector as a function of time.
2. The velocity and acceleration can be found by successive differentiation with respect to time.
3. The centre of mass is the mass-weighted average of the positions of the objects.
4. Elastic collisions obey both energy and momentum conservation.
5. Inelastic collisions obey only momentum conservation.
6. Polar coordinates, and the derivatives of the polar unit vectors.

Dynamics

1. Force as a vector, and Newton's first law.
2. Newton's second law: $\mathbf{F} = \dot{\mathbf{p}}$.
3. Newton's third law of motion: action equals opposite reaction.
4. Conservative forces as the spatial derivative of a potential energy.
5. The $1/r$ potential and Newton's law of gravitation.
6. Equations of motion as differential equations determined by force balance and potential.
7. Work as force applied along the distance of travel.
8. Work-energy theorem.

Lecture 7 Polar coordinates and circular motion

Key concept: Rotational motion is described more easily in polar coordinates. The cross product with the angular velocity ω then becomes a convenient shortcut for the time derivative in uniform circular motion.

7.1 Polar coordinates

When an object moves in a circle in the xy-plane around the origin O, the x and y coordinates as a function of time are given by

$$x(t) = a\cos\omega t \qquad \text{and} \qquad y(t) = a\sin\omega t\,, \tag{7.1}$$

where in this case the object lies on the x-axis at time $t = 0$, and the radius of the circle is a. The *angular* velocity of the particle is ω, to which we will return in a moment. The x and y coordinates are components of a two-dimensional vector

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \tag{7.2}$$

as we encountered before. While the Cartesian coordinates x and y (and z in three dimensions) are often very convenient, for circular motion there are other coordinates that are more convenient, namely *polar* coordinates r and θ. Here r is the distance to the origin, and θ is the angle between the x-axis and the line from the origin to the object's position. The reason these polar coordinates are more convenient is that for the circular motion above the distance to the origin stays constant ($r = a$), so we should be able to reduce the two-coordinate problem (x and y) to a single-coordinate problem (θ only).

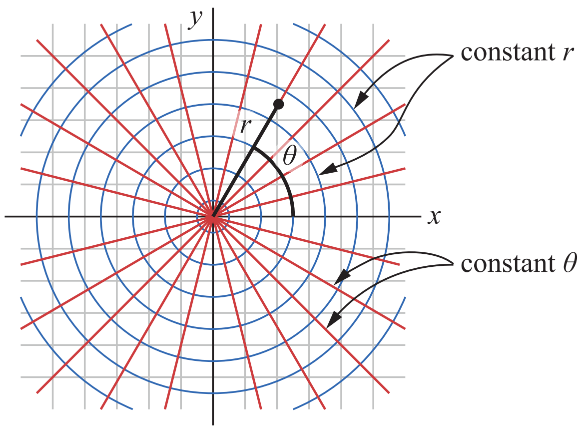


Figure 8 – Polar coordinates against a grid of Cartesian coordinates.

The relation between the Cartesian coordinates and the polar coordinates is given by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta\,. \tag{7.3}$$

Note that these are always true, and carry a completely different meaning than the expressions in equation (7.1), which look superficially similar. Equation (7.1) describes the motion of a particle over time, while equation (7.3) describes the relation between two coordinate systems.

We can invert equation (7.3) to get r and θ in terms of x and y:

$$r = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \theta = \arctan\left(\frac{y}{x}\right). \tag{7.4}$$

This means that r runs from zero to infinity (we take the positive square root), and θ runs from zero to 2π[2]:

$$r \in [0, \infty) \qquad \text{and} \qquad \theta \in [0, 2\pi)\,.$$

We can write the motion of the object in equation (7.1) in terms of polar coordinates as

$$r(t) = a = \text{constant} \qquad \text{and} \qquad \theta(t) = \omega t\,. \tag{7.5}$$

This means, as noted before, that we have to worry only about θ as a variable with respect to time. More general motion will again require two variables r and θ, or x and y. As a rule of thumb, if the motion looks more like linear motion (but not quite), Cartesian coordinates may be better, and if it is closer to circular motion polar coordinates are likely better.

Returning to our example, we have to consider the basis vectors of our coordinate systems. For the Cartesian system this is simple:

$$\mathbf{r}(t) = \begin{pmatrix} a\cos\omega t \\ a\sin\omega t \end{pmatrix} = a\cos\omega t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a\sin\omega t \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a\cos\omega t\; \hat{\mathrm{e}}_x + a\sin\omega t\; \hat{\mathrm{e}}_y\,, \tag{7.6}$$

where we introduced the basis vectors

$$\hat{\mathrm{e}}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \hat{\mathrm{e}}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{7.7}$$

Wherever we are in the xy-plane, the basis vector $\hat{\mathrm{e}}_x$ always points in the direction of the positive x-axis, and the basis vector $\hat{\mathrm{e}}_y$ always points in the direction of the positive y-axis. However, the basis vectors $\hat{\mathrm{e}}_r$ and $\hat{\mathrm{e}}_\theta$ point in different directions at different locations in the plane. The vector $\hat{\mathrm{e}}_r$ always points away from the origin, while $\hat{\mathrm{e}}_\theta$ is orthogonal to $\hat{\mathrm{e}}_r$ and lies tangential to the circle centred around the origin and intersecting the point (r, θ). It points in the counter-clockwise direction.

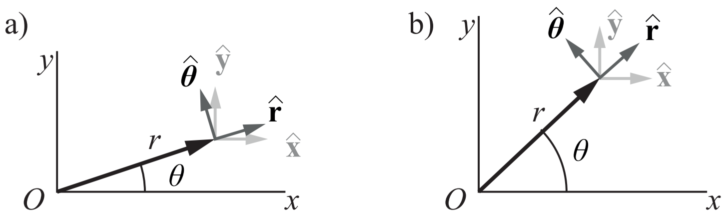


Figure 9 – Changing basis vectors in polar coordinates.

If you want to know the basis vectors $\hat{\mathrm{e}}_r$ and $\hat{\mathrm{e}}_\theta$ in terms of $\hat{\mathrm{e}}_x$ and $\hat{\mathrm{e}}_y$, you have to use trigonometry. You can readily infer from figure 9 that this gives

$$\hat{\mathrm{e}}_r = \cos\theta\; \hat{\mathrm{e}}_x + \sin\theta\; \hat{\mathrm{e}}_y \qquad \text{and} \qquad \hat{\mathrm{e}}_\theta = -\sin\theta\; \hat{\mathrm{e}}_x + \cos\theta\; \hat{\mathrm{e}}_y\,. \tag{7.8}$$

We will use these relations a lot in what follows. The changing directions of $\hat{\mathrm{e}}_r$ and $\hat{\mathrm{e}}_\theta$ become particularly important when we take derivatives, for example when we want to work out the velocity and acceleration of an object. We will show how this works in a moment. First, what does a vector $\mathbf{r}$ look like in polar coordinates? It is remarkably compact:

$$\mathbf{r} = r\,\hat{\mathrm{e}}_r = a\,\hat{\mathrm{e}}_r\,, \tag{7.9}$$

where the second equality is specifically for our example. You should convince yourself by drawing $\mathbf{r}$ and $\hat{\mathrm{e}}_r$ that there is no $\hat{\mathrm{e}}_\theta$ contribution in the expression of the general vector $\mathbf{r}$ in polar coordinates.

[2]This is another difference between equation (7.1) and equation (7.3): the mapping between coordinate systems has to make sure the plane is covered exactly once by each coordinate system, whereas ωt can take values much greater than 2π as it describes the trajectory of a particle going around in circles indefinitely.

7.2 Derivatives of polar coordinates

For the object in circular motion in equation (7.1), we want to know what is its velocity. In Cartesian coordinates this follows the usual procedure. We use the recipe from the third lecture and work out the time derivative of $\mathbf{r}$:

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a\cos\omega t \\ a\sin\omega t \end{pmatrix} = \begin{pmatrix} -a\omega\sin\omega t \\ a\omega\cos\omega t \end{pmatrix} . \tag{7.10}$$

We have to take derivatives of sines and cosines, but we know how to do that. The velocity of the particle has a constant magnitude

$$\begin{aligned} v = |\mathbf{v}(t)| &= \sqrt{(-a\omega\sin\omega t)^2 + (a\omega\cos\omega t)^2} = a\omega\sqrt{\sin^2\omega t + \cos^2\omega t} \\ &= a\omega . \end{aligned} \tag{7.11}$$

This is how the *angular* velocity ω relates to the regular velocity v. The greater the radius a of the circle, the larger the velocity v must be to keep up with the same angular velocity ω. The direction of $\mathbf{v}$ changes over time (as you can clearly visualise for circular motion), and in terms of Cartesian coordinates we can write this as

$$\mathbf{v} = -v\sin\omega t\ \hat{\mathsf{e}}_x + v\cos\omega t\ \hat{\mathsf{e}}_y , \tag{7.12}$$

where $\hat{\mathsf{e}}_x$ and $\hat{\mathsf{e}}_y$ are again (still!) constant at every position, and therefore constant over time, as our object moves from point to point in the plane. From equation (7.8) we can see that the velocity of the circular motion always points in the tangential θ direction:

$$\mathbf{v} = v\,\hat{\mathsf{e}}_\theta . \tag{7.13}$$

This reflects the fact that the basis vector $\hat{\mathsf{e}}_\theta$ changes over time, since the direction of $\mathbf{v}$ changes over time but the magnitude v is constant. This is a special case of the more general result we found in lecture 3, where the velocity is always tangent to the path of a particle.

Next, let's consider the velocity of the object in polar coordinates. We take the time derivative of $\mathbf{r}$ in equation (7.9) to get

$$\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = a\frac{\mathrm{d}\hat{\mathsf{e}}_r}{\mathrm{d}t} , \tag{7.14}$$

where we used that a is a constant. Clearly, we need to know how to take the time derivative of the basis vector $\hat{\mathsf{e}}_r$. It cannot be zero, since that would make the velocity zero, and that is obviously wrong. The time derivative needs to be expressed in terms of $\hat{\mathsf{e}}_r$ and $\hat{\mathsf{e}}_\theta$ themselves (which we can always do, because r and θ form a proper coordinate system for any vector). We can start with equation (7.8) and use the fact that $\hat{\mathsf{e}}_x$ and $\hat{\mathsf{e}}_y$ do not change and thus their time derivative is zero:

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathsf{e}}_r}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\cos\theta\ \hat{\mathsf{e}}_x + \sin\theta\ \hat{\mathsf{e}}_y\right) = -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\ \hat{\mathsf{e}}_x + \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\ \hat{\mathsf{e}}_y \\ &= \frac{\mathrm{d}\theta}{\mathrm{d}t}\left(-\sin\theta\ \hat{\mathsf{e}}_x + \cos\theta\ \hat{\mathsf{e}}_y\right) = \frac{\mathrm{d}\theta}{\mathrm{d}t}\ \hat{\mathsf{e}}_\theta . \end{aligned} \tag{7.15}$$

Similarly,

$$\begin{aligned} \frac{\mathrm{d}\hat{\mathsf{e}}_\theta}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(-\sin\theta\ \hat{\mathsf{e}}_x + \cos\theta\ \hat{\mathsf{e}}_y\right) = -\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\ \hat{\mathsf{e}}_x - \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\ \hat{\mathsf{e}}_y \\ &= -\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\cos\theta\ \hat{\mathsf{e}}_x + \sin\theta\ \hat{\mathsf{e}}_y\right) = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\ \hat{\mathsf{e}}_r . \end{aligned} \tag{7.16}$$

We collect these two important results in the following equation:

$$\frac{\mathrm{d}\hat{\mathrm{e}}_r}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\,\hat{\mathrm{e}}_\theta \qquad \text{and} \qquad \frac{\mathrm{d}\hat{\mathrm{e}}_\theta}{\mathrm{d}t} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\hat{\mathrm{e}}_r\,. \tag{7.17}$$

These are the same relations between $\hat{e}_T$ and $\hat{e}_N$ in the third lecture. Remembering that $\theta(t) = \omega t$, and substituting equation (7.17) in the expression for $\mathbf{v}$, we have

$$\mathbf{v}(t) = a\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\hat{\mathrm{e}}_\theta = a\omega\,\hat{\mathrm{e}}_\theta\,, \tag{7.18}$$

as required. Notice how the actual calculation in equation (7.14) using equation (7.17) and equation (7.5) is computationally very simple—much simpler than the derivatives of the sines and cosines! Here it is again in a streamlined version:

> The polar coordinates of the object are given by
>
> $$r = a \qquad \text{and} \qquad \theta = \omega t\,.$$
>
> The velocity $\mathbf{v}$ of the object is
>
> $$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = a\frac{\mathrm{d}\hat{\mathrm{e}}_r}{\mathrm{d}t} = a\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\hat{\mathrm{e}}_\theta = a\omega\,\hat{\mathrm{e}}_\theta\,.$$
>
> That's it. From this we can read off immediately that the speed is $v = a\omega$.

This calculation is so much simpler because the coordinate system matches the shape of the motion. You may object that equation (7.17) was hard to find, but that is a *general* result, which holds for any motion in polar coordinates. You may still rather do the familiar calculus with sines and cosines in this case, but when we have non-uniform acceleration around the circle the calculation in Cartesian coordinates quickly become intractable. For example, consider the case where the rotation speed increases linearly with time ($\theta = \frac{1}{2}\alpha t^2$). Calculate the vector $\mathbf{v}(t)$ in Cartesian coordinates and in polar coordinates and see the difference. Another example is the Archimedean spiral, where the radius increases linearly with θ.

7.3 Circular motion

For uniform rotations in *three* dimensions, where the angular velocity ω is constant, there is another convenient and often-used description of circular motion that involves the *cross product*. The cross product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is written as

$$\mathbf{a}\times\mathbf{b} = (a_y b_z - a_z b_y)\hat{\mathrm{e}}_x + (a_z b_x - a_x b_z)\hat{\mathrm{e}}_y + (a_x b_y - a_y b_x)\hat{\mathbf{z}}\,. \tag{7.19}$$

This can be written as

$$\mathbf{a}\times\mathbf{b} = ab\sin\theta\,\hat{e}_N\,, \tag{7.20}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{e}_N$ is the vector normal to the plane spanned by $\mathbf{a}$ and $\mathbf{b}$. This plane has an orientation: if the fingers on your right hand point in the direction as you rotate the vector $\mathbf{a}$ to vector $\mathbf{b}$, your thumb will point in the direction of $\hat{e}_N$.

Define the angular velocity ω as

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}\,, \tag{7.21}$$

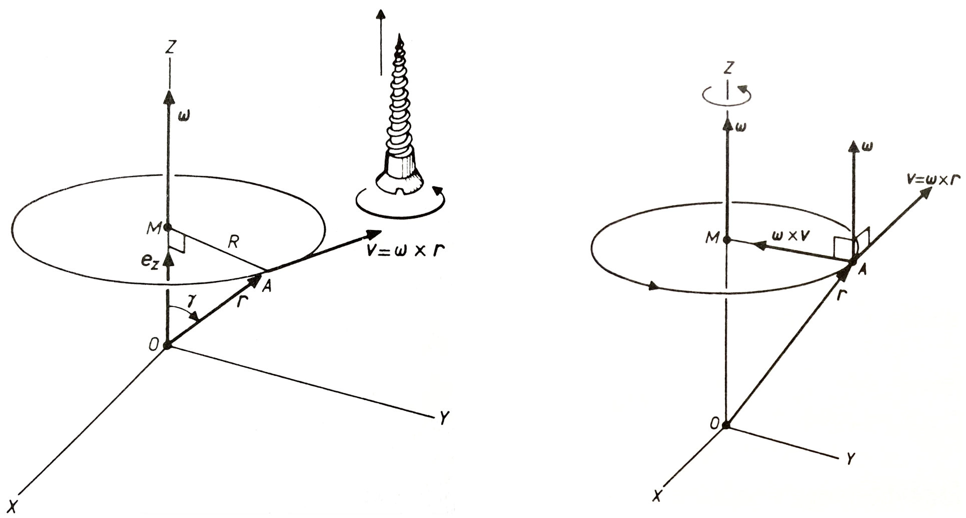


Figure 10 – Lewft: vector relation between (constant) angular velocity $\boldsymbol{\omega}$, linear velocity $\mathbf{v}$ and position $\mathbf{r}$. Right: acceleration towards the centre M.

as before. The speed along a circle with radius R is then

$$v = \omega R\,. \tag{7.22}$$

However, in three dimensions not all possible points lie in the xy-plane, and we want a relation between $\mathbf{v}$, $\mathbf{r}$, and ω. To this end we orient ω along the z-axis, so we can write $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, and we pick a point A along a circle with radius R above the xy-plane. The position vector pointing to A is $\mathbf{r}$. This is shown in Fig. 10.

From the figure, $R = r\sin\gamma$, with γ the angle between the vector $\hat{\mathbf{z}}$ and $\mathbf{r}$. Hence, the linear speed around the circle is

$$v = \omega r \sin\gamma\,. \tag{7.23}$$

However, this can be written as a cross product between $\boldsymbol{\omega}$ and $\mathbf{r}$:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}\,. \tag{7.24}$$

This is true only for circular motion with constant r and γ. While this is neat and compact, it is not so clear yet why this is a nice form. However, it allows us to treat the time derivative $\mathrm{d}/\mathrm{d}t$ in the case of constant circular motion as the cross product with the angular velocity: $\boldsymbol{\omega} \times \cdot$. We can find the acceleration from the time derivative of $\mathbf{v}$, and therefore for circular motion with constant ω we obtain

$$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{v}\,. \tag{7.25}$$

In its most useful form, we have

$$\mathbf{a} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,. \tag{7.26}$$

7.4 Spherical and cylindrical coordinates

In three dimensions, there are two main sets of coordinates associated with rotational motion: cylindrical and spherical. Their names give you a clue when you want to use what. These coordinate systems will be discussed in detail in the rest of your degree, so here we only give their relation to the Cartesian coordinates.

Cylindrical coordinates are very much like polar coordinates, but supplemented with a "Cartesian" third dimension, z. Instead of r as the distance to the origin, we typically use ρ and ϕ. Do not confuse ρ with a density!

$$\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \\ z &= z \,. \end{aligned} \tag{7.27}$$

The last one feels a bit redundant, but it reminds us that there is a third dimension. You should work out ρ and ϕ in terms of x, y, and z.

Spherical coordinates are often the coordinates of choice when the potential depends only on r, the distance to the origin. They are defined by the relations

$$\begin{aligned} x &= r \cos\phi \sin\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\theta \,. \end{aligned} \tag{7.28}$$

You should construct the inverse relation, specifying r, θ, and ϕ in terms of x, y, and z.

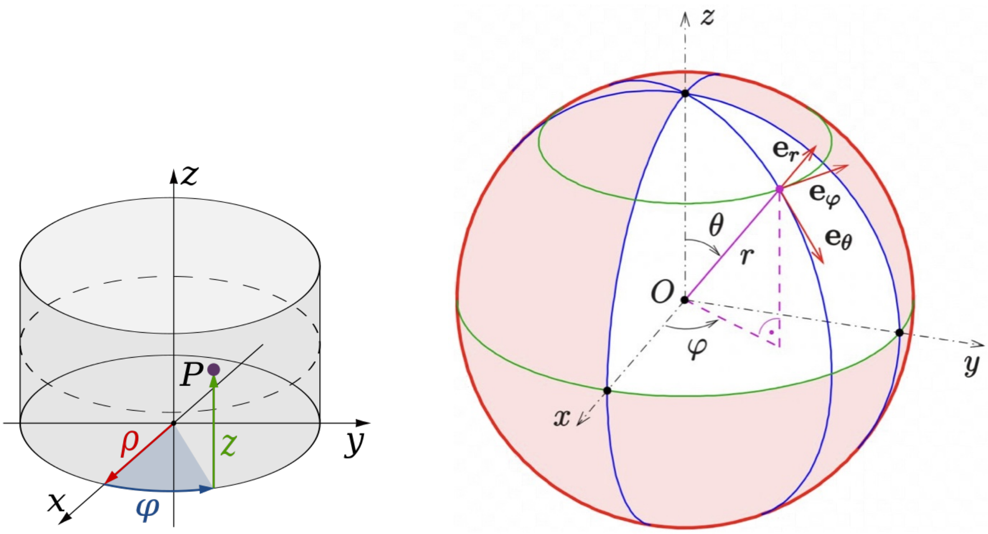


Figure 11 – Cylindrical and spherical coordinates.

Lecture 8 Angular momentum

Key concept: Angular momentum is the amount of motion associated with rotations. It is conserved in systems that experience only central forces, like gravity or electrical forces in atoms. It can be used to analyse scattering problems and orbital mechanics.

As we have seen, the momentum $\mathbf{p}$ of a body with mass m is the amount of *linear* motion of the body. To change this amount of linear motion, we need to apply a force $\mathbf{F} = \mathrm{d}\mathbf{p}/\mathrm{d}t$. The situation changes somewhat when we consider *rotational* motion. We cannot merely apply a force to set a body in motion, we need a *moment*, or *torque* $\boldsymbol{\tau}$. You are already familiar with torque: a lever or see-saw has two forces (e.g., gravity at the end of the see-saw and the normal force at the fulcrum), in opposite direction and with a distance d apart. You have been writing this as $\tau = Fd$. Now we will give this expression its proper vector form[3].

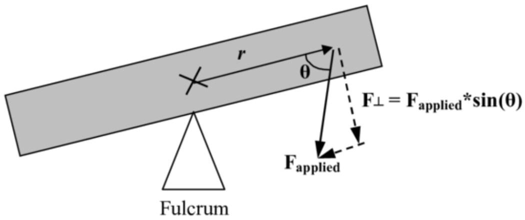


Figure 12 – Torque as a combination of vectors $\mathbf{r}$ and $\mathbf{F}_{\text{applied}}$.

Consider the see-saw. Let $\mathbf{F}$ be the force down on the tip of the see-saw, and let $\mathbf{r}$ be the distance vector from the fulcrum to the top of the see-saw where the force $\mathbf{F}$ is applied. This will create a torque $\boldsymbol{\tau}$ that points in the direction of the angular velocity $\boldsymbol{\omega}$. All these vectors are perpendicular, so we must use the cross product to relate them. When we work out the directions (do this!) we find

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{8.1}$$

We can write the time derivative of $\mathbf{p}$ in terms of the chain rule on the whole cross product $\mathbf{r} \times \mathbf{p}$ using

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times \mathbf{p} + \mathbf{r} \times \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{8.2}$$

The term $\dot{\mathbf{r}} \times \mathbf{p}$ is zero because $\dot{\mathbf{r}} = \mathbf{v}$ and $\mathbf{p} = m\mathbf{v}$ are parallel, and we can therefore write

$$\tau = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \times \mathbf{p}) \equiv \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \qquad \text{with} \qquad \mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{8.3}$$

The quantity $\mathbf{L}$ is called *angular momentum*, and it is the rotational equivalent to the linear momentum $\mathbf{p}$. The expression $\boldsymbol{\tau} = \dot{\mathbf{L}}$ is the analog of $\mathbf{F} = \dot{\mathbf{p}}$ for rotational motion.

Just like the case where an absence of external forces leads to conservation of momentum:

$$\mathbf{F} = 0 = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \qquad \Rightarrow \qquad \mathbf{p} \text{ is constant in time,} \tag{8.4}$$

so does an absence of torque lead to conservation of angular momentum:

$$\boldsymbol{\tau} = 0 = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} \qquad \Rightarrow \qquad \mathbf{L} \text{ is constant in time.} \tag{8.5}$$

[3]Not least because in this form Fd could be mistaken for work.

One example of this is planetary motion. The only force on a planet orbiting a star is gravity, which always points along the line connecting the planet and the star at every moment in time. Therefore, the cross product $\mathbf{r}\times\mathbf{F} = \boldsymbol{\tau}$ is always zero ($\mathbf{F}$ is in the direction of $\mathbf{r}$). As a consequence, the angular momentum of a planet in orbit around a star is conserved. Specifically the angular momentum at the aphelion is equal to the angular momentum at the perihelion:

$$\mathbf{L} = \mathbf{r}_{\text{ap}} \times \mathbf{p}_{\text{ap}} = \mathbf{r}_{\text{peri}} \times \mathbf{p}_{\text{peri}} \,. \tag{8.6}$$

At both aphelion and perihelion the position and momentum vectors are perpendicular, so we can ignore the $\sin\theta$ term in the cross product. We then have

$$r_{\text{ap}} p_{\text{ap}} = r_{\text{peri}} p_{\text{peri}} \,, \tag{8.7}$$

where we use the convention that non-bold symbols signify the magnitude.

It is because of this conservation law that angular momentum is so important and ubiquitous in physics. The orbital motion of electrons in atoms is classified by energy and angular momentum, and the entire periodic table is a direct consequence of the allowed energy and angular momentum values of the electrons. Rotating black holes have angular momentum, and this creates interesting frame dragging effects near their horizons. Gyroscopes keep their orientation in space because of conservation of angular momentum, and this allows for navigation at sea in the absence of external signals (such as the position of the stars on cloudy nights). These are just a few examples of the central importance of angular momentum in physics. You will keep encountering this topic throughout your studies.

The expression $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ depends explicitly on $\mathbf{r}$, and therefore $\mathbf{L}$ must always be given relative to the coordinate origin of $\mathbf{r}$. Often the choice of origin is so obvious that people don't specify it, but you should always make sure that you know which origin is used to calculate the angular momentum. In atomic and planetary systems it is typically the centre of mass.

As an example, let's consider simple circular motion, where $\mathbf{r}$ is the position vector of a particle of mass m moving in a circle around the origin, and $\mathbf{p}$ is the momentum that is directed in the tangent direction to the circle. The origin of $\mathbf{r}$ is the centre of the circle. Since $\mathbf{r}$ and $\mathbf{p}$ are perpendicular, we can write the magnitude of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ as

$$L = rp = mrv = mr^2\omega \,, \tag{8.8}$$

where we preferred ω to v because the angular velocity is more natural to use here ($v = r\omega$). From the cross product we deduce that $\mathbf{L}$ is in the same direction as $\boldsymbol{\omega}$, so we can write

$$\mathbf{L} = mr^2\boldsymbol{\omega} \,. \tag{8.9}$$

Next, consider a more general curved motion, shown in Fig. 13. At any point along the path the velocity $\mathbf{v}$ will have a radial and a transversal component, $\mathbf{v}_r$ and $\mathbf{v}_\theta$, which sum together to form $\mathbf{v} = \mathbf{v}_r + \mathbf{v}_\theta$. The angular momentum can then be written as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v} = m\mathbf{r} \times (\mathbf{v}_r + \mathbf{v}_\theta) = m\mathbf{r} \times \mathbf{v}_r + m\mathbf{r} \times \mathbf{v}_\theta = m\mathbf{r} \times \mathbf{v}_\theta \,. \tag{8.10}$$

The last equality holds because $\mathbf{v}_r$ is parallel to $\mathbf{r}$, by definition, and the cross product between the two must therefore be zero. Hence $\mathbf{L}$ always captures only the transversal part of the momentum. Also, since $v_\theta = r\mathrm{d}\theta/\mathrm{d}t$, we can write the magnitude in the most general form

$$L = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \,. \tag{8.11}$$

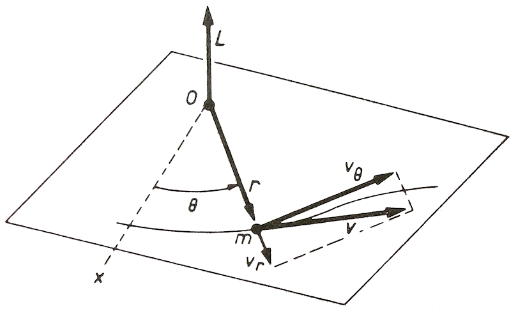


Figure 13 – Angular momentum of a particle moving along a general curved path.

When the angular velocity is constant, this reduces to equation (8.8).

Another consequence of the conservation of angular momentum is that the direction of **L** does not change. This means that the motion must be confined to a plane perpendicular to **L**. Translating this to the orbital motion of planets, each planet must always move in a plane. Without external disturbances, no planet can drift out of this plane as it orbits the Sun.

Finally, we consider the important example of the scattering of particles from a central potential, shown in Fig. 14. We assume that the central potential falls off with $1/r$ (for example the electrostatic potential of a point charge), so that the force is given by $F = k/r^2$. We want to know how the scattering angle ϕ depends on the scattering parameter b, the mass m of the particle, and its initial velocity v_0. First, we calculate the angular momentum of the particle. In point A in Fig. 14, the angular momentum is (see weekly practice problems):

$$L = mv_0 b\,. \tag{8.12}$$

Then, at some other (arbitrary) point M, according to equation (8.11) the angular momentum must be equal to

$$L = mr^2 \frac{\mathrm{d}\theta}{\mathrm{d}t}\,. \tag{8.13}$$

Conservation of angular momentum applies, since F is a central force, so

$$mr^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} = mv_0 b\,. \tag{8.14}$$

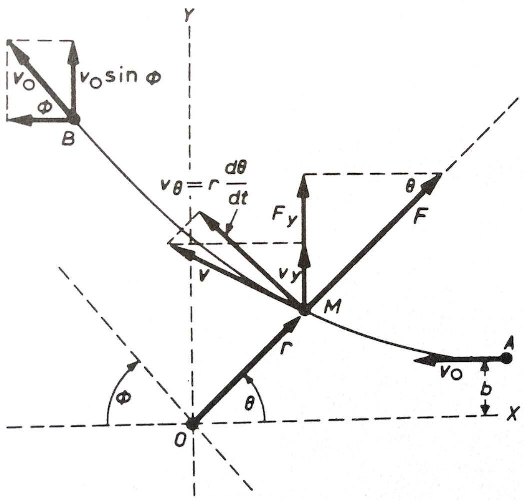


Figure 14 – Scattering from a $1/r$ potential

Next, we consider the vertical force

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = F_y = F\sin\theta = \frac{k\sin\theta}{r^2}\,. \tag{8.15}$$

This allows us to eliminate r^2 from equation (8.14):

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{k\sin\theta}{mv_0 b}\frac{\mathrm{d}\theta}{\mathrm{d}t}\,. \tag{8.16}$$

As the particle comes in from $t \to -\infty$, scatters, and whizzes off to $t \to +\infty$, it changes direction from a horizontal trajectory to an asymptotic trajectory at an angle ϕ. We can therefore integrate both sides of equation (8.19) from the initial value of v_y and θ to their final values:

$$\int_0^{v_0\sin\phi} \mathrm{d}v_y = \frac{k}{mv_0 b}\int_0^{\pi-\phi} \sin\theta\,\mathrm{d}\theta\,. \tag{8.17}$$

Note the limits of integration: as time runs from $-\infty$ to $+\infty$, the vertical component of the velocity v_y goes from zero to $v_0\sin\phi$, and the angle θ goes from zero to $\pi - \phi$ (see Fig. 14). Solving these integrals is straightforward, and yields

$$v_0\sin\phi = \frac{k}{mv_0 b}(1+\cos\phi)\,. \tag{8.18}$$

Extracting ϕ from this is not entirely trivial, but can be done using the co-tangens:

$$\cot\frac{\phi}{2} = \frac{mv_0^2 b}{k}\,. \tag{8.19}$$

Hence the potential $V = k/r$ predicts a very specific scattering angle for the various velocities v_0 and scattering parameters b. A different potential, for example $V = k/r^n$, would have given rise to a different relationship between the scattering angle and v_0 and b. This is why scattering experiments are used in particle physics to understand the inner workings of subatomic particles. It allowed Rutherford to deduce from his scattering experiments that the atom consists of a massive nucleus surrounded by light electrons.

A note on energy conservation: We could not solve this problem using energy conservation, since the balance between kinetic and potential energy depends on the distance of the particle to the centre of the potential. Energy is still conserved, but here it did not give us the information we needed. This underscores the importance of angular momentum.

Lecture 9 Rotating rigid bodies

Key concept: The shape of a body has influence on its resistance against changes in rotational motion. This captured by the momentum of inertia. It is the rotational analog of mass as the resistance against changes in linear motion.

9.1 Rigid bodies and the moment of inertia

A rigid body is a body that has some finite extended shape—so not a point particle, but a shape that does not deform under reasonable forces. When we want to describe the motion of such bodies, particularly rotations, their shape becomes important. As an example, consider a uniform solid cylinder with height h, radius R, and mass M. We can set it to rotate with angular velocity ω around its symmetry axis, namely the axis through the centre perpendicular to the circular cross section. This requires a torque, and the rotating cylinder has some angular momentum L relative to the origin at the centre of mass that lies on the rotation axis. The direction of L is along the rotation axis.

Next, consider a *hollow* cylinder with otherwise identical specs: height h, radius R, and mass M. Now all the mass is concentrated in the outside layer. For simplicity we assume that the end caps are also hollow, so the object is a tube or pipe. When we set this object spinning around the same axis with the same angular velocity ω, does this object gain the same angular momentum as the solid cylinder? You may already intuit that the answer has to be 'no'. The solid cylinder has some of its mass closer to the rotation axis, requiring a smaller $\mathbf{r}$ in $\mathbf{L} = \mathbf{r}\times\mathbf{p}$, and a correspondingly lower tangential velocity $\mathbf{v}$. We need to find out how we can calculate this.

We can think of a rigid body as a collection of small masses put together. Each of these masses m_j has its own angular momentum

$$\mathbf{L}_j = m_j r_j^2 \boldsymbol{\omega}\,. \tag{9.1}$$

For simplicity, assume that $\boldsymbol{\omega}$ is pointing in the z-direction. Then $L_j = m_j r_j^2 \omega$. Note that ω does not depend on j; all elements move at the same angular velocity. This is what makes it a rigid body. When we put all the small masses together, the total angular momentum is the sum of all the individual angular momenta:

$$L = \sum_j L_j = \left(\sum_j m_j r_j^2\right)\omega \equiv I\omega\,. \tag{9.2}$$

Here we defined a new quantity, the *moment of inertia* I that captures the effect of the shape of the rotating object on the angular momentum.

Example 1: *The figure skater*. You have likely seen the effect of a spinning ice skater who pulls in her arms to spin faster. We can easily explain this effect using the changing moment of inertia. In its most basic form, the angular momentum of the skater is conserved, since there are no torques acting on her. When she has her arms spread wide, the angular momentum is

$$L = I_{\text{wide}}\omega_{\text{wide}}\,,$$

and when she has her arms tucked in close, her angular momentum is

$$L = I_{\text{close}}\omega_{\text{close}}\,.$$

Conservation of angular momentum then determines that

$$I_{\text{wide}}\omega_{\text{wide}} = I_{\text{close}}\omega_{\text{close}} \qquad \text{or} \qquad \omega_{\text{close}} = \frac{I_{\text{wide}}}{I_{\text{close}}}\omega_{\text{wide}}\,. \tag{9.3}$$

When $I_{\text{wide}} > I_{\text{close}}$, the angular velocity ω_{close} *increases*.

Before we calculate I for a variety of shapes, consider the parallels between $\mathbf{p} = m\mathbf{v}$ and $\mathbf{L} = I\boldsymbol{\omega}$. Both $\mathbf{p}$ and $\mathbf{L}$ measure the amount of motion of a body, linear and rotational, respectively. The linear and angular velocities $\mathbf{v}$ and $\boldsymbol{\omega}$ also clearly play analogous roles. This suggests an interpretation of I by analogy. Recall that m is the inertial mass, i.e., the resistance of a body to change its (linear) motion. Similarly, I is the resistance of a rigid body to a change in rotational motion. Hence the name 'moment of inertia'.

9.2 Calculating the moment of inertia

For a collection of particles with mass m_j that all rotate around the same axis with angular velocity ω, the moment of inertia is immediately calculated by summing the masses times the distance-squared to the rotation axis:

$$I = \sum_j m_j r_j^2 \,. \tag{9.4}$$

Things become a bit more involved when we have a solid body like a disk, cylinder or sphere. Let's assume that the density of the object at a position $\mathbf{r}$ is given by $\rho(\mathbf{r})$. Then a 'small mass' can be put in infinitesimal form as

$$\mathrm{d}m(\mathbf{r}) = \rho(\mathbf{r})\mathrm{d}V \,, \tag{9.5}$$

where $\mathrm{d}V$ is an infinitesimal volume around the position $\mathbf{r}$. Summation now becomes integration (the summing of tiny bits!), and we can write for the moment of inertia:

$$I = \int r^2\,\mathrm{d}m = \int \rho(\mathbf{r})\, r^2\,\mathrm{d}V \,. \tag{9.6}$$

Note that we take r as the distance to the rotation axis, not the actual distance from $\mathrm{d}m$ to the origin, since $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ means that we need to take only the part of $\mathbf{r}$ that is perpendicular to $\mathbf{L}$. And given that $\mathbf{L}$ lies along the rotation axis, only the component perpendicular to the rotation axis contributes to I.

Example 2: *The moment of inertia for a ring*. Consider a thin ring of radius a and uniform mass distribution, rotating around its symmetry axis. Since the mass is concentrated in a one-dimensional circle, we can write $\rho = M/2\pi a$. The sum over all masses then becomes the integral around the ring over an angle between 0 and 2π, where all radii are equal to a:

$$I_{\text{ring}} = \int_0^{2\pi} \rho a^2\, a\,\mathrm{d}\theta = \int_0^{2\pi} \frac{Ma^2}{2\pi}\,\mathrm{d}\theta = Ma^2 \,. \tag{9.7}$$

Note that the integral around the ring is equivalent to summing over infinitesimal lengths $a\mathrm{d}\theta$.

Example 3: *The moment of inertia for a disk*. We can use the moment of inertia of a ring $I_{\text{ring}} = Ma^2$ to calculate the moment of inertia of a disk rotating around its symmetry axis. Remember that calculating the moment of inertia is like summing lots of small parts. We can think of the disk as a collection of thin rings that we need to sum. Each ring has an area $2\pi r\mathrm{d}r$, and the mass in this ring is the area times the mass density per area, $\sigma = M/\pi a^2$. Summing these rings together then amounts to the integral

$$I_{\text{disk}} = \int_0^a r^2\mathrm{d}m(r) = \int_0^a r^2 \frac{M}{\pi a^2} 2\pi r\,\mathrm{d}r = \frac{2M}{a^2}\int_0^a r^3\,\mathrm{d}r = \frac{1}{2}Ma^2 \,. \tag{9.8}$$

From this you see that the moment of inertia of a ring is larger than the moment of inertia of a disk, if both have the same mass.

Example 4: Consider two masses, a distance d apart, rotating around their centre of mass in circular orbits. This could for example be a planet of mass m orbiting a star of mass M, or a binary star system. We make no assumptions about which mass is bigger. The distances of m and M to the centre of mass are:

$$r_m = \frac{Md}{m+M} \qquad \text{and} \qquad r_M = \frac{md}{m+M} \,. \tag{9.9}$$

We consider only the magnitudes of $\mathbf{r}_m$ and $\mathbf{r}_M$ here. The moment of inertia for this system is

$$I = mr_m^2 + Mr_M^2 = \frac{mM^2d^2}{(m+M)^2} + \frac{Mm^2d^2}{(m+M)^2} = \frac{mMd^2}{m+M} \equiv \mu d^2 \,, \tag{9.10}$$

where we again encountered the reduced mass μ for two masses.

Lecture 10 Applications of angular momentum

Key concept: Applications of Angular Momentum

The inertial resistance to rotation is measured by the moment of inertia.

10.1 Kinetic energy and rolling motion

The kinetic energy T in a rotating rigid body of mass M and moment of inertia I can easily be determined by establishing the kinetic energy of its constituent parts:

$$T_{\text{rot}} = \sum_j \frac{1}{2} m_j v_j^2 = \frac{1}{2} \left(\sum_j m_j r_j^2 \right) \omega^2 \; = \frac{1}{2} I \omega^2 , \tag{10.1}$$

where we again used that ω is the same for all elements in a rigid body. This kinetic energy is purely due to the rotation around an axis. In addition, we can set the rigid body in linear motion, where the centre of mass has velocity $\mathbf{v}$. This leads to a translational kinetic energy

$$T_{\text{trans}} = \frac{1}{2} M v^2 . \tag{10.2}$$

The total kinetic energy is the sum of these:

$$T = T_{\text{trans}} + T_{\text{rot}} = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 . \tag{10.3}$$

This becomes important when we want to analyse rolling motion, for example when a cylinder rolls down a ramp. Previously, we considered frictionless sliding blocks that convert their potential energy entirely into kinetic energy: $mgh = \frac{1}{2} m v^2$, so the final velocity of the block is $v = \sqrt{2gh}$. However, now we can consider the effect of the energy going into the rotational motion of the cylinder. We assume the no-slipping condition, which says that the friction between the ramp and the cylinder is large enough so that the cylinder rolls and does not slide. This gives us a relation between the angular speed of the rotation and the translational speed:

$$v = R\omega , \tag{10.4}$$

where R is the radius of the cylinder. Substituting this into equation (10.3) and equating $T = Mgh$, we obtain

$$Mgh = \frac{1}{2} M v^2 + \frac{1}{2} \frac{I}{R^2} v^2 = \frac{1}{2} \left(M + \frac{I}{R^2} \right) v^2 . \tag{10.5}$$

Solving for v, we find

$$v = \sqrt{\frac{2Mgh}{M + I/R^2}} . \tag{10.6}$$

This is smaller than the sliding block case, and reduces to it when $I \to 0$. Note that this is true for cylinders with any mass distribution, not just the uniformly solid cylinder and the open tube. We can even consider spherical objects or egg-shaped rigid bodies. As long as it can obey the no-slipping condition this formula holds. For example, you can do rolling experiments with raw and hard-boiled eggs. Would you expect a difference in the final centre of mass velocity?

Here is a useful thing to remember. For every quantity pertaining to linear motion, there is a corresponding quantity for rotational motion. This is captured in table 1.

	linear	circular
position	$\mathbf{r}$	$\boldsymbol{\theta}$
velocity	$\mathbf{v} = \dot{\mathbf{r}}$	$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$
acceleration	$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$	$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \ddot{\boldsymbol{\theta}}$
mass	m	$I = mr^2$
momentum	$\mathbf{p}$	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$
force	$\mathbf{F} = \dot{\mathbf{p}}$	$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \dot{\mathbf{L}}$
kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$

Table 1 – Translating linear motion to circular motion.

10.2 Scattering problems

Let's consider the role of angular momentum in the energy of a planet orbiting a star. The total energy for a planet with mass m in the gravitational potential of a star with mass M can be written as

$$E = \frac{1}{2}mv^2 + V(r)\,. \tag{10.7}$$

We will keep $V(r)$ for now to keep the notation simple and clear. We will also use polar coordinates, r and θ, in which v^2 becomes

$$v^2 = v_r^2 + v_\theta^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\,. \tag{10.8}$$

From equation (8.11) we deduce that

$$r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \frac{L^2}{(mr)^2}\,, \tag{10.9}$$

where L is the angular momentum of the planet. The direction of $\mathbf{L}$ is perpendicular to the plane of the orbital motion. Substituting v^2 into the kinetic energy of the planet, we obtain

$$E = \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{L^2}{2mr^2} + V(r)\,. \tag{10.10}$$

It is worth pausing at this important expression. The angular momentum L is conserved for any central potential $V(r)$, so L is just another constant in this expression. We managed to eliminate θ from E altogether, which means that there is a one-to-one relation between r and θ. If we know the coordinate r of the planet, we should be able to work out the coordinate θ. Once we know $r(t)$ we can deduce $\theta(t)$, and from that we can find v_r and v_θ by differentiation. In addition, for a central potential we know that the resulting force will always be in the radial direction (we assume that the origin is chosen at the centre of mass). Hence, the acceleration of the planet will be in the direction of the centre of mass. Therefore, we have reduced the problem of planetary motion to a one-dimensional problem for r. It now makes sense to define a new *effective* potential that includes the angular momentum term:

$$E = \frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}(r) \qquad \text{with} \qquad V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2}\,. \tag{10.11}$$

The angular momentum term provides a *repulsive* potential, since it is a positive quantity, and is sometimes called the *centrifugal* potential energy. It is purely a result of describing the problem in polar coordinates instead of cartesian coordinates. This is very similar to the fictitious centrifugal force, which appeared when we described circular motion in polar coordinates. In both cases this centrifugal component is an expression of the tendency of massive bodies to move in a straight line.

When we look at the case of the gravitational potential, we see that the centrifugal potential drops off faster, i.e., with $1/r^2$, than $V(r)$, which drops off with $1/r$. That means that for large distances $V(r)$ is the dominant potential, but very close to the central mass the centrifugal force will dominate.

Lecture 11 Orbital mechanics

Key concept: The inverse-square-law of gravitation leads to Kepler's laws of orbital motion. To show this, we make use of conservation of energy and angular momentum.

We now arrive at the capstone of this lecture series: Kepler's laws of planetary motion. They can be stated as follows:

1. Planets orbit the Sun in elliptical orbits.
2. The planet sweeps out equal areas in equal time.
3. The orbital period squared is proportional to the semi-major axis cubed.

We will prove these in turn, but first we need to explore the mathematics of ellipses in more detail. An elliptical orbit in cartesian coordinates is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{11.1}$$

where a and $b \leq a$ are the semi-major and semi-minor axes, respectively. It is convenient to define the eccentricity of the ellipse e as a measure of how the shape of the ellipse deviates from a circle:

$$e = \sqrt{1 - \frac{b^2}{a^2}} \qquad \text{with} \qquad 0 \leq e \leq 1\,. \tag{11.2}$$

When $e = 0$ the ellipse is a circle (i.e., the deviation from a circle is zero), and when $e = 1$ the ellipse is a line. However, we are solving the problem of planetary motion in polar coordinates, so we need to express the ellipse in terms of r and θ. Using $x = r\cos\theta$ and $y = r\sin\theta$, we find that

$$r(\theta) = \frac{b^2/a}{1 + e\cos\theta}\,. \tag{11.3}$$

The origin coincides with one of the foci of the ellipse.

Kepler's Second Law

It is easiest to first consider Kepler's second law. From equation (8.11) we know that

$$r^2\dot{\theta} = \frac{L}{m} = \text{constant.} \tag{11.4}$$

However, $r^2\dot{\theta}$ has a clear geometric meaning, as shown in Fig. 15. The shaded area is a triangle when $r\mathrm{d}\theta$ is infinitesimally small, and the area of this triangle is

$$\mathrm{d}A = \frac{1}{2} r\, r\mathrm{d}\theta = \frac{1}{2} r^2 \mathrm{d}\theta\,. \tag{11.5}$$

Therefore,

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2 \dot{\theta} = \text{constant.} \tag{11.6}$$

In other words, the rate of the area A swept out by the line connecting the planet and the origin is constant over time, and this is precisely Kepler's second law. You see that it is an expression of the conservation of angular momentum.

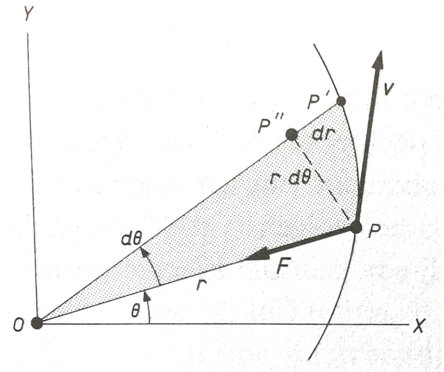


Figure 15 – The area of the shaded triangle is $\simeq \frac{1}{2} r\, r d\theta$.

Kepler's First Law

For Kepler's first law, we want to show that the orbit of a planet is an ellipse, namely

$$r(\theta) = \frac{b^2/a}{1 + e\cos\theta},$$

for some values of a, b, and e that we need to determine. We start with the energy in equation (10.11) and determine the force in the radial direction from the potential V_{eff}:

$$F = m\ddot{r} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r} = \frac{L^2}{mr^3} - \frac{GmM}{r^2}. \tag{11.7}$$

We can divide by m and reintroduce $\dot{\theta} = L/mr^2$ to obtain

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2}. \tag{11.8}$$

This is not a differential equation we can easily solve, but fortunately we can cast it in a much more amenable form. If we make the substitution $\rho = 1/r$, we can write

$$\frac{\mathrm{d}\rho}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\frac{1}{r^2}\dot{r}\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{m\dot{r}}{L}. \tag{11.9}$$

However, equation (11.8) requires the second time derivative $\ddot{r}$. We can get this via a circuitous way, namely from the second derivative of ρ with respect to θ:

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\theta^2} = -\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{m\dot{r}}{L} = -\frac{m}{L}\frac{\mathrm{d}\dot{r}}{\mathrm{d}\theta} = -\frac{m}{L}\ddot{r}\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{m^2r^2}{L^2}\ddot{r}. \tag{11.10}$$

We can solve for $\ddot{r}$ and substitute this into equation (11.8). This leads to

$$-\frac{L^2\rho^2}{m^2}\frac{\mathrm{d}^2\rho}{\mathrm{d}\theta^2} = \frac{L^2\rho^3}{m^2} - GM\rho^2. \tag{11.11}$$

Dividing this equation by ρ^2L^2/m^2, we obtain the differential equation:

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}\theta^2} + \rho - \frac{GMm^2}{L^2} = 0, \tag{11.12}$$

This is mathematically identical to the differential equation for a harmonic oscillator, whose solutions can be written as

$$\rho(\theta) = \frac{GMm^2}{L^2} + B\cos(\theta + \phi_0), \tag{11.13}$$

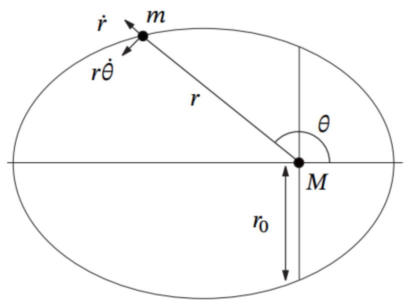


Figure 16 – A planet of mass m orbiting on an elliptical path around a larger mass M. The distance between the two masses changes throughout the orbit. The velocity of the small mass is resolved into two components, one pointing radially and the other perpendicular to this.

where B is the 'amplitude' of the solution, and we can set $\phi_0 = 0$ as our initial condition. Solving for $r(\theta) = 1/\rho(\theta)$, we find

$$r(\theta) = \frac{L^2/Gm^2M}{1 + L^2B/Gm^2M\cos\theta}\,. \tag{11.14}$$

This is an elliptical orbit with

$$e = \frac{L^2B}{Gm^2M} \qquad \text{and} \qquad \frac{b^2}{a} = \frac{L^2}{Gm^2M}\,. \tag{11.15}$$

This proves Kepler's first law. The eccentricity becomes zero for $B = 0$, or where ρ is a constant for all angles θ.

Kepler's Third Law

The area of an ellipse is πab, and the rate of sweeping out area in elliptical orbits is a constant $L/2m$. The time T to complete an orbit is equal to the area divided by the rate of area swept. In other words,

$$T = \frac{\pi ab}{L/2m} = \frac{2m\pi ab}{L}\,. \tag{11.16}$$

We now need to relate T to the semi-major axis a. This is quite straightforward if we extract b from equation (11.15) and use it to eliminate b from equation (11.16).

From equations (11.15) and (11.16) we see that b^2 is equal to

$$b^2 = \frac{aL^2}{Gm^2M} = \frac{T^2L^2}{4m^2\pi^2a^2}\,. \tag{11.17}$$

Hence

$$\frac{aL^2}{Gm^2M} = \frac{T^2L^2}{4m^2\pi^2a^2} \qquad \text{or} \qquad \frac{T^2}{a^3} = \frac{4\pi^2}{GM} = \text{constant}\,. \tag{11.18}$$

This proves Kepler's third law.

The derivation of Kepler's laws was a triumph for Newtonian mechanics. It demonstrated that the inverse-square law of gravity was capable of reproducing Kepler's experimentally well-established laws from simple principles (Newton's three laws), even though the *application* of Newton's laws is not necessarily all that straightforward. The conservation of energy and angular momentum played a central role in understanding where Kepler's laws come from.

Mechanics is a very mathematical subject that requires familiarity with derivatives and integrals, vectors, and non-cartesian coordinate systems. However, mechanics lies at the heart of all classical physics, and these techniques are a prerequisite for understanding a wide range of topics further in your studies.